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ARTICLE HISTORY

ABSTRACT
After (Lizárraga, 2004) it is known that for nonholonomic systems it is impossible to design a *universal* controller able to asymptotically stabilize *any feasible* reference trajectory. In this paper we present a smooth time-varying controller able to stabilize a wide class of reference trajectories that include converging (*parking problem*) and persistently exciting (*tracking problem)* ones, as well as *set-points*. To the best of our knowledge, for the first time in the literature, we establish uniform global asymptotic stability for the origin of the closed-loop system in the kinematics state space. We also show that the kinematics controller renders the system robust to perturbations in the sense of integral-input-to-state stability. Then, we show that for the case in which the velocity dynamics equations are also considered (full model), *any* velocity-tracking controller with the property that the error velocities are square integrable may be used to ensure global tracking or stabilization. This modularity and robustness of our controller, added to the strength of our stability statements, renders possible the extension of our main results to the difficult scenario of control under parametric uncertainty.

KEYWORDS
Persistency of excitation, autonomous vehicles, Lyapunov's method, input-to-state stability

1. Introduction

In the well-known paper Brockett (1983) necessary conditions for asymptotic set-point stabilization of nonlinear systems via continuous controllers are given. In particular, it is showed that nonholonomic systems are not stabilizable to a point via continuous autonomous feedback. The seminal paper Lizárraga (2004) extends the results of Brockett (1983) for smooth control systems by establishing sufficient conditions for the *non* existence of *universal* continuous stabilizers (even time-varying) of arbitrary feasible trajectories. A particular but fundamental contribution of Lizárraga (2004) is that for nonholonomic systems it is beyond reach to design a universal controller (even time-varying) capable of stabilizing an arbitrary feasible trajectory.

The work of Mohamed Maghenem was carried out while he was with Univ Paris-Saclay, Saclay, France.
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†The work of E. Panteley is partially suported by the Government of the Russian Federation under grant 074-U01

The results of Brockett (1983) certainly triggered the interest of the research community on different stabilization and tracking control problems for nonholonomic systems and, in accordance with Lizárraga (2004), available results apply either to the set-point control problem —see *e.g.* Astolfi (1996); Bayat, Mobayen, and Javadi (2016), to the so-called parking control problem —see *e.g.* Lee, Song, Lee, and Teng (2001); Morin and Samson (2003), or to other kinds of restricted time-varying trajectories —see *e.g.* Dixon, Dawson, Zergeroglu, and Behal (2001); Kanayama, Kimura, Miyazaki, and Naguchi (1990); Loría, Dasdemir, and Alvarez-Jarquin (2016). Tracking control in the sense defined in Kanayama et al. (1990) is addressed, *e.g.*, via nonlinear backstepping control in Jiang and Nijmeijer (1997). There also exist several remarkable works under stringent conditions: for instance, parametric uncertainty is coped with via adaptive control in Dixon, de Queiroz, Dawson, and Flynn (2004); Fukao, Nakagawa, and Adachi (2000); Huang, Wen, Wang, and Jiang (2014), control under input constraints is addressed *e.g.* in Consolini, Morbidi, Prattichizzo, and Tosques (2008); Jiang, Lefeber, and Nijmeijer (2001); Lee et al. (2001).

In Panteley, Lefeber, Loría, and Nijmeijer (1998) a simple *linear* time-varying controller was proposed and, for the first time in the literature, persistency of excitation was explicitly imposed as a necessary and sufficient condition for stabilization of a time-varying trajectory. The underlying ideas, however, are already present in early work by C. Samson de Wit, Khennouf, Samson, and Sørdalen (1993). Ever since, persistency of excitation has been recurrently used in the literature to design smooth controllers for autonomous vehicles —see *e.g.*, Dixon, Dawson, Zhang, and Zergeroglu (2000); Lefeber (2000); Miao and Wang (2015); Wang, Miao, Zhong, and Pan (2015) to mention a few. However, controllers relying on persistency-of-excitation conditions on the reference trajectories fail in other stabilization tasks, as for instance, in the case that the reference velocities tend to zero, as in the case of the parking-control problem Lee et al. (2001). In the latter, as also in Cao and Tian (2007), nonlinear time-varying controllers are designed to allow for reference velocity trajectories that converge to zero. Furthermore, it is worth to emphasize that Lee et al. (2001) covers the case when both the angular and forward velocity may converge to zero.

For the set-point stabilization problem it is also possible to use persistency of excitation as a stabilization mechanism, but the property needs to be redefined for functions that are state-dependent. To the best of our knowledge, this was done for the first time in Loría, Panteley, and Teel (1999) where, inspired by the seminal paper Samson (1995), the so-called δ-persistently-exciting controllers were introduced[1]. See also Wang et al. (2015) for more recent references.

Designing a unique controller capable of stabilizing bounded converging or diverging (*i.e.*, persistently exciting), reference velocity trajectories is not only a major challenge but, based on Lizárraga (2004), it may not appear as an overstatement to say that it is close to the broadest solvable control problem for nonholonomic systems. To the best of our knowledge, it has been addressed only in Dixon et al. (2004); Do, Jiang, and Pan (2004b); Lee et al. (2001); Miao and Wang (2015); Morin and Samson (2003); Wang et al. (2015).

The problem is solved for particular cases of the reference velocity trajectories in Lee et al. (2001), *e.g.*, it is required that either the forward or the angular reference velocities is separated from zero. In Morin and Samson (2003) the elegant transverse-functions approach is presented; a unified velocity controller for generic chained-form

[1] Persistency of excitation in the sense defined in Loría et al. (1999) is also implicitly used in the earlier reference Samson (1993)

systems (hence beyond the unicycle particular system) is proposed. A similar result is presented in Dixon et al. (2004) using an adaptive-control approach. In the latter two references convergence of the tracking errors in a practical sense is established. The method of Wang et al. (2015), which is based on the design of δ-persistently exciting controllers, is appealing in the sense that it consists in using the combination of a tracking controller and a stabilization controller carefully weighted by a function that depends on the leader velocities, thereby favoring the action of either controller; a similar controller, assuming parametric uncertainty, is presented in Miao and Wang (2015). In Do et al. (2004b) a Lyapunov-based unified controller is proposed in order to make the tracking error converge to zero in either of the *tracking* and the *parking* scenarios.

In this paper we address the simultaneous tracking-stabilization problem, based on the approach of Wang et al. (2015). To the best of our knowledge for the first time in the literature, we establish uniform global asymptotic stability (UGAS), as opposed to the weaker property of non-uniform convergence. The importance of UGAS for time-varying systems cannot be overestimated; only this property guarantees robustness of the system with respect to bounded disturbances in the sense of *total stability* Malkin (1944) —a concept better known nowadays as *local* input-to-state stability.

Furthermore, our proofs of UGAS and iISS rely on Lyapunov's direct method, which is fundamental to establish asymptotic tracking in the case when the full model (comprising the dynamics and the kinematics equations) is considered. Hence, another contribution of this paper is to establish that *any* velocity controller that guarantees velocity tracking, including under parametric uncertainty, may be easily incorporated. Such a statement is not possible to obtain without guaranteeing uniform global asymptotic stability for the kinematics model.

The design of our Lyapunov functions follows the efficient *Mazenc construction* method Mazenc (2003) which, loosely speaking, consists in designing a strict Lyapunov function upon a preliminary non-strict one —see Malisoff and Mazenc (2009) for further detail. For the robustness properties that we establish we also appeal to technical results in Angeli, Ingalls, Sontag, and Wang (2004); Angeli, Sontag, and Wang (2000). Although Lyapunov's first method has been used, *e.g.*, in Do et al. (2004b) the conditions in the latter reference are more conservative.

The rest of the paper is organized as follows. In Section 2 we formulate the control problem and we present our main theoretical findings. In Section 3 we present the proofs of our main results. Simulations that illustrate our theoretical findings are presented in Section 4 and concluding remarks are given in Section 5.

2. Problem formulation and its solution

2.1. *Problem statement*

Let us consider the following dynamical model of a force-controlled nonholonomic vehicle:

$$\begin{cases} \dot{x} &= v\cos\theta \\ \dot{y} &= v\sin\theta \\ \dot{\theta} &= \omega \end{cases} \tag{1}$$

$$\begin{cases} \dot{v} &= f_1(t,v,\omega,z) + g_1(t,v,\omega,z)u_1 \\ \dot{\omega} &= f_2(t,v,\omega,z) + g_2(t,v,\omega,z)u_2 \end{cases} \tag{2}$$

where v and ω denote the forward and angular velocities respectively, the first two elements of $z := [x\ y\ \theta]^\top$ correspond to the Cartesian coordinates of a point on the robot with respect to a fixed reference frame, and θ denotes the robot's orientation with respect to the same frame. The functions f_1 and f_2 are assumed to possess the minimal properties for existence and uniqueness of (Caratheodory) solutions, and u_1 and u_2 correspond to the two control inputs (proportional to wheel torques). Thus, the equations (1) correspond to the kinematics model while (2) correspond to the force-balance equations.

The tracking-control problem consists in making the robot follow a fictitious reference vehicle modeled by

$$\begin{aligned} \dot{x}_r &= v_r \cos\theta_r && \text{(3a)} \\ \dot{y}_r &= v_r \sin\theta_r && \text{(3b)} \\ \dot{\theta}_r &= \omega_r, && \text{(3c)} \end{aligned}$$

that moves about with reference velocities $v_r(t)$ and $\omega_r(t)$. More precisely, it is desired to steer the differences between the Cartesian coordinates to some values d_x, d_y, and to zero the orientation angles and the velocities of the two robots, that is, the quantities

$$p_\theta := \theta_r - \theta, \quad p_x := x_r - x - d_x, \quad p_y := y_r - y - d_y.$$

The distances d_x, d_y define the position of the robot with respect to the (virtual) leader and are assumed to be constant. Then, as it is customary, we transform the error coordinates $[p_\theta\ p_x\ p_y]$ of the leader robot from the global coordinate frame to local coordinates fixed on the robot, that is, we define

$$\begin{bmatrix} e_\theta \\ e_x \\ e_y \end{bmatrix} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} p_\theta \\ p_x \\ p_y \end{bmatrix}. \tag{4}$$

In these new coordinates, the error kinematics equations become

$$\begin{aligned} \dot{e}_\theta &= \omega_r(t) - \omega && \text{(5a)} \\ \dot{e}_x &= \omega e_y - v + v_r(t)\cos(e_\theta) && \text{(5b)} \\ \dot{e}_y &= -\omega e_x + v_r(t)\sin(e_\theta). && \text{(5c)} \end{aligned}$$

The complete system also includes Eqs (2).

Generally speaking, the control problem consists in steering the error trajectories $e(t)$, which are solutions of (5), to zero via the inputs u_1 and u_2 in (2). A natural method consists in designing, first, virtual control laws w^* and v^* so that,

$$\lim_{t\to\infty} e(t) = 0, \qquad e = [e_\theta\ e_x\ e_y]^\top. \tag{6}$$

Then, to design control inputs u_1 and u_2 such that

$$\lim_{t\to\infty} (\tilde{v}, \tilde{w}) = (0, 0) \tag{7}$$

where

$$\tilde{v} := v - v^*, \quad \tilde{\omega} := \omega - \omega^*. \tag{8}$$

Depending on the conditions on the reference trajectories v_r and ω_r we identify the following mutually-exclusive scenarios:
Tracking scenario (S1): it is assumed that there exist T and $\mu > 0$ such that

$$\int_t^{t+T} \left[|v_r(\tau)|^2 + |\omega_r(\tau)|^2 \right] d\tau \geq \mu \quad \forall t \geq 0. \tag{9}$$

Stabilization scenario (S2): it is assumed that $|v_r(t)| + |\omega_r(t)| \to 0$ and there exists $\beta > 0$ such that, for all $t \geq t_\circ$:

$$\int_{t_\circ}^{t} \left[|v_r(\tau)| + |\omega_r(\tau)| \right] d\tau < \beta, \quad \forall t \geq t_\circ. \tag{10}$$

2.2. *Main result*

Under the conditions described above, we design a controller that achieves the trajectory tracking objective (6), (7) in either of the two mutually-exclusive scenarios described above. Our contributions are the following:

- we propose a class of control inputs v^* and ω^* that extends the controller proposed in Wang et al. (2015) and we ensure uniform global asymptotic stability of the origin for (5);
- for the velocity error kinematics in closed loop, we establish integral input-to-state stability with respect to the error velocities $[\tilde{v}, \tilde{\omega}]$;
- for *any* control inputs u_1 and u_2 ensuring that $\tilde{v} \to 0$ and $\tilde{\omega} \to 0$, we establish global attractivity of the origin provided that the error velocities converge sufficiently fast (they are square-integrable).

The control laws that ensure the properties above are:

$$\begin{aligned} v^* &:= v_r(t)\cos(e_\theta) + k_x e_x, & (11)\\ \omega^* &:= \omega_r + k_\theta e_\theta + k_y e_y v_r \phi(e_\theta) + \rho(t) k_y f(t, e_x, e_y) & (12) \end{aligned}$$

where ϕ is the so-called 'sinc' function defined by

$$\phi(e_\theta) := \frac{\sin(e_\theta)}{e_\theta},$$

$$\rho(t) := \exp\left(-\int_0^t \left[|v_r(\tau)| + |\omega_r(\tau)| \right] d\tau \right), \tag{13}$$

and $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^2 \to \mathbb{R}$ is a continuously differentiable function defined such that the following hypotheses hold.

A1. There exist a non-decreasing function $\sigma_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $\sigma_2 > 0$ such that

$$\begin{aligned} \max\left\{\frac{\partial f}{\partial t}, \frac{\partial f}{\partial e_x}, \frac{\partial f}{\partial e_y}\right\} &\leq \sigma_1(|\,[e_x\ e_y]\,|) & (14)\\ |f(t, e_x, e_y)| &\leq \sigma_2|\,[e_x\ e_y]\,|. & (15)\end{aligned}$$

A2. For the function

$$f_\circ(t, e_y) := f(t, 0, e_y) \tag{16}$$

we assume that $\partial f_\circ/\partial t$ is uniform $\delta-$persistently exciting with respect to e_y that is, for any $\delta > 0$ there exist μ_δ and $T_\delta > 0$ such that

$$|e_y| \geq \delta \implies \int_t^{t+T_\delta} \left|\frac{\partial f_\circ}{\partial t}(\tau, e_y)\right| d\tau \geq \mu_\delta \quad \forall\, t \geq 0 \tag{17}$$

—*cf.* (Loría, Panteley, Popovic, & Teel, 2002, Definition 3). Roughly speaking, the purpose of the function f is to excite the e_y–dynamics as long as $|e_y|$ is separated from zero.

The controller (12), which achieves both the tracking and the stabilization control goals, is a weighted sum of the tracking controller

$$\omega^*_{tra} := \omega_r + k_\theta e_\theta + k_y e_y v_r \phi(e_\theta),$$

and the stabilization controller

$$\omega^*_{stab} := \omega_r + k_\theta e_\theta + k_y f(t, e_x, e_y)$$

—*cf.* Miao and Wang (2015); Wang et al. (2015). The weight function $\rho(t)$ promotes the application of either ω^*_{tra} or ω^*_{stab}, depending on the task scenario **S1** or **S2**. More precisely, from (13) we see that ρ satisfies

$$\dot\rho = -\left[\,|v_r(t)| + |\omega_r(t)|\,\right]\rho \tag{18}$$

and $\rho \to 0$ exponentially fast if (9) holds. Hence, the tracking scenario **S1** is favoured. If, instead, (10) holds, the reference velocities converge and $\rho(t) > \exp(-\beta)$. Hence, the action of the stabilization controller is enhanced.

Remark 1. The idea of so merging the two controllers for the two scenarios **S1** and **S2** was introduced in Wang et al. (2015). The class of controllers satisfying **A1**–**A2** covers those in Wang et al. (2015); in particular, the function f is not necessarily globally bounded and may depend only on e_y. A more significant contribution with respect to the literature is that we establish uniform global asymptotic stability for (5) in closed-loop with $(v, \omega) = (v^*, \omega^*)$; this is in contrast with Wang et al. (2015) and Do, Jiang, and Pan (2004a) where it is proved that the convergence property (6) holds. In addition, we establish integral-input-to-state stability of (5) with respect to $[\tilde v, \tilde\omega]$.

Proposition 1 (Main result)**.** *Consider the system* (5) *with* $v = \tilde v + v^*$, $\omega = \tilde\omega + \omega^*$, *and the virtual inputs* (11) *and* (12)*. Let* k_x*,* k_θ*, and* $k_y > 0$*. Assume that there exist*

$\bar{\omega}_r$, $\bar{\dot{\omega}}_r$, $\bar{v}_r$, $\bar{\dot{v}}_r > 0$ *such that*[2]

$$|\omega_r|_\infty \leq \bar{\omega}_r, \quad |\dot{\omega}_r|_\infty \leq \bar{\dot{\omega}}_r, \quad |v_r|_\infty \leq \bar{v}_r, \quad |\dot{v}_r|_\infty \leq \bar{\dot{v}}_r.$$

Furthermore, assume that ***A1-A2*** *hold.*

Then, if either (9) *or* (10) *hold the closed-loop system resulting from* (5), (8), (11), *and* (12) *has the following properties:*

(P1) *if* $\tilde{v} = \tilde{\omega} = 0$, *the origin* $\{e = 0\}$ *is uniformly globally asymptotically stable;*

(P2) *the closed-loop system is integral input-to-state stable with respect to* $\eta := [\tilde{v}\ \tilde{\omega}]^\top$*;*

(P3) *if* $\eta \to 0$ *and* $\eta \in \mathcal{L}_2$, *then* (6) *holds.*

The proof is presented in Section 3. Below, we present an example of an adaptive controller that ensures that $\tilde{v}$, $\tilde{\omega} \to 0$ for any once continuously differentiable v^*, ω^*.

2.3. *Example*

As in Do (2007), we consider mobile robots modeled by

$$\dot{z} = J(z)\nu \tag{19a}$$

$$M\dot{\nu} + C(\dot{z})\nu = \tau \tag{19b}$$

where $z := [x\ y\ \theta]$ contains the Cartesian coordinates (x, y) and the orientation θ of the robot, $\tau \in \mathbb{R}^2$ corresponds to the (torque) control input; $\nu := [\nu_1\ \nu_2]$ stands for the angular velocities corresponding to the two robot's wheels, M is the inertia matrix, which is constant, symmetric and positive definite, and $C(\dot{z})$ is the matrix of Coriolis forces, which is skew-symmetric. In addition, we use the coordinate transformation matrix

$$J(z) = \frac{r}{2}\begin{bmatrix} \cos(\theta) & \cos(\theta) \\ \sin(\theta) & \sin(\theta) \\ 1/b & -1/b \end{bmatrix}$$

where r is the radius of either steering wheel and b is the distance from the center of either wheel to the Cartesian point (x, y). The relation between the wheels' velocities, ν, and the robot's velocities in the fixed frame, $\dot{z}$, is given by

$$\begin{bmatrix} v \\ \omega \end{bmatrix} := \frac{r}{2b}\begin{bmatrix} b & b \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} = \frac{1}{r}\begin{bmatrix} 1 & b \\ 1 & -b \end{bmatrix}\begin{bmatrix} v \\ \omega \end{bmatrix} \tag{20}$$

which may be used in (19a) to obtain the familiar model (1).

We assume that the inertia parameters and the constants contained in $C(\dot{z})$ are unknown while r and b are considered to be known. Let $\hat{M}$ and $\hat{C}$ denote, respectively, the estimates of M and C. Furthermore, let $\nu^* := [\nu_1^*\ \nu_2^*]^\top$,

$$\begin{bmatrix} \nu_1^* \\ \nu_2^* \end{bmatrix} = \frac{1}{r}\begin{bmatrix} 1 & b \\ 1 & -b \end{bmatrix}\begin{bmatrix} v^* \\ \omega^* \end{bmatrix}, \tag{21}$$

[2]For a continuous function $t \mapsto \varphi$ we define $|\varphi(t)|_\infty := \sup_{t\geq 0} |\varphi(t)|$.

and let us introduce the certainty-equivalence control law

$$\tau^* := \hat{M}\dot{\nu}^* + \hat{C}(\dot{z})\nu^* - k_d\sigma(\tilde{\nu}), \quad k_d > 0 \tag{22}$$

where $\tilde{\nu} := \nu - \nu^*$ and $\sigma(\,\cdot\,)$ is a continuous locally linear odd function for which there exist ϵ_1 and $\epsilon_2 > 0$ such that $\sigma(s)^\top s \geq \epsilon_1 |s|^2$ for all $|s| \leq \epsilon_2$ as, for instance, $\sigma(s) = s$, $\sigma(s) = [\tanh(s_1) \ \cdots \ \tanh(s_n)]^\top$, and many other saturation functions.

Then, let us define $\tilde{M} := \hat{M} - M$ and $\tilde{C} := \hat{C} - C$, so

$$\tau^* := M\dot{\nu}^* + C(\dot{z})\nu^* - k_d\sigma(\tilde{\nu}) + \tilde{M}\dot{\nu}^* + \tilde{C}\nu^* \tag{23}$$

and, setting $\tau = \tau^*$ in (19b), we obtain the closed-loop equation

$$M\dot{\tilde{\nu}} + C(\dot{z})\tilde{\nu} + k_d\sigma(\tilde{\nu}) = \Psi(\dot{z}, \dot{\nu}^*, \nu^*)^\top \tilde{\Theta} \tag{24}$$

where $\Theta \in \mathbb{R}^m$ is a vector of constant (unknown) lumped parameters in M and C, $\hat{\Theta}$ denotes the estimate of Θ, $\tilde{\Theta} := \hat{\Theta} - \Theta$ is the vector of estimation errors, and $\Psi : \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^{m\times 2}$ is a continuous known function. To obtain (24), we used the property that (19b) is linear in the constant lumped parameters. In addition, we use the passivity-based adaptation law $-$*cf.* Ortega and Spong (1989),

$$\dot{\hat{\Theta}} = -\gamma\Psi(\dot{z}, \dot{\nu}^*, \nu^*)\tilde{\nu}, \qquad \gamma > 0. \tag{25}$$

Then, a direct computation shows that the total derivative of

$$V(\tilde{\nu}, \tilde{\Theta}) := \frac{1}{2}\left[|\tilde{\nu}|^2 + \frac{1}{\gamma}|\tilde{\Theta}|^2\right]$$

along the trajectories of (24), (25), yields

$$\dot{V}(\tilde{\nu}, \tilde{\Theta}) \leq -k_d\sigma(\tilde{\nu})^\top \tilde{\nu}$$

which implies that $\dot{V}(\tilde{\nu}(t), \tilde{\Theta}(t)) \leq 0$. Integrating the latter from 0 to infinity, we obtain that $\tilde{\Theta}$, $\tilde{\nu} \in \mathcal{L}_\infty$. Then, there exists $\epsilon_2 > 0$ such that $|\tilde{\nu}(t)| \leq \epsilon_2$ for all $t \geq 0$ and, therefore, there also exists $\epsilon_1(\epsilon_2) > 0$ such that

$$\dot{V}(\tilde{\nu}(t), \tilde{\Theta}(t)) \leq -k_d|\tilde{\nu}(t))|^2.$$

Integrating the latter from 0 to infinity we obtain that $\tilde{\nu} \in \mathcal{L}_2$. It follows, *e.g.*, from (Ioannou & Sun, 1996, Lemma 3.2.5), that $\tilde{\nu} \to 0$ and, in view of (20),

$$\lim_{t\to\infty} |\tilde{v}(t)| + |\tilde{\omega}(t)| = 0. \tag{26}$$

Also, in view of (20), $\tilde{\nu} \in \mathcal{L}_2$ implies that $\eta \in \mathcal{L}_2$.

3. Proof of the main result

For each scenario, **S1** and **S2** we establish uniform global asymptotic stability for the closed-loop kinematics equation (5) restricted to $\eta = 0$ (**P1**). Then, we establish the iISS with respect to η (**P2**) by showing that the closed-loop trajectories are bounded, under the condition that η is square integrable —*cf.* Angeli et al. (2004).

3.1. *Under Scenario S1*

The proof of Proposition 1 under condition (9) is constructive; we provide a strict Lyapunov function. To that end, we start by observing that the error system (5), (8), (11) and (12) takes the form

$$\dot{e} = A_{v_r}(t,e)e + B_1(t,e)\rho(t) + B_2(e)\eta, \tag{27}$$

where $\rho(t)$ is defined in (13),

$$\begin{aligned} A_{v_r}(t,e) &:= \begin{bmatrix} -k_\theta & 0 & -v_r(t)k_y\phi(e_\theta) \\ 0 & -k_x & \varphi(t,e) \\ v_r(t)\phi(e_\theta) & -\varphi(t,e) & 0 \end{bmatrix}, \\ B_1(t,e) &:= \begin{bmatrix} -k_y f(t,e_x,e_y) \\ k_y f(t,e_x,e_y)e_y \\ -k_y f(t,e_x,e_y)e_x \end{bmatrix}, \ B_2(e) := \begin{bmatrix} 0 & -1 \\ -1 & e_y \\ 0 & -e_x \end{bmatrix} \end{aligned} \tag{28}$$

where $\varphi(t,e) := \omega_r(t) + k_\theta e_\theta + k_y e_y v_r(t)\phi(e_\theta)$. Writing the closed-loop dynamics as in (27) is convenient to stress that the "nominal" system $\dot{e} = A_{v_r}(t,e)e$ has a familiar structure encountered in model reference adaptive control. Moreover, defining

$$V_1(e) := \frac{1}{2}\left[e_x^2 + e_y^2 + \frac{1}{k_y}e_\theta^2\right], \tag{29}$$

we obtain, along the trajectories of $\dot{e} = A_{v_r}(t,e)e$,

$$\dot{V}_1(e) \le -k_x e_x^2 - \frac{k_\theta}{k_y}e_\theta^2.$$

This is a fundamental first step for the design of a strict Lyapunov function for the "perturbed" system (27), using the Mazenc construction method.

Now, to establish the proof in the case of scenario **S1**, we follow the following steps:

Step 1) We build a strict Lyapunov function $V(t,e)$ for the nominal system $\dot{e} = A_{v_r}(t,e)e$. This establishes **P1**.

Step 2) We construct a function $W(t,e)$ for the perturbed system $\dot{e} = A_{v_r}(t,e)e + B_1(t,e)\rho$.

Step 3) We use $W(t,e)$ to prove integral ISS of (27) with respect to η (*i.e.*, **P2**) as well as the boundedness of the trajectories under the assumption that $\eta \in \mathcal{L}_2$. This and the assumption that $\eta \to 0$ implies (6), *i.e.*, **P3**.

Step 1. We establish UGAS for the nominal system

$$\dot{e} = A_{v_r}(t,e)e \tag{30}$$

via Lyapunov's direct method[3]. Let $F_{[3]}, S_{[3]} : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$, and $P_{[k]} : \mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$

[3]This proof of uniform stability replaces the corresponding one proposed in Maghenem, Loría, and Panteley (2017), which is incorrect.

be smooth polynomials in V_1 with strictly positive and bounded coefficients of degree 3 and k respectively. After (Maghenem, Loría, & Panteley, 2016, Proposition 1), there exists a positive definite radially unbounded function $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$ defined as

$$\begin{aligned} V(t,e) \quad := \quad & P_{[2]}(t, V_1)V_1(e) - \omega_r(t)e_x e_y \\ & +v_r(t)P_{[1]}(t, V_1)e_\theta e_y, \end{aligned} \tag{31}$$

and such that

$$F_{[3]}(V_1) \leq V(t,e) \leq S_{[3]}(V_1), \tag{32}$$

where V_1 is defined in (29), It is showed in Maghenem et al. (2016) that the total derivative of V along the trajectories of (30) satisfies

$$\dot{V}(t,e) \leq -\frac{\mu}{T}V_1(e) - k_x e_x^2 - \frac{k_\theta}{k_y}e_\theta^2. \tag{33}$$

Hence uniform global asymptotic stability of the null solution of (30) follows.

Step 2. Now we construct a strict Lyapunov function for the system

$$\dot{e} = A_{v_r}(t,e)e + B_1(t,e)\rho(t). \tag{34}$$

To that end, we start by "reshaping" the function V defined in (31) to obtain a particular negative bound on its time derivative. Let

$$Z(t,e) := Q_{[3]}(V_1)V_1(e) + V(t,e) \tag{35}$$

where $Q_{[3]}(V_1)$ is a third order polynomial of V_1, with strictly positive coefficients. Then, in view of (33), the total derivative of Z along the trajectories of (30) satisfies

$$\dot{Z}(t,e) \leq -\frac{\mu}{T}V_1(e) - Q_{[3]}(V_1)\big[k_x e_x^2 + \frac{k_\theta}{k_y}e_\theta^2\big]. \tag{36}$$

Next, we recall that in view of (9) $\rho(t)$, which satisfies (18) is uniformly integrable. Therefore, for any $\gamma > 0$, there exists $c > 0$ such that

$$G(t) := exp\left(-\gamma \int_0^t \rho(s)ds\right) \geq c > 0 \quad \forall t \geq 0 \tag{37}$$

and, since $Z(t,e)$ and $V(t,e)$ are positive definite radially unbounded —see (32) and (35), so is the function

$$W(t,e) \quad := \quad G(t)Z(t,e); \tag{38}$$

indeed, we have

$$exp\left(-\gamma \int_0^\infty \rho(s)ds\right) Z(t,e) \leq W(t,e) \leq Z(t,e).$$

Now, the time derivative of W along trajectories of (34) verifies

$$\begin{aligned}
\dot{W}(t,e) &\leq -Y(t,e) + \dot{G}(t)Z(t,e) \\
&\quad + G(t)\frac{\partial\left(V + Q_{[3]}(V_1)V_1\right)}{\partial e}B_1(t,e)\rho(t) \qquad (39)\\
Y(t,e) &:= G(t)\Big[\frac{\mu}{T}V_1(e) + Q_{[3]}(V_1)\big[k_x e_x^2 + \frac{k_\theta}{k_y}e_\theta^2\big]\Big]. \qquad (40)
\end{aligned}$$

Note that, in view of (37), $Y(t,e)$ is positive definite. We proceed to show that the rest of the terms bounding $\dot{W}$ are negative semi-definite. To that end, we develop (dropping the arguments of $f(t,e_x,e_y)$)

$$\begin{aligned}
\frac{\partial V}{\partial e}B_1(t,e) &= \frac{\partial V}{\partial V_1}\frac{\partial V_1}{\partial e}B_1(t,e) - \omega_r k_y f(\cdot)\left[e_x + e_y^2\right] \\
&\quad - v_r P_{[1]}(t,V_1)k_y f(\cdot)\left[e_\theta e_x + e_y\right] \qquad (41)
\end{aligned}$$

and

$$\frac{\partial\left(Q_{[3]}(V_1)V_1\right)}{\partial e}B_1(t,e) = \frac{\partial\left(Q_{[3]}(V_1)V_1\right)}{\partial V_1}\frac{\partial V_1}{\partial e}B_1(t,e), \qquad (42)$$

and we decompose $B_1(t,e)$ into

$$B_1(t,e) = \begin{bmatrix} -k_y f(\cdot) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & k_y f(\cdot) \\ 0 & -k_y f(\cdot) & 0 \end{bmatrix} e.$$

Then, since

$$\frac{\partial V_1}{\partial e}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & k_y f(\cdot) \\ 0 & -k_y f(\cdot) & 0 \end{bmatrix} e = 0,$$

it follows that

$$\frac{\partial V_1}{\partial e}B_1(t,e) = -\frac{\partial V_1}{\partial e_\theta}k_y f(\cdot) = -e_\theta f(\cdot).$$

Thus, using the latter equation, we obtain

$$\begin{aligned}
\dot{W}(t,e) \leq &- Y(t,e) + \dot{G}(t)Z(t,e) \\
&- G(t)\rho(t)f(\cdot)\frac{\partial\left(V + Q_{[3]}(V_1)V_1\right)}{\partial V_1}e_\theta \\
&+ v_r f(\cdot)G(t)\rho(t)P_{[1]}(t,V_1)\left[-k_y e_\theta e_x - k_y e_y\right] \\
&+ \omega_r G(t)\rho(t)f(\cdot)\left[-k_y e_x + k_y e_y^2\right]. \qquad (43)
\end{aligned}$$

In view of (15) and the boundedness of v_r and ω_r, there exists a polynomial $R_{[3]}(V_1)$

with non-negative coefficients, such that

$$
\begin{aligned}
R_{[3]}(V_1)V_1 \geq & -f(\cdot)\frac{\partial\left(V + Q_{[3]}(V_1)V_1\right)}{\partial V_1}e_\theta \\
& + \omega_r f(\cdot)\left[-k_y e_x + k_y e_y^2\right] \\
& + v_r f(\cdot)P_{[1]}(t, V_1)\left[-k_y e_\theta e_x - k_y e_y\right]. \qquad (44)
\end{aligned}
$$

Hence, since $V(t,e) \geq F_{[3]}(V_1)$ —see (32), we obtain

$$
\dot{W} \quad \leq \quad -Y(t,e) + \dot{G}(t)F_{[3]}(V_1)V_1 + G(t)\rho(t)R_{[3]}(V_1)V_1.
$$

On the other hand, in view of (37), $\dot{G}(t) \leq -\gamma G(t)\rho(t)$ for any $\gamma > 0$ and the coefficients of $F_{[3]}(V_1)$ are strictly positive. Therefore, there exists $\gamma > 0$ such that

$$
\gamma F_{[3]}(V_1) \geq R_{[3]}(V_1)
$$

and, consequently, $\dot{W}(t,e) \leq -Y(t,e)$ for all $t \geq 0$ and all $e \in \mathbb{R}^3$. Uniform global asymptotic stability of the null solution of (34) follows.

Step 3. In order to establish iISS with respect to η and boundedness of the closed-loop trajectories subject to $\eta \in \mathcal{L}_2$, we proceed as in (Maghenem et al., 2016, Proposition 4). Let

$$
W_1(t,e) := \ln\left(1 + W(t,e)\right). \qquad (45)
$$

The derivative of W_1 along trajectories of (27) satisfies

$$
\dot{W}_1 \quad \leq \quad -G_m \frac{\frac{\mu}{T}V_1(e) + Q_{[3]}\left[k_x e_x^2 + \frac{k_\theta}{k_y}e_\theta^2\right]}{1 + W(t,e)} + \frac{\left|\frac{\partial W}{\partial e}B_2\eta\right|}{1 + W(t,e)} \qquad (46)
$$

with $G_m := exp\left(-\gamma\int_0^\infty \rho(t)dt\right)$.

Next, we decompose $B_2(e)\eta$ introduced in (27) into

$$
B_2(e)\eta := B_{21}(\eta) + B_{22}(\eta)e
$$

where

$$
B_{21}(\eta) := \begin{bmatrix} -\tilde{\omega} \\ -\tilde{v} \\ 0 \end{bmatrix}, \quad B_{22}(\eta) := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \tilde{\omega} \\ 0 & -\tilde{\omega} & 0 \end{bmatrix}.
$$

Then, we use the fact that $\frac{\partial V_1}{\partial e}B_{22}(\eta)e = 0$ and $|G(t)| \leq 1$, and we define

$$
H(e, V_1) := Q_{[3]} + P_{[3]} + \frac{\partial Q_{[3]}}{\partial V_1}V_1 + \frac{\partial P_{[3]}}{\partial V_1}V_1 + \bar{v}_r\left|e_\theta\right|\left|e_y\right|\frac{\partial P_{[1]}}{\partial V_1}
$$

and

$$\xi = \begin{bmatrix} \frac{e_\theta}{k_y} \\ e_x \end{bmatrix}, \tag{47}$$

to obtain

$$\begin{aligned}
\left|\frac{\partial W}{\partial e}B_2\eta\right| \;&\leq\; H(e,V_1)|\xi||\eta| + \bar{\omega}_r|e_y||\eta| + \bar{v}_r P_{[1]}|e_y||\eta| \\
&\qquad + \bar{\omega}_r V_1|\eta| + \bar{v}_r P_{[1]}|e_\theta||e_x||\eta| \\
&\leq\; H(e,V_1)\Big[\frac{1}{2\epsilon}|\xi|^2 + \frac{\epsilon}{2}|\eta|^2\Big] + \bar{\omega}_r\Big[\frac{1}{2\epsilon}V_1 + \frac{\epsilon}{2}V_1|\eta|^2\Big] \\
&\qquad + \bar{\omega}_r\Big[\frac{1}{2\epsilon}V_1 + \frac{\epsilon}{2}|\eta|^2\Big] + \bar{v}_r\Big[\frac{1}{2\epsilon}V_1 + \frac{\epsilon}{2}P_{[1]}^2|\eta|^2\Big] \\
&\qquad + \bar{v}_r P_{[1]}\Big[\frac{1}{2\epsilon}V_1|e_\theta|^2 + \frac{\epsilon}{2}|\eta|^2\Big] \\
&\leq \big[H(e,V_1) + \bar{v}_r P_{[1]}k_y^2 V_1\big]\,\frac{1}{2\epsilon}|\xi|^2 + [2\bar{\omega}_r + \bar{v}_r]\,\frac{1}{2\epsilon}V_1 \\
&\qquad + \frac{\epsilon}{2}|\eta|^2\Big[H(e,V_1) + \bar{\omega}_r V_1 + \bar{\omega}_r + \bar{v}_r P_{[1]}^2 + \bar{v}_r P_{[1]}\Big].
\end{aligned}$$

Next, we choose $\epsilon > 0$ such that

$$\begin{aligned}
\frac{H + \bar{v}_r P_{[1]}k_y^2 V_1}{\epsilon}|\xi|^2 \;&\leq\; G_m Q_{[3]}\big[k_x e_x^2 + \frac{k_\theta}{k_y}e_\theta^2\big], \\
\frac{2\bar{\omega}_r + \bar{v}_r}{\epsilon} \;&\leq G_m\frac{\mu}{T}.
\end{aligned}$$

Such $\epsilon > 0$ exists because $Q_{[3]}$ is a third order polynomial of V_1 with strictly positive coefficients. So (46) becomes

$$\begin{aligned}
\dot{W}_1 \;\;&\leq\;\; -\frac{G_m}{2}\frac{\frac{\mu}{T}V_1(e) + Q_{[3]}\big[k_x e_x^2 + \frac{k_\theta}{k_y}e_\theta^2\big]}{1 + W(t,e)} \\
&\qquad + \;\frac{D_{[3]}(V_1)}{1 + W(t,e)}\frac{\epsilon}{2}\,|\eta|^2
\end{aligned} \tag{48}$$

where $D_{[3]}(V_1)$ is a third order polynomial satisfying

$$H + \bar{\omega}_r V_1 + \bar{\omega}_r + \bar{v}_r P_{[1]}^2 + \bar{v}_r P_{[1]} \leq D_{[3]}.$$

From the positivity of V, (32), and the definition of W in (38), we have

$$G_m Q_{[3]}(V_1)V_1 \leq W(t,e) \leq S_{[3]}(V_1)V_1 \tag{49}$$

hence,

$$\dot{W}_1 \leq -\frac{G_m}{2}\frac{\frac{\mu}{T}V_1 + Q_{[3]}(V_1)\left[k_x e_x^2 + \frac{k_\theta}{k_y}e_\theta^2\right]}{1+S_{[3]}(V_1)} + \frac{D_{[3]}(V_1)}{1+G_m Q_{[3]}(V_1)V_1}\frac{\epsilon}{2}|\eta|^2 . \tag{50}$$

This implies the existence of a positive constant $c > 0$ and a positive definite function $e \mapsto \alpha$ such that

$$\dot{W}_1 \leq -\alpha(e) + c|\eta|^2 . \tag{51}$$

The result follows from Angeli et al. (2000).

3.2. *Under the scenario S2:*

The proof of Proposition 1 under condition (10) relies on arguments for stability of cascaded systems as well as on tools tailored for systems with persistency of excitation. We start by rewriting the closed-loop equations in a convenient form for the analysis under the conditions of Scenario 2. To that end, to compact the notation, let us introduce

$$f_\rho(t, e_x, e_y) := \rho(t) f(t, e_x, e_y) \tag{52}$$

$$\Phi(t, e_\theta, e_x, e_y) = k_\theta e_\theta + k_y f_\rho(t, e_x, e_y) \tag{53}$$

Then, the closed-loop equations become

$$\dot{e} = f_e(t,e) + g(t,e)\eta, \quad \eta = [\tilde{v}\ \tilde{\omega}]^\top, \tag{54}$$

where

$$f_e(t,e) := \begin{bmatrix} -k_\theta e_\theta - k_y f_\rho - k_y v_r \phi(e_\theta) e_y \\ -k_x e_x + \Phi e_y + \left[\omega_r + k_y v_r \phi(e_\theta) e_y\right] e_y \\ -\Phi e_x - \left[\omega_r + k_y v_r \phi(e_\theta) e_y\right] e_x + v_r \sin e_\theta \end{bmatrix},$$

$$g(t,e) := \begin{bmatrix} 0 & -1 \\ -1 & e_y \\ 0 & -e_x \end{bmatrix}.$$

Following the proof-lines of (Panteley & Loría, 2001, Lemma 1) for cascaded systems, we establish the following for the system (54):
Claim 1. The solutions are uniformly globally bounded subject to $\eta \in \mathcal{L}_2$,
Claim 2. The origin of $\dot{e} = f_e(t,e)$ is uniformly globally asymptotically stable (*i.e.*, **P1**).

After Angeli et al. (2004) the last two claims together imply integral ISS with respect to η (*i.e.*, **P2**). Moreover, Claim 1 implies the convergence of the closed-loop trajectories to the origin provided that the input η tends to zero and is square integrable (*i.e.*, **P3**).

3.2.1. Proof of Claim 1

Let

$$W(e) \;:=\; \ln(1+V_1(e)), \qquad V_1(e) := \frac{1}{2}\big[\,e_x^2 + e_y^2\,\big]. \tag{55}$$

The total derivative of V_1 above along the trajectories of (54) yields

$$\dot{V}_1(e) \le -k_x e_x^2 + |e_x||\tilde{v}| + |v_r||\sin(e_\theta)||e_y| \tag{56}$$

hence,

$$\dot{W}(e) \;\le\; \frac{1}{1+V_1}\Big[-\frac{k_x}{2}e_x^2 + |v_r||e_y| + \frac{\tilde{v}^2}{2k_x}\Big] \tag{57}$$

$$\le\; \frac{|e_y|}{1+V_1}|v_r| + \frac{1}{2k_x[1+V_1]}\tilde{v}^2. \tag{58}$$

Integrating on both sides of (58) along the trajectories, from 0 to t, and invoking the integrability of v_r and the square integrability of η we see that $W(e(t))$ is bounded for all $t \ge 0$. Boundedness of $e_x(t)$ and $e_y(t)$ follows since W is positive definite and radially unbounded in (e_x, e_y).

Next, we observe that the $\dot{e}_\theta$–equation in (54) corresponds to an exponentially stable system with bounded input $u(t) = -k_y v_r(t)\phi(e_\theta(t))e_y(t) - k_y f_\rho(t, e_x(t), e_y(t)) - \tilde{\omega}(t)$ hence, we also have $e_\theta \in \mathcal{L}_\infty$.

Remark 2. For further development, we also emphasize that proceeding as above from Inequality (57) we conclude that $e_x \in \mathcal{L}_2$, uniformly in the initial conditions.

3.2.2. Proof of Claim 2

We split the drift of the nominal system $\dot{e} = f_e(t,e)$ into the output injection form:

$$f_e(t,e) = F(t,e) + K(t,e) \tag{59}$$

where

$$F(t,e) := \begin{bmatrix} -k_\theta e_\theta - k_y f_\rho(t, e_x, e_y) \\ -k_x e_x + \Phi(t, e_\theta, e_x, e_y)e_y \\ -\Phi(t, e_\theta, e_x, e_y)e_x \end{bmatrix}$$

and

$$K(t,e) := \begin{bmatrix} -k_y v_r \phi(e_\theta)e_y \\ \big[\omega_r + k_y v_r \phi(e_\theta)e_y\big]e_y \\ -\big[\omega_r + k_y v_r\phi(e_\theta)e_y\big]e_x + v_r \sin e_\theta \end{bmatrix}. \tag{60}$$

Then, to establish UGAS for the origin of $\dot{e} = f_e(t,e)$ we invoke the output-injection statement (Panteley, Loría, & Teel, 2001, Proposition 3). According to the latter, UGAS follows if:

a) there exist: an "output" y, non decreasing functions k_1, k_2, and β: $\mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$, a class $\mathcal{K}_\infty$ function k, and a positive definite function γ such that, for all $t \ge 0$ and all

$e \in \mathbb{R}^3$,

$$
\begin{aligned}
|K(t,e)| &\le k_1(|e|)k(|y|) && (61)\\
|y(t,e)| &\le k_2(|e|) && (62)\\
\int_\circ^\infty \gamma\big(|y(t)|\big)dt &\le \beta(|e(0)|); && (63)
\end{aligned}
$$

b) the origin of $\dot e = f_e(t,e)$ is uniformly globally stable;
c) the origin of $\dot e = F(t,e)$ is UGAS.

Condition a. Using (60), a direct computation shows that there exists $c > 0$ such that

$$
|K(t,e)| \le c\big[\,|e|^2 + |e|\,\big]\,|\,[v_r\ \omega_r]\,|, \tag{64}
$$

so (61) holds with $k_1(s) := c(s^2 + s)$, $k(s) := s$, and $y := [v_r\ \omega_r]$. Moreover, (62) and (63) hold with $\gamma(s) = s$, since $[v_r\ \omega_r] \in \mathcal{L}_1$, for a constant functions β and k_2 which, moreover, are independent of the initial state.

Condition b. Uniform global stability is tantamount to uniform stability and uniform global boundedness of the solutions —see Hahn (1967). The latter was established already for the closed-loop system under the action of the "perturbation" η hence, it holds all the more in this case, where $\eta = 0$.

In order to establish uniform stability, we use Lyapunov's direct method. Let $R > 0$ be arbitrary but fixed.

We claim that, for the system $\dot e = F(t,e)$, there exists a Lyapunov function candidate $V : \mathbb{R}_{\ge 0} \times \mathbb{R}^3 \to \mathbb{R}_{\ge 0}$ and positive constants α_1, α_2, and α_3 such that

$$
\begin{aligned}
\alpha_1 |e|^2 \le V(t,e) \le \alpha_2 |e|^2 & \qquad \forall t \ge 0,\ e \in \mathbb{R}^3 && (65)\\
\left|\tfrac{\partial V(t,e)}{\partial e}\right| \le \alpha_3 |e| & \qquad \forall t \ge 0,\ e \in \mathbb{R}^3 && (66)\\
\tfrac{\partial V}{\partial t} + \tfrac{\partial V}{\partial e}F(t,e) \le 0 & \qquad \forall t \ge 0,\ e \in B_R && (67)
\end{aligned}
$$

where $B_R := \{e \in \mathbb{R}^3\ :\ |e| \le R\}$. Furthermore, from (64) it follows that

$$
|K(t,e)| \le c(R+1)\big[|v_r| + |\omega_r|\big]|e| \quad \forall\, t \ge 0,\ e \in B_R.
$$

Then, evaluating the time derivative of V along the trajectories of (59), we obtain

$$
\begin{aligned}
\dot V(t,e) &\le \frac{\partial V(t,e)}{\partial e}K(t,e) \le \alpha_3 c(R+1)\big[|v_r| + |\omega_r|\big]|e|^2\\
&\le \frac{\alpha_3 c[R+1]}{\alpha_1}\big[|v_r| + |\omega_r|\big]V(t,e) \quad \forall\, t \ge 0,\ e \in B_R.
\end{aligned}
$$

Defining $v(t) := V(t,e(t))$ and invoking the comparison lemma, we conclude that

$$
v(t) \le \exp\left(\frac{c\alpha_3[R+1]}{\alpha_1}\int_{t_\circ}^\infty \big[\,|v_r(s)| + |\omega_r(s)|\,\big]ds\right)v(t_\circ)
$$

for all initial conditions $t_\circ \ge 0$ and $e(t_\circ)$ generating trajectories $e(t) \in B_R$. In view of

(10), we obtain

$$|e(t)|^2 \leq \frac{\alpha_2}{\alpha_1} \exp\left(\frac{\alpha_3 c[R+1]}{\alpha_1}\beta\right) |e(t_\circ)|^2$$

so uniform stability of (59) follows.

It is left to construct a Lyapunov function candidate V for the system $\dot{e} = F(t, e)$, that satisfies the conditions (65)-(67). To that end, consider the coordinates

$$e_z = e_\theta + g(t, e_y) \tag{68}$$

where $g : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by

$$g(t, e_y) := e^{-k_\theta(t-t_\circ)} g(t_\circ, e_y) + \int_{t_\circ}^{t} k_y e^{-k_\theta(t-s)} f(s, 0, e_y) ds$$

and, for further development we observe that

$$\frac{\partial g}{\partial t}(t, e_y) = -k_\theta g(t, e_y) + k_y f_\rho(t, 0, e_y). \tag{69}$$

Let $g(t_\circ, e_y)$ be such that $|g(t_\circ, e_y)| \leq |e_y|$ which implies, using Assumption **A1**, that

$$|g(t, e_y)| \leq (1 + k_y \sigma_2) |e_y|. \tag{70}$$

In the new coordinates, we obtain

$$\dot{e}_z = -k_\theta e_z - \frac{\partial g}{\partial e_y} \Phi e_x - k_y \tilde{f}(t, e_x, e_y)$$

where $\tilde{f}(t, e_x, e_y) := f_\rho(t, e_x, e_y) - \tilde{f}_\rho(t, 0, e_y)$. Then, Assumption **A1** implies that for any $R > 0$ there exists a positive constant $c_R > 0$ such that

$$\max_{e \in B_R} \left\{ \sup_{t \geq 0} \left| \tilde{f}_\rho(t, e_x, e_y) \right|, \quad \sup_{t \geq 0} \left| \frac{\partial g}{\partial e_y} \Phi e_x \right| \right\} \leq c_R |e_x|.$$

Thus, consider the following Lyapunov function candidate

$$V(t, e) := \left[\frac{1}{2} \frac{c_R^2}{k_\theta k_x} + (1 + k_y \sigma_2)^2 \right] \left[e_x^2 + e_y^2 \right] + \frac{1}{2} e_z^2 \tag{71}$$

which trivially satisfies (66). Its total time derivative is

$$\begin{aligned} \dot{V}(t, e) &= -\frac{c_R^2}{k_\theta} e_x^2 - e_z \left[k_\theta e_z + \frac{\partial g}{\partial e_y} \Phi e_x + k_y \tilde{f}(t, e_x, e_y) \right] \\ &\leq -\frac{c_R^2}{k_\theta} e_x^2 - k_\theta e_z^2 - c_R |e_z||e_x| \leq 0, \quad \forall e \in B_R, \end{aligned}$$

so (67) holds. Using (70) and the inequalities

$$e_z^2 \geq e_\theta^2 - 2|e_\theta||g(t,e_y)| + |g(t,e_y)|^2 \geq \frac{1}{2}e_\theta^2 - (1+k_y\sigma_2)^2|e_y|^2.$$

$$e_z^2 \leq e_\theta^2 + 2|e_\theta||g(t,e_y)| + |g(t,e_y)|^2 \leq 2e_\theta^2 + 2(1+k_y\sigma_2)^2|e_y|^2,$$

we see that the following bounds on V follow

$$V(t,e) \geq \frac{1}{2}\frac{c_R^2}{k_\theta k_x}\left[e_x^2 + e_y^2\right] + \frac{1}{4}e_\theta^2$$

$$V(t,e) \leq \left[\frac{1}{2}\frac{c_R^2}{k_\theta k_x} + 2(1+k_y\sigma_2)^2\right]\left[e_x^2 + e_y^2\right] + e_\theta^2.$$

Thus the inequalities in (65) also hold.

Condition c. Since the solutions are uniformly globally bounded, for any $r > 0$, there exists $R > 0$ such that $|e(t)| \leq R := \{|e| \leq R\}$ for all $t \geq t_\circ$, all $e_\circ \in B_r$, and all $t_\circ \geq 0$. It is only left to establish uniform global attractivity. To that end, we observe that the nominal $\dot{e} = F(t,e)$ has the form

$$\dot{e}_\theta = -k_\theta e_\theta - k_y f_\rho(t, e_x, e_y) \tag{72a}$$

$$\begin{bmatrix}\dot{e}_x \\ \dot{e}_y\end{bmatrix} = \begin{bmatrix} -k_x & \Phi_\theta(t,e_x,e_y) \\ -\Phi_\theta(t,e_x,e_y) & 0\end{bmatrix}\begin{bmatrix}e_x \\ e_y\end{bmatrix} \tag{72b}$$

where, for each $e_\theta \in B_R$, we define the smooth *parameterized* function $\Phi_\theta : \mathbb{R}_{\geq 0} \times \mathbb{R}^2 \to \mathbb{R}$ as

$$\Phi_\theta(t,e_x,e_y) := \Phi(t,e_\theta,e_x,e_y).$$

Then, the system (72) may be regarded as a cascaded system —*cf.* Loría (2008). Moreover, the system (72a) is input-to-state stable and the perturbation term $k_y f_\rho(t, e_x(t), e_y(t))$ is uniformly bounded. Therefore, in order to apply a statement for cascaded systems, we must establish that the origin of (72b) is globally asymptotically stable, uniformly in the initial conditions $(t_\circ, e_{x\circ}, e_{y\circ}) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^2$ and in the "parameter" $e_\theta \in B_R$. For this, we invoke (Loría et al., 2002, Theorem 3) as follows. Since $k_x > 0$ there is only left to show that $\Phi_\theta^\circ(t,e_y)e_y$, where

$$\Phi_\theta^\circ(t,e_y) := \Phi_\theta(t,0,e_y),$$

is uniformly δ-persistently exciting with respect to e_y, uniformly for any $\theta \in B_R$ —*cf.* (Loría et al., 2002, Definition 3). Since Φ_θ° is smooth, it suffices to show that for any $|e_y| \neq 0$ and r, there exist T and μ such that

$$|e_y| \neq 0 \implies \int_t^{t+T} \left|\tilde{\Phi}_\theta^\circ(\tau,e_y)\right| d\tau \geq \mu \quad \forall\, t \geq 0 \tag{73}$$

—see (Loría et al., 2002, Lemma 1).

Remark 3. In general, μ depends both on e_θ and on e_y, but since $e_\theta \in B_R$ and B_R is compact, by continuity, one can always choose the smallest qualifying μ, for each fixed e_y. Therefore, as in Loría et al. (2002), μ may be chosen as a class $\mathcal{K}$ function dependent of $|e_y|$ only.

Now, we show that (73) holds under Assumption **A2**. To that end, we remark that

$$\Phi_\theta^\circ(t, e_y) = k_\theta e_\theta + k_y \rho(t) f_\circ(t, e_y)$$

−*cf.* Eq. (53), satisfies

$$\dot{\Phi}_\theta^\circ \quad = \quad -k_\theta \Phi + k_y \dot{\rho} f_\circ + k_y \rho \frac{\partial f_\circ}{\partial t} - k_y \rho \frac{\partial f_\circ}{\partial e_y} \Phi e_x$$

where we used $\dot{e}_\theta = -\Phi$ and $\dot{e}_y = \Phi e_x$. Therefore, defining

$$K_\Phi(t, e) \quad := \quad k_\theta [\Phi_\theta^\circ - \Phi] - k_y \rho \frac{\partial f_\circ}{\partial e_y} \Phi e_x$$

we obtain

$$\dot{\Phi}_\theta^\circ \quad = \quad -k_\theta \Phi_\theta^\circ - k_y \rho \frac{\partial f_\circ}{\partial t} + k_y \dot{\rho} f_\circ + K_\Phi(t, e).$$

The latter equation corresponds to that of a linear filter with state Φ_θ° and input

$$\Psi(t, e_y) := -k_y \rho(t) \frac{\partial f_\circ}{\partial t}(t, e_y) + k_y \dot{\rho}(t) f_\circ(t, e_y) + K_\Phi(t, e(t));$$

therefore, Φ_θ° is uniformly δ-PE with respect to e_y, if so is Ψ (see *e.g.* Ioannou and Sun (1996)). Now, from Assumption **A1** and uniform global boundedness of the solutions, for any r there exists $c > 0$ such that

$$\left| k_y \dot{\rho}(t) f_\circ(t, e_y(t)) + K_\Phi(t, e(t)) \right| \leq c(r) \left[\, |e_x(t)| + |\dot{\rho}(t)| \, \right]$$

Therefore, uniform δ-PE with respect to e_y of Ψ follows from Assumption **A2** and the fact that $\dot{\rho}$ and e_x are uniformly square integrable. That $\dot{\rho} \in \mathcal{L}_2$, with a bound uniform in the initial times, follows from (18) because v_r, ω_r, and ρ are bounded and $|v_r| + |\omega_r|$ is uniformly integrable. That e_x is uniformly $\mathcal{L}_2$ follows from (57) —see Remark 2.

This concludes the proof of UGAS for the nominal system $\dot{e} = f_e(t, e)$ hence, **Claim 2.** is proved.

This completes the proof of Proposition 1. ■

4. Simulations

To illustrate our main theoretical results we performed some simulation tests under Simulink™ of Matlab™, according to the two scenarios described previously.

The robot's physical parameters are taken from Fukao et al. (2000):

$$M = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_1 \end{bmatrix}, \quad C(\dot{z}) = \begin{bmatrix} 0 & c\omega \\ -c\omega & 0 \end{bmatrix},$$

with $m_1 = 0.6227$, $m_2 = -0.2577$, $c = 0.2025$, $r = 0.15$, and $b = 0.5$. We used the control law (22) with $\sigma(\tilde{\nu}) = [\tanh(\tilde{\nu}_1)\ \tanh(\tilde{\nu}_2)]^\top$.

For the purpose of the first scenario, we define the reference velocities v_r and ω_r as periodic functions (hence persistently exciting) —see Figure 1. Such references generate a "staircase-shappe" path that is asymptotically followed by the vehicle —see Figure 5, where we show the simulation results for two different values of the vehicle's initial conditions. The initial conditions for the reference vehicle are set to $[x_r(0), y_r(0), \theta_r(0)] = [0, 0, 0]$ and those for the adaptation law (25) are set to $\hat{\Theta}(0) = (\hat{m}_1, \hat{m}_2, \hat{c}) = (0, 0, 0)$.

The desired distance between the actual vehicle and the reference is obtained by setting the desired orientation offset to zero and defining $[d_{x_r}, d_{y_r}] := [0, 0]$. The control gains are set to $k_x = 1$, $k_y = .2$, $k_\theta = 0.1$, $k_d = 20$, and $\gamma = 1 \times 10^{-5}$. The function f which verifies the assumptions **A1** and **A2** is defined as $f(t, e_x, e_y) := p(t)|e_{xy}|$ with $p(t) = 50\sin(0.5t) + 5$; we notice that both $p(t)$ and $\dot{p}$ are persistently exciting signals. Therefore, the conditions (14), (15) and (17) hold.

The tracking position errors are depicted in Figure 2 while in Figure 3 are showed the vehicle's and the reference velocities. The input torques at the wheels are depicted in Figure 4. To avoid graphical saturation we provide the curves only for the case in which the initial conditions are $x(0) = 2$, $y(0) = 1$ and $\theta(0) = 0$.

For the stabilization scenario **S2**, we use exponentially-fast decaying reference trajectories —see Figure 6 and the control gains $k_d = 30$, $k_y = 1$. In Figure 10 we show the path followed by vehicle starting from two different points in the plane, as well as the reference path generated by the fictitious vehicle, which comes to a full stop. The tracking position and velocity errors are depicted in Figures 7 and 8 respectively, the input torques at the wheels are depicted in Figure 9 for the case in which the initial conditions of the vehicle are $x(0) = 1$, $y(0) = 1$ and $\theta(0) = 0$.

The controller's performance may be compared, for instance, to that of the controllers in Do et al. (2004b); Lee et al. (2001).

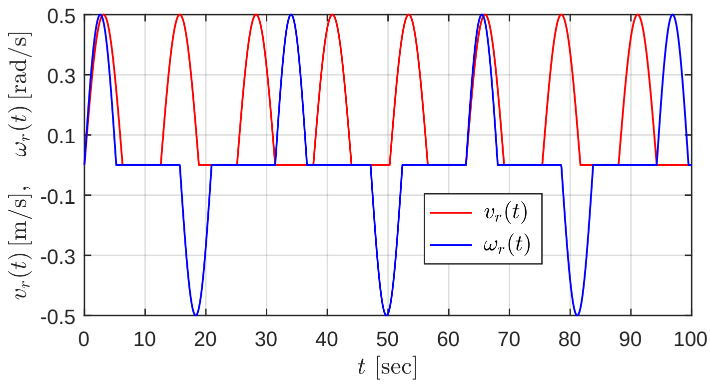

Figure 1. Persistently-exciting reference velocities v_r and ω_r for the scenario **S1**

Acknowledgements

The authors are grateful to H. Khalil for his keen technical remarks on the material presented in this paper.

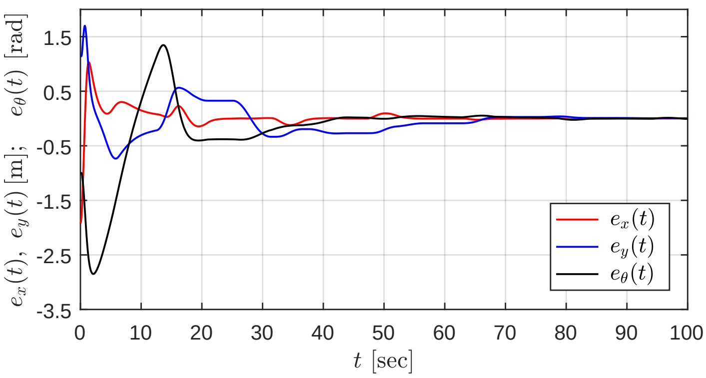

Figure 2. Tracking errors under the scenario **S1**

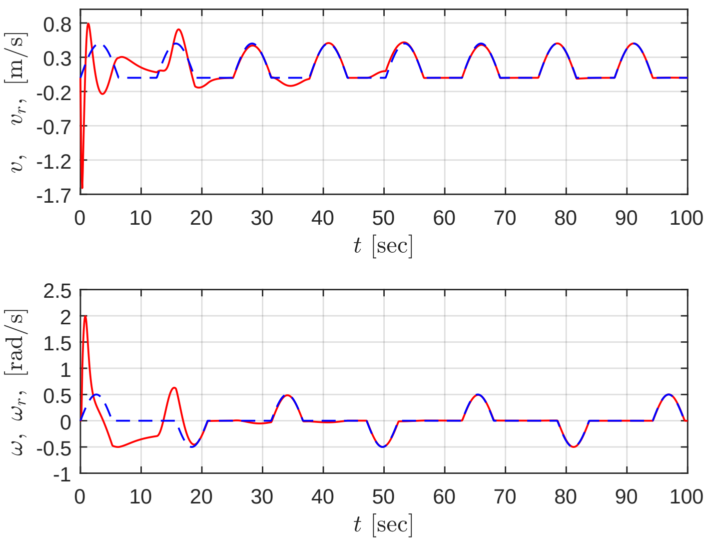

Figure 3. Vehicle's and reference velocities under the scenario **S1**

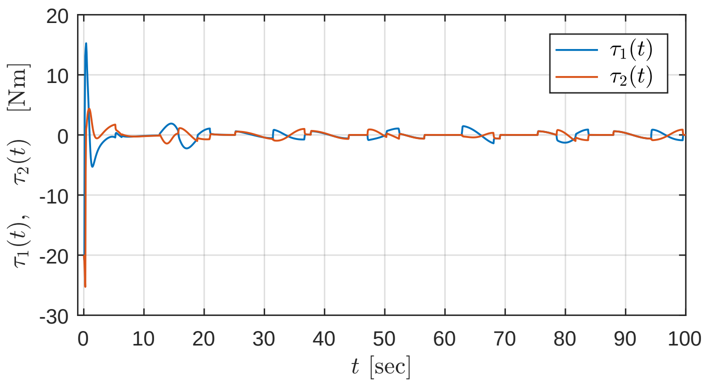

Figure 4. Input torques under the scenario **S1**

5. Conclusion

We presented a unique controller for nonholonomic vehicles with a generic dynamic model that achieves uniform global asymptotic stability in closed loop, for a large variety of reference trajectories. The simplicity and modularity of our design seems

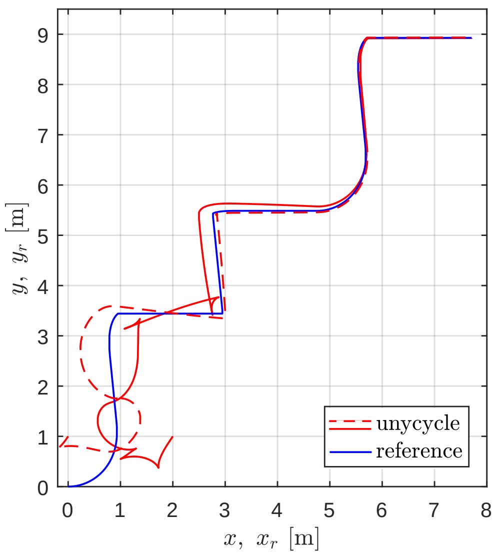

Figure 5. Path followed under the scenario **S1** considering two different sets of initial conditions

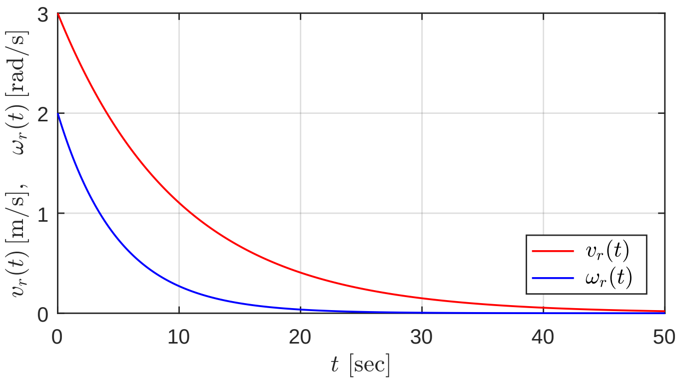

Figure 6. Exponentially-fast decaying reference velocities v_r and ω_r for the scenario **S2**

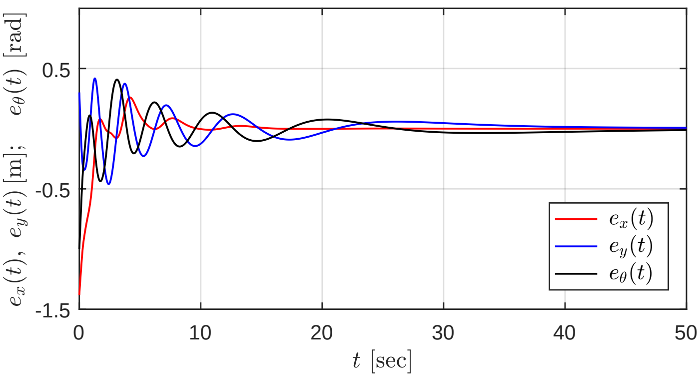

Figure 7. Tracking errors under the scenario **S2**

promising to broach other scenarios such as control under input constraints.

Our proofs are constructive for the tracking-control scenario; moreover, the construction of strict Lyapunov functions makes it possible to extend our designs to the cases of output feedback and parametric uncertainty. While an example of the latter is given, the former is under study. Furthermore, current research is being carried

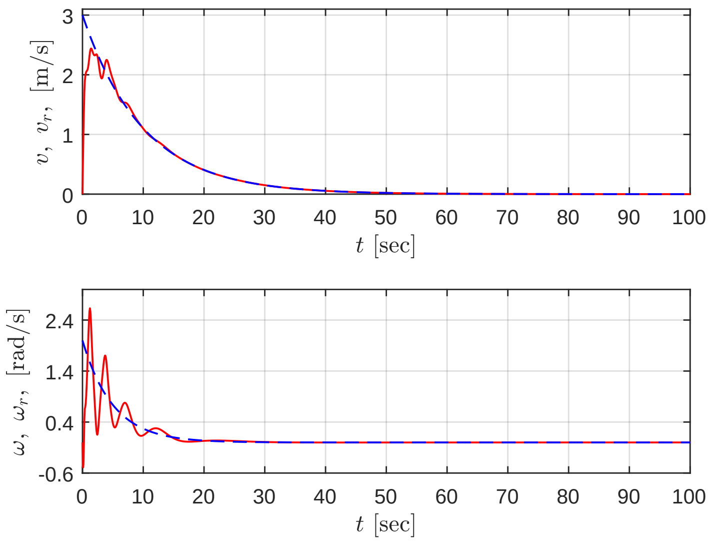

Figure 8. Vehicle and reference velocities under the scenario **S2**

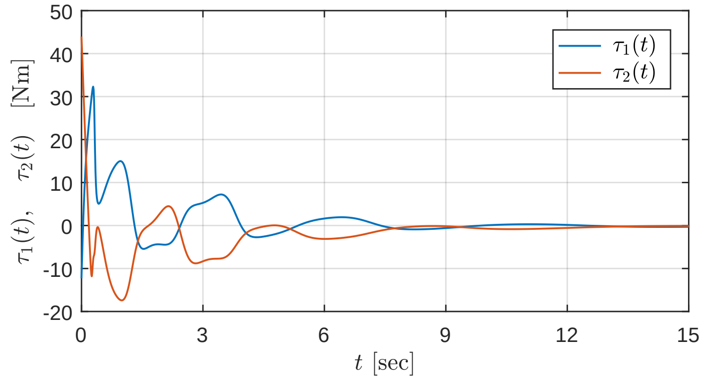

Figure 9. Input torques under the scenario **S2**

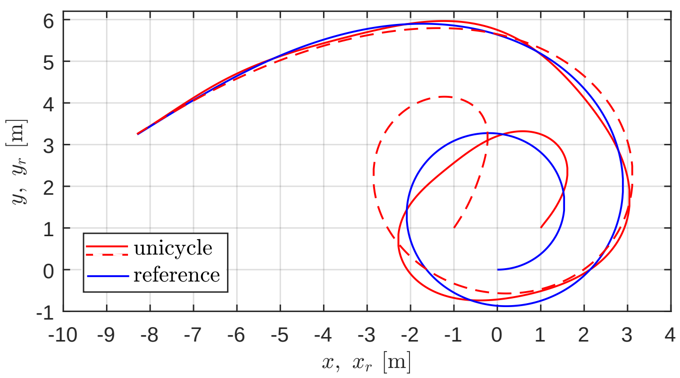

Figure 10. Path followed under the scenario **S2** under different initial conditions

out to relax the standing assumption of integrability of the reference velocities in the stabilization scenario, to allow for slowly-converging reference velocities.

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