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Matrix formulation in Acoustics: the Transfer Matrix Method

I. Herrero-Durá, A. Cebrecos, V. Romero-García and L.M. Garcia-Raffi

Abstract

Matrices are introduced in mathematical subjects related to vector spaces and linear Algebra, being disconnected with their applications in other fields of Science and Engineering studies. The transmission of this knowledge is done, in many occasions, in a purely theoretical manner and posing problems to students that are sometimes abstract, lacking application to real world. In this paper we present an application of matrix formalism in Acoustics: The Transfer Matrix Method. It is a simple method widely used in transmission problems. For a system composed of a waveguide and an aluminum plate, subjected to the pressure of an acoustic wave, we can establish a simple connection between the matrix formalism, the use of vectors and the physical magnitudes

1 Introduction

Matrix formalism is part of the basic mathematical education of scientists and technologists The term "matrix"was introduce by Sylvester but was a collaborator of him, Calley (1821—1895), who used a form of matrices in the work "On the Theory of Linear Transformations"using the term "matrix"in its modern mathematical sense (Keiner, 2007). Usually, matrices are introduced in mathematical subjects related to vector spaces and linear algebra, being disconnected form their applications in other fields of science. Of course, matrices are used in the classroom in many subjects but sometimes students are not able to establish the correct connection between certain mathematical objects and its application. In many cases, classical applications such as those relating to mechanics, especially those relating to diagonalization and its relationship with the determination of principal moments and axes of inertia of a body are the only ones treated in the math classroom. The other typical example is related to the matrix formulation of quantum mechanics, but its interpretation is much more complex for students.

Acoustics, one of the disciplines of classical physics, is often absent in the general physics syllabus. However, it is a discipline that provides facts

of physics that in many occasions can be checked by the student in a simple way, using very simple setups. The theoretical description of acoustics can be complex involving the resolution of Navier-Stokes equations, but in many cases acoustic phenomena can be described using a simple mathematical setting. This is the case of the Transfer Matrix Method (TMM). This method is a matrix formalism used in many field of physics (optics, acoustics, quantum mechanics) to analyze wave propagation in a 1-D setting (see for example Markoš and P., Soukoulis, C.M., 2008). This method gives teachers the possibility to pose simple problems to students in the field of physics, where diagonalization, eigenvectors and eigenvalues have a simple interpretation. The example proposed in this article starts with a simple problem, easily understandable for students with basic knowledge in physics, and follows with building up of the mathematical formalism based on matrices. After that, the resolution of the problem is connected with finding eigenvalues and eigenvectors of a matrix, giving a physical meaning to them. Under the perspective of didactics, the activity proposed can be considered a modeling eliciting activity (Lesh, R. and Doerr, H. M. ,2003).

First, we introduce the TMM, that has a transversal character from a teaching perspective, as the same formalism can be applied at different levels in science and technology, covering from a basic course on Physics to a more specialized one. The system under analysis is a transmission line (a waveguide) with an obstacle (a clamped plate). TMM relates the acoustic pressure and the particle velocity in the left side of the obstacle to those on the right side. Transmission and reflection coefficients will be defined from the transfer matrix formalism. Next, we will introduce the scattering matrix, which describes the outgoing waves form the obstacle in terms of the ingoing waves and is directly related to the transfer matrix. Eigenvalues and eigenvectors of this scattering matrix will be calculated and interpreted in terms of the physical behavior of the system. Finally, an experimental activity in acoustics will be proposed in order to bring mathematics to the laboratory. All the concepts of matrix algebra that appear in this paper can be found in several textbooks (see for example, Lay, D. C., 1997 or Strang, G., 2016).

2 Transfer matrix method

We start considering a square aluminum clamped plate placed into a tube (see Figure 1), in other words, a resonant inhomogeneity in a waveguide. We consider that the thickness of the plate, h, is much smaller than the wavelength of the incoming waves, λ. Imagine we want to transmit a wave through it, for example an acoustic wave. We place a sound source (loudspeaker) at the beginning of the tube that emits a signal containing several frequencies that will be our range of study, always with frequencies lower

than the cutoff frequency of the waveguide, which correspond to the range of frequencies where only plane waves can propagate. Thus, the system can be considered one dimensional. The incident plane wave impinges the plate, inducing the vibration of it. As a result of this interaction, one part of the energy is reflected back and other part is transmitted through the plate. If no losses are considered, by energy conservation, all the energy that is not reflected is transmitted. If losses are considered, part of the incident energy is reflected, part absorbed by heating the plate, and part is transmitted.

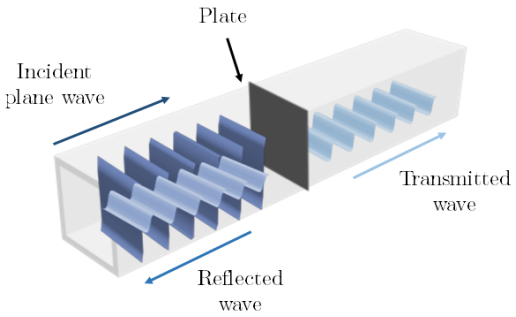


Figure 1: 3-D scheme of the problem in transmission. Dark blue represents the incident plane wave generated by the source, medium blue represents the sound wave that is reflected by the plate and soft blue represents the transmitted wave. The plate is represented in grey.

The acoustic process can be characterized by the state vector $\vec{s}$ constructed from the sound pressure, p, that is related with the force applied to the medium and the normal particle velocity, $\vec{v} = v\vec{e_x}$, with $\vec{e_x}$ the unitary vector along the x-direction. The particle velocity is related with the reaction of the medium to the application of this force, i.e., related with the opposition of the medium to be perturbed or acoustic impedance. This state vector is in $\mathbb{R}^2$, and can be represented as,

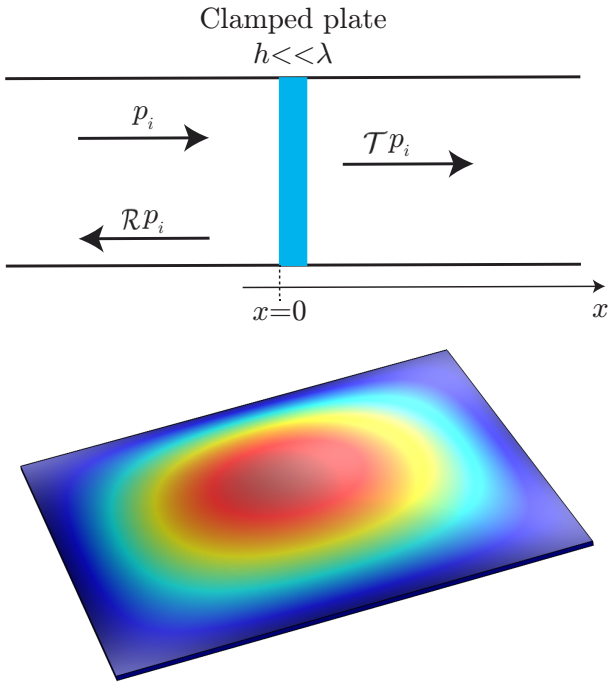
$$\vec{s} = \begin{pmatrix} p \\ v \end{pmatrix}. \tag{1}$$

The TMM relates the sound pressure, p, and normal acoustic particle velocity, v, between the two faces of a system extending from $x = 0$ to the thickness $x = h$ by means of the transfer matrix $\mathbf{T}$, defined as follows

$$\begin{pmatrix} p \\ v \end{pmatrix}_{x=0} = \mathbf{T} \begin{pmatrix} p \\ v \end{pmatrix}_{x=h}, \tag{2}$$

In other words, in the math language at classroom, the matrix $\mathbf{T}$ is a linear mapping.

We consider waves having a wavelength much longer than the diameter of the tube. In this case only plane waves are excited in the tube and this can be considered as a 1-D (x-axis) problem (see Figure 2(a)) where the reflected and transmitted waves can be easily related to the incident wave through the reflection and transmission coefficients. Under this conditions, we are only interested in the first resonant mode of the clamped plate, i.e., the "piston mode"as can be seen in Figure 2(b) (see for example Hazell, C. R. and MitchellI, A. K., 1986),



(a) (b)

Figure 2: (a) 1-D scheme of the problem in transmission. (b) First mode (piston mode) of the plate.

The acoustic behavior of the clamped plate can be characterized by its acoustic impedance, Z_p, which is a classical resonant function expressed as follows,

$$Z_p = \left(\frac{i\Gamma\omega}{\omega_0^2 - \omega^2 + i\sigma\omega} \right)^{-1}, \tag{3}$$

where Γ and σ are the admittance parameter and the dissipation term, respectively. ω_0 is the frequency of the first resonant mode of the clamped

plate that can be obtained from the following approximated equations,

$$C_p = \frac{3.73\ 10^{-4}a^6}{D}, \quad M_p = \frac{2.06\rho h}{a^2}, \tag{4}$$

$$\omega_0 = \sqrt{\frac{1}{M_p C_p}}, \tag{5}$$

where ρ the density of the plate, and $D = Eh^3/12(1-\nu^2)$, the bending rigidity, with E the Young's modulus and ν the Poison's ratio; a represents the side of the square clamped plate. In this work we will use $a = 0.1$ m, $E = 60$ GPa, $\rho = 2700$ kg/m^3, $\nu = 0.3$ and $h = 0.5$ mm, $\Gamma = 0.5$ m/Pa s^{-2} and σ (s^{-1}) will vary depending on the analyzed case ($\sigma = 0$ represents the lossless case and $\sigma \neq 0$ the lossy case). Notice that in this work we will use the temporal dependence $e^{i\omega t}$.

On the other hand, we can consider that the clamped plate is a locally resonant elements in series in a main waveguide. As $h << \lambda$, we consider the plate as punctual resonator in the x direction. Defining the pressure at both sides, $p_{x=0}$, $p_{x=h}$, and the corresponding particle velocity $v_{x=0}$, $v_{x=h}$, the effect of the plate on the acoustic field is twofold: (i) the velocity is the same on both sides due to continuity boundary conditions, i.e., $v_d = v_u$; (ii) there is a pressure drop between each side due to the presence of the plate, and we can relate it at both sides of the punctual resonator using the impedance of the resonator as

$$Z_p = \frac{\Delta p}{v} = \frac{p_{x=0} - p_{x=h}}{v_{x=h}}. \tag{6}$$

Then, the equations relating pressure and velocity at both sides are

$$p_{x=0} = p_{x=h} + Z_p v_{x=h}, \tag{7}$$

$$v_{x=0} = v_{x=h}. \tag{8}$$

Relating the state vectors $(p_{x=0}, v_{x=0})$ and $(p_{x=h}, v_{x=h})$, the transfer matrix, can be obtained as

$$\mathbf{T}_p = \begin{bmatrix} 1 & Z_p \\ 0 & 1 \end{bmatrix}, \tag{9}$$

where Z_p is the acoustic impedance of the clamped plate given by Eq. 3.

This matrix represents the transfer matrix of the clamped plate. But, at the beginning of the section we talk about the scattering coefficients. Hence, which are the reflection and transmission coefficients of our system?

2.1 Reflection and Transmission coefficients

We consider here the general case of a scattering problem by an slab of material of thickness L and represented by a transfer matrix T. Then,

applying Eq. 2 we obtain:

$$\begin{aligned} p(0) &= T_{11}p(h) + T_{12}v(h), \\ v(0) &= T_{21}p(h) + T_{22}v(h). \end{aligned} \tag{10}$$

Due to the geometry of the considered system, the scattering is symmetrical, meaning that the reflection coefficients obtained from both sides are identical. This condition implies that the elements of the transfer matrix must fulfill this equation $T_{11} = T_{22}$. Moreover, we are in the linear regime and the scattering process should be reciprocal, meaning that transmission is identical in both senses. This implies that the determinant of the transfer matrix is unity, $T_{11}T_{22} - T_{12}T_{21} = 1$. Notice that Eq. 9 fulfills both conditions because a clamped plate in the linear regime creates a reciprocal and symmetric 1D scattering problem. For more details, we refer to reference (Song, B.H and Bolton, J.S., (1999)).

Under these conditions, for a wave of amplitude one traveling from left to the right, the pressure in the left-hand side of the plate will be the result of the incident pressure plus the reflected one, that is:

$$p(0) = 1 + \mathcal{R},$$

and for the velocity we have to subtract the part corresponding to the reflected wave that travels in the opposite sense than the incident wave and divide by the impedance of the air in the tube, that is:

$$v(0) = \frac{1 - \mathcal{R}}{Z_0}.$$

In the same manner, the wave in the right part will be the transmitted one:

$$p(h) = \mathcal{T}e^{\imath kh},$$

where we have taken into account the phase $e^{\imath kh}$ corresponding to the plane wave traveling to the right part $e^{\imath kx}$ evaluated at $x = h$. For the velocity

$$v(h) = \frac{\mathcal{T}e^{\imath kh}}{Z_0}.$$

Combining these equations with Equation 10 we obtain:

$$\begin{aligned} 1 + \mathcal{R} &= \mathcal{T}e^{\imath kh}(T_{11} + \frac{T_{12}}{Z_0}), \\ 1 - \mathcal{R} &= \mathcal{T}e^{\imath kh}(Z_0T_{21} + T_{22}), \end{aligned} \tag{11}$$

and then adding both equations and solving for $\mathcal{T}$

$$\mathcal{T} = \frac{2e^{-\imath kh}}{T_{11} + \frac{T_{12}}{Z_0} + Z_0T_{21} + T_{22}}. \tag{12}$$

Dividing equations in Eq. 11, we obtain

$$\frac{1+\mathcal{R}}{1-\mathcal{R}} = \frac{T_{11} + \frac{T_{12}}{Z_0}}{Z_0 T_{21} + T_{22}},$$

from which we can solve

$$\mathcal{R} = \frac{T_{11} + \frac{T_{12}}{Z_0} - Z_0 T_{21} - T_{22}}{T_{11} + \frac{T_{12}}{Z_0} + Z_0 T_{21} + T_{22}}. \tag{13}$$

Now, we can particularize Eqs. 12 and 13 with the transfer matrix of the clamped plate, given by Eq. 9, then simple equations for the reflection and transmission coefficients can be obtained:

$$\mathcal{T} = \frac{2e^{-ikh}}{(2 + Z_p/Z_0)} \simeq \frac{2}{(2 + Z_p/Z_0)}, \tag{14}$$

$$\mathcal{R} = \frac{Z_p/Z_0}{2 + Z_p/Z_0}. \tag{15}$$

We notice that in the lossless case $|\mathcal{R}|+|T| = 1$. In the lossy case, $|\mathcal{R}|+|\mathcal{T}| \neq 1$, and an absorption coefficient should be defined as $\alpha = 1 - |\mathcal{R}|^2 - |\mathcal{T}|^2$.

2.2 Scattering of waves by a clamped plate

Figure 3 represents the acoustic impedance and the scattering coefficients of both the lossless and the lossy cases. In the lossless case, we can observe that at the resonant frequency $\omega/\omega_0 = 1$, the condition $\mathrm{Im}(Z_p)$=0 with $\mathrm{Re}(Z_p) = 0$ is fulfilled (see Fig. 3(a)), which represents the resonant condition. In fact, if we look at the scattering coefficients, Fig. 3(b), we can see that at this particular frequency, the transmission is enhanced producing perfect transmission of waves through the plate with $R = 0$. However as soon as the losses are introduced in the plate, at the resonant frequency, the real part of the impedance is no longer zero (see Fig. 3(c)). This dramatically impacts the reflection and transmission coefficients. Figure 3(d) shows the enhancement of the absorption at the resonant frequency as a consequence of the presence of losses in the plate.

3 Scattering matrix: eigenvalues and eigenvectors

As we have mentioned above, when the acoustic wave is transmitted along the tube, it arrives to the plate, makes it to vibrate and some part of the wave is transmitted and some part is reflected. This process is intimately related with the fact that the plate scatters the wave: is the scattering problem. This interaction is in general too complex to be easily described, but in many cases it is sufficient to know the wave function in the left and the right part of the resonant inhomogeneity.

The waves in the left and the right-hand side of the plate can be defined in terms of plane waves, which in mathematical terms are simple complex exponentials (see Figure 4):

$$p_L^+(x) = Ae^{-\imath kx} \quad p_R^+(x) = Ce^{-\imath kx}, \tag{16}$$

$$p_L^-(x) = Be^{\imath kx} \quad p_R^-(x) = De^{\imath kx}. \tag{17}$$

$$\tag{18}$$

Then the total pressures at both sides of the structure are given by:

$$p(x) = Ae^{-\imath kx} + Be^{+\imath kx} \text{ for } \quad x < x_p, \tag{19}$$

$$p(x) = Ce^{-\imath kx} + De^{+\imath kx} \text{ for } \quad x > x_p + h, \tag{20}$$

being x_p the position of the plate.

In general, the relationship between incoming and outgoing amplitudes remains:

$$\begin{pmatrix} C \\ B \end{pmatrix} = S \begin{pmatrix} A \\ D \end{pmatrix} = \begin{pmatrix} \mathcal{T} & \mathcal{R} \\ \mathcal{R} & \mathcal{T} \end{pmatrix} \begin{pmatrix} C \\ B \end{pmatrix} \tag{21}$$

where $\mathcal{T}$ and $\mathcal{R}$ are the transmission and reflection coefficients calculated above.

As it has been mentioned above, this system is symmetric and reciprocal, and the resulting scattering matrix is symmetric and diagonalizable (moreover it is unitary $SS^\dagger = I$ where $\dagger$ is the conjugated transposed operation and I is the identity matrix, and $det(S) = 1$).

By using the matrix diagonalization, it is possible to obtain the eigenvalues and eigenvectors, which as we will see have relevance from a physical point of view.

3.1 Eigenvalues

A scalar λ is called an eigenvalue of A if there is a non-trivial solution $\vec{x}$ of $A\vec{x} = \lambda\vec{x}$; where $\vec{x}$ is called eigenvector corresponding to λ (Lay, D.C. 1997 or Strang, G. 2016). In order to obtain the eigenvalues of a matrix, it is necessary to solve the following equation

$$(S - \lambda I)\,\vec{x} = 0,$$

or calculate the *Kernel* of the mapping, $Ker\,(S - \lambda I)$.

Obtaining a solution implies that the determinant should be equal to zero, that is:

$$|S - \lambda I| = 0, \tag{22}$$

where S is the scattering matrix, λ is the eigenvalue that we want to determine and I is the identity matrix.

Equation 22 results in a second-degree polynomial which solutions are the two eigenvalues of the system (a problem in $\mathbb{R}^2$)

$$(\mathcal{T} - \lambda)^2 - \mathcal{R}^2 = 0, \tag{23}$$

$$\lambda_{1,2} = \mathcal{T} \mp \mathcal{R}. \tag{24}$$

Figure 5 shows the dependence of the eigenvalues of the scattering matrix with frequency. Two cases are represented: the first one without losses ($\sigma = 0$) (Figs. 5(a)) and the second one with losses ($\sigma = 416$) s^{-1} (Figs. 5(b)). In the lossless case the eigenvalues take the same value at the resonance frequency of the system $\lambda_{1,2} = 1$ as a consequence of the energy conservation, i.e., $|R|^2 + |T|^2 = 1$. In fact, at the resonant frequency, $|R| = 0$ and $|T| = 1$, then $\lambda_1 = \lambda_2 = 1$. In the lossy case, the situation is a little bit more complicated, as losses induce absorption of energy. In this case, we can observe that for the chosen case, $R = T = 0.5$ (see Fig. 3(d)). As a consequence, we have one of the eigenvalues that has a zero value at the resonant frequency and the other one that is 1.

We can evaluate the dependence of the scattering coefficients and the eigenvalues of the scattering matrix on the losses of the system, σ. Figure 6(a) represents the absorption of the system at the resonance frequency as the losses increase. We can observe that the absorption cannot pass the threshold of $\alpha(\omega_0) = 0.5$. In fact, this particular point correspond to the analyzed case in Fig. 5(b). This particular situation is due to the fact that only one of the two eigenvalues of the scattering matrix is zero at the resonant frequency, meaning that only half of the energy can be efficiently absorbed. In such kind of problems, with single resonators, only one mode can be excited in the plate, and only one of the two eigenvalues can be used to dissipate the energy; the other one remains almost invariable. In this work, this situation corresponds to the case with $\sigma = 416$ s^{-1} (see Fig. 6(c)). If we have smaller or bigger values of losses than this value, the efficiency in absorption is reduced, as shown in Figs. 6(b) and (d) respectively.

3.2 Eigenvectors

As in any other mathematical problem of diagonalization of a symmetric matrix, we are interested also in calculating the eigenvectors. Both, eigenvalues and eigenvectors are identified with resonances of the system. The eigenvector $\vec{v_1}$ corresponding to the eigenvalue λ_1 must satisfy the equation $A\vec{v_1} = \lambda_1\vec{v_1}$ for $\vec{v_1}$ unknown (Lay, D.C. 1997 or Strang, G. 2016). In fact we are dealing with obtaining a base element of the corresponding subspace associated to the eigenvalue λ_1. The equation of this subspace is given by:

$$(S - \lambda_1 I)\vec{v_1} = 0, \tag{25}$$

where S is the scattering matrix, λ_1 is one of the two eigenvalues calculated above and $\vec{v_1}$ is the eigenvector to be determined.

Solving the previous equation for λ_1 and λ_2,i.e., the two previous eigenvalues calculated, one gets the two eigenvectors of the system (one corresponding to each eigenvalue).

$$\vec{v_1} = [R, -R], \quad \vec{v_2} = [R, R]. \tag{26}$$

At this point, due to the fact that we are dealing with a symmetric matrix, we can show for example that, if $\lambda_1 \neq \lambda_2$ eigenvectors corresponding to different eigenvalues are ortogonal. This can be easily checked using the standard euclidean inner product:

$$\vec{v_1} \cdot \vec{v_2} = [R, -R]\,[R, R]^t = R^2 - R^2 = 0, \tag{27}$$

where the symbol t represent the matrix operation *transpose*.

4 Proposal of action: How to connect with maths in the acoustics lab

Considering the information given along this work, it is possible to present a proposal of a practical lesson to be applied in the first course of engineering and science degrees. This lecture consists of solving the proposed problem by using the analytical tools and comparing the results obtained with experimental measurements.

The students, having a previous background in mathematics, especially regarding to matrix diagonalization, complex exponentials and calculation of eigenvalues and eigenvectors, can check and compare the results of the model with experimental measurements. Also a basic knowledge on acoustics is necessary at a level of basic physics syllabus. The fundamentals of acoustics and TMM can be given to the students in a previous session. Once this step has been completed, the practical case that has been previously developed in this article will be explained and the students will have to obtain its analytical solution by using the given tools. The activity will be completed with a lab session in which students will perform experimental measurements with the device shown in Fig. 7. These measurements are performed by using the transfer-function method, detailed in the standard ISO 10534-2:2002. Figure 7 shows a picture with some elements necessary for the experimental measurement.

5 Conclusions

In this article we present the development of the formalism corresponding to the Transfer Matrix Method. This method gives the opportunity to use elements of matrix algebra that appear in math subjects in the first course of practically all degrees in science and technology in a real problem. The

activity in the math classroom can be translated to the laboratory where, without the need of a sophisticated experimental setup, students can compare the predictions of a mathematical model with experimental results, and increase their understanding of the involved physical phenomena. This gives the opportunity not only to put mathematics in context, but also organize a multi-disciplinary activity where mathematical and physical concepts go together. In summary, the activity should contribute not only to a better understanding of the physical phenomena and their interpretation in terms of the analytical results, but also it should also anchor the mathematic knowledge via its relation with real problems.

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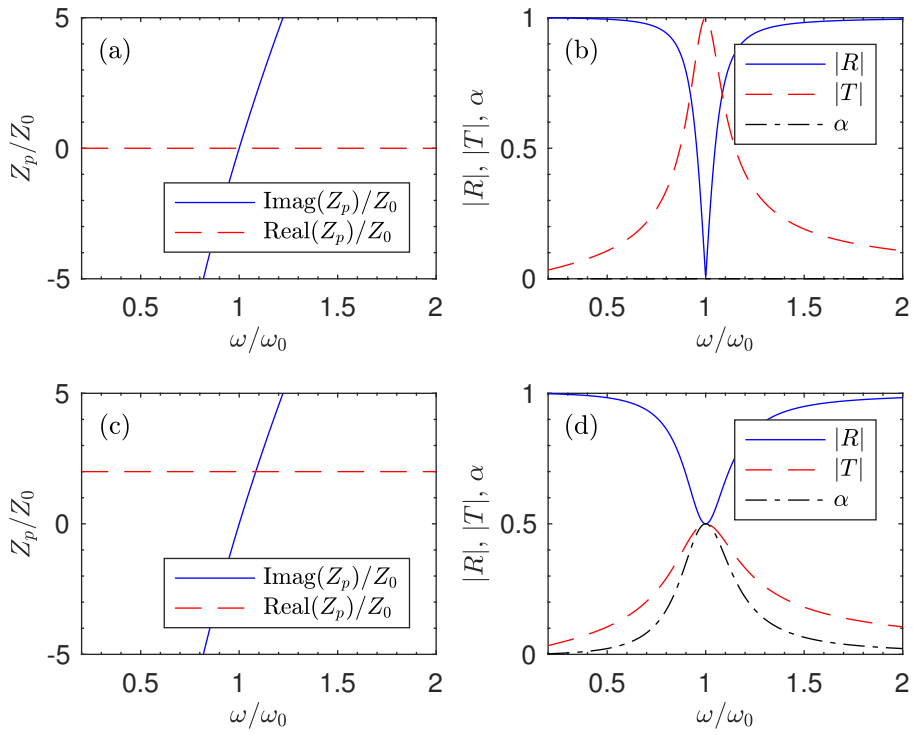


Figure 3: Analysis of the acoustic impedance of the clamped plate and the scattering coefficients. (a) and (b) represent the case without losses ($\sigma = 0$) and (c) and (d) represent the case with losses ($\sigma = 416$ s^{-1}). (a) and (b) represent the imaginary (blue continuous line) and the real (red dashed line) part of the normalized impedance of the clamped plate. (b) and (c) represent the absolute value of the scattering coefficients, i.e., reflection (blue continuous line), transmission (red dashed line) and absorption (black dash-dotted line). In all the cases the dependence is shown with respect to the normalized frequency with respect to the resonant frequency of the plate, ω_0.



Figure 4: Forward and backward waves in the scattering process.

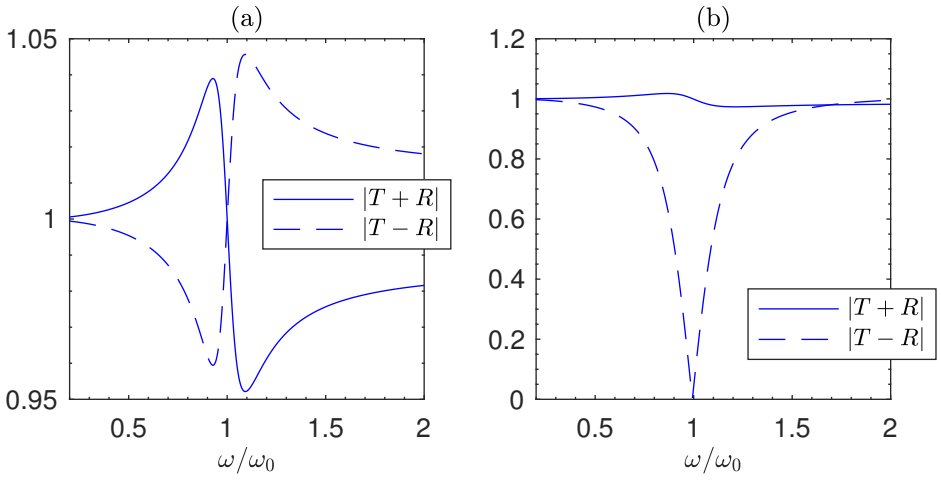


Figure 5: Analysis of the eigenvalues of the system. (a) represents the case without losses ($\sigma = 0$) and (b) represents the case with losses ($\sigma = 416$ s^{-1}). (a) [(b)] represents the absolute value of the both eigenvalues of the system, $|T+R|$ (blue continuous line) and $|T-R|$ (blue dashed line) for the lossless [lossy] case. In all the cases the dependence is shown with respect to the normalized frequency with respect to the resonant frequency of the plate, ω_0.

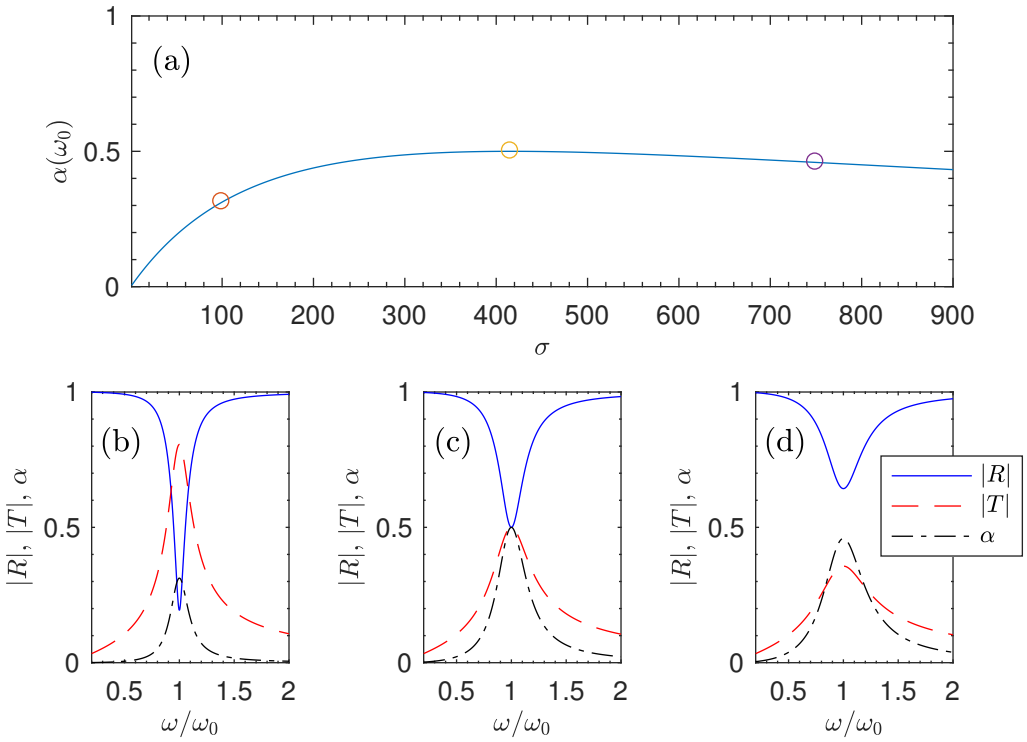


Figure 6: Analysis of the scattering parameters on the losses of the system. (a) represents the absorption of the system at the resonance frequency of the system as the losses increases. Dots in (a) represents the analyzed cases in (b), (c) and (d). (b), (c) and (d) show the scattering coefficients for three particular values of the losses: $\sigma = 100$ s^{-1}, $\sigma = 416$ s^{-1}, $\sigma = 750$ s^{-1} respectively.

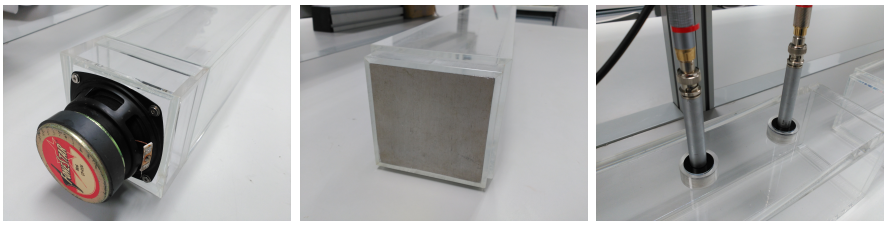


Figure 7: Experimental set up for measurements.