Using computer algebraic systems to teach mathematics: A didactic perspective

Michèle Artigue

GEOMETRY IN
MOUVEMENT

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USING COMPUTER ALGEBRAIC SYSTEMS TO TEACH MATHEMATICS : A DIDACTIC PERSPECTIVE

Michèle ARTIGUE, Equipe DIDIREM
Université Paris VII et IUFM de Reims

Summary : In France, for several years, a number of groups connected with IREM have been considering the potential for teaching secondary mathematics offered by computer algebraic systems (C.A.S.), as well as the problems associated with integrating computer based work into the mathematics curriculum. This research is supported by the National Ministry of Education's initiative DITEN[1], which has itself set up a working group. This is by no means an easy task: there is a persistent gap between the discourse and enthusiastic writings of innovators in the field of information technology and the current weak integration of the use of computers into mathematics teaching.

What exactly are the real potentialities for the use of C.A.S. to support mathematics teaching, and for what level? What conditions favour its use? Where do obstacles lie, and are they located in the hardware or the software, at the cognitive level, in didactic practice, or at the institutional level? With regard to all these questions, our understanding is as yet very limited. In order to make progress we need to begin by collating the work that has already been done on using C.A.S., and to consider innovations that have been introduced.

This is the work we have been doing since 1993 in the DIDIREM[2] group, working in close cooperation with the DITEN group referred to above. In this article I give a brief survey of our work. This is followed by a summary of what the literature claims are the advantages offered by using C.A.S. to teach mathematics. I then present what we have learned from our observations of the teaching sessions we have run, and this leads me to identify a number of points which seem to require particular vigilance on the part of teachers and teacher trainers.

I - THE RESEARCH

The aim of the research was to contribute to the study of the impact of using C.A.S., and more particularly on the use of DERIVE, on the work done by students in the secondary school. The research used two methodologies:

- an external methodology by which, through the use of questionnaires (one for the teacher and one for the students), we attempted to determine the extent to which DERIVE is at present in use in our collèges and lycées, and the effects of this use on the work done by the students in mathematics.

- an internal methodology based on our observation and analysis of a series of lessons in which DERIVE was used.

These two methodologies are complementary: the first provides us with an overview of the extent to which DERIVE is currently being used in secondary education. But we are well aware that, through using questionnaires, we shall only obtain *a posteriori* rationalisations from users of this software, and that a great deal of bias may be associated with this type of methodology. The second methodology provides a much more accurate approach but it has the disadvantage of remaining extremely local. By combining these two methodologies, external and internal, we hope to be able to compensate, at least partially, for the limitations inherent in each of them.

The questionnaires were sent out to about fifty mathematics teachers throughout France who were willing to cooperate with the project, as well as to their students. The class observations were carried out in the mathematics classes of the DITEN working group referred to above. The teachers in this group suggested sequences of lessons which, in their view, were able to demonstrate the value of using DERIVE in teaching mathematics. As a rule we did not interfere with the planning of these lessons, since the teachers were the 'experts' in using DERIVE. On the other hand, we did ask the teachers to provide us with a prior analysis of what they expected from the lessons, using a questionnaire which had been negotiated with the group. Ten lessons were observed, from the last year of collège (*3rd*) and the last year of lycée (*Terminale*).

To conclude this brief summary of the research procedure, we would like to underline what we see as the particular value in working with expert teachers:

- first, it seems to us important to present the expertise that these experts have, and which they are often not good at making explicit, in order to be able to pass this on for the benefit of training. It is also important to show to those involved in training the degree of expertise that is needed for efficient management of classrooms when using computers.

- second, the difficulties encountered by the experts, and the difference between their expectations and what actually occurs in their classes, seem to us to be particularly important in trying to understand what may prevent the integration of computer learning into mathematics teaching, and in identifying the questions which merit particular attention for those engaged in research in this area.

II - PERCEIVED ADVANTAGES OF C.A.S

The literature concerning the use of C.A.S. in schools attributes to it a number of advantages, the majority of which are not specific to any particular software. These advantages relate to the nature of the mathematical tasks that can be offered to students, as well as to the students' cognitive functioning.

Thus, a systematic search through the bibliography in this area (Artigue et al., 1993) shows that the use of C.A.S. is most frequently presented as[3]:

- *Permitting a more effective development of an experimental approach to mathematics* with its phases of exploration, formulation and testing of conjectures and validation, at least pragmatically. This claim is most commonly supported by referring to the user-friendliness of the system (particularly for DERIVE), the considerable reduction in effort needed to carry out explorations compared with traditional methods, as well as certain characteristics of the user/machine interface: speed, frequency of interaction, exploiting feedback from the machine in the case of error.

- *Allowing for the exploration of problems which are more interesting than those usually encountered, and not strictly within the school syllabus,* through using calculation aids (numerical, algebraic and graphical) provided by the computer. Moreover, many of the publications relating to DERIVE up to recently have been essentially concerned with illustrating this thesis through proposing such activities.

- *Providing a more user-friendly atmosphere for teaching and more suited to the needs of the particular learner.* This aspect is most commonly argued with reference to the characteristics of machine/student interface compared with teacher/student interaction. It is also claimed that, in general, an error made by the student carries less penalty when working with the computer and that the teacher, released by the machine from routine work, is more free to turn his attention to those problems that really require his help and that he can be seen as a resource to be called on, and as a partner in solving the tasks that have been set.

It is clear that these different potential benefits are seen as leading to an increase in student motivation for the subject and for working on tasks.[4] The use of software like DERIVE is also presented, at the cognitive level, as having the benefit of

- *Compensating, up to a point, for the mathematical difficulties that some students encounter and permitting them to continue to learn without being hindered by these obstacles.* For software such as DERIVE it is claimed that it can compensate for the difficulties involved in numerical calculations and elementary algebra (Kutzler, 1994).

- *Providing for more reflective, strategic and conceptual functioning by freeing the student from technique.* Comprehension, meaning and verification take precedence over simple technical procedures. This point is without any doubt the one most frequently cited and is, in general, associated with a criticism of traditional teaching, which is presented as concerned with the acquisition of purely technical competencies. Software programs such as DERIVE are thus seen as possible vectors for the desired change in education.

- *Providing for the development of mental images, through the possibilities of visualisation, and for an improved understanding of the* *links between the fields of number, algebra and graphs through using them conjointly in mathematical tasks.* In the case of software like DERIVE this supposed impact of visualisation on comprehension is most frequently illustrated by referring to the teaching of analysis: examples are the concept of a derivative, polynomial approximations, Fourier series... As regards the interaction between graphs and algebra, the illustrations offered are essentially concerned with the study of elementary functions.[5]

- *Promoting work on syntax in algebra, recognition of patterns and generalisation of the status of different objects being handled, through their existence as requirements of the 'milieu' and not just of the didactic contract.* Syntactical aspects are shown to be particularly valuable in the use of DERIVE at the beginning of learning algebra (Hirlimann, 1994).

- *Contributing to a 'mathematisation' of algebraic manipulations, by favouring their explicit and operational character through instructions or sequences of instructions given to the computer.* This aspect is again more commonly cited in connection with the beginnings of learning algebra, in particular when dealing with the solution of equations and systems of equations where it is known that the manipulations are not easy to grasp and where they are not seen to have a real mathematical status for the student.

- *Providing for a change in the way the students conceive of numbers through the possibility of carrying out exact calculations.* At the cognitive level, this aspect is particularly seen as enabling the questioning of certain erroneous conceptions of number, reinforced by the widespread use of calculators.

All these claims are theoretical. In the majority of the texts, they are illustrated by examples and analyses that are *a priori*, later carried out in predetermined classroom experimentation, but they are in fact hardly questioned. One comes away from reading the literature with the impression that the work in class more or less runs itself, and that a teacher sufficiently familiar with the software will encounter no particular difficulty (above all with DERIVE, which is judged to be particularly accessible). However, studies which are beginning to be made, in particular those which go beyond the classical control/experimental group methodology, tend to show that when teaching with C.A.S., as well as with other types of software, the results obtained are not always as beneficial as was expected. (*cf.* for example Monaghan, 1994).

In the rest of this article, we shall try to indicate some approaches for analysis and reflection relevant to teaching and teacher training, taken from our own research observations. Our approach is didactic. A class session is thus analysed as an interactive system consisting of three parts: students, the teacher and the mathematics material. The system is analysed through the interactions of these three parts with themselves and with the 'milieu', which here includes computers. For an intended situation, the possible dynamics envisaged a priori are known: what we do is to study the dynamics of what happens, in order to understand the relations between teaching and learning in the real situation, and what determines them. We also want to determine what changes in the command variables of the situation may lead to an optimisation of these relations, in the light of the intended learning goals. Finally, more synthetically, a search is made through the characteristics of each particular analysis, in order to identify consistent features which are transferable. It is at this more synthetic level that we are reporting here, while making reference to particular observations.

III - PUTTING THEORY INTO PRACTICE

We shall deal here with what seem to be to us the key points for putting theory into practice. For clarity, we shall deal with them separately, but in the real situation they inter-relate in a relatively complex way. Within the limits of this article, we have chosen to look at the following two points:

- technical, procedural and conceptual operations,
- strategies used in DERIVE, strategies that are favoured by using DERIVE, strategies envisaged by the mathematics syllabus.

III.1 - Technical, procedural and conceptual operations

As we mentioned earlier, the use of software like DERIVE is generally seen as allowing the student to be able to stand back from purely procedural and technical mathematical processes in order to engage in more reflective and conceptual activity. Since DERIVE can quickly and efficiently carry out calculations and draw graphs, the student is less likely, it is thought, to lose his way in working through the solution of a problem. Further, the student's work does not require technical manipulations, but organisation, testing and interpretation, in other words it is more reflective and conceptual.

The justification for this can also be based on research in didactics such as that done by A. Sfard and E. Dubinsky, although this research is not directly concerned with learning using computers. These authors base their work on the duality of mathematical concepts - concepts which can be seen at the same time as both processes and objects (Sfard, 1991 and Dubinsky, 1991) and they show that:

- for a very large number of concepts, the first understandings of them, which come from doing, mean they are initially thought of as dynamic processes,
- the ability to conceive of these processes as entities in themselves, as static objects, which can be used in other processes at a higher level, is an operation of interiorisation and encapsulation which is cognitively complex,
- being able to do mathematics efficiently supposes a good flexibility between these two aspects: process and object.

In the literature referring to these theoretical considerations, the use of computers is seen as a way of favouring conceptualisation through aiding the transition from process to object. Ed Dubinsky, in the article referred to above, writes:

> For example, it seems that if a student implements a process on a computer, by using software that does not introduce programming distractions (such as complex syntax, constructs that do not relate to mathematical ideas, etc.), then the student will, as a result of the work with computers, tend to interiorise the process. If that same process, once implemented, can be treated on the computer as an object on which operations can be performed, then the student is likely to encapsulate the process.[6]

C.A.S. have the potential for doing this, even those like DERIVE whose programming possibilities are limited.

What do we conclude from the observations that we have made of the subject?

That things are more complex than is supposed by what has been said above. It would seem that we are instead faced with a didactic system which is subject to contradictory forces:

- the first tending to reflective and conceptual actions, as indicated,
- the second, conversely, tending to the division of the solution of a problem into a multiplicity of actions, whose global coherence is not evident, and a reduction of reflection in favour of action.

We have been able to consider a number of different cases of observed classes, which were run according to the hypotheses indicated above, but there was nothing systematic about them.

To illustrate the claim made in the preceding paragraph, the example that we have chosen is one of the first we observed. It took place in 1993 in a *Seconde classe* (first year of lycée). The students had already used DERIVE in an earlier session in which they had solved a number of equations, constructed a general formula for solving equations of the type $ax + b = 0$, and tested the formula with a number of examples. In the session we observed, their task was to use the general formula they had found in order to generalise a method for solving a linear system of two equations with two unknowns. The purpose of this work, was to get them to think about the particular numerical methods for solving equations that they had been taught at collège.

The hypotheses that were made in an a priori analysis of the value of using DERIVE here conformed entirely with those set out earlier, since they are essentially the following (Artigue et al., 1993):

a) DERIVE, by freeing the student from having to carry out calculations, can enable him to concentrate on the solution processes, and not get lost in the process, and it can help him to develop the desired reflective attitude.

b) On the other hand, having to make the steps of the solution procedure explicit through the use of commands in DERIVE such as soLve, Manage Substitute, etc., can help students to make explicit certain implicit manipulations in the solution, and to give these manipulations more the status of operators. The explicit nature of the commands will also require the students to distinguish between unknowns (variables) and parameters (coefficients), and to realise that an equation can be solved for several unknowns and not just for x.

c) We also have the hypothesis that this situation, with its relatively formal nature, is a situation which would not be easy to handle with paper and pencil, but conversely, by using DERIVE, producing a general solution, through generalising already existing commands, will seem a more natural one for the students.

d) Further, the work involves a complex situation where the students need to be able to manipulate algebraic expressions comprising 8 distinct letters and we can imagine that many students would have difficulty with the calculations without the aid of DERIVE.

e) The availability of immediate feed-back in a situation that is considered a priori as one having difficulty, is considered to be an asset of DERIVE. Nonetheless, the verification carried out by DERIVE is only partial. Unless the students take the trouble to verify their formulas by substituting them in the equations - and this seems unlikely - DERIVE will not 'say' whether or not the formulas that have been found are true.

What did we find from our observations? The students got involved with the problem (getting them started presented no difficulty), and despite the difficulties encountered, none of them gave up. Purely technical problems to do with the computer were marginal. But although there was undeniable student activity, and using the computer did not present any problems, the a priori hypotheses hardly matched up to the most 'optimistic' a posteriori analysis.

The problem posed was, for all of the students, a real problem: they had no initial strategy for solving the problem, and for some no reliable strategy for finding the numerical solution of a system of equations. Inevitably the students used a number of features of DERIVE which were easily available but were not the ones that were needed to deal with the intended question, and were even counter-productive: in the absence of any strategy, the ease of carrying out trials with DERIVE led to a multiplication of trials in the hope that something positive might emerge. This phenomenon we called 'fishing', and it is one which might well have a reasonable chance of being successful when working with computers. The ease with which many attempts can be made, and the large number of possible choices initially available, meant that for many students, and this was confirmed in later observations, they did not feel the need to interpret feed-back, which in any case would require them to think hard about the problem. They preferred instead simply to register the fact that an attempt proved negative and then went on to make another choice. What followed for some groups was an erratic exploration of possibilities, which ended up being successful, following the intervention of the teacher or through the diffusion of the results obtained by neighbouring groups, without ever giving the impression that they had achieved the expected reflective distance from the particulars of the problem.

Being able to focus on the processes involved in solving the problem was also affected by a number of other factors:

- insufficient confidence in using DERIVE functions, which led to some of them becoming lost in the verification procedures,
- simple unnoticed key-board errors which led to bizarre results, which was even more of a problem for those students whose grasp of the solution procedure was too weak to enable them to distinguish between different types of wrong results ,
- more mathematical errors linked to DERIVE syntax, in particular in situations requiring the use of multiple brackets, or errors made through misunderstanding the behaviour of the soLve command when using variables (entering coefficients as variables, for example!). It is clear that a better knowledge of the software, notably at the level of the commands Build and Manage Substitute could reduce the number of such errors and so reduce the number of re-writes.

Further, although by the end of the session the students had obtained a schema for solving a pair of simultaneous equations along the lines of:

- solve the two equations for x
- equate these two expressions
- solve for y
- start again reversing the roles of x and y

it is difficult to know what conceptual anchorage this schema has.[7] The difficulties encountered by many groups, in particular those who could not extricate themselves from the vicious circle of a double solution for x and y, or those who, having obtained y this way did not know what to do to find x, leave one thinking that, for these students, solving an equation is simply means a process by which one arrives at $x = \dots$ after carrying out a certain number of codified manipulations. We are left with the impression that the work carried out with DERIVE in this session, while producing a new schema, has done nothing to challenge that view.[8]

In contrast, an analysis of our observations of other groups shows the use of DERIVE in an entirely positive light. This was the case, for example, for a group who used a strategy of linear combinations of the numbers for the numerical examples but who, when it came to generalise, reverted to ineffective simple differences of equations. The teacher then advised the two members of the group to translate their numerical procedure using DERIVE. This translation, which was not self-evident but which the group was able to do on their own, produced the reflective attitude. Following this, a generalisation was seen to be an easy adaptation of work that had already been done. Moreover we were to see these same two students working first with paper and pencil and not moving on to use DERIVE until they became stuck by the formal complexity of the equation: $y(eb - fa) = ce - ga$ which they had found by eliminating x using linear combinations. They used the same mixed approach to find x. We should point out that the phenomenon observed here: a qualitative change allowed by DERIVE simulating a process 'known by its action', is entirely in agreement with the quotation by Dubinsky given above.

This example, briefly reported, seems to us to show the need for vigilance in regard to two particular points which are present throughout this work:

- When the claim is made that DERIVE frees the mind to be able to operate at the conceptual level, it is implicit that functioning at the conceptual can exist, independently of beginning an technical work. This is doubtless true for someone working with previously well mastered objects, but at the conceptualisation phase, the technical level and the conceptual level are not independent. On the contrary, situations which completely take over the technical operations, or make the procedural operations invisible, may provide an obstacle to conceptualisation, and may even lead to the construction of what we may call 'pseudo-objects', which are named and labelled but lack all the necessary flexibility for use at the procedural level to which they apply. From this point of view, it can easily be seen that working with DERIVE, if it is not carefully thought out and adapted for the students, can maintain the illusion that the students are operating at a conceptual level that is higher than their actual level, simply because they are using commands which globalise complex processes.

- We can also see in this situation that, although the students may feel that reflective and strategic functioning makes too heavy demands on them, the ease of using DERIVE provides escape routes which may well be successful (judged only by outcome). In other words, results can be successfully achieved without engaging in the conceptual functioning that the teacher intended. We should point out that if they achieve reasonable results, the students will be even more convinced of their methods![9]

It seems therefore necessary, when developing situations for using DERIVE, to take account of these characteristics, so as to achieve an adequate equilibrium between these opposing forces which are necessarily present, and which will be different for individual students.

To close this section, we should mention that in 1993-94 the research group designed a new approach to solving linear equation systems, this time for collège students, which incorporated the relationships between technique and concept, using the specific features of DERIVE. This allowed the hypotheses made about the potential of DERIVE to be realised by the great majority of students.

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III.2 - Strategies used in DERIVE

We wish to mention here two points of which we became progressively more aware as we conducted our research. The first relates specifically to the way DERIVE carries out its operations, and the problems that may come from them being hidden from the user. The second concerns the techniques and solution strategies favoured by the use of DERIVE and the possible difference between these and those which the teacher intends the students to learn.

As before, we shall introduce our thoughts about this by referring to two examples of classroom observation. The first was in a class of *Première Scientifique* (2nd year of lycée) at the end of their work on polynomials. The problem posed was an open one, and it was hoped that the students would be able to reinforce what they had learnt in this area of work. It would also, according to the teacher's view beforehand, have an heuristic objective and would help the students to develop research attitudes. The activity was based on one which has been already written up (Hirlimann et al., 1993).

The problem presented to the students was the following: find the factors of $X^n - 1$. The problem can be tackled without the use of DERIVE but one would not expect a great deal to come from such an approach. Factorisation is connected with the roots 1, and if n is even, -1, found by inspection or from division by $(X - 1)$ and $(X^2 - 1)$, and generalising the identity $(X^2 - 1) = (X - 1)(X + 1)$ to $(X^{2n} - 1) = (X^n - 1)(X^n + 1)$. DERIVE is, without doubt, an indispensable tool here for opening up the field of possibilities, creating possible conjectures, testing them easily, as well as in helping the student when carrying out paper and pencil calculations alongside the computer based investigation. Five factorisation possibilities are offered by DERIVE: Trivial, Squarefree, Rational, raDical and Complex. The students were quick to limit themselves to rational factorisation which is the most productive, as well as the most meaningful, at their level.

Mathematically speaking, rational factorisation is governed by the theorems about cyclotonic polynomials. The DERIVE algorithm goes through intermediary factorisations in Z/pZ. These two levels of analysis are not accessible to the students for whom DERIVE operates as a black box. But even so, it would appear a priori that the students could carry out an interesting investigation, which may lead to calculations and arguments about polynomials that could usefully extend what is usually done.

During the course of the observation, two distinct ways of seeing the task emerged: one consisted in producing general conjectures with the help of DERIVE, the other consisted in producing conjectures about the factorisations that were offered by DERIVE. These two tasks are profoundly different.

The students who made the first interpretation of the task went on to produce about ten conjectures during the session, correct and incorrect (errors involved in using the index notation being the most common, like:

$$X^n - 1 = (X - 1)(X + 1)\,(X^{n-2} + X^{n-4} + \mathrm{K} + X + 1)$$

for n even). The production of these conjectures was not automatic, since they do not correspond precisely to the factorisations given by DERIVE. The standard factorisation of $(X^n - 1) = (X - 1)(X^{n-1} + X^{n-2} + \mathrm{K} + 1)$ for example is only given for prime n and the correct ver-

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sion of the erroneous one given above is only given for $n = 4$. In order to test their conjectures, the students need to combine terms in order to perform partial products, either mentally, on paper or using DERIVE (which to be done efficiently requires a good level of competence with this software). They could also perform the corresponding divisions using DERIVE and ask for the result to be simplified. This leads to thinking about polynomials and manipulating them which involves handling polynomials in a way that is more flexible and more complex than is usual at this level.

Those (rather fewer) students who chose the second interpretation of the task had considerable difficulty in formulating conjectures, the conjectures collapsing as they were tested on increasing values of n. At this level, one could not hope for other than partial conjectures, and may reasonably assume that even if the students limit themselves in this way, they are still going to encounter certain difficulties. In fact the easiest to find are the factorisations of $(X^n - 1)$ in the cases where n is prime, or double a prime, or where n is a power of 2 (see appendix 1). Recognising and formulating the properties of divisibility play little part in the mathematics syllabuses followed today and they do not correspond to the categorisations of numbers familiar to the students. Our observations confirm these difficulties, as well as attesting to the richness of the work on polynomials carried out by these students.

On reflection, it may be that it was not a good idea for the students to engage in this second interpretation of the task, although it resulted in real endeavour, since they were not able to get very far with it. On the other hand, an investigation of the problem of forecasting what DERIVE might produce for factorisations would make an interesting extension, and could be done after work using the first interpretation of the task. This would not only involve making conjectures, for the reasons advanced above, but also finding factorisation methods to produce certain of the DERIVE factorisations, and justifying them. For reasonable values of n (up to 30, say) this would be an activity that would only employ a small number of distinct methods.[10]

The investigation posed here seems to us an interesting one since, through the use of DERIVE, the students are able to work on it productively at their own level. The outcome should not be judged solely by number of conjectures the students are able to formulate. The work also requires them to develop a research attitude, with all that entails, towards the mathematics, and it provides an opportunity for revision of what they have learned in the officially prescribed syllabus. We also consider that it is not sufficient to be content to remain at the discovery phase, but to extend the session, as was done in the mathematics class example here, by going back over the discovery phase, with the help of DERIVE, in order to justify the results without using DERIVE. The observations we made seem to show that this is certainly possible at their level (the equivalence of algebraic expressions of quite different forms is not yet self evident for these students) and that tackling the work is facilitated by contrasting different conjectures, some correct and some incorrect.

In the example we have been looking at, DERIVE functions as a black box whose operational strategies are inaccessible, but this inaccessibility does not pose any specific problems. In fact, the results obtained are reliable, means for validating them without using DERIVE are accessible to the students and, finally, it is possible in a not insignificant number of cases to rework the DERIVE results by use of elementary factorisation procedures (albeit different from those employed by DERIVE). These characteristics are sufficient for us to be able to set up and run an investigation which is an altogether consistent research activity. We should point out however, that the methods that arise naturally from this activity are not those that are particularly intended for this level of education, for example the use of the theorem that P(X) has a factor ($X - a$) when a is a root of the polynomial P(X). Our observation shows that formal identification procedures that were the ones most used.

The fact that the operational strategies of DERIVE are hidden, and that there is a difference between the strategies that arise naturally from DERIVE and those that students can be expected to know at any particular stage of their secondary education, can sometimes be a problem. We should like to illustrate this by reporting on another class, this time from the final year of lycée (*Terminale C*). The session involved a study of the expression $\cos x + \cos 3x + \cos 5x$, determining equivalent ways of writing this, solving the equation $\cos x + \cos 3x + \cos 5x = 0$ and, for the more advanced students, studying the function associated with this expression. The students could choose whether or not to use DERIVE, but they had to produce a written answer to the question in a 'standard' way. The session was presented as a revision session in trigonometry. The purpose was to revise the fact that a trigonometric expression can be dressed up in many forms, and altered according to meet the solution needs of a particular problem, as well as to revise the specific way of expressing the multiple roots of a trigonometric equation. A prior analysis suggests that using DERIVE here will have a number of benefits: easily obtaining different forms of the given expression, noticing that solving an equation with DERIVE requires a choice of the form of the adapted equation, help with the calculations for students working essentially with paper and pencil methods using standard formulas, promoting reflection about multiple roots given that DERIVE gives several solutions but not all of them (see appendix 2), and the opportunity of drawing a graph of the associated function to allow conjectures about the solutions of the equation[11] which will be useful for later study of the function.

The observation of the session showed that the students were more hindered than helped by using DERIVE, and this for a number of reasons. First of all, the Manage Trigonometry command in DERIVE allows a number of different options. We can choose: Direction: Auto, Collect, Expand and combine this with Toward: Auto, Sines, Cosines. This gives nine possibilities for which the commands Simplify, Expand, Factor and soLve can act differently. The differences are not systematic, the terms describing these options do not allow us easily to predict the best choices. Finally, a not insignificant number of the choices only produce a change in the order of the terms in the expression. In these conditions, the student is thrown into an environment without any accessible reference system, and only has recourse to random 'fishing'. To this we should add that there is an entirely satisfactory, and much more economical, paper and pencil strategy for solving the equation, which will appear therefore to be the 'natural' one to the student. This is done by changing $\cos x + \cos 5x$ into $2\cos 3x \cos 2x$, or changing $\cos 3x + \cos 5x$ into $2\cos x \cos 4x$, either of which can be used to factorise the initial expression, giving two equations of the type $\cos nx = a$ to solve. The students can do this provided they avoid the natural temptation of grouping the two first terms. Now it happens that this factorisation is particularly difficult to achieve using DERIVE (see appendix 2) and the students, who would expect the computer to change the given expression directly into one that can be used, will find this particularly difficult.

Those who do not abandon using DERIVE will nonetheless finally end up with quintic polynomial expressions in cos$x$ which DERIVE can easily factorise and solve. A problem now arises with respect to the last part of the question: how can the hidden procedures used by DERIVE be identified so as to show that the initial expression is indeed equivalent to the quintic polynomial that has been found? The students would not be able to use DERIVE in the sophisticated way that is needed to identify all the intermediate steps, and so would need to go back to their standard formulas and paper and pencil work. Many in fact did not do this part, or abandoned an attempt to produce a final answer containing an expression and factorisation which DERIVE allowed.

On the other hand, our observations showed that for those students who worked with DERIVE to find a solution, the problem of interpreting the results provided a real mathematical task. However, the contents of the pages handed in at the end of the session showed that most often this work had not been successful in the sense that it was intended. The students were often content to identify just two solutions of opposite sign for the elementary subordinate equation of the type cos$x$ = a. Finally, as for the use of graphs, very few used this facility to help with solving the equation (and no student went so far as to study the behaviour of the function). It was noticed that, once again, the tendency was for the great majority of the students to abandon the use of DERIVE in favour of using their graphic calculators.

From these two examples of classroom observation it seems to us that the teacher needs to be careful about, at least, two points:

- The operational methods used by DERIVE may well be different from those which the student could use, at any particular level of mathematical development, or may not be the ones that the teacher wishes to be worked on. Also the hidden nature of DERIVE's operational system makes it difficult for the student to recreate the intermediary steps in any solution. The hidden nature of DERIVE's methods and the difference between these methods and those available to the student, or intended by the teacher, are not of themselves to the detriment of using DERIVE. As we have tried to show these are variables which can be used to stimulate questioning and mathematical activity on the part of the student. But it needs to be said firstly, that this is not always possible and secondly, that our observations show that this is too complex to be able to be correctly improvised in the real classroom context during the running of a session. It needs to be taken account of beforehand at the planning stage.

- The students cannot be expected, in the way it is used at present, to become experts with DERIVE, even where they use DERIVE almost continuously over a number of months, as was the case in the two classes we have reported on. Further they do not have the sort of mathematical background knowledge that helps us to interpret what DERIVE produces. Nor can they use DERIVE as an aid in the way we want, nor can they find ways to make the hidden steps become apparent. This was very clear in the session on trigonometry.

In preparing a session using DERIVE, and thinking about what the students may do, we are liable to forget the extent to which this double expertise, in mathematics and in using DERIVE, differs between us and the student, both in operating DERIVE and in validating and interpreting its results. Indeed, the user-friendliness of DERIVE can lead us to underestimate the role that DERIVE expertise plays in the solution of any particular problem. All this, without doubt, leads to a not inconsiderable degree of unwarranted optimism about what may be expected of a session.

A situation like the trigonometry session shows us what may happen. It also shows us that, to enable coherent and reflective work on the part of the student, some restrictions need to be placed on DERIVE's considerable freedom of operation. It can be seen that, contrary to expectation, instead of DERIVE encouraging systematic research behaviour, the student, confronted by an uncontrollable environment, simply resorts to random operating.

We also wish to point out that what we have said shows that learning to operate with DERIVE necessarily requires some part of mathematical learning and training. We have too often under-estimated it because we have been unable to notice it, the processes that it involves not being like those that appear in the usual context of learning mathematics due to the specific role played, for example, by perception and by formal analogies. Integrating DERIVE into the teaching of mathematics, requires accepting the need for this training and finding ways of doing this and exploiting it to the full. We must also be ready to parry the simplistic view that DERIVE will simply do the mathematics instead of the student doing it, since we teachers know how to get DERIVE to do it.

III.4 - In conclusion

In this article we have drawn attention to two aspects of using DERIVE in the mathematics classroom. These are not the only possible ones: other observers in the same situations might well have noticed other important features. The choice we made, more or less consciously, was connected with our own experience and understanding of DERIVE. This is not that of the specialist in C.A.S. but that of a mathematics educator. To this extent, in our observations of teaching sessions using DERIVE, we are influenced by categories of analysis which belong to didactics rather than to mathematics. Here, DERIVE is an essential element in the teaching and learning environment, an element of the 'didactic milieu', and we are trying to understand what effect this element has on the mathematics being taught, the ways the students relate to the mathematics, and the ways of managing the teaching environment. We are attempting to assess to what extent the anticipated consequences of using DERIVE actually happen, and to identify which are the properties of DERIVE that are effective in this, and which hinder the desired outcome, as well as trying to identify suitable tasks. We are trying to demonstrate consistent features which may transcend a particular situation and have global application, and then see if they apply to the observations we have already made. Considering the student's learning, as a process of adaptation to situations which are 'problematic' for him, we have tried to find out what it is in the sessions we observed that may make them problematic for the student, and the means of adaptation which he has available, whether these are 'mathematical' or 'didactic' (that is in him using his knowledge of the didactic system), without making the hypothesis that the former should necessarily take priority over the latter. This analysis allows us to make hypotheses about what one might expect to happen in a given situation, or to conceive of changes that are likely to make a given situation more profitable. This is certainly a very partial look at the use of DERIVE in education which needs to be put alongside other research.

I would like to restate what our work has to tell us: first it reinforces the conviction that DERIVE can be an effective aid for teaching and learning mathematics, that its and graphs), as well as its accessibility, constitutes a real asset at the secondary education level; but also that these capabilities in themselves are not able to be an effective tool for education at this level; that at the same time we teachers need to carry out an analysis of what the teaching session is meant to achieve. Using DERIVE in a well thought out context can, without doubt, support learners of mathematics, and help to provide interesting mathematics activities in schools. But the use of DERIVE could also lead to a lowering of the level of the students' mathematical activity (by taking over the usual technical work the student does without moving him on to other levels of activity) or it may reinforce a view of mathematics as a purely formal activity of symbol manipulation, a game of rules with no meaning. Only a careful conception of situations for using DERIVE can tilt the balance in the right direction. We hope that our research will provide a number of analytical approaches that can be used for guidance.

It is essential, if we really wish to see an effective integration of computers into education, to be convinced of the potential benefit of this integration. But conviction without insight will be no help in overcoming obstacles.

Thanks

I would like to express my thanks to the members of the DITEN C.A.S. group for all they have taught me about DERIVE and C.A.S. programs, for the valuable help they have given to this research, and most of all for the kindness with which they have received my intrusions into their teaching and the disturbance this has caused, without the least guarantee that it may one day produce something useful.

APPENDIX 1

Some rational factorisations of x^{n-1} for n less than 20, and an example of an elementary derivation of a given factorisation

$x^2 - 1 = (x-1)(x+1)$

$x^3 - 1 = (x-1)(x^2+x+1)$

$x^4 - 1 = (x-1)(x+1)(x^2+1)$

$x^6 - 1 = (x-1)(x+1)(x^2-x+1)(x^2+x+1)$

$x^8 - 1 = (x-1)(x+1)(x^2+1)(x^4+1)$

$x^9 - 1 = (x-1)(x^2+x+1)(x^6+x^3+1)$

$x^{10} - 1 = (x-1)(x+1)(x^4-x^3+x^2-x+1)(x^4+x^3+x^2+x+1)$

$x^{12} - 1 = (x-1)(x+1)(x^2-x+1)(x^2+x+1)(x^2+1)(x^4-x^2+1)$

$x^{14} - 1 = (x-1)(x+1)(x^6-x^5+x^4-x^3+x^2-x+1)(x^6+x^5+x^4+x^3+x^2+x+1)$

$x^{15} - 1 = (x-1)(x^2+x+1)(x^4+x^3+x^2+x+1)(x^8-x^7+x^5-x^4+x^3-x+1)$

$x^{16} - 1 = (x-1)(x+1)(x^2+1)(x^4+1)(x^8+1)$

$x^{18} - 1 = (x-1)(x+1)(x^2-x+1)(x^2+x+1)(x^6-x^3+1)(x^6+x^3+1)$

$x^{20} - 1 = (x-1)(x+1)(x^2+1)(x^4-x^3+x^2-x+1)(x^4+x^3+x^2+x+1)(x^8-x^6+x^4-x^2+1)$

example of the derivation of x^{18-1}:

$x^{18} - 1 = (x^6-1)(x^{12}+x^6+1)$

by changing the variable $x \to x^6$ in the factorisation of $x^3 - 1$ (procedure: 'multiple')

$x^6 - 1 = (x^3-1)(x^3+1)$

by procedure: known identity

$x^3 - 1 = (x-1)(x^2+x+1)$

by procedure: standard factorisation

$x^3 + 1 = (x+1)(x^2-x+1)$

by change of variable $x \to -x$ in the standard factorisation of $x^3 - 1$

$x^{12} + x^6 + 1 = (x^6-x^3+1)(x^6+x^3+1)$

by using the factorisation: $x^4 + x^2 + 1 = (x^2-x+1)(x^2+x+1)$ with a change of variable $x \to x^3$.

APPENDIX 2

DERIVE and the treatment of the expression cosx + cos3x + cos5x

1. In the Direction field Auto or Collect, whichever Toward field is chosen, the commands Simplify, Expand and Factor have no effect on the expression, which is simply rearranged as cos5x + cos3x + cos$x$.

2. In the Direction field: Expand and the Toward field: Auto we get the following:

- Simplfy: $16\cos^5 x - 16\cos^3 x + 3\cos x$
- Expand: the same
- Factor Rational: $\cos x(1 - 4\sin^2 x)(2\cos x - 1)(2\cos x + 1)$
- Factor raDical: $\cos x(\cos x - 1/2)(\cos x + 1/2)(\cos x - \sqrt{3}/2)(\cos x + \sqrt{3}/2)$

3. In the Direction field: Expand and the Toward field: Cosine we get the same results.

4. In the Direction field: Expand and the Toward field: Sine we get:

- Simplify: $\cos x(16\sin^4 x - 16\sin^2 x + 3)$
- Expand: $16\sin^4 x\cos x - 16\sin^2 x\cos x + 3\cos x$
- Factor Rational: $\cos x(2\sin x - 1)(2\sin x + 1)(4\sin^2 x - 3)$
- Factor raDical: $16\cos x(\sin x - 1/2)(\sin x + 1/2)(\sin x - \sqrt{3}/2)(\sin x + \sqrt{3}/2)$

5. Starting with $\cos x(16\cos^4 x - 16\cos^2 x + 3)$, using Direction: Collect, Toward: Auto, and selecting the seconf factor, the command Simplfy gives $\cos x(1 + 2\cos 4x)$!

6. In the Direction field: Expand, the command soLve applied to the original expression lets you find directly 15 different solutions:

- in Toward fields: Auto and Cosine, the solutions between $-5\pi/6$ and $11\pi/6$,
- in Toward field: Sine, the solutions between $-4\pi/3$ and $3\pi/2$.

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NOTES

[1] DITEN: Direction de l'Information et des Technologies Nouvelles

[2] M. Abboud, J. P. Drouhard and J. B. Lagrange have been working with me on this project.

[3] The order in which these advantages are presented does not have any hierarchical significance. We have tried to group them first according to the advantages claimed for the activities and their management, and second for the potential they may have in aiding the student's thinking (cognitive functioning).

[4] The literature often contrasts, somewhat simplistically, traditional teaching where the student is passive, all activity being reserved to the teacher who transmits information, with a computer (or calculator) environment where it is the student who is active and the teacher is all but passive.

[5] A DEA memoir by J. F. Canet specifically deals with this aspect (Canet, 1994), through an analysis of cognitive interaction, using semiotic interaction grids developed by R. Duval and his students (Duval, 1993).

[6] page 123.

[7] We note that this schema, using particularly few DERIVE commands, and not requiring the need to look for and to know how to use the hidden command Substitute, does not exactly correspond to the two methods classically taught: substitution and elimination (through linear combination).

[8] We should point out that a complementary approach representing the solution algebraically and graphically, or by developing a functional approach to the solution of the equations so as to go beyond these purely formal aspects, is entirely possible with DERIVE.

[9] In fact, it would be doubtless futile to wish to eliminate this type of functioning at all costs: we ourselves do not avoid it when we work in a computer environment. This would be to deny the characteristics of such environments and the specifics of the processes of adaptation that can be developed there.

[10] Such a classroom experiment was carried in 1994 with interesting results.

[11] In fact, since this expression is equivalent to $\sin(6x)/2\sin x$, the graph shows the regularity of the solutions and, by using the cursor, these can be found using the approximate numerical values given by the computer.

DIDACTIC ENGINEERING AND THE ACQUISITION OF MATHEMATICAL KNOWLEDGE IN THE SECONDARY SCHOOL

Régine DOUADY
Irem de Paris VII

INTRODUCTION

In this chapter, I am concerned with the relationship between what the teacher intends to teach to a mathematics class and what the students are capable of learning effectively. The words, *teach*, *learn* and *know* can carry different meanings. I shall clarify the meaning I shall give them here.

The working out of a problem is a *step* in didactic engineering. In this context, the term *didactic engineering* designates a set of class sessions conceived, organised and articulated, in a coherent way, in a time frame, by a *master-engineer* with the purpose of achieving the learning of a certain mathematical content for a certain student population. Thus, didactic engineering is, at the same time, a *product*: the result of an *a priori* analysis, and a *process*: the result of adapting the product of the dynamic conditions of the class to carrying out the work.

In what follows, I am principally concerned with the relations between the *construction of meaning* and the *acquisition of knowledge* in mathematics. In directing the didactic experiences of the class, and with the inevitable institutional constraints and various pressures he is under, is it possible for a teacher to take these relations into account? And what margin of manoeuvre does he have?

However, beyond these concerns, there exists a crucial question, of a sociological nature, but which conditions the choice and the conduct of the envisaged didactic actions:

In the school setting, what is the place of knowledge for the teacher, and for the students? Is it something that is at stake in the didactic relation?

Instituts de Recherche sur l'enseignement des Mathématiques

Teaching Mathematics :

the relationship between Knowledge, Curriculum and Practice

This work considers the interconnections between knowledge, curriculum and practice in mathematics education. The teaching of mathematics lies at the centre of a complex system ; the teacher is a guardian of knowledge, is obliged to pay regard to the official curriculum, and also has to be skilled in classroom practice. In order to get a grasp on this complex situation, and to provide a position from which we can view different perspectives, we have chosen a starting point for looking at the teaching of mathematics : our focus is on meaning.

"If a large number of students find learning mathematics so difficult, it is because what they have to learn is too abstract ; it makes no sense to them, and at the very best all they do is to learn and recite rules. We do not really exploit their intelligence !" This is a common enough remark to be heard and there is certainly truth in it, but it is only one part of a complex mixture.

Abstract does not necessarily mean without meaning, just as concrete cannot for certain be said to be meaningful. The question is to identify what we mean by possessing meaning in respect to mathematics :

— does it refer to the internal consistency of the corpus of mathematics between its constituent parts, made up of concepts, theorems, algorithms, etc. ?

— does it refer to the relevance and conditions of validity of a mathematical model applied to a non mathematical situation we are hoping to solve ?

— does it refer to the meaning that a mathematician gives to a problem he is researching ?

— does it refer to the meaning that students give to mathematics questions prepared by a teacher at the end of a course ?

Research for meaning involves a positive quest for underlying factors and here, in particular, a search to identify and understand the choices relating to mathematics content, — what to teach, and the choices facing teachers, — how to teach. This work consists of fourteen articles written by teachers or those involved in teacher training who identify with this goal. It is hoped that this [illegible] of articles will be prove to be useful for the mathematics teacher, and [illegible] the teacher in training, as well as giving the reader an indication of [illegible] thought about in French mathematics teaching today.

TOPIQUES éditions

Publication des
I.R.E.M.
sous la direction de
E. Barbin et R. Douady

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