The W,Z/ν,δ Paradigm for the First Passage of Strong Markov Processes without Positive Jumps

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Article

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Abstract: As well-known, the benefit of restricting to Lévy processes without positive jumps is the "W, Z scale functions paradigm", by which the knowledge of the scale functions W, Z extends immediately to other risk control problems (see for example [1–5]). The same is true largely for strong Markov processes X_t, with the notable distinctions that a) it is more convenient to use as "basis" differential exit functions ν, δ introduced in [6], and that b) it is not yet known how to compute ν, δ or W, Z beyond the Lévy, diffusion, and a few other cases. The unifying framework outlined in this paper suggests however via an example that the spectrally negative Markov and Lévy cases are very similar (except for the level of work involved in computing the basic functions ν, δ). We illustrate the potential of the unified framework by introducing a new objective (33) for the optimization of dividends, inspired by the de Finetti problem of maximizing expected discounted cumulative dividends until ruin, where we replace ruin by an optimally chosen Azema-Yor/generalized drawdown/regret/trailing stopping time. This is defined as a hitting time of the "drawdown" process $Y_t = \sup_{0\leq s\leq t} X_s - X_t$ obtained by reflecting X_t at its maximum (see [7] for an application to the Skorokhod embedding problem, and [8–11] for applications to mathematical finance and risk theory). This new variational problem has been solved in the parallel paper [12].

Keywords: first passage; drawdown process; spectrally negative process; scale functions; dividends; de Finetti valuation objective; variational problem

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Contents

0. A brief review of first passage theory for strong Markov processes without positive jumps and their drawdowns

Motivation. First passage times intervene in the control of reserves/risk processes. The rough idea is that when below low levels a, the reserves should be replenished at some cost, and when above high levels b, the reserves should be invested to yield dividends – see for example [13]. There is a wide variety of first passage control problems (involving absorption, reflection and other boundary mechanisms), and it has been known for a long while that these problems are simpler in the "completely asymmetric" case when all jumps go in the same direction. In recent years it became furthermore clear that most first passage problems can be reduced to the two basic problems of going up before down, or viceversa, and that their answers may usually be ergonomically expressed in terms of two basic "scale functions" W, Z [1–3,5,6,9–11,14–21]. The proofs require typically not much more than the strong Markov property; it is natural therefore to develop extensions to strong Markov processes. This has been achieved already in particular spectrally negative cases like random walks [4], Markov additive processes [3], Lévy processes with Ω state dependent killing [3], certain Lévy processes with state dependent drift [22], and is in fact possible in general. However, characterizing the functions W, Z is still an open problem, even for simple classic processes like the Ornstein-Uhlenbeck and the Feller branching diffusion with jumps.

Let X_t denote a one dimensional strong Markov process without positive jumps, defined on a filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Denote its first passage times above and below by

$$T_{b,+} = T_{b,+}(X) = \inf\{t \geq 0 : X_t > b\}, \quad T_{a,-} = T_{a,-}(X) = \inf\{t \geq 0 : X_t < a\},$$

with $\inf \emptyset = +\infty$.

Recall that first passage theory for diffusions and spectrally negative or spectrally positive Lévy processes is considerably simpler than that for processes which may jump both ways. For these two families, a large variety of first passage problems may be reduced to the computation of two monotone "scale functions" W, Z (by simple arguments like the strong Markov property). See [1,3,5,14–21] for the introduction and applications of W, Z in the Lévy case. For diffusions, the most convenient basic functions are the monotone solutions φ_+, φ_- of the Sturm-Liouville equation – see [23]. Finally, for spectrally negative or spectrally positive Lévy processes and diffusions, off-shelf computer programs could easily produce the answer to a large variety of problems, once approximations for the basic functions associated to the process have been produced. This continues to be true in principle for non-homogeneous Markov processes with one-sided jumps (by a simple application of the strong Markov property at the smooth crossing exit from an interval). However, there are very few papers proposing methods to compute W, Z for non-Lévy processes (see though [22], and [24], where the case of Ornstein-Uhlenbeck processes with phase-type jumps is studied).

The two sided exit functions. The most important first passage functions are the solutions of the two-sided upward and downward exit problems from a bounded interval $[a, b]$:

$$\begin{cases} \overline{\Psi}^b_{q,\theta}(x,a) := \mathbb{E}_x \left[e^{-qT_{b,+}-\theta(X_{T_{b,+}}-b)} 1_{\{T_{b,+}<T_{a,-}\}} \right] \\ \Psi^b_{q,\theta}(x,a) := \mathbb{E}_x \left[e^{-qT_{a,-}+\theta(X_{T_{a,-}}-a)} 1_{\{T_{a,-}<T_{b,+}\}} \right] \end{cases} \quad q,\ \theta \geq 0,\ a \leq x \leq b. \tag{1}$$

We will also call them killed **survival** and **ruin** first passage probabilities, respectively. Note that these are functions of five variables, very hard to compute in general. For processes with one sided jumps, one of the exits must be smooth (without overshoot); in this case, the parameter θ is unnecessary and will be omitted. Also, when $a = 0$, it will be omitted, to simplify the notation.

For diffusions and Lévy processes with one sided jumps, the two sided exit functions have well-known explicit formulas.

For **spectrally negative Lévy processes**, the simplest is the smooth survival probability, which factors:

$$\overline{\Psi}_q^b(x,a) \quad = \frac{W_q(x-a)}{W_q(b-a)} = e^{-\int_x^b \nu_q(s-a)ds}. \tag{2}$$

$W_q(x)$ is called the scale function [14,25] [1]. We will assume throughout that W_q is differentiable (see [26] for information on the smoothness of scale functions). Then, $\nu_q(s) = \frac{W_q'(s)}{W_q(s)}$ is the logarithmic derivative of W_q, and may be interpreted as the "survival function of excursions lengths" [25]. The non-smooth ruin probability has a more complicated explicit formula involving a second scale function Z_q [1] – see remark 1 below.

The drawdown/regret/loss/process. Motivated by applications in statistics, mathematical finance and risk theory, there has been increased interest recently in the study of the running maximum and of the drawdown/regret/loss/process reflected at the maximum, defined by

$$Y_t = \overline{X}_t - X_t, \quad \overline{X}_t := \sup_{0 \le t' \le t} X_s.$$

Of equal interest is the infimum, and the drawup/gain/process reflected at the infimum, defined by

$$\underline{Y}_t = X_t - \underline{X}_t, \quad \underline{X}_t = \inf_{0 \le t' \le t} X_s.$$

See [27–29] for references to the numerous applications of drawdowns and drawups.

Drawdown and drawup times are first passage times for the reflected processes:

$$\begin{aligned} \tau_d &:= \inf\{t \ge 0 : \overline{X}_t - X_t > d\}, \\ \underline{\tau}_d &:= \inf\{t \ge 0 : X_t - \underline{X}_t > d\}, \ d > 0. \end{aligned} \tag{3}$$

Such times turn out to be optimal in several stopping problems, in statistics [30] in mathematical finance/risk theory – see for example [1,31–34] – and in queueing. More specifically, they figure in risk theory problems involving capital injections or dividends at a fixed boundary, and idle times until a buffer reaches capacity in queueing theory.

Remark 1. The second scale function Z [1,3,35] useful for solving the spectrally negative non-smooth ruin probability (and many other problems) is best defined via the solution of the **non-smooth total discounted "regulation" problem**.

Let $X_t^{[0} = X_t + L_t$ denote the process X_t modified by Skorohod reflection at 0, with regulator $L_t = -\underline{X}_t$, let $E_x^{[0}$ denote expectation for this process and let

$$T_b^{[0} = T_{b,+} \, \mathbb{1}_{\{T_{b,+} < T_{0,-}\}} + \underline{\tau}_b \, \mathbb{1}_{\{T_{0,-} < T_{b,+}\}} \tag{4}$$

denote the first passage to b of $X_t^{[0}$.

[1] The fact that the survival probability has the multiplicative structure (2) is equivalent to the absence of positive jumps, by the strong Markov property.

a) The Laplace transform of the total regulation ("capital injections/bailouts") into the process reflected non-smoothly at 0, until the first smooth up-crossing of a level b, may be factored as [3, Thm. 2]:

$$\mathbb{E}_x^{[0}\left[e^{-qT_b^{[0}-\theta L_{T_b^{[0}}}\right]=\begin{cases}\frac{Z_{q,\theta}(x)}{Z_{q,\theta}(b)}, & \theta<\infty\\ \mathbb{E}_x\left[e^{-qT_b^{[0}};\,T_{b,+}<T_{0,-}\right]=\frac{W_q(x)}{W_q(b)}, & \theta=\infty\end{cases}, \tag{5}$$

with $Z_{q,\theta}(x)$ determined up to a multiplying constant.

b) Decomposing (5) at $\min(T_b^+,T_{0,-})$ yields a formula (1) for the ruin probability [3]. Indeed:

$$\mathbb{E}_x^{[0}\left[e^{-qT_b^{[0}-\theta L_{T_b^{[0}}}\right]=\frac{Z_{q,\theta}(x)}{Z_{q,\theta}(b)}=\frac{W_q(x)}{W_q(b)}+\mathbb{E}_x\left[e^{-qT_{0,-}+\theta X_{T_{0,-}}};T_{0,-}<T_{b,+}\right]\frac{Z_{q,\theta}(0)}{Z_{q,\theta}(b)}\Longrightarrow \tag{6}$$

$$\Psi_{q,\theta}^b(x)Z_{q,\theta}(0)=\mathbb{E}_x\left[e^{-qT_{0,-}+\theta X_{T_{0,-}}};T_{0,-}<T_{b,+}\right]Z_{q,\theta}(0)=Z_{q,\theta}(x)-W_q(x)W_q(b)^{-1}Z_{q,\theta}(b). \tag{7}$$

To simplify this formula, it is customary to choose $Z_{q,\theta}(0)=1$.

For non-homogeneous spectrally negative Markov processes, it is possible [5] to extend the equalities (2), (7) to analogue expressions involving scale functions of two variables

$$\overline{\Psi}_q^b(x,a)=\frac{W_q(x,a)}{W_q(b,a)},\qquad \Psi_{q,\theta}^b(x,a)=Z_{q,\theta}(x,a)-W_q(x,a)W_q(b,a)^{-1}Z_{q,\theta}(b,a). \tag{8}$$

However, it is simpler to start, following [6], with differential versions, whose existence will be assumed throughout this paper.

Assumption 1. *For all $q,\theta\geq 0$ and $y\leq x$ fixed, assume that $\overline{\Psi}_q^b(x,y)$ and $\Psi_{q,\theta}^b(x,y)$ are differentiable in b at $b=x$, and in particular that the following limits exist:*

$$\nu_q(x,y):=\lim_{\varepsilon\downarrow 0}\frac{1-\overline{\Psi}_q^{x+\varepsilon}(x,y)}{\varepsilon} \tag{9}$$

and

$$\delta_{q,\theta}(x,y):=\lim_{\varepsilon\downarrow 0}\frac{\Psi_{q,\theta}^{x+\varepsilon}(x,y)}{\varepsilon} \tag{10}$$

Remark 2. A necessary condition for Assumption 1 to hold is that X is upward regular and creeping upward at every x in the state space – see [6, Rem. 3.1]. Within this class, it seems difficult to provide examples where Assumption 1 is not satisfied.

It turns out that the differentiability of the two-sided ruin and survival probabilities as functions of the upper limit provides a method for computing other first passage quantities; for example, (12), (23) below may be computed by solving the first order ODE's in Theorem 3. Informally, we may say that the pillar of first passage theory for spectrally negative Markov processes is proving the existence of ν,δ.

In the Lévy case note that by (2) $\nu_q(x,y)=\frac{W_q'(x-y)}{W_q(x-y)}=\nu_q(x-y)$, and $\delta_{q,\theta}(x,y)=\delta_{q,\theta}(x-y)$ where [5]

$$\delta_{q,\theta}(x):=Z_{q,\theta}(x)-W_q(x)\frac{Z_{q,\theta}'(x)}{W_q'(x)}. \tag{11}$$

Remark 3. For diffusions, $W_q(x,a)$ is a certain Wronskian–see for example [23]. Also, for Langevin type processes with decreasing state-dependent drifts, $W_q(x,a)$ solves a certain renewal equation [22]. The case of Ornstein-Uhlenbeck/Segerdahl-Tichy processes with exponential jumps is currently under study in [36]. Some information about the generalization to Ornstein-Uhlenbeck processes with phase-type jumps can be found in [24]. Beyond that, computing $W_q(x,a)$ or $\nu_q(x,a)$ is an open problem. This is an important problem, and we conjecture that the method of [24] may be extended, at least to affine diffusions with phase-type jumps, and possibly to all diffusions with phase-type jumps.

The drawdown exit functions. Recently, control results with drawdown times τ_d replacing classic first passage times started being investigated – see for example [27,28]. Two natural objects of interest for studying τ_d are the two sided exit times

$$T_{b+,d} = \min(\tau_d, T_{b,+}), \qquad T_{a-,d} = \min(\tau_d, T_{a,-}).$$

In terms of the two dimensional process $t \mapsto (X_t, Y_t)$, these are the first exit times from the regions $(-\infty, b] \times [0,d]$ and $[a, \infty) \times [0,d]$.

Fundamental in the study of say $T_{b+,d}$ are the following two Laplace transforms UbD/DbU (up-crossing before drawdown/drawdown before up-crossing), which are analogues of the killed survival and ruin probabilities :

$$\begin{aligned} UbD^b_{q,\theta,d}(x) &= \mathbb{E}_x\left[e^{-qT_{b,+}-\theta(X_{T_{b,+}}-b)}; T_{b,+} < \tau_d\right] = \mathbb{E}_x\left[e^{-qT_{b,+}-\theta(X_{T_{b,+}}-b)}; \overline{X}_{\tau_d} > b\right] \\ DbU^b_{q,\theta,d}(x) &= \mathbb{E}_x\left[e^{-q\tau_d-\theta(Y_{\tau_d}-d)}; \tau_d < T_{b,+}\right] = \mathbb{E}_x\left[e^{-q\tau_d-\theta(Y_{\tau_d}-d)}; \overline{X}_{\tau_d} < b\right]. \end{aligned} \tag{12}$$

For **spectrally negative Lévy processes**, these have again simple formulas:

1. $$UbD^b_{q,d}(x) := \mathbb{E}_x\left[e^{-qT_{b,+}}; T_{b,+} \le \tau_d\right] = e^{-(b-x)\frac{W_q'(d)}{W_q(d)}}, \tag{13}$$

2. The function DbU may be obtained by integrating the fundamental law [27, Thm 1], [28, Thm 3.1] [2]

$$\begin{aligned} \delta_{q,\theta}(d,x,s) &:= \mathbb{E}_x\left[e^{-q\tau_d-\theta(Y_{\tau_d}-d)}; \overline{X}_{\tau_d} \in \mathrm{d}s\right] = \left(\nu_q(d)\, e^{-\nu_q(d)(s-x)_+}\,\mathrm{d}s\right)\delta_{q,\theta}(d) \\ \Leftrightarrow \mathbb{E}_x\left[e^{-q\tau_d-\theta(Y_{\tau_d}-d)-\vartheta(\overline{X}_{\tau_d}-x)}\right] &= \frac{\nu_q(d)}{\vartheta+\nu_q(d)}\delta_{q,\theta}(d) \end{aligned} \tag{14}$$

where $\delta_{q,\theta}(d)$ is given by (11). Integrating yields

$$DbU^b_{q,\theta,d}(x) = \left(1 - e^{-(b-x)\frac{W_q'(d)}{W_q(d)}}\right)\delta_{q,\theta}(d). \tag{15}$$

Remark 4. The probabilistic interpretation of ν_q, the logarithmic derivative of W_q. Taking $a=0$ for simplicity, the last formula in (2) has the interesting interpretation as the probability that no arrival

[2] Note that [27, Thm 1] give a more complicated "sextuple law" with two cases, and that [28, Thm 3.1] use an alternative to the function $Z_q(x,\theta)$, so that some computing is required to get (14), (11).

has occurred between times x and b, for a nonhomogeneous Poisson process of rate $\nu_q(s), s \in [x, b]$. Alternatively, differentiating (2) yields

$$\frac{d}{ds}\overline{\Psi}_q^b(s) - \nu_q(s)\overline{\Psi}_q^b(s) = 0, \qquad \overline{\Psi}_q^b(b) = 1. \tag{16}$$

This equation coincides the Kolmogorov equation for the probability that a deterministic process $\widetilde{Y}_s = s$, killed at rate $\nu_q(s)$, reaches b before killing, when starting at s. It turns out, by excursion theory, that such a process $\widetilde{Y}_s$ may be constructed by excising the negative excursions from X_t, and by taking the running maximum s as time parameter.

The logarithmic derivative $\nu_q(s)$ will be needed below in the de Finetti problem (17), where we will use the fact that the expected dividends $v_q(b)$ paid at a fixed barrier b, starting from b, equal the expected discounted time until killing, which is exponential with parameter $\nu_q(b)$, being therefore simply the reciprocal of the killing parameter $\nu_q(b)$:

$$v_q(b) := \mathbb{E}_b\left[\int_0^{T_{0,-}^{b]}} e^{-qt} d(\overline{X}_t - b)\right] = \nu_q(b)^{-1}. \tag{17}$$

We see in the equation above and others that ν_q may serve as a convenient alternative characteristic of a spectrally negative Markov process, replacing W_q. Just as W_q, it may be extended to the case of generalized drawdown killing introduced in [9,10].

Contents. We start in Section 1 by presenting a pedagogic first passage example illustrating the W, Z paradigm: the first time

$$T_R = T_{a,b,d} = T_{a,-} \wedge T_{b,+} \wedge \tau_d. \tag{18}$$

when (X, Y) with X Lévy leaves a rectangular region $R = [a, b] \times [0, d]$.

Remark 5. Note that letting $a \to -\infty, b \to \infty$ reduces $T_{a,b,d}$ to τ_d, and letting $d \to \infty, b \to \infty$ reduces $T_{a,b,d}$ to $T_{a,-}$. Hence both classic first passage and drawdown times appear as special cases of $T_{a,b,d}$. For finite a, b, d, our region has two classic and one drawdown exit boundary.[3]

In Section 2 we provide geometric considerations which reduce computations of the Laplace transforms of the "three-sided" exit times of (X, Y) to that of Laplace transforms of two-sided exit problems involving $T_{a,-}$, $T_{b,+}$ and τ_d (like (1), (12)) – see Figure 1.

Only the strong Markov property is used; however, for the sake of simple notations we restricted the exposition to the family of Lévy processes (which have also the convenient feature that the scale functions W, Z may be computed by inverting Laplace transforms [1–3,17,25]).

In Section 3 we enlarge the framework to that of generalized drawdown times [9,10]. This immediately entails that ν, δ become functions of two variables defined in (9), (10), and the extension to the spectrally negative Markov case becomes natural. We turn therefore to exits from certain trapezoidal-type regions in Section 4, under the spectrally negative Markov model.

In Section 5 we consider processes reflected at an upper barrier and formulate a Finetti's optimal dividends type objective with combined ruin and generalized drawdown stopping; this involves adding one reflecting vertex to our trapezoidal region. Included here is a new variational problem for de Finetti's dividends with generalized drawdown stopping (33); since the solution is not immediate even in the Lévy case, this has been provided in the parallel paper [12].

[3] Choosing a, b, d optimally in various control problems involving optimal dividends and capital injections should be of interest, and will be pursued in further work.

1. Geometric considerations concerning the joint evolution of a Lévy process and its drawdown in a rectangle

In order to study the process (X_t, Y_t), it is useful to start with its evolution in a rectangular region $R := [a, b] \times [0, d] \subset \mathbb{R} \times \mathbb{R}_+$, where $a < b$ and $d > 0$. Define

$$T_R = T_{a,b,d} := \inf\{t : (X_t, Y_t) \notin R\} = \tau_d \wedge T_{a,-} \wedge T_{b,+}.$$

A sample path of (X, Y), where X is chosen to be a spectrally negative Lévy process, and the region R is depicted in Figure 1.

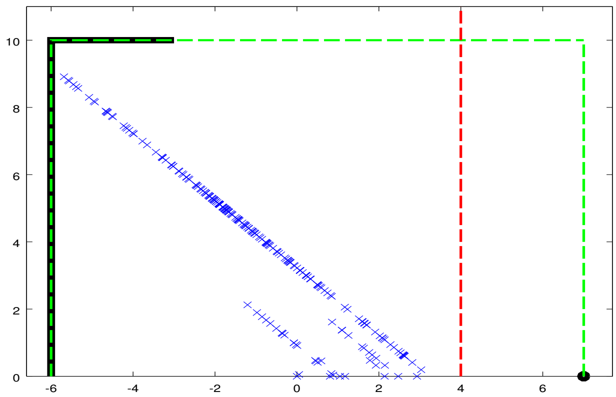

Figure 1. A sample path of (X, Y) with X a spectrally negative Lévy process. The region R has $d = 10$, $a = -6$ and $b = 7$; the dark boundary shows the possible exit points of (X, Y) from R. The base of the red line separates R in two parts with different behavior

As is clear from the figure and from its definition, the process (X, Y) has very particular dynamics on R: away from the boundary $\partial_1 := \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}_+ : x_2 = 0\}$ it oscillates during negative excursions from the maximum on line segments $l_{\overline{X}_t}$ where, for $c \in \mathbb{R}$, $l_c := \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}_+ : x_1 + x_2 = c\}$.

As $\overline{X}_t$ increases, the line segment $l_{\overline{X}_t}$ on which (X, Y) oscillates advances to the right – continuously, in the spectrally negative case, and in general possibly with jumps.

On ∂_1, we observe the Markovian upward ladder process, i.e. the maximum $\overline{X}$ with downward excursions excised, with extra spatial killing upon exiting R. If only time killing was present, with $d = \infty$, this would be a killed drift subordinator, with Laplace exponent $\kappa(s) = s + \Phi_q$ (as a consequence of the Wiener-Hopf decomposition [2]). In the rectangle, in the spectrally negative case, the ladder process becomes a killed drift with generator $\mathcal{G}\varphi(s) := \varphi'(s) - \nu_q(d)\varphi(s)$ [9,37]. Finally, with generalized drawdown (when the upper boundary is replace by one determined by certain parametrizations $(\widehat{d}(s), d(s))$ – see below), the generator will have state dependent killing:

$$\mathcal{G}\varphi(s) := \varphi'(s) - \nu_q(d(s))\varphi(s). \tag{19}$$

Several functionals (ruin, dividends, tax, etc.) of the original process may be expressed as functionals of the killed ladder process. This explains the prevalence of first order ODE's – see (25) for one example – when working with spectrally negative processes. Several implications for T_R are immediately clear from these dynamics: for example, the process (X, Y) can leave R only through $\partial R \cap \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}_+ : x_1 \leq b - d\}$ or through the point $(b, 0)$ (see the shaded region in Figure 1). Also,

1. If $b \le a + d$, it is impossible for the process to leave R through the upper boundary of ∂R and for these parameter values T_R reduces to $T_{a,-} \wedge T_{b,+}$. Here it suffices to know the functions (1) in order to obtain the Laplace transform of T_R.
2. If $a + d \le x$, it is impossible for the process to leave R through the left boundary of ∂R, and T_R reduces to $T_{b,+} \wedge \tau_d$. Here it suffices to apply the spectrally negative drawdown formulas provided in [27,28].
3. In the remaining case $x \le a + d \le b$, both drawdown and classic exits are possible. For the latter case, see Figure 1. The key observation here is that drawdown [classic] exits occur iff X_t does [does not] cross the line $x_1 = d + a$. The final answers will combine these two cases.

2. The three Laplace transforms of the exit time out of a rectangle for Lévy processes without positive jumps

In this section we provide Laplace transforms of T_R and of the eventual overshoot at T_R. One can break down the analysis of T_R to nine cases, depending on which of the three exit boundaries $T_{a,-}$, $T_{b,+}$ or τ_d occurred, and on the three relations between x, a, b and d described above.

The following results are immediate applications of the strong Markov property and of known first passage and drawdown results.

Theorem 1. *Consider a spectrally negative Lévy process X with differentiable scale function W_q. Then, for fixed $d \ge 0$ and $a \le x \le b$, letting UbD, DbU denote the functions defined in (13), (15), we have:*

	$a+d \le x \le b$	$x \le a+d \le b$	$b \le a+d$
$\mathbb{E}_x\left[e^{-qT_{b,+}}; T_{b,+} \le \min(\tau_d, T_{a,-})\right] =$	$UbD^b_{q,d}(x)$	$\overline{\Psi}^{(a+d)}_q(x,a)UbD^b_{q,d}(a+d)$	$\overline{\Psi}^b_q(x,a)$
$\mathbb{E}_x\left[e^{-qT_{a,-}+\theta(X_{T_{a,-}}-a)}; T_{a,-} \le \min(\tau_d, T_{b,+})\right] =$	0	$\Psi^{(a+d)}_{q,\theta}(x,a)$	$\Psi^b_{q,\theta}(x,a)$
$\mathbb{E}_x\left[e^{-q\tau_d-\theta(Y_{\tau_d}-d)}; \tau_d \le \min(T_{b,+}, T_{a,-})\right] =$	$DbU^b_{q,\theta,d}(x)$	$\overline{\Psi}^{(a+d)}_q(x,a)DbU^b_{q,\theta,d}(a+d)$	0

(20)

Proof: Note that in the third column the d boundary is invisible and does not appear in the results, and in the first column the a boundary is invisible and does not appear in the results. These two cases follow therefore by applying already known results.

The middle column holds by breaking the path at the first crossing of $a + d$. The main points here are that

1. the middle case may happen only if X_t visits a before $a + d$;
2. the first case (exit through b) and the third case (drawdown exit) may happen only if X_t visits first $a + d$, with the drawdown barrier being invisible, and that subsequently the lower first passage barrier a becomes invisible.

The results follow then due to the smooth crossing upward and the strong Markov property.

Proof: Let us check the first and third row of the second column. Applying the strong Markov property at $T_{a+d,+}$ yields

$$\mathbb{E}_x\left[e^{-qT_{b,+}}; T_{b,+} \le \min(\tau_d, T_{a,-})\right] = \mathbb{E}_x\left[e^{-qT_{b,+}}; T_{a+d,+} \le T_{a,-}\right] \mathbb{E}_{a+d}\left[e^{-qT_{b,+}}; T_{b,+} \le \tau_d\right]$$
$$= \frac{W_q(x-a)}{W_q(d)} e^{-(b-a-d)\frac{W_q'(d)}{W_q(d)}}$$

and

$$\mathbb{E}_x\left[e^{-q\tau_d-\theta(Y_{\tau_d}-d)};\tau_d\le\min(T_{b,+},T_{a,-})\right]=\mathbb{E}_x\left[e^{-q\tau_d-\theta(Y_{\tau_d}-d)};T_{a+d,+}\le T_{a,-}\right]\mathbb{E}_{a+d}\left[e^{-q\tau_d-\theta(Y_{\tau_d}-d)};\tau_d\le T_{b,+}\right]$$
$$=\frac{W_q(x-a)}{W_q(d)}\delta_{q,\theta}(d)\left(1-e^{-(b-a-d))\frac{W_q'(d)}{W_q(d)}}\right).$$

3. Generalized drawdown stopping for processes without positive jumps

Generalized drawdown times appear naturally in the Azema Yor solution of the Skorokhod embedding problem [7], and in the Dubbins-Shepp-Shiryaev, and Peskir-Hobson-Egami optimal stopping problems [38–41]. Importantly, they allow a unified treatment of classic first passage and drawdown times (see also [11] for a further generalization to taxed processes)–see [9,10]. The idea is to replace the upper side of the rectangle R by a parametrized curve

$$(x_1,x_2)=(\widehat{d}(s),d(s)),\quad \widehat{d}(s)=s-d(s),$$

where $s=x_1+x_2$ represents the value of $\overline{X}_t$ during the excursion which intersects the upper boundary at (x_1,x_2) (see Figure 2). Alternatively, parametrizing by x yields

$$y=h(x),\quad h(x)=\widehat{d}^{-1}(x)-x.$$

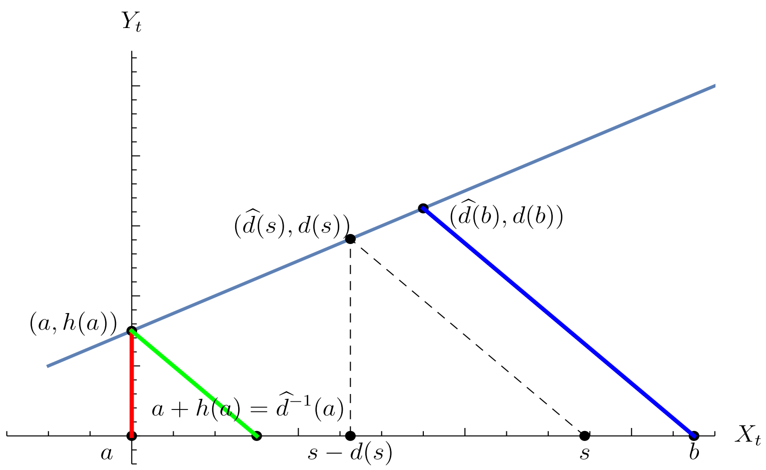

Figure 2. Affine drawdown exit of (X,Y) $d(s)=\frac{1}{3}s+1$

Definition 2. [10] For any function $d(s)>0$ such that $\widehat{d}(s)=s-d(s)$ is nondecreasing, a **generalized drawdown** time is defined by

$$\tau_{\widehat{d}(\cdot)}:=\inf\{t\ge 0:Y_t>d(\overline{X}_t)\}=\inf\left\{t\ge 0:X_t<\widehat{d}(\overline{X}_t)\right\}. \tag{21}$$

Such times provide a natural unification of classic and drawdown times.

Introduce

$$\widetilde{Y}_t:=Y_t-d(\overline{X}_t),\ t\ge 0$$

to be called drawdown type process. Note that we have $\widetilde{Y}_0=-\widehat{d}(X_0)<0$, and that the process $\widetilde{Y}_t$ is in general non-Markovian. However, it is Markovian during each negative excursion of X_t, along one of the oblique lines in the geometric decomposition sketched in Figure 1.

Example 1. With affine functions

$$d(s) = (1-\xi)s + d \Leftrightarrow \widehat{d}(s) = \xi s - d, \quad \xi \in [0,1], d > 0, \tag{22}$$

we obtain the affine drawdown/regret times studied in [9].

Affine drawdown times reduce to a classic drawdown time (3) when $\xi = 1$, $d(s) = d$, and to a ruin time when $\xi = 0$, $\widehat{d}(s) = -d$, $d(s) = s + d$. When ξ varies, we are dealing with the pencil of lines passing through $(x_1, x_2) = (-d, d)$. In particular, for $\xi = 1$ we obtain the rectangle case from section 2, and for $\xi = 0$ we have an infinite strip with a vertical boundary at $x_1 = -d$.

One of the merits of affine drawdown times is that they allow unifying the classic first passage theory with the drawdown theory [9]; in particular, the generalized drawdown functions (23) below unify the classic and drawdown survival and ruin probabilities (and have relatively simple formulas as well – see [5]).

Introduce now generalized drawdown analogues of the drawdown survival and ruin probabilities (12) for which we will use the same notation:

$$\begin{aligned} UbD^b_{q,\widehat{d}(\cdot)}(x) &= \mathbb{E}_x\left[e^{-qT_{b,+}}; T_{b,+} \le \tau_{\widehat{d}(\cdot)}\right] \\ DbU^b_{q,\theta,\widehat{d}(\cdot)}(x)) &= \mathbb{E}_x\left[e^{-q\tau_{\widehat{d}(\cdot)}-\theta \widetilde{Y}_{\tau_{\widehat{d}(\cdot)}}}; \tau_{\widehat{d}(\cdot)} < T^+_b\right]. \end{aligned} \tag{23}$$

Remark 6. In the spectrally negative case, these functions may be represented as integrals:

$$\begin{aligned} UbD^b_{q,\widehat{d}(\cdot)}(x) &= e^{-\int_x^b \nu_q(s,\widehat{d}(s))ds}, \\ DbU^b_{q,\theta,\widehat{d}(\cdot)}(x) &= \int_x^b e^{-\int_x^y \nu_q(s,\widehat{d}(s))\mathrm{d}s}\nu_q(y,\widehat{d}(y))\delta_{q,\theta}(y,\widehat{d}(y))dy, \end{aligned} \tag{24}$$

where $\nu_q(y,\widehat{d}(y))$, $\delta_{q,\theta}(y,\widehat{d}(y))$ are defined in (9), (10).

This is already apparent in [6, Cor 3.1], and may be easily understood probabilistically from figure 2: the first equation is the probability of no occurrence in a nonhomogeneous Poisson process, and the second decomposes the transform of the deficit, by conditioning on the point $y \in [x,b]$ where the maximum occurred.

We provide now a heuristic proof valid for the Lévy case when $\nu_q(y,\widehat{d}(y)) = \nu_q(y-\widehat{d}(y)) = \nu_q(d(y))$ and $\delta_{q,\theta}(y,\widehat{d}(y)) = \delta_{q,\theta}(y-\widehat{d}(y)) = \delta_{q,\theta}(d(y))$.

1. Due to creeping, *UbD* is a product of infinitesimal events

   $$\overline{\Psi}^{y+\epsilon}_q(y, y-d(y)) = \frac{W_q(d(y))}{W_q(d(y)+\epsilon)} \sim 1 - \epsilon\nu_q(d(y)) \sim e^{-\epsilon\nu_q(d(y))}.$$

   Taking product, with $\epsilon = \mathrm{d}y$, yields (24).
2. Informally, we condition on the density $\overline{X}_t \in \mathrm{d}y$. The integrand of *DbU* is obtained multiplying survival infinitesimal events up to level y by an infinitesimal termination event in $[y, y+\mathrm{d}y]$. The probability of this event, conditioned on survival up to y, is given by the deficit formula

   $$\begin{aligned} \Psi^{y+\epsilon}_{q,\theta}(y, y-d(y)) &= Z_{q,\theta}(d(y)) - W_q(d(y))\frac{Z_{q,\theta}(d(y)+\epsilon)}{W_q(d(y)+\epsilon)} \\ &\sim \epsilon(-Z'_{q,\theta}(d(y)) + \nu_q(d(y))Z_{q,\theta}(d(y)) = \epsilon\nu_q(d(y))\delta_{q,\theta}(d(y)) \end{aligned}$$

   For a rigorous (rather intricate) proof, see [11].

The end result for generalized drawdown times is [11, Thm1]:

Theorem 3. *Consider a process X for which the functions $\Psi, \overline{\Psi}$ are differentiable in the upper variable b. Assume $d(x) > 0$ and $\widehat{d}(x) = x - d(x)$ nondecreasing. Then, $\forall q, \theta \geq 0, b \in \mathbb{R}$, the functions $UbD(x) = UbD_q^b(x, \widehat{d}(\cdot))$, $DbU(x) = DbU_{q,\theta}^b(x, \widehat{d}(\cdot))$ satisfy (24). Alternatively, they satisfy the ODE's*

$$UbD'(y) - \nu_q(y, \widehat{d}(y))UbD(y) = 0, \quad UbD(b) = 1, \tag{25}$$

$$DbU'(y) - \nu_q(y, \widehat{d}(y))DbU(y) + \delta_{q,\theta}(y, \widehat{d}(y)) = 0, \quad DbU(b) = 0. \tag{26}$$

Remark 7. The operator involved in the ODE's above is the generator of the upward ladder process, under time and spatial killing, and with the downward excursions excised. Once this known, variations involving different boundary conditions are easily obtained as well.

4. The three Laplace transforms of the exit time out of a curved trapezoid, for processes without positive jumps

We will replace now the classic drawdown time in section 2 by a generalized one. Similar geometric considerations, with $d(\cdot), a + h(a)$ replacing $d, a + d$ in Theorem 1, yield:

Theorem 4. *Consider a spectrally negative Lévy process X with differentiable scale function W_q. Then, for $a \leq x \leq b$ and $d(\cdot)$ satisfying the conditions of Definition 2, we have:*

	$a + h(a) \leq x$	$x \leq a + h(a) \leq b$	$b \leq a + h(a)$
$\mathbb{E}_x\left[e^{-qT_{b,+}}; T_{b,+} \leq \min(\tau_{\widehat{d}(\cdot)}, T_{a,-})\right] =$	$UbD^b_{q,\widehat{d}(\cdot)}(x)$	$\overline{\Psi}_q^{a+h(a)}(x,a)UbD^b_{q,\widehat{d}(\cdot)}(a+h(a))$	$\overline{\Psi}_q^b(x,a)$
$\mathbb{E}_x\left[e^{-qT_{a,-}+\theta(X_{T_{a,-}}-a)}; T_{a,-} \leq \min(\tau_{\widehat{d}(\cdot)}, T_{b,+})\right] =$	0	$\Psi_{q,\theta}^{a+h(a)}(x,a)$	$\Psi_{q,\theta}^b(x,a)$
$\mathbb{E}_x\left[e^{-q\tau_{\widehat{d}(\cdot)}-\theta(Y_{\tau_{\widehat{d}(\cdot)}}-d)}; \tau_{\widehat{d}(\cdot)} \leq \min(T_{b,+}, T_{a,-})\right] =$	$DbU^b_{q,\theta,\widehat{d}(\cdot)}(x)$	$\overline{\Psi}_q^{a+h(a)}(x,a)DbU^b_{q,\theta,\widehat{d}(\cdot)}(a+h(a))$	0

Proof: Note that if $b \leq a + h(a)$ (narrow band), it is again impossible for the process to leave R through the upper boundary of ∂R, and T_R reduces to $T_{a,-} \wedge T_{b,+}$, and nothing changes. Similarly, if $a + h(a) \leq x$ (flat band), it is impossible for the process to leave R through the left boundary of ∂R, and T_R reduces to $T_{b,+} \wedge \tau_d$. Finally, the two zones in the intermediate case are separated by $a + h(a)$ (instead of $a + d$). □

5. De Finetti's optimal dividends for spectrally negative Markov processes with generalized drawdown stopping

In this section we revisit the de Finetti's optimal dividend problem for spectrally negative Markov processes with the point b becoming a reflecting boundary, instead of absorbing, as it was in section 2.

Define the Skorokhod reflected/constrained process at first passage times below or above by:

$$X_t^{[a} = X_t + L_t, \quad X_t^{b]} = X_t - U_t. \tag{27}$$

Here

$$L_t = L_t^{[a} = -(\underline{X}_t - a)_-, \quad U_t = U_t^{b]} = \left(\overline{X}_t - b\right)_+ \tag{28}$$

are the minimal "Skorohod regulators" constraining X_t to be bigger than a, and smaller than b, respectively.

Let now

$$V^{b]}(x) = V^{b]}_{q,\widehat{d}(\cdot)}(x) := \mathbb{E}_x \left[\int_0^{\tau_{\widehat{d}(\cdot)} \wedge T_{a,-}} e^{-qt} dU^{b]}_t \right] \tag{29}$$

denote the present value of all dividend payments at b, until the the first passage time either below a, or below the drawdown boundary for the process $X^{b]}_t$ reflected at b, starting from $x \le b$ (a generalization of the famous de Finetti objective). By the strong Markov property, it holds that

$$V^{b]}(x) = \mathbb{E}_x \left[e^{-qT_{b,+}}; T_{b,+} \le \min(\tau_{\widehat{d}(\cdot)}, T_{a,-}) \right] v(b), \quad v(b) = v_q(b, \widehat{d}(b)) := \mathbb{E}_b \left[\int_0^{\tau_{\widehat{d}(\cdot)}} e^{-qt} dU^{b]}_t \right]. \tag{30}$$

Remark 8. The function $v(b)$, the expected discounted time until killing for the reflected process, when starting from b, equals the time the process reflected at b spends at point $(b, 0)$ in Figure 2, before a downward excursion beyond $\widehat{d}(b)$ kills the process. In the Lévy case, it is well-known [2] that this time is exponential with parameter $\nu_q(b, \widehat{d}(b))$, and thus its expectation is the reciprocal of the killing parameter $\nu_q(b, \widehat{d}(b))$, i.e.

$$v(b) = \nu_q(b, \widehat{d}(b))^{-1} \tag{31}$$

Excursion theoretic arguments show that (31) continues to hold in the spectrally negative Markov case (for a proof under a similar setup, see [42, Sec 4]).

Furthermore, by [11, Thm1] included above as (24), it holds that

$$\mathbb{E}_x \left[e^{-qT_{b,+}} 1_{\{T_{b,+} < \tau_{d(\cdot)}\}} \right] = e^{-\int_x^b \nu_q(z, \widehat{d}(z)) \mathrm{d}z}. \tag{32}$$

When $a = -\infty$, we arrive finally to an explicit formula

$$V^{b]}(x) = \frac{e^{-\int_x^b \nu_q(y, \widehat{d}(y)) \mathrm{d}s}}{\nu_q(b, \widehat{d}(b))} \tag{33}$$

expressing the expected dividends in terms of $\nu_q(y, \widehat{d}(y))$. Note that in the Lévy case the equation (33) simplifies to:

$$V^{b]}(x) = \frac{W_q(d(x))}{W_q(d(b))} \nu_q(d(b))^{-1}$$

(using $x - l(x) = d(x)$), which checks with [43, Lem. 3.1-3.2].

The problem of choosing a drawdown boundary to optimize dividends in (33) is solved in [12] via Pontryagin's maximum principle.

6. Example: Affine drawdown stopping for Brownian motion

Consider optimizing expected dividends $V^{b]}(x)$ given in Equation (29) with respect to the optimal dividend barrier b for Brownian motion with drift $X(t) = \sigma B_t + \mu t$ and with affine drawdown stopping $d(x) = (1 - \xi)x + d$, where $\xi \in [0, 1]$, $d \ge 0$, $a \le x \le b$.

Note that if $a + h(a) > b$, where $h(x) = d(x)/\xi$, then the drawdown constraint is invisible and the problem reduces to the classical de Finetti objective. Hence, we consider $a + h(a) \le b$.

The scale function of Brownian motion is

$$W_q(x) = \frac{2e^{-\mu x/\sigma^2}}{\Delta} \sinh(x\Delta/\sigma^2) = \frac{1}{\Delta}[e^{(-\mu+\Delta)x/\sigma^2} - e^{-(\mu+\Delta)x/\sigma^2}],$$

where $\Delta = \sqrt{\mu^2 + 2q\sigma^2}$. Assume that $x \geq a + h(a) = a + \frac{d(a)}{\xi} = \frac{a+d}{\xi}$, then as a special case of spectrally negative Levy process, the expected dividends for Brownian motion equals

$$V^{b]}(x) = \mathbb{E}_x\left[e^{-qT_{b,+}}; T_{b,+} \leq \min(\tau_{\widehat{d}(\cdot)}, T_{a,-})\right] v(b) = \left(\frac{W_q(d(x))}{W_q(d(b))}\right)^{\frac{1}{1-\xi}} \frac{W_q(d(b))}{W_q'(d(b))}, \tag{34}$$

see [9, Thm. 1.1], with tax parameter $\gamma = 0$, and [9, Rem. 7], with tax parameter $\gamma = 1$. The barrier influence function which must be optimized in b becomes

$$BI(b,d,\xi) = \frac{W_q((1-\xi)x+d)^{1-\frac{1}{1-\xi}}}{W_q'((1-\xi)x+d)} = \frac{\sigma^2}{2}\frac{e^{x\mu/\sigma^2}\operatorname{csch}\left(x\sqrt{\mu^2+2q\sigma^2}/\sigma^2\right)}{\coth\left((d+x-x\xi)\sqrt{\mu^2+2q\sigma^2}/\sigma^2\right) - \mu/\sqrt{\mu^2+2q\sigma^2}}. \tag{35}$$

The critical point b^* satisfies

$$\frac{W_q''W_q}{(W_q')^2}((1-\xi)b^*+d) = -\frac{\xi}{1-\xi}, \tag{36}$$

that is b^* satisfies

$$-\frac{q\sigma^2+\mu^2+\mu\sqrt{2q\sigma^2+\mu^2}\sinh\left(\frac{2b^*\sqrt{2q\sigma^2+\mu^2}}{\sigma^2}\right) - (q\sigma^2+\mu^2)\cosh\left(\frac{2b^*\sqrt{2q\sigma^2+\mu^2}}{\sigma^2}\right)}{\left(\sqrt{2q\sigma^2+\mu^2}\cosh\left(\frac{b^*\sqrt{2q\sigma^2+\mu^2}}{\sigma^2}\right) - \mu\sinh\left(\frac{b^*\sqrt{2q\sigma^2+\mu^2}}{\sigma^2}\right)\right)^2} = -\frac{\xi}{1-\xi}.$$

In Figure 3 given below, we have an illustration of plot of barrier influence function and its derivative for Brownian motion with drift $\mu = 1/2$ and $\sigma = 1$.

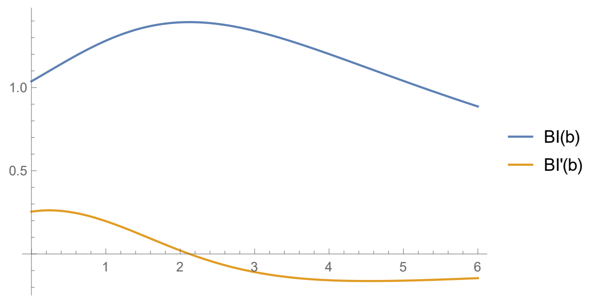

Figure 3. Optimizing dividends with affine drawdown stopping where $\mu = 1/2$, $q = 1/10$, $\sigma = 1$, $\xi = 1/3$, $b = 20$, $d = 1$. The critical point $b^* = 2.12445$.

Remark 9. Note that once ξ is fixed, we get nontrivial results for the optimal barrier. However, if we maximize over ξ as well, the optimum is achieved by the classical de Finetti solution $\xi = 0 \Longrightarrow W_q''(b^*+d) = 0$, corresponding to forced stopping below $-d$ (d is just a shift of the origin, with respect to the classical solution $W_q''(b^*) = 0$) [12]. In the diffusion case, it is not yet known whether examples in which the generalised De Finetti problem improves on the classic De Finetti solution are possible.

Remark 10. Let us note now that the equation (36) holds in fact for any spectrally negative Lévy process. Similar computations may be therefore performed for any spectrally negative Levy process, by plugging exact or approximate formulas for the scale function into the function

$$\frac{W_q'' W_q}{(W_q')^2} \tag{37}$$

which is required to solve (36).

The easiest case is the Cramér-Lundberg process with phase-type claims, since in this case the scale function is a sum of exponentials. For example, for a Cramér-Lundberg process with premium rate $c > 0$, Poisson arrivals of intensity λ and exponential claims with mean $1/\mu$, the scale function is $W_q(x) = c^{-1}(\frac{\mu+\Delta_+}{\Delta_+-\Delta_-}e^{\Delta_+ x} - \frac{\mu+\Delta_-}{\Delta_+-\Delta_-}e^{\Delta_- x}), \quad x \geq 0$, where $\Delta_\pm = \frac{q+\lambda-\mu c \pm \sqrt{(q+\lambda-\mu c)^2+4cq\mu}}{2c}$, and similar computations may be performed (see also [43, Example 5.2]).

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