Leaderless Consensus-based Formation Control of Multiple Nonholonomic Mobile Robots with Interconnecting Delays

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***Abstract*— The control objective in the leaderless consensus-based formation control is to ensure that the Cartesian positions of the nonholonomic mobile robots converge to a given position in a formation pattern and the barycentre of such formation is agreed, in a decentralized manner, by all the robots, while their orientations converge to a common value. The main problem behind this stabilization objective is that for robots that exhibit nonholonomic restrictions, due to the Brocket's condition, the controller has to be designed such that it is discontinuous or it is non-autonomous (time-varying). In this work we propose a simple Proportional plus damping (P+d) smooth controller that solves the aforementioned objective and we provide a sufficient condition on the damping gain to ensure robustness with respect to variable time-delays in the interconnection. Simulations are provided to show the effectiveness of our control proposal.**

I. Introduction

The control objective in the consensus of multiple dynamical agents is to ensure that the states of all agents agree on a common value by sharing (part of) their state with their corresponding neighbors [1], [2], [3], [4]. There are mainly two different consensus problems: the leader-follower, in which the agreement value is given as a desired value to a set of (follower) agents in the network; and the leaderless, in which such common value if found among the agents [5], [6], [7]. In this work we deal with the consensus-based formation control of a class of nonholonomic mobile robots, where the control objective is to drive the Cartesian positions of all the robots to a given position in a formation pattern and the barycentre of such formation is agreed upon by all the robots in a decentralized fashion. Furthermore, the robot orientations converge to a common consensus value.

Consensus of nonholonomic mobile robots has been studied, for instance, in [8] where a decentralized feedback control that drives a system of multiple nonholonomic unicycles to a rendezvous point in terms of both position and orientation is proposed, the control law is discontinuous and time-invariant. In [9] necessary and sufficient conditions for the feasibility of a class of position formations are laid. In [10] the position/orientation formation control problem for multiple nonholonomic agents using a time-varying controller that leads the agents to a given formation using only their orientation is proposed. In [11] a distributed consensus control law for a network of nonholonomic agents in the presence of bounded disturbances with unknown dynamics in all inputs channels is presented. In [12] a smooth distributed formation control law using a consensus-based approach to drive a group of agents to a desired geometric pattern is proposed. The latter result is extended in [13] by introducing a Proportional plus damping (P+d) controller for the velocity dynamics. In [14] the consensus-based formation problem is solved using a time-varying controller and the solution is enabled with collision avoidance capabilities. However, in all of these references it is assumed that the communications are reliable and a simplified kinematics model of the nonholonomic mobile robots is used.

In contrast to the latter, in [15], [16] and [17] a more realistic torque-controlled second-order dynamics model is used, but it is assumed that communications are reliable (without delays). The consensus problem is also addressed in [18] via a cooperative control law that is robust to constant communication delays. However, in the latter reference, a first-order kinematics model is used.

In this article we solve the consensus-based formation control problem for a class of nonholonomic mobile robots interconnected through unreliable communication channels that exhibit variable time-delays. In addition, we use a second-order torque-controlled model for the robots. Our controller is simple to implement, as it consists in a smooth controller of the Proportional plus damping (P+d) type. To the best of the authors' knowledge this is the first work on consensus of mobile robots, in which the full second-order dynamic model is used and variable time-delays in the communications are taken into account.

Notation. $\mathbb{R} := [-\infty, \infty]$, $\mathbb{R}_{>0} := (0, \infty]$, $\mathbb{R}_{\geq 0} := [0, \infty]$. $\|\mathbf{x}\|$ stands for the standard Euclidean norm of vector $\mathbf{x}$. $\mathbf{I}_n$ represents the identity matrix of size $n \times n$. $\mathbf{1}_k$ and $\mathbf{0}_k$ represent column vectors of size k with all entries equal to one and to zero, respectively. $\otimes$ represents the standard Kronecker product. For any function $\mathbf{f} : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, the $\mathcal{L}_\infty$-norm is defined as $\|\mathbf{f}\|_\infty := \sup_{t \geq 0} |\mathbf{f}(t)|$, $\mathcal{L}_2$-norm as $\|\mathbf{f}\|_2 := (\int_0^\infty |\mathbf{f}(t)|^2 dt)^{1/2}$. The $\mathcal{L}_\infty$ and $\mathcal{L}_2$ spaces are defined as the sets $\{\mathbf{f} : \mathbb{R}_{\geq 0} \to \mathbb{R}^n | \ \|\mathbf{f}\|_\infty < \infty\}$ and $\{\mathbf{f} : \mathbb{R}_{\geq 0} \to \mathbb{R}^n | \ \|\mathbf{f}\|_2 < \infty\}$, respectively. The set $\bar{N}$ is defined as $\bar{N} := \{1, \ldots, N\}$, where N is a positive natural number.

II. Problem Setting

We consider a swarm of N nonholonomic vehicles modeled as differential drive robots that move in the Cartesian $xy-$plane with three degrees of freedom, two translations and one rotation. Let us define $\mathbf{z}_i := [x_i, y_i]^\top \in \mathbb{R}^2$

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and $\theta_i \in \mathbb{R}$ as the translation and rotation coordinates, respectively, of the ith-robot. Let us also define $\boldsymbol{\delta}_i \in \mathbb{R}^2$ as the relative desired translation of the ith-robot with regards to the barycentre of a given formation pattern. Thus, the corresponding position of the ith-robot translated to the barycentre of the desired formation is $\bar{\mathbf{z}}_i := \mathbf{z}_i - \boldsymbol{\delta}_i$.

The interconnection of the mobile robots is modelled using the Laplacian matrix $\mathbf{L} := [\ell_{ij}] \in \mathbb{R}^{N\times N}$, whose elements are defined as

$$\ell_{ij} = \begin{cases} \sum\limits_{k\in\mathcal{N}_i} a_{ik} & i = k \\ -a_{ik} & i \neq k \end{cases} \tag{1}$$

where $i \in \bar{N}$ and $\mathcal{N}_i$ is the set of transmitting information to the ith robot, $a_{ik} > 0$ if $k \in \mathcal{N}_i$ and $a_{ik} = 0$ otherwise.

Similar to passivity-based (energy-shaping) synchronization [19], [20], [21], the following assumption is used in this paper:

A1. The interconnection graph is *undirected, static and connected*.

Remark 1: Note that, by construction, $\mathbf{L}$ has a zero row sum, *i.e.*, $\mathbf{L}\mathbf{1}_N = \mathbf{0}_N$. Moreover, Assumption **A1**, ensures that $\mathbf{L}$ is symmetric, has a single zero-eigenvalue and the rest of the spectrum of $\mathbf{L}$ is strictly positive. Thus, $\text{rank}(\mathbf{L}) = N - 1$. Therefore, the only vector living in the kernel of $\mathbf{L}$ is $\text{span}(\mathbf{1}_N)$. △

The information exchange between the vehicles is subjected to time-delays that satisfy the following assumption.

A2. The communication from the jth to the ith robot is subject to a variable time-delay denoted $T_{ji}(t)$ which is bounded by a known upper-bound ${}^*T_{ji} \geq 0$. Further, $T_{ji}(t)$ has, up to the fourth, bounded time-derivatives.

The control objective in this paper is the following:

(LFP) Leaderless Formation Problem. Consider a swarm of N nonholonomic vehicles modeled as differential drive robots. Under Assumptions **A1** and **A2**, design a decentralized controller such that, given a desired formation pattern, all the robots (globally and asymptotically) agree on their relative Cartesian positions and orientations, *i.e.*, for all $\mathbf{z}_i \in \mathbb{R}^2$ and $\theta_i \in \mathbb{R}$

$$\lim_{t\to\infty} \begin{bmatrix} \bar{\mathbf{z}}_i(t) \\ \theta_i(t) \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{z}}_c \\ \theta_c \end{bmatrix},$$

where $\bar{\mathbf{z}}_c \in \mathbb{R}^2$ is the Cartesian consensus position of the barycentre of the formation and $\theta_c \in \mathbb{R}$ is the consensus orientation of the robots. ◁

III. Dynamic Model

We assume that for each robot, the geometrical center and the center of mass are located at the same point $\mathbf{Q} := [x_i, y_i]^\top$. Then, the corresponding dynamics of the ith-robot is given by

$$\begin{aligned} \dot{\mathbf{z}}_i &= \begin{bmatrix} \cos(\theta_i) \\ \sin(\theta_i) \end{bmatrix} v_i, \\ \dot{\theta}_i &= \omega_i, \\ \begin{bmatrix} \dot{v}_i \\ \dot{\omega}_i \end{bmatrix} &= \begin{bmatrix} \frac{1}{m_i} & 0 \\ 0 & \frac{1}{I_i} \end{bmatrix} \mathbf{B}_i \boldsymbol{\tau}_i \end{aligned} \tag{2}$$

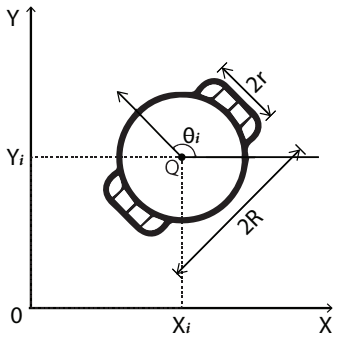


Fig. 1. Schematics of a differential wheeled mobile robot.

where v_i and ω_i are the linear and the angular velocities of the center of mass, respectively; m_i is the mass; I_i is the moment of inertia; R is the distance between point $\mathbf{Q}$ and the wheels; and r the radius of the latter. $\boldsymbol{\tau}_i$ is the control input torque of the left and right wheels, *i.e.*, $\boldsymbol{\tau}_i = [\tau_{il}, \tau_{ir}]^\top$, and $\mathbf{B}_i = \frac{1}{r}\begin{bmatrix} 1 & 1 \\ R & -R \end{bmatrix}$ —see Fig. 1.

A first step in the controller design is to propose the following inner control-loop

$$\boldsymbol{\tau}_i = \mathbf{B}_i^{-1}\mathbf{u}_i = \frac{r}{2}\begin{bmatrix} 1 & \frac{1}{R} \\ 1 & -\frac{1}{R} \end{bmatrix}\begin{bmatrix} u_{vi} \\ u_{\omega i} \end{bmatrix}, \tag{3}$$

where the extra input term $\mathbf{u}_i \in \mathbb{R}^2$ will be defined in the next section. The closed-loop (2) and (3) yields

$$\begin{aligned} \dot{\mathbf{z}}_i &= \boldsymbol{\varphi}(\theta_i) v_i, \\ \dot{\theta}_i &= \omega_i, \\ \dot{v}_i &= \frac{1}{m_i} u_{vi} \\ \dot{\omega}_i &= \frac{1}{I_i} u_{\omega i}, \end{aligned} \tag{4}$$

where

$$\boldsymbol{\varphi}(\theta_i) = \begin{bmatrix} \cos(\theta_i) \\ \sin(\theta_i) \end{bmatrix}. \tag{5}$$

IV. Distributed Consensus-based Formation Control

The extra input term $\mathbf{u}_i$ of the decentralized controller that solves the **(LFP)** problem is composed of two smooth P+d elements that drive the linear and angular accelerations, respectively. The first element is

$$u_{vi} = -p_{vi}\boldsymbol{\varphi}^\top(\theta_i)\mathbf{e}_i - d_{vi}v_i, \tag{6}$$

where $p_{vi}, d_{vi} > 0$ are the proportional and the damping gains, respectively; and $\mathbf{e}_i$ is the Cartesian position error of the ith-robot, with regards to its neighbors, defined as

$$\mathbf{e}_i := \sum_{j\in\mathcal{N}_i} a_{ij}(\bar{\mathbf{z}}_i - \bar{\mathbf{z}}_j(t - T_{ji}(t))). \tag{7}$$

The second element is

$$u_{\omega i} = -p_{\omega i}e_{\theta i} - d_{\omega i}\omega_i + \alpha_i(t, \theta_i, \mathbf{e}_i), \tag{8}$$

where $p_{\omega i}, d_{\omega i} > 0$ are the proportional and the damping gains, respectively; $e_{\theta i}$ is the orientation error of the ith-robot, with regards to its neighbors, given by

$$e_{\theta i} := \sum_{j\in\mathcal{N}_i} a_{ij}(\theta_i - \theta_j(t - T_{ji}(t))), \tag{9}$$

and $\alpha_i \in \mathbb{R}$ is a smooth time-varying function that is included to satisfy Brockett's condition and that will be designed later.

The complete closed-loop system given by (2), (3), (6) and (8) is

$$\begin{aligned}
\Sigma_v \left\{ \begin{array}{ll} \dot{\mathbf{z}}_i = & \boldsymbol{\varphi}(\theta_i)v_i \\ \dot{v}_i = & -\frac{1}{m_i}\left[p_{vi}\boldsymbol{\varphi}^\top(\theta_i)\mathbf{e}_i + d_{vi}v_i\right] \end{array} \right. & \quad \forall i \in \bar{N}, \\
\Sigma_\omega \left\{ \begin{array}{ll} \dot{\theta}_i = & \omega_i \\ \dot{\omega}_i = & -\frac{1}{I_i}\left[p_{\omega i}e_{\theta i} + d_{\omega i}\omega_i - \alpha_i(t,\theta_i,\mathbf{e}_i)\right] \end{array} \right. & \quad \forall i \in \bar{N}.
\end{aligned} \tag{10}$$

Next, we establish some important properties for Σ_ω, regarded as a linear autonomous system with variable time-delays and driven by the "input" $\alpha_i(t,\theta_i,\mathbf{e}_i)$.

Proposition 1: Suppose that Assumptions **A1** and **A2** hold. Set the damping gain $d_{\omega i}$ such that

$$2d_{\omega i} > p_{\omega i}\sum_{j\in\mathcal{N}_i} a_{ij}\left(\beta_i + \frac{{}^*T_{ij}^2}{\beta_j}\right), \quad \forall i \in \bar{N}, \tag{11}$$

for some $\{\beta_1,\ldots,\beta_N\} > 0$. In this scenario:

1) if $\alpha_i(t,\theta_i,\mathbf{e}_i) = 0$ then, for Σ_ω in (10), there exists $\theta_c \in \mathbb{R}$ such that $(\omega_i,\theta_i) = (0,\theta_c)$ is Globally Asymptotically Stable (GAS);
2) if $\alpha_i \in \mathcal{L}_\infty$ then $\dot{\omega}_i, \omega_i \in \mathcal{L}_\infty$;
3) if $\alpha_i \in \mathcal{L}_\infty$ and $\lim_{t\to\infty}\alpha_i(t,\theta_i(t),\mathbf{e}_i(t)) = 0$ then $\lim_{t\to\infty}\omega_i(t) = 0$ and $\lim_{t\to\infty}\theta_i(t) = \theta_c$. ◁

Proof. To establish item 1) of the proposition we set $\alpha_i = 0$ in Σ_ω. Consider the Lyapunov-Krasovskii functional

$$\mathcal{V} := \sum_{i=1}^N \left[\frac{I_i}{2p_{\omega i}}\omega_i^2 + \frac{1}{4}\sum_{j\in\mathcal{N}_i} a_{ij}(\theta_i-\theta_j)^2 + c_i\sum_{j\in\mathcal{N}_i} a_{ij}\int_{-{}^*T_{ji}}^0\int_{t+\sigma}^t \omega_j^2(\eta)d\eta d\sigma\right],$$

where c_i is a positive number. $\dot{\mathcal{V}}$ is given by

$$\dot{\mathcal{V}} = \sum_{i=1}^N\left[\frac{I_i}{p_{\omega i}}\omega_i\dot{\omega}_i + \frac{1}{2}\sum_{j\in\mathcal{N}_i} a_{ij}(\dot{\theta}_i-\dot{\theta}_j)(\theta_i-\theta_j) + c_i\sum_{j\in\mathcal{N}_i} a_{ij}\left({}^*T_{ji}\omega_j^2 - \int_{t-{}^*T_{ji}}^t \omega_j^2(\sigma)d\sigma\right)\right].$$

Evaluating $\dot{\mathcal{V}}$ along Σ_ω, in (10), yields

$$\dot{\mathcal{V}} = -\sum_{i=1}^N\left[\frac{d_{\omega i}}{p_{\omega i}}\omega_i^2 + \sum_{j\in\mathcal{N}_i} a_{ij}\omega_i\int_{t-T_{ji}(t)}^t \omega_j(\sigma)d\sigma - c_i\sum_{j\in\mathcal{N}_i} a_{ij}\left({}^*T_{ji}\omega_j^2 - \int_{t-{}^*T_{ji}}^t \omega_j^2(\sigma)d\sigma\right)\right],$$

where, to obtain this equation, we have used the fact that

$$\theta_j - \theta_j(t-T_{ji}(t)) = \int_{t-T_{ji}(t)}^t \dot{\theta}_j(\sigma)d\sigma = \int_{t-T_{ji}(t)}^t \omega_j(\sigma)d\sigma$$

and we have also invoked **A1** and the properties of the Laplacian matrix to prove that

$$\sum_{i=1}^N\sum_{j\in\mathcal{N}_i} a_{ij}(\dot{\theta}_i + \dot{\theta}_j)(\theta_i-\theta_j) = 0.$$

It also holds that, for any $\beta_i > 0$,

$$\begin{aligned}
-\omega_i\int_{t-T_{ji}(t)}^t \omega_j(\sigma)d\sigma &\le \frac{\beta_i}{2}\omega_i^2 + \frac{1}{2\beta_i}\left|\int_{t-T_{ji}(t)}^t \omega_j(\sigma)d\sigma\right|^2 \\
&\le \frac{\beta_i}{2}\omega_i^2 + \frac{{}^*T_{ji}}{2\beta_i}\int_{t-{}^*T_{ji}}^t \omega_j^2(\sigma)d\sigma.
\end{aligned}$$

Setting $c_i = \frac{{}^*T_{ji}}{2\beta_i}$ returns

$$\dot{\mathcal{V}} \le -\sum_{i=1}^N\left[\frac{d_{\omega i}}{p_{\omega i}}\omega_i^2 - \sum_{j\in\mathcal{N}_i} a_{ij}\left(\frac{\beta_i}{2}\omega_i^2 + \frac{{}^*T_{ji}^2}{2\beta_i}\omega_j^2\right)\right]$$

and using $l_{ii} := \sum_{j\in\mathcal{N}_i} a_{ij}$, where l_{ii} is the diagonal element of the Laplacian, we obtain

$$\dot{\mathcal{V}} \le -\sum_{i=1}^N\left[\left(\frac{d_{\omega i}}{p_{\omega i}} - \frac{\beta_i}{2}l_{ii}\right)\omega_i^2 - \sum_{j\in\mathcal{N}_i} a_{ij}\frac{{}^*T_{ji}^2}{2\beta_i}\omega_j^2\right].$$

As in [19], defining $\mathbf{W} := \mathrm{col}(\omega_1^2,...,\omega_N^2)$ and

$$\boldsymbol{\Psi} = \begin{bmatrix} \frac{d_{\omega 1}}{p_{\omega 1}} - \frac{\beta_1}{2}l_{11} & -\frac{{}^*T_{21}^2}{2\beta_1}a_{12} & \cdots & -\frac{{}^*T_{N1}^2}{2\beta_1}a_{1N} \\ -\frac{{}^*T_{12}^2}{2\beta_2}a_{21} & \frac{d_{\omega 2}}{p_{\omega 2}} - \frac{\beta_2}{2}l_{22} & \cdots & -\frac{{}^*T_{N2}^2}{2\beta_2}a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{{}^*T_{1N}^2}{2\beta_N}a_{N1} & -\frac{{}^*T_{2N}^2}{2\beta_N}a_{N2} & \cdots & \frac{d_{\omega N}}{p_{\omega N}} - \frac{\beta_N}{2}l_{NN} \end{bmatrix},$$

we can write $\dot{\mathcal{V}} \le -\mathbf{1}_N^\top\boldsymbol{\Psi}\mathbf{W}$ or, equivalently,

$$\dot{\mathcal{V}} \le -\sum_{i=1}^N\left(\frac{d_{\omega i}}{p_{\omega i}} - \sum_{j\in\mathcal{N}_i} a_{ij}\left(\frac{\beta_i}{2} + \frac{{}^*T_{ij}^2}{2\beta_j}\right)\right)\omega_i^2.$$

Then, setting $d_{\omega i}$ such that (11) holds, it follows that there exists $\lambda_i > 0$ such that $\dot{\mathcal{V}} \le -\sum_{i=1}^N \lambda_i\omega_i^2$. Thus $\omega_i \in \mathcal{L}_2$. Further, since $\mathcal{V}$ is positive definite and radially unbounded with regards to ω_i and $\theta_i - \theta_j$, then these signals are bounded. Which, in turn, imply that $\dot{\omega}_i \in \mathcal{L}_\infty$. Invoking Barbalat's Lemma it is established that $\lim_{t\to\infty}\omega_i(t) = 0$. Obviously, from Σ_ω in (10), if we can prove that $\lim_{t\to\infty}\dot{\omega}_i(t) = 0$ then $\lim_{t\to\infty} e_{\theta i}(t) = 0$. This last, in turn, implies that

$$\lim_{t\to\infty}\sum_{j\in\mathcal{N}_i} a_{ij}\left(\theta_i(t) - \theta_j(t) + \int_{t-T_{ji}(t)}^t \omega_j(\sigma)d\sigma\right) = 0.$$

Since ω_i converges for all $i \in \{1,\dots,N\}$ and $T_{ji}(t)$ is globally bounded, then

$$\lim_{t\to\infty} \sum_{j\in\mathcal{N}_i} a_{ij}(\theta_i(t)-\theta_j(t)) = 0.$$

Defining $\boldsymbol{\theta} := [\theta_1,\dots,\theta_N]^\top \in \mathbb{R}^N$, then we have that $\lim_{t\to\infty} \mathbf{L}\boldsymbol{\theta}(t) = \mathbf{0}$ which implies that $\lim_{t\to\infty} \theta_i(t) = \theta_c$.

Now, since

$$\lim_{t\to\infty} \int_0^t \dot{\omega}_i(\sigma)d\sigma = \lim_{t\to\infty} \omega_i(t) - \omega_i(0) = \omega_i(0),$$

then such limit exists. Moreover,

$$\ddot{\omega}_i = -\frac{1}{I_i}\Big[p_{\omega i} \sum_{j\in\mathcal{N}_i} a_{ij}\big(\omega_i - (1-\dot{T}_{ji})\omega_j(t-T_{ji}(t))\big) + d_{\omega i}\dot{\omega}_i\Big].$$

which, in view of Assumption **A2** and the fact that $\dot{\omega}_i, \omega_i \in \mathcal{L}_\infty$ imply that $\dot{\omega}_i$ is uniformly continuous. Thus, by Barbalat's Lemma ω_i asymptotically converges to zero. This completes the proof of item 1).

Next, we establish item 2). To that end, we first recall that without input the equilibrium $(\omega_i,\theta_i) = (0,\theta_c)$ is GAS. Further Σ_ω is a marginally stable linear time-varying system with uniformly bounded time-delays. Hence, by Proposition 3 in [22] if $\alpha_i(t,\theta_i,\mathbf{e}_i)$ is bounded then $\dot{\omega}_i, \omega_i \in \mathcal{L}_\infty$.

Item 3) is established also by Proposition 3 in [22]. This finishes the proof of the proposition. □

In what follows we establish the fact that $(v_i, \boldsymbol{\varphi}^\top(\theta_i)\mathbf{e}_i) = (0,0)$ is GAS, for system Σ_v in (10).

Proposition 2: Suppose that Assumptions **A1** and **A2** hold. Set the damping gain d_{vi} such that

$$2d_{vi} > p_{vi} \sum_{j\in\mathcal{N}_i} a_{ij}\left(\beta_i + \frac{{}^*T_{ij}^2}{\beta_j}\right), \quad \forall i \in \bar{N}, \tag{12}$$

for some $\{\beta_1,\dots,\beta_N\} > 0$. Then $v_i \in \mathcal{L}_\infty \cap \mathcal{L}_2$, $\|\bar{\mathbf{z}}_i - \bar{\mathbf{z}}_j\| \in \mathcal{L}_\infty$ and $\lim_{t\to\infty} v_i(t) = 0$. If, additionally, $\omega_i \in \mathcal{L}_\infty$ then $\lim_{t\to\infty} \boldsymbol{\varphi}^\top(\theta_i(t))\mathbf{e}_i(t) = 0$. ◁

Proof. Consider the Lyapunov-Krasovskii functional

$$\mathcal{W} := \sum_{i=1}^N \left[\frac{m_i}{2p_{vi}}v_i^2 + \frac{1}{4}\sum_{j\in\mathcal{N}_i} a_{ij}\|\bar{\mathbf{z}}_i - \bar{\mathbf{z}}_j\|^2 + c_i \sum_{j\in\mathcal{N}_i} a_{ij}\int_{-{}^*T_{ji}}^0 \int_{t+\sigma}^t v_j^2(\eta)d\eta d\sigma\right].$$

whose total derivative yields

$$\dot{\mathcal{W}} = \sum_{i=1}^N \left[\frac{m_i}{p_{vi}}v_i\dot{v}_i + \frac{1}{2}\sum_{j\in\mathcal{N}_i} a_{ij}(\dot{\mathbf{z}}_i - \dot{\mathbf{z}}_j)^\top(\bar{\mathbf{z}}_i - \bar{\mathbf{z}}_j) + c_i \sum_{j\in\mathcal{N}_i} a_{ij}\left({}^*T_{ji}v_j^2 - \int_{t-{}^*T_{ji}}^t v_j^2(\sigma)d\sigma\right)\right].$$

Since $\boldsymbol{\varphi}^\top(\theta_i)\boldsymbol{\varphi}(\theta_i) = 1$ then $\|\dot{\mathbf{z}}_i\|^2 = v_i^2$. This last, setting $c_i = \frac{{}^*T_{ji}}{2\beta_i}$ and using the same arguments as in the proof of item 1) in Proposition 1, we get

$$\dot{\mathcal{W}} \le -\sum_{i=1}^N \left(\frac{d_{vi}}{p_{vi}} - \sum_{j\in\mathcal{N}_i} a_{ij}\left(\frac{\beta_i}{2} + \frac{{}^*T_{ij}^2}{2\beta_j}\right)\right) v_i^2.$$

Therefore, for all $i \in \bar{N}$, setting d_{vi} satisfying (12) ensures that $v_i \in \mathcal{L}_\infty \cap \mathcal{L}_2$ and $\|\bar{\mathbf{z}}_i - \bar{\mathbf{z}}_j\| \in \mathcal{L}_\infty$. This implies that $\mathbf{e}_i \in \mathcal{L}_\infty$ and, since $\boldsymbol{\varphi}(\theta_i)$ is bounded, $\dot{v}_i \in \mathcal{L}_\infty$. Therefore, by Barbalat's Lemma, $\lim_{t\to\infty} v_i(t) = 0$. This completes the first part of the proof.

Since

$$\lim_{t\to\infty} \int_0^t \dot{v}_i(\sigma)d\sigma = \lim_{t\to\infty} v_i(t) - v_i(0) = v_i(0),$$

we only need to prove that $\dot{v}_i$ is uniformly continuous to establish convergence of $\boldsymbol{\varphi}^\top(\theta_i)\mathbf{e}_i$ to zero. First note that

$$\ddot{v}_i = -\frac{1}{m_i}\big[p_{vi}\omega_i\boldsymbol{\varphi}^{\perp\top}(\theta_i)\mathbf{e}_i + p_{vi}\boldsymbol{\varphi}^\top(\theta_i)\dot{\mathbf{e}}_i + d_{vi}\dot{v}_i\big], \tag{13}$$

where we have defined $\boldsymbol{\varphi}^\perp$ as the annihilator of $\boldsymbol{\varphi}(\theta_i)$, *i.e.*, $\boldsymbol{\varphi}^\top(\theta_i)\boldsymbol{\varphi}^\perp(\theta_i) = \boldsymbol{\varphi}^{\perp\top}(\theta_i)\boldsymbol{\varphi}(\theta_i) = 0$, and it is given by

$$\boldsymbol{\varphi}^\perp(\theta_i) = \begin{bmatrix} -\sin(\theta_i) \\ \cos(\theta_i) \end{bmatrix}. \tag{14}$$

Moreover, since $\dot{\theta}_i = \omega_i$, $\dot{\boldsymbol{\varphi}}(\theta_i) = \omega_i\boldsymbol{\varphi}^\perp(\theta_i)$ and $\dot{\boldsymbol{\varphi}}^\perp(\theta_i) = -\omega_i\boldsymbol{\varphi}(\theta_i)$.

The fact that $v_i \in \mathcal{L}_\infty \cap \mathcal{L}_2$, $\|\bar{\mathbf{z}}_i - \bar{\mathbf{z}}_j\| \in \mathcal{L}_\infty$ and (by assumption) $\omega_i \in \mathcal{L}_\infty$ ensure that $\ddot{v}_i \in \mathcal{L}_\infty$ and thus $\dot{v}_i$ is uniformly continuous. This completes the proof. □

Remark 2: A key step in the solution of the **(LFP)** problem is the design of the function $\alpha_i(t,\theta_i,\mathbf{e}_i)$. On one hand, from Proposition 1, α_i has to be a bounded time-varying function to ensure that $\omega_i \in \mathcal{L}_\infty$. Since $\lim_{t\to\infty} \boldsymbol{\varphi}^\top(\theta_i(t))\mathbf{e}_i(t) = 0$ does not imply that $\lim_{t\to\infty} \mathbf{e}_i(t) = 0$, on the other hand, α_i has to be designed such that if α_i and $\boldsymbol{\varphi}^\top(\theta_i)\mathbf{e}_i$ converge to zero then $\mathbf{e}_i$ converges to zero. △

In order to accomplish the second objective in Remark 2, we first note that

$$\begin{bmatrix} \boldsymbol{\varphi}^\top(\theta_i) \\ \boldsymbol{\varphi}^{\perp\top}(\theta_i) \end{bmatrix} = \begin{bmatrix} \cos(\theta_i) & \sin(\theta_i) \\ -\sin(\theta_i) & \cos(\theta_i) \end{bmatrix}$$

is a full-rank matrix. Proposition 2 establishes the conditions under which $\lim_{t\to\infty} \boldsymbol{\varphi}^\top(\theta_i(t))\mathbf{e}_i(t) = 0$. Hence, if we establish that $\lim_{t\to\infty} \boldsymbol{\varphi}^{\perp\top}(\theta_i(t))\mathbf{e}_i(t) = 0$ then we will have that

$$\lim_{t\to\infty} \begin{bmatrix} \boldsymbol{\varphi}^\top(\theta_i(t)) \\ \boldsymbol{\varphi}^{\perp\top}(\theta_i(t)) \end{bmatrix} \mathbf{e}_i(t) = \mathbf{0}_2,$$

whose only possible solution is $\lim_{t\to\infty} \mathbf{e}_i(t) = \mathbf{0}_2$.

After such reasoning, we design of α_i as

$$\alpha_i(t,\theta_i,\mathbf{e}_i) := k_{\alpha i} f_i(t)\boldsymbol{\varphi}^{\perp\top}(\theta_i)\mathbf{e}_i, \tag{15}$$

where $k_{\alpha i} > 0$ and $f_i(t)$ is any time-varying function such that $f_i \in \mathcal{C}^2$, $f_i, \dot{f}_i, \ddot{f}_i \in \mathcal{L}_\infty$, $\lim_{t\to\infty} f_i(t) \neq 0$ and $\lim_{t\to\infty} \dot{f}_i(t) \neq 0$.

Remark 3: There exist several functions that satisfy the conditions on $f_i(t)$, one trivial example is: $f_i(t) =$

$A_i \sin(\omega_i^\star t)$, where $A_i, \omega_i^\star \in \mathbb{R}$ are the magnitude and the frequency, respectively. $\triangle$

We are now ready to establish the main result of this work.

Theorem 1: Controller (3), (6), (8) and (15) solves the **(LFP)** problem, provided that (11) and (12) hold. $\triangleleft$

Proof. First we invoke the first part of Proposition 2 to show that $v_i \in \mathcal{L}_\infty \cap \mathcal{L}_2$, $\|\bar{\mathbf{z}}_i - \bar{\mathbf{z}}_j\| \in \mathcal{L}_\infty$ and $\lim_{t\to\infty} v_i(t) = 0$.

These bounded signals imply that $\mathbf{e}_i \in \mathcal{L}_\infty$. Hence, the function α_i defined in (15) is bounded. Therefore, we may invoke item 2) of Proposition 1 to conclude that $\dot{\omega}_i, \omega_i \in \mathcal{L}_\infty$. Hence, following the second part of Proposition 2, it holds that $\lim_{t\to\infty} \boldsymbol{\varphi}^\top(\theta_i(t))\mathbf{e}_i(t) = 0$.

Moreover, in Proposition 2, it has also been proved that $\lim_{t\to\infty} \dot{v}_i(t) = 0$. Hence

$$\lim_{t\to\infty} \int_0^t \ddot{v}_i(\sigma)d\sigma = \lim_{t\to\infty} \dot{v}_i(t) - \dot{v}_i(0) = -\dot{v}_i(0).$$

So, if we prove that $\ddot{v}_i$ is uniformly continuous then $\lim_{t\to\infty} \ddot{v}_i(t) = 0$. Using (13) we obtain $\dddot{v}_i$ as

$$\begin{aligned}\frac{m_i}{p_{vi}}\dddot{v}_i = &-\dot{\omega}_i\boldsymbol{\varphi}^{\perp\top}(\theta_i)\mathbf{e}_i + \omega_i^2\boldsymbol{\varphi}^\top(\theta_i)\mathbf{e}_i - 2\omega_i\boldsymbol{\varphi}^{\perp\top}(\theta_i)\dot{\mathbf{e}}_i \\ &- \boldsymbol{\varphi}^\top(\theta_i)\ddot{\mathbf{e}}_i - \frac{d_{vi}}{p_{vi}}\ddot{v}_i.\end{aligned} \tag{16}$$

Assumption **A2** and all the bounded signals imply that $\dddot{v}_i \in \mathcal{L}_\infty$. Thus, $\lim_{t\to\infty} \ddot{v}_i(t) = 0$. Using the previous signal chasing method and invoking systematically Barbalat's Lemma, it can also be concluded that $\lim_{t\to\infty} \dddot{v}_i(t) = 0$ and that $\lim_{t\to\infty} \frac{d}{dt}\dddot{v}_i(t) = 0$.

Now, $\frac{d}{dt}\dddot{v}_i$ is given by

$$\begin{aligned}\frac{m_i}{p_{vi}}\frac{d}{dt}\dddot{v}_i = &-\ddot{\omega}_i\boldsymbol{\varphi}^{\perp\top}(\theta_i)\mathbf{e}_i - 3\dot{\omega}_i\boldsymbol{\varphi}^{\perp\top}(\theta_i)\dot{\mathbf{e}}_i + \omega_i^3\boldsymbol{\varphi}^{\perp\top}(\theta_i)\mathbf{e}_i \\ &- 3\omega_i\left[\boldsymbol{\varphi}^{\perp\top}(\theta_i)\ddot{\mathbf{e}}_i - \dot{\omega}_i\boldsymbol{\varphi}^\top(\theta_i)\mathbf{e}_i - \omega_i\boldsymbol{\varphi}^\top(\theta_i)\dot{\mathbf{e}}_i\right] \\ &- \boldsymbol{\varphi}^\top(\theta_i)\dddot{\mathbf{e}}_i - \frac{d_{vi}}{p_{vi}}\dddot{v}_i.\end{aligned} \tag{17}$$

Since $v_i, \dot{v}_i, \ddot{v}_i, \dddot{v}_i, \frac{d}{dt}\dddot{v}_i$ and $\boldsymbol{\varphi}^\top(\theta_i)\mathbf{e}_i$ converge to zero, they imply that the terms $\omega_i\boldsymbol{\varphi}^{\perp\top}(\theta_i)\mathbf{e}_i$, $\dot{\omega}_i\boldsymbol{\varphi}^{\perp\top}(\theta_i)\mathbf{e}_i$ and $\ddot{\omega}_i\boldsymbol{\varphi}^{\perp\top}(\theta_i)\mathbf{e}_i$ also converge to zero. Substituting $\ddot{\omega}_i$, given by

$$\frac{I_i}{k_{\alpha i}}\ddot{\omega}_i = -\frac{p_{\omega i}}{k_{\alpha i}}\dot{e}_{\theta i} - \frac{d_{\omega i}}{k_{\alpha i}}\dot{\omega}_i - f_i\omega_i\boldsymbol{\varphi}^\top\mathbf{e}_i + f_i\boldsymbol{\varphi}^{\perp\top}\dot{\mathbf{e}}_i + \dot{f}_i\boldsymbol{\varphi}^{\perp\top}\mathbf{e}_i,$$

in the first right hand side term of (17), *i.e.*, $\ddot{\omega}_i\boldsymbol{\varphi}^{\perp\top}(\theta_i)\mathbf{e}_i$, supports the fact that the resulting term $\dot{f}_i\left(\boldsymbol{\varphi}^{\perp\top}\mathbf{e}_i\right)^2$ also converges to zero. The design of $f_i(t)$ guarantees that $\dot{f}_i(t)$ does not vanish. Hence $\lim_{t\to\infty} \boldsymbol{\varphi}^{\perp\top}(\theta_i(t))\mathbf{e}_i(t) = 0$, as required. This establishes the proof that $\lim_{t\to\infty} \mathbf{e}_i(t) = 0$ and this, in turn, implies that $\lim_{t\to\infty} \bar{\mathbf{z}}_i(t) = \bar{\mathbf{z}}_c$. The proof is completed by invoking item 3) of Proposition 1. $\square$

V. Simulations

In order to show the effectiveness of the proposed scheme, this section presents simulation results with six differential drive robots, whose communication topology and formation pattern are depicted in Fig. 2. For simplicity, the physical parameters are the same for all the robots, the mass is $m_i = 10kg$, the moment of inertia is $I_i = 3Kgm^2$, the distance between point **Q** and the wheels is $R = 0.3m$ and the radius of the wheels is $r = 0.05m$. These values are the same as those used in [23].

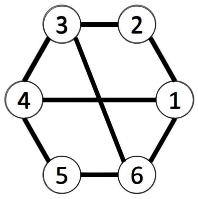


Fig. 2. Communication topology and formation pattern.

TABLE I

Initial positions for the robots and desired relative position.

	1	2	3	4	5	6
$x_i(0)$	5	7	7	3	1	1
$y_i(0)$	2	5.5	3.5	2	3.5	5.5
$\theta_i(0)$	0	$-\frac{\pi}{4}$	$-\frac{\pi}{2}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{\pi}{4}$
δ_{xi}	2	1	-1	-2	-1	1
δ_{yi}	0	2	2	0	-2	-2

The initial velocities are all set to zero and the initial positions, together with the relative desired positions, are shown in Table I. The controller gains have been set to $p_{vi} = 300$, $d_{vi} = 600$, $p_{\omega i} = 30$, $d_{\omega i} = 60$, $k_{\alpha i} = 150$ these gains satisfy the bounds (11) and (12) with an upper-bound ${}^*T_{ji} = 0.33$. The transmission delays, depicted in Fig. 3, follow a normal Gaussian distribution with a mean of 0.2 and a variance of 0.001. The time-varying function $f_i(t)$ is the same for all robots and it is given by $f_i(t) = \frac{5}{2} + 2\pi\sin(t) + \pi\sin(2t) + \frac{2}{3}\sin(3t)$.

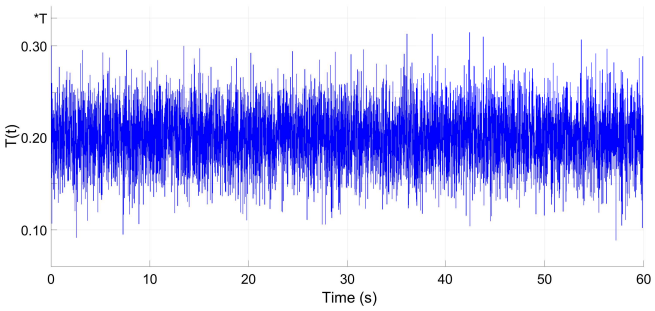


Fig. 3. Variable time-delay with an upper-bound at ${}^*T$.

The position trajectories of the non-holonomic robots are shown in Fig. 4, from which one can see that the desired hexagon formation is achieved. The orientation behavior is presented in Fig. 5, it takes about 30 seconds for the network of robots to reach a consensus. Hence, the **(LFP)** is solved, as desired.

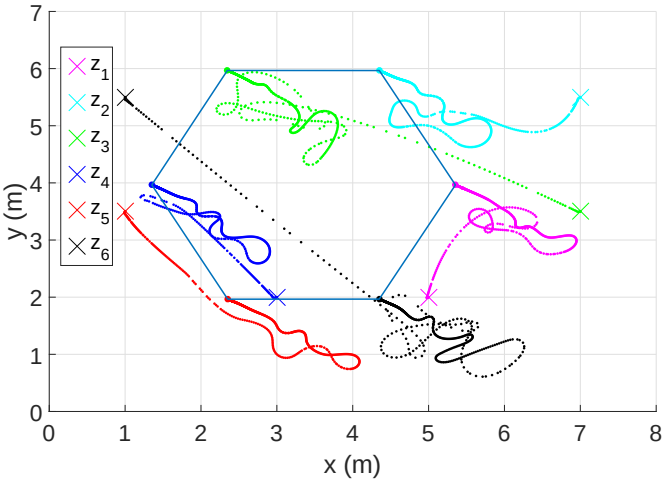


Fig. 4. Position trajectories in the Cartesian xy-plane.

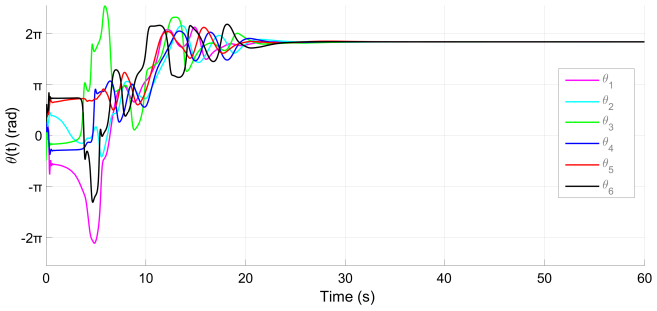


Fig. 5. Robot orientations.

VI. Conclusions

In this work we solve the leaderless consensus-based formation control of swarms of a class of nonholonomic mobile robots. We consider the real scenario where the robots are modeled by second order dynamics and where unreliable communications that exhibit variable time-delays arise. The proposed smooth time-varying controller is a simple to implement Proportional plus damping (P+d) scheme.

Future work include the solution to the leader-follower formation control and the design of a controller that does not rely on velocity measurements. In order for the robots not to collide while they find their respective position in the formatin, we also plan to include a collision avoidance term in the proposed controller.

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