# Set-membership computation of integrals with uncertain endpoints

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Numerical integration is one of the fundamental tool of scientific computation. Providing a reliable result to such problem is important for validated simulation [1] or for global optimization with continuous objective function [2]. An important work on inclusion methods for integral equations can be found in [3]. We propose an efficient **guaranteed** method for the computation of the **integral** of a nonlinear continuous function $f$ between two **interval endpoints** $[x_1]$ and $[x_2]$, define by:

$$\int_{[x_1]}^{[x_2]} f(x)dx = \left\{ \int_{x_1}^{x_2} f(x)dx \, : x_1 \in [x_1], x_2 \in [x_2] \right\} \tag{1}$$

Two cases can occur whether the intervals $[x_1]$ and $[x_2]$ intersect or not. If the two intervals do not intersect, we produce an outer approximation of the set of possible values for the integral proving the following proposed theorem:

**Theorem 1.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. For $[x_1] = \left[\underline{x_1}, \overline{x_1}\right]$ and $[x_2] = \left[\underline{x_2}, \overline{x_2}\right]$ two intervals in $\mathbb{IR}$. Then the minimum and the maximum of set (1) can be defined by*

$$\int_{[x_1]}^{[x_2]} f(x)dx = \left[ \min \int_{\mathcal{X}_{1-}^* \cup \{\overline{x_1}\}}^{\mathcal{X}_{2+}^* \cup \{\underline{x_2}\}} f(x)dx, \max \int_{\mathcal{X}_{1+}^* \cup \{\overline{x_1}\}}^{\mathcal{X}_{2-}^* \cup \{\underline{x_2}\}} f(x)dx \right] \tag{2}$$

*using*
$\mathcal{X}_{i-}^* = \{x \in [x_i] : f(x) = 0, f'(x) < 0\}$, $\mathcal{X}_{i+}^* = \{x \in [x_i] : f(x) = 0, f'(x) > 0\}$.

When the two intervals intersect, Second Fundamental Theorem of Calculus is applied. An implementation of the computation of the bounds (2) is introduced using interval analysis [4] and tested on several examples.

## References

[1] Alexandre dit Sandretto, J. and Chapoutot, A. (2016) *Validated Explicit and Implicit Runge-Kutta Methods.* Reliable Computing.
[2] Hansen, Eldon R. (2003) *Global Optimization Using Interval Analysis.* CRC Press.
[3] Corliss, G. (1987) *Computing narrow inclusions for definite integrals.* Computer-arithmetic (eds. E. Kaucher et al.), pp. 150—179. Wiley-Teubner, Stuttgart.
[4] Moore, Ramon E. (1966) *Interval analysis.* Prentice-Hall Englewood Cliffs, NJ.