Unification in epistemic logics

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ABSTRACT
Epistemic logics are essential to the design of logical systems that capture elements of reasoning about knowledge. In this paper, we study the computability of unifiability and the unification types in several epistemic logics.

KEYWORDS
Epistemic logics; unification problem; unification type

1. Introduction

Epistemic logics are essential to the design of logical systems that capture elements of reasoning about knowledge. There exist variants of these logics with one or several agents, with or without common knowledge, etc. The logical problems addressed in their setting usually concern their axiomatisability and their decidability (see Fagin, Halpern, Moses, & Vardi, 1995). Epistemic logics can have a number of other desirable properties which one should establish whenever possible. Such properties concern, for example, the admissibility problem and the unifiability problem. About the admissibility problem, an inference rule $\frac{\phi_1,\ldots,\phi_n}{\psi}$ is admissible in an epistemic logic L if for all instances $\frac{\phi'_1,\ldots,\phi'_n}{\psi'}$ of the inference rule, if $\phi'_1,\ldots,\phi'_n$ are in L then ψ' is in L too (see Rybakov, 1997; Wolter & Zakharyaschev, 2008). About the unifiability problem, a formula ϕ is unifiable in an epistemic logic L if there exists an instance ϕ' of the formula such that ϕ' is in L (see Baader & Ghilardi, 2011; Ghilardi, 2000).

When an epistemic logic L is axiomatically presented, its admissible inference rules can be added to its axiomatical presentation without changing the set of its theorems. As a result, in order to improve the efficiency of automated theorem provers for epistemic logics, methods for deciding the admissibility of inference rules can be used (see Babenyshev, Rybakov, Schmidt, & Tishkovsky, 2010). The unifiability problem is easily reducible to the admissibility problem, seeing that the formula ϕ is unifiable in L iff the inference rule $\frac{\phi}{\bot}$ is non-admissible in L. In some cases, when L's unification type is finitary, the admissibility problem is reducible to the unifiability problem (see Dzik, 2007; Gencer & de Jongh, 2009; Ghilardi, 2000). Therefore, in order to improve the efficiency of automated theorem provers for epistemic logics, methods for deciding the unifiability of formulas can be used as well. In this paper, we study the computability of unifiability and the unification types in several epistemic logics.

The unifiability problem has been already considered in restricted fragments of epistemic logics where the unique modal connective is the one of common knowledge (see Rybakov,

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2002, 2011). Much remains to be done, seeing that the computability of unifiability and the unification types are unknown in most epistemic logics. In this paper, given an epistemic logic L, we examine the following questions. Is it computable whether a given formula is unifiable in L? When the answer is 'yes', how complex is the problem? When a formula is unifiable in L, has it a minimal complete set of unifiers? When the answer is 'yes', how large is this set? A final word about epistemic logics before entering into the details. Many propositional logics deserve to be called 'epistemic logics'. In this paper, we will only interest in the normal modal logics $K45$, $KD45$ and $S5$ and their multi-agent versions. Moreover, in order to avoid non-essential definitions, we will sometimes make no explicit difference between a class of frames and the normal modal logic it gives rise to, for instance: the class of all transitive frames and the normal modal logic $K4$, the class of all reflexive and transitive frames and the normal modal logic $S4$, etc.

The paper is organised as follows. In Sections 2 and 3, we present the syntax and the semantics of ordinary modal logic. Section 4 is concerned with the computability of the unifiability problem in the normal modal logics $K45$, $KD45$ and $S5$. In Section 5, we establish the unification types of some of these normal modal logics. Sections 6 and 7 are devoted to the unification problem in epistemic logics with parameters and in multi-agent epistemic logics. The unifiability problem has been studied since the beginning of logic (Boole, Löwenheim, etc.). But recent years have seen an increase of interest, important results have been obtained. In this paper, we survey what is known in the computability of unifiability and the unification types in epistemic logics. Some of the results presented below are simple adaptations of already known results whereas other results are new. We hope the reader will find it useful as a starting point for further research on unification in epistemic logics.

2. Syntax

It is now time to meet the modal language we will be working with. Let VAR be a countable set of atomic formulas called variables (with typical members denoted x, y, etc.). The formulas are defined by the rule

- $\phi ::= x \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid \Box\phi$.

We follow the standard rules for omission of the parentheses whereas we adopt the standard definitions for the remaining Boolean operations. We also write $\Box^+\phi$ for $\phi \wedge \Box\phi$. Let

- $\Diamond\phi ::= \neg\Box\neg\phi$.

We also write $\Diamond^+\phi$ for $\neg\Box^+\neg\phi$. We write $\phi(x_1, \dots, x_n)$ to denote a formula whose variables form a subset of $\{x_1, \dots, x_n\}$. The result of the replacement of $x_1, \dots, x_n$ in their places in ϕ with formulas $\psi_1, \dots, \psi_n$ will be denoted $\phi(\psi_1, \dots, \psi_n)$. A substitution is a function σ associating to each variable x a formula $\sigma(x)$. We shall say that a substitution σ is closed if for all variables x, $\sigma(x)$ is a variable-free formula. For all formulas $\phi(x_1, \dots, x_n)$ let $\sigma(\phi)$ be $\phi(\sigma(x_1), \dots, \sigma(x_n))$. The composition $\sigma \circ \tau$ of the substitutions σ and τ associates to each variable x the formula $\tau(\sigma(x))$. Obviously, this 'composition' operation on substitutions is associative.

3. Semantics

A frame is a structure of the form $\mathcal{F} = (W, R)$ where W is a nonempty set of states and R is a binary relation on W. In this paper, we will consider the following important properties of a frame $\mathcal{F} = (W, R)$:

- $\mathcal{F}$ is Euclidean when for all $s, t, u \in W$, if sRt and sRu then tRu,
- $\mathcal{F}$ is reflexive when for all $s \in W$, sRs,
- $\mathcal{F}$ is serial when for all $s \in W$, there exists $t \in W$ such that sRt,
- $\mathcal{F}$ is transitive when for all $s, t, u \in W$, if sRt and tRu then sRu.

A model based on a frame $\mathcal{F} = (W, R)$ is a triple $\mathcal{M} = (W, R, V)$ where V is a function associating a subset $V(x)$ of W to each $x \in VAR$. We define the notion of a formula ϕ being true in model $\mathcal{M} = (W, R, V)$ at a state s in W (in symbols $\mathcal{M}, s \models \phi$) as follows:

- $\mathcal{M}, s \models x$ iff $s \in V(x)$,
- $\mathcal{M}, s \not\models \bot$,
- $\mathcal{M}, s \models \neg\phi$ iff $\mathcal{M}, s \not\models \phi$,
- $\mathcal{M}, s \models \phi \vee \psi$ iff either $\mathcal{M}, s \models \phi$, or $\mathcal{M}, s \models \psi$,
- $\mathcal{M}, s \models \Box\phi$ iff for all $t \in W$, if sRt then $\mathcal{M}, t \models \phi$.

As a result,

- $\mathcal{M}, s \models \Diamond\phi$ iff there exists $t \in W$ such that sRt and $\mathcal{M}, t \models \phi$.

A formula ϕ is globally true in a model $\mathcal{M} = (W, R, V)$ (in symbols $\mathcal{M} \models \phi$) if for all $s \in W$, $\mathcal{M}, s \models \phi$. A formula ϕ is valid on a frame $\mathcal{F}$ (in symbols $\mathcal{F} \models \phi$) if for all models $\mathcal{M}$ based on $\mathcal{F}$, $\mathcal{M} \models \phi$. A formula ϕ is valid on a class $\mathcal{C}$ of frames (in symbols $\mathcal{C} \models \phi$) if for all frames $\mathcal{F}$ in $\mathcal{C}$, $\mathcal{F} \models \phi$. Let $\mathcal{C}$ be a class of frames. A substitution σ is $\mathcal{C}$-equivalent to a substitution τ (in symbols $\sigma \simeq_{\mathcal{C}} \tau$) if for all variables x, $\mathcal{C} \models \sigma(x) \leftrightarrow \tau(x)$. A substitution σ is more $\mathcal{C}$-general than a substitution τ (in symbols $\sigma \preceq_{\mathcal{C}} \tau$) if there exists a substitution μ such that $\sigma \circ \mu \simeq_{\mathcal{C}} \tau$. Obviously, this 'more $\mathcal{C}$-general than' relation between substitutions is transitive. In this paper, we will mainly interest in the classes $\mathcal{C}_{K45}$ of all transitive and Euclidean frames, $\mathcal{C}_{KD45}$ of all serial, transitive and Euclidean frames and $\mathcal{C}_{S5}$ of all reflexive, transitive and Euclidean frames underlying the normal modal logics $K45$, $KD45$ and $S5$.

4. Unifiability problem

Let $\mathcal{C}$ be a class of frames. A formula ϕ is $\mathcal{C}$-unifiable if there exists a substitution σ such that $\mathcal{C} \models \sigma(\phi)$. In this case, σ is a $\mathcal{C}$-unifier of ϕ.

Example: The formula $\phi = \Box x \vee \Box\neg x$ is $\mathcal{C}_{K45}$-unifiable, the substitution σ such that $\sigma(x) = \Box x$ being one of its $\mathcal{C}_{K45}$-unifiers. Since $\mathcal{C}_{K45}$ contains $\mathcal{C}_{KD45}$ and $\mathcal{C}_{S5}$, ϕ is $\mathcal{C}_{KD45}$-unifiable and $\mathcal{C}_{S5}$-unifiable as well. Moreover, the $\mathcal{C}_{K45}$-unifiers of ϕ are also $\mathcal{C}_{KD45}$-unifiers and $\mathcal{C}_{S5}$-unifiers of ϕ.

Example: The formula $\phi = \Diamond x \vee \Diamond\neg y$ is $\mathcal{C}_{KD45}$-unifiable, the substitution σ such that $\sigma(x) = x$ and $\sigma(y) = \Box x$ being one of its $\mathcal{C}_{KD45}$-unifiers. Since $\mathcal{C}_{KD45}$ contains $\mathcal{C}_{S5}$, ϕ is $\mathcal{C}_{S5}$-unifiable as well. Moreover, the $\mathcal{C}_{KD45}$-unifiers of ϕ are also $\mathcal{C}_{S5}$-unifiers of ϕ. Note that ϕ is not $\mathcal{C}_{K45}$-unifiable.

Given a class $\mathcal{C}$ of frames, we study the computability of the following decision problem:

input: a formula ϕ,
output: determine whether ϕ is $\mathcal{C}$-unifiable.

Lemma 1: *If ϕ possesses a $\mathcal{C}$-unifier, then ϕ possesses a closed $\mathcal{C}$-unifier.*

Proof: This follows from the fact that for all $\mathcal{C}$-unifiers σ of ϕ and for all closed substitutions τ, $\sigma \circ \tau$ is a closed $\mathcal{C}$-unifier of ϕ. $\square$

Lemma 2: *Let $\phi(x_1, \ldots, x_n)$ be a Boolean formula. The following conditions are equivalent:*

(1) $\phi(x_1, \ldots, x_n)$*, considered as a Boolean formula, is satisfiable.*
(2) $\phi(x_1, \ldots, x_n)$*, considered as a modal formula, is $\mathcal{C}$-unifiable.*

Proof: Suppose $\phi(x_1, \ldots, x_n)$, considered as a Boolean formula, is satisfiable. Hence, there exists formulas $\psi_1, \ldots, \psi_n$ in $\{\bot, \top\}$ such that $\phi(\psi_1, \ldots, \psi_n)$ is classically equivalent to $\top$. Thus, $\phi(\psi_1, \ldots, \psi_n)$ is $\mathcal{C}$-equivalent to $\top$. Consequently, $\phi(x_1, \ldots, x_n)$, considered as a modal formula, is $\mathcal{C}$-unifiable.

Reciprocally, suppose $\phi(x_1, \ldots, x_n)$, considered as a modal formula, is $\mathcal{C}$-unifiable. Let σ be a $\mathcal{C}$-unifier of $\phi(x_1, \ldots, x_n)$. Let $\mathcal{M} = (W, R, V)$ be a $\mathcal{C}$-model and $s \in W$. Since σ is a $\mathcal{C}$-unifier of $\phi(x_1, \ldots, x_n)$, therefore $\mathcal{M}, s \models \phi(\sigma(x_1), \ldots, \sigma(x_n))$. Let $\psi_1, \ldots, \psi_n$ be formulas in $\{\bot, \top\}$ such that for all $i \in \{1, \ldots, n\}$, if $\mathcal{M}, s \models \sigma(x_i)$ then $\psi_i = \top$ else $\psi_i = \bot$. Since $\mathcal{M}, s \models \phi(\sigma(x_1), \ldots, \sigma(x_n))$, therefore $\phi(\psi_1, \ldots, \psi_n)$ is classically equivalent to $\top$. Hence, $\phi(x_1, \ldots, x_n)$, considered as a Boolean formula, is satisfiable. $\square$

Lemma 3: *If either $\mathcal{C}$ is $\mathcal{C}_{K45}$, or $\mathcal{C}$ is $\mathcal{C}_{KD45}$, or $\mathcal{C}$ is $\mathcal{C}_{S5}$ then the following decision problem is in P:*

input: *a variable-free formula ϕ,*
output: *determine whether $\mathcal{C} \models \phi$.*

Proof: This is a well-known property. $\square$

Lemma 4: *If either $\mathcal{C}$ is $\mathcal{C}_{KD45}$, or $\mathcal{C}$ is $\mathcal{C}_{S5}$ then every variable-free formula is either $\mathcal{C}$-equivalent to $\bot$, or $\mathcal{C}$-equivalent to $\top$.*

Proof: This is a well-known property. $\square$

Lemma 5: *If either $\mathcal{C}$ is $\mathcal{C}_{KD45}$, or $\mathcal{C}$ is $\mathcal{C}_{S5}$ then every closed substitution is $\mathcal{C}$-equivalent to a substitution σ such that for each variable x, either $\sigma(x) = \bot$, or $\sigma(x) = \top$.*

Proof: By Lemma 4. $\square$

Lemma 6: *Let ϕ be a formula. If either $\mathcal{C}$ is $\mathcal{C}_{KD45}$, or $\mathcal{C}$ is $\mathcal{C}_{S5}$ then the following conditions are equivalent:*

(1) *ϕ is $\mathcal{C}$-unifiable.*
(2) *There exists a $\mathcal{C}$-unifier σ of ϕ such that for all variables x, either $\sigma(x) = \bot$, or $\sigma(x) = \top$.*

Proof: By Lemmas 1 and 5. $\square$

Proposition 1: *If either $\mathcal{C}$ is $\mathcal{C}_{KD45}$, or $\mathcal{C}$ is $\mathcal{C}_{S5}$ then the $\mathcal{C}$-unifiability problem is NP-complete.*

Proof: Suppose either $\mathcal{C}$ is $\mathcal{C}_{KD45}$, or $\mathcal{C}$ is $\mathcal{C}_{S5}$. In order to determine whether a given formula $\phi(x_1, \ldots, x_n)$ is $\mathcal{C}$-unifiable, let us consider the following procedure:

procedure $UNI(\phi(x_1, \ldots, x_n))$
begin
guess a tuple $(\psi_1, \ldots, \psi_n)$ of formulas in $\{\bot, \top\}$
$bool := BG(\phi(x_1, \ldots, x_n), (\psi_1, \ldots, \psi_n))$
if *bool* then accept else reject
end

The function $BG(\cdot)$ takes as input a formula $\phi(x_1, \ldots, x_n)$ and a tuple $(\psi_1, \ldots, \psi_n)$ of formulas in $\{\bot, \top\}$. It returns the Boolean value $\top$ if $\mathcal{C} \models \phi(\psi_1, \ldots, \psi_n)$. Otherwise, it returns the Boolean value $\bot$. By Lemma 3, it can be implemented as a deterministic Turing machine working in polynomial time. By Lemma 6, the procedure $UNI(\cdot)$ accepts its input $\phi(x_1, \ldots, x_n)$ iff $\phi(x_1, \ldots, x_n)$ is $\mathcal{C}$-unifiable. It can be implemented as a nondeterministic Turing machine working in polynomial time. Hence, the $\mathcal{C}$-unifiability problem is in *NP*. As for the *NP*-hardness of the $\mathcal{C}$-unifiability problem, it follows from Lemma 2. □

Lemma 7: *Every variable-free formula is either $\mathcal{C}_{K45}$-equivalent to $\bot$, or $\mathcal{C}_{K45}$-equivalent to $\top$, or $\mathcal{C}_{K45}$-equivalent to $\Box\bot$, or $\mathcal{C}_{K45}$-equivalent to $\Diamond\top$.*

Proof: This is a well-known property. □

Lemma 8: *Every closed substitution is $\mathcal{C}_{K45}$-equivalent to a substitution σ such that for each variable x, either $\sigma(x) = \bot$, or $\sigma(x) = \top$, or $\sigma(x) = \Box\bot$, or $\sigma(x) = \Diamond\top$.*

Proof: By Lemma 7. □

Lemma 9: *Let ϕ be a formula. The following conditions are equivalent:*

(1) *ϕ is $\mathcal{C}_{K45}$-unifiable.*
(2) *There exists a $\mathcal{C}_{K45}$-unifier σ of ϕ such that for all variables x, either $\sigma(x) = \bot$, or $\sigma(x) = \top$, or $\sigma(x) = \Box\bot$, or $\sigma(x) = \Diamond\top$.*

Proof: By Lemmas 1 and 8. □

Proposition 2: *The $\mathcal{C}_{K45}$-unifiability problem is NP-complete.*

Proof: In order to determine whether a given formula $\phi(x_1, \ldots, x_n)$ is $\mathcal{C}_{K45}$-unifiable, let us consider the following procedure:

procedure $UNI_{45}(\phi(x_1, \ldots, x_n))$
begin
guess a tuple $(\psi_1, \ldots, \psi_n)$ of formulas in $\{\bot, \top, \Box\bot, \Diamond\top\}$
$bool := BG_{45}(\phi(x_1, \ldots, x_n), (\psi_1, \ldots, \psi_n))$
if *bool* then accept else reject
end

The function $BG_{45}(\cdot)$ takes as input a formula $\phi(x_1, \ldots, x_n)$ and a tuple $(\psi_1, \ldots, \psi_n)$ of formulas in $\{\bot, \top, \Box\bot, \Diamond\top\}$. It returns the Boolean value $\top$ if $\mathcal{C}_{K45} \models \phi(\psi_1, \ldots, \psi_n)$. Otherwise, it returns the Boolean value $\bot$. By Lemma 3, it can be implemented as a deterministic Turing machine working in polynomial time. By Lemma 9, the procedure $UNI_{45}(\cdot)$ accepts its input $\phi(x_1, \ldots, x_n)$ iff $\phi(x_1, \ldots, x_n)$ is $\mathcal{C}_{45}$-unifiable. It can be implemented as a nondeterministic Turing machine working in polynomial time. Hence, the $\mathcal{C}_{45}$-unifiability problem is in *NP*. As for the *NP*-hardness of the $\mathcal{C}_{45}$-unifiability problem, it follows from Lemma 2. □

In Rybakov, Terziler, and Gencer (1999), syntactic characterisations have been given for the unifiability problem in normal modal logics like $KD4$ and $S4$. Later on, in Gencer and de Jongh (2009), similar syntactic characterisations have been given for the unifiability problem in normal modal logics like GL and $K4.3$. Now, we give syntactic characterisations for the unifiability problem in normal modal logics like $K45$, $KD45$ and $S5$. As for $S5$, the syntactic characterisation looks like those considered in Gencer and de Jongh (2009); Rybakov et al. (1999).

Proposition 3: *Let $\phi(x_1, \ldots, x_n)$ be a formula. The following conditions are equivalent:*

(1) *$\phi(x_1, \ldots, x_n)$ is $\mathcal{C}_{S5}$-unifiable.*
(2) $\mathcal{C}_{S5} \not\models \phi(x_1, \ldots, x_n) \to \bigvee\{\Diamond x_i \wedge \Diamond\neg x_i : 1 \leq i \leq n\}$.

Proof: Suppose $\phi(x_1, \ldots, x_n)$ is $\mathcal{C}_{S5}$-unifiable and $\mathcal{C}_{S5} \models \phi(x_1, \ldots, x_n) \to \bigvee\{\Diamond x_i \wedge \Diamond\neg x_i : 1 \leq i \leq n\}$. By Lemma 6, let $\psi_1, \ldots, \psi_n$ be formulas in $\{\bot, \top\}$ such that $\mathcal{C}_{S5} \models \phi(\psi_1, \ldots, \psi_n)$. Since $\mathcal{C}_{S5} \models \phi(x_1, \ldots, x_n) \to \bigvee\{\Diamond x_i \wedge \Diamond\neg x_i : 1 \leq i \leq n\}$, therefore $\mathcal{C}_{S5} \models \phi(\psi_1, \ldots, \psi_n) \to \bigvee\{\Diamond\psi_i \wedge \Diamond\neg\psi_i : 1 \leq i \leq n\}$. Since $\mathcal{C}_{S5} \models \phi(\psi_1, \ldots, \psi_n)$, therefore $\mathcal{C}_{S5} \models \bigvee\{\Diamond\psi_i \wedge \Diamond\neg\psi_i : 1 \leq i \leq n\}$. Since for all formulas ψ in $\{\bot, \top\}$, the formula $\Diamond\psi \wedge \Diamond\neg\psi$ is $\mathcal{C}_{S5}$-equivalent to $\bot$, therefore $\mathcal{C}_{S5} \models \bot$: a contradiction.

Reciprocally, suppose $\mathcal{C}_{S5} \not\models \phi(x_1, \ldots, x_n) \to \bigvee\{\Diamond x_i \wedge \Diamond\neg x_i : 1 \leq i \leq n\}$. Let $\mathcal{M} = (W, R, V)$ be a $\mathcal{C}_{S5}$-model and $s \in W$ be such that $\mathcal{M}, s \models \phi(x_1, \ldots, x_n)$ and for all $i \in \{1, \ldots, n\}$, either $\mathcal{M}, s \models \Box x_i$, or $\mathcal{M}, s \models \Box\neg x_i$. Without loss of generality, we can assume that $\mathcal{M}$ is generated from s. Let $\psi_1, \ldots, \psi_n$ be formulas in $\{\bot, \top\}$ such that for all $i \in \{1, \ldots, n\}$, if $\mathcal{M}, s \models \Box x_i$ then $\psi_i = \top$ else $\psi_i = \bot$. Since $\mathcal{M}, s \models \phi(x_1, \ldots, x_n)$, therefore $\mathcal{M}, s \models \phi(\psi_1, \ldots, \psi_n)$. Since $\phi(\psi_1, \ldots, \psi_n)$ is variable-free, therefore $\mathcal{C}_{S5} \models \phi(\psi_1, \ldots, \psi_n)$. Thus, $\phi(x_1, \ldots, x_n)$ is $\mathcal{C}_{S5}$-unifiable. □

Concerning $KD45$, the syntactic characterisation is similar to the one for $S5$.

Proposition 4: *Let $\phi(x_1, \ldots, x_n)$ be a formula. The following conditions are equivalent:*

(1) *$\phi(x_1, \ldots, x_n)$ is $\mathcal{C}_{KD45}$-unifiable.*
(2) $\mathcal{C}_{KD45} \not\models \phi(x_1, \ldots, x_n) \to \bigvee\{(x_i \vee \Diamond x_i) \wedge (\neg x_i \vee \Diamond\neg x_i) : 1 \leq i \leq n\}$.

Proof: Similar to the proof of Proposition 3. □

About $K45$, things are different: the syntactic characterisation uses the universal modality. We will make use of the following abbreviation where $[U]$ is universal modality:

- $(\phi \equiv \psi) ::= [U](\Box\bot \vee (\phi \leftrightarrow \psi) \vee \Diamond(\phi \leftrightarrow \psi)) \vee [U](\Diamond\top \vee (\phi \leftrightarrow \psi) \vee \Diamond(\phi \leftrightarrow \psi))$.

The universal modality is interpreted in models by the universal relation. More precisely, a formula $[U]\phi$ is true in model $\mathcal{M} = (W, R, V)$ at a state s in W iff ϕ is globally true in $\mathcal{M}$. See (Goranko & Passy, 1992) for details about the extension of the ordinary language of modal logic by means of the universal modality.

Proposition 5: *Let $\phi(x_1, \ldots, x_n)$ be a formula. The following conditions are equivalent:*

(1) *$\phi(x_1, \ldots, x_n)$ is $\mathcal{C}_{K45}$-unifiable.*
(2) $\mathcal{C}_{K45} \not\models [U]\phi(x_1, \ldots, x_n) \to \bigvee\{\bigwedge\{x_i \equiv \psi : \psi \in \{\bot, \top, \Box\bot, \Diamond\top\}\} : 1 \leq i \leq n\}$.

Proof: Suppose $\phi(x_1, \ldots, x_n)$ is $\mathcal{C}_{K45}$-unifiable and $\mathcal{C}_{K45} \models [U]\phi(x_1, \ldots, x_n) \to \bigvee\{\bigwedge\{x_i \equiv \psi : \psi \in \{\bot, \top, \Box\bot, \Diamond\top\}\} : 1 \leq i \leq n\}$. By Lemma 9, let $\psi_1, \ldots, \psi_n$ be formulas in $\{\bot, \top, \Box\bot, \Diamond\top\}$ such that $\mathcal{C}_{K45} \models \phi(\psi_1, \ldots, \psi_n)$. Since $\mathcal{C}_{K45} \models [U]\phi(x_1, \ldots, x_n) \to \bigvee\{\bigwedge\{x_i \equiv \psi : \psi \in \{\bot, \top, \Box\bot, \Diamond\top\}\} : 1 \leq i \leq n\}$, therefore $\mathcal{C}_{K45} \models [U]\phi(\psi_1, \ldots, \psi_n) \to$

$\bigvee\{\bigwedge\{\psi_i \equiv \psi : \psi \in \{\bot, \top, \Box\bot, \Diamond\top\}\} : 1 \leq i \leq n\}$. Since $\mathcal{C}_{K45} \models \phi(\psi_1, \ldots, \psi_n)$,
therefore $\mathcal{C}_{K45} \models \bigvee\{\bigwedge\{\psi_i \equiv \psi : \psi \in \{\bot, \top, \Box\bot, \Diamond\top\}\} : 1 \leq i \leq n\}$. Since for all $i \in \{1, \ldots, n\}$, the formula $\bigwedge\{\psi_i \equiv \psi : \psi \in \{\bot, \top, \Box\bot, \Diamond\top\}\}$ is $\mathcal{C}_{K45}$-equivalent to the formula $[U]\Box\bot \vee [U]\Diamond\top$, therefore $\mathcal{C}_{K45} \models [U]\Box\bot \vee [U]\Diamond\top$: a contradiction.

Reciprocally, suppose $\mathcal{C}_{K45} \not\models [U]\phi(x_1, \ldots, x_n) \to \bigvee\{\bigwedge\{x_i \equiv \psi : \psi \in \{\bot, \top, \Box\bot, \Diamond\top\}\} : 1 \leq i \leq n\}$. Let $\mathcal{M} = (W, R, V)$ be a $\mathcal{C}_{K45}$-model and $s \in W$ be such that $\mathcal{M}, s \models [U]\phi(x_1, \ldots, x_n)$ and for all $i \in \{1, \ldots, n\}$, $\mathcal{M}, s \not\models \bigwedge\{x_i \equiv \psi : \psi \in \{\bot, \top, \Box\bot, \Diamond\top\}\}$. This time, because we are using formulas containing the universal modality, we cannot assume that $\mathcal{M}$ is generated from s. Let $\psi_1, \ldots, \psi_n$ be formulas in $\{\bot, \top, \Box\bot, \Diamond\top\}$ such that for all $i \in \{1, \ldots, n\}$, $\mathcal{M}, s \models \langle U\rangle(\Box\bot \wedge (x_i \leftrightarrow \psi_i) \wedge \Box(x_i \leftrightarrow \psi_i)) \wedge \langle U\rangle(\Diamond\top \wedge (x_i \leftrightarrow \psi_i) \wedge \Box(x_i \leftrightarrow \psi_i))$. Since $\mathcal{M}, s \models [U]\phi(x_1, \ldots, x_n)$, therefore $\mathcal{M}, s \models \langle U\rangle(\Box\bot \wedge \phi(\psi_1, \ldots, \psi_n)) \wedge \langle U\rangle(\Diamond\top \wedge \phi(\psi_1, \ldots, \psi_n))$. Since $\phi(\psi_1, \ldots, \psi_n)$ is a variable-free formula, therefore $\mathcal{C}_{K45} \models \phi(\psi_1, \ldots, \psi_n)$. Thus, $\phi(x_1, \ldots, x_n)$ is $\mathcal{C}_{K45}$-unifiable. □

5. Unification types

Let $\mathcal{C}$ be a class of frames. A $\mathcal{C}$-unifier σ of a formula ϕ is a most $\mathcal{C}$-general unifier if for all $\mathcal{C}$-unifiers τ of ϕ, $\sigma \preceq_{\mathcal{C}} \tau$. A set Σ of $\mathcal{C}$-unifiers of a $\mathcal{C}$-unifiable formula ϕ is complete if for all $\mathcal{C}$-unifiers σ of ϕ, there exists a $\mathcal{C}$-unifier τ of ϕ in Σ such that $\tau \preceq_{\mathcal{C}} \sigma$. In some cases, every $\mathcal{C}$-unifiable formula possesses a most $\mathcal{C}$-general unifier. This is the case if $\mathcal{C}$ is the class $\mathcal{C}_{S5}$. See Proposition 6. Moreover, in some other cases, every $\mathcal{C}$-unifiable formula possesses a finite minimal complete set of $\mathcal{C}$-unifiers. This is the case if $\mathcal{C}$ is the class $\mathcal{C}_{K4}$ (see Ghilardi, 2000). Finally, in some other cases, there exists $\mathcal{C}$-unifiable formulas possessing no minimal complete set of $\mathcal{C}$-unifiers at all. This is the case if $\mathcal{C}$ is the class $\mathcal{C}_K$ of all frames (see Jeřábek, 2015). Hence, now, the question is: When a formula is $\mathcal{C}$-unifiable, has it a minimal complete set of $\mathcal{C}$-unifiers? When the answer is 'yes', how large is this set? Given a $\mathcal{C}$-unifiable formula, these questions of the existence and, when it exists, the cardinality of a minimal complete set of $\mathcal{C}$-unifiers for this formula is central (see Baader & Ghilardi, 2011; Dzik, 2003, 2007; Ghilardi, 2000; Jeřábek, 2015). It can be formalised as follows. Let ϕ be a $\mathcal{C}$-unifiable formula. We will say that

- ϕ is of type unitary (1) for $\mathcal{C}$ iff ϕ possesses a most $\mathcal{C}$-general unifier,
- ϕ is of type finitary (ω) for $\mathcal{C}$ iff there exists a finite minimal complete set of $\mathcal{C}$-unifiers of ϕ but ϕ does not possess a most $\mathcal{C}$-general unifier,
- ϕ is of type infinitary (∞) for $\mathcal{C}$ iff there exists a minimal complete set of $\mathcal{C}$-unifiers of ϕ but there exists no such a set with finite cardinality,
- ϕ is of type nullary (0) for $\mathcal{C}$ iff there exists no minimal complete set of $\mathcal{C}$-unifiers of ϕ.

As for the class $\mathcal{C}$, we will say that

- $\mathcal{C}$ is unitary if every $\mathcal{C}$-unifiable formula is of type unitary,
- $\mathcal{C}$ is finitary if there exists a $\mathcal{C}$-unifiable formula of type finitary and every $\mathcal{C}$-unifiable formula is either of type unitary, or of type finitary,
- $\mathcal{C}$ is infinitary if there exists a $\mathcal{C}$-unifiable formula of type infinitary and every $\mathcal{C}$-unifiable formula is either of type unitary, or of type finitary, or of type infinitary,
- $\mathcal{C}$ is nullary if there exists a $\mathcal{C}$-unifiable formula of type nullary.

Another interesting notion related to the existence and, when they exist, the cardinality of minimal complete sets of $\mathcal{C}$-unifiers for $\mathcal{C}$-unifiable formulas is the notion of directedness.

We will say that the class $\mathcal{C}$ is directed iff for all $\mathcal{C}$-unifiable formulas ϕ and for all $\mathcal{C}$-unifiers σ, τ of ϕ, there exists a $\mathcal{C}$-unifier μ of ϕ such that $\mu \leq_{\mathcal{C}} \sigma$ and $\mu \leq_{\mathcal{C}} \tau$.

Lemma 10: *If $\mathcal{C}$ is directed then either $\mathcal{C}$ is unitary, or $\mathcal{C}$ is nullary.*

Proof: Suppose $\mathcal{C}$ is directed and neither $\mathcal{C}$ is unitary, nor $\mathcal{C}$ is nullary. Hence, either $\mathcal{C}$ is finitary, or $\mathcal{C}$ is infinitary. Let ϕ be a $\mathcal{C}$-unifiable formula either of type finitary, or of type infinitary. Let Γ be a minimal complete set of $\mathcal{C}$-unifiers of ϕ. Since ϕ is either of type finitary, or of type infinitary, therefore $Card(\Gamma) \geq 2$. Let σ, τ in Γ be such that $\sigma \neq \tau$. Such σ, τ in Γ exists because $Card(\Gamma) \geq 2$. Let μ be a $\mathcal{C}$-unifier of ϕ such that $\mu \leq_{\mathcal{C}} \sigma$ and $\mu \leq_{\mathcal{C}} \tau$. Such $\mathcal{C}$-unifier of ϕ exists because $\mathcal{C}$ is directed. Let υ in Γ be such that $\upsilon \leq_{\mathcal{C}} \mu$. Such υ in Γ exists because Γ is a complete set of $\mathcal{C}$-unifiers of ϕ. Since $\mu \leq_{\mathcal{C}} \sigma$ and $\mu \leq_{\mathcal{C}} \tau$, therefore $\upsilon \leq_{\mathcal{C}} \sigma$ and $\upsilon \leq_{\mathcal{C}} \tau$. Since Γ is minimal, therefore $\upsilon = \sigma$ and $\upsilon = \tau$. Hence, $\sigma = \tau$: a contradiction. $\square$

Consider a formula ϕ and a substitution σ. Let τ_ϕ^σ be the substitution defined by $\tau_\phi^\sigma(x) = (\Box\phi \wedge x) \vee (\Diamond\neg\phi \wedge \sigma(x))$.

Lemma 11: *Let ψ be a formula.*

(1) $\mathcal{C}_{S5} \models \Box\phi \rightarrow (\tau_\phi^\sigma(\psi) \leftrightarrow \psi)$.
(2) $\mathcal{C}_{S5} \models \Diamond\neg\phi \rightarrow (\tau_\phi^\sigma(\psi) \leftrightarrow \sigma(\psi))$.

Proof: (1) The proof is done by induction on ψ. The case when $\psi = x$ is easy whereas the Boolean cases are left to the reader. Thus, we only give the proof of the case $\psi = \Box\psi'$. By induction hypothesis, we know that

$\mathcal{C}_{S5} \models \Box\phi \rightarrow (\tau_\phi^\sigma(\psi') \leftrightarrow \psi')$. Then,
$\mathcal{C}_{S5} \models \Box\Box\phi \rightarrow \Box(\tau_\phi^\sigma(\psi') \leftrightarrow \psi')$. Since,
$\mathcal{C}_{S5} \models \Box\phi \rightarrow \Box\Box\phi$, therefore,
$\mathcal{C}_{S5} \models \Box\phi \rightarrow \Box(\tau_\phi^\sigma(\psi') \leftrightarrow \psi')$. Since,
$\mathcal{C}_{S5} \models \Box(\tau_\phi^\sigma(\psi') \leftrightarrow \psi') \rightarrow (\Box\tau_\phi^\sigma(\psi') \leftrightarrow \Box\psi')$, therefore,
$\mathcal{C}_{S5} \models \Box\phi \rightarrow (\Box\tau_\phi^\sigma(\psi') \leftrightarrow \Box\psi')$. Consequently,
$\mathcal{C}_{S5} \models \Box\phi \rightarrow (\tau_\phi^\sigma(\Box\psi') \leftrightarrow \Box\psi')$.

(2) Similar to the proof of (1), this time using the fact that $\mathcal{C}_{S5} \models \Diamond\neg\phi \rightarrow \Box\Diamond\neg\phi$. $\square$

Lemma 12: *If σ is a $\mathcal{C}_{S5}$-unifier of ϕ then τ_ϕ^σ is a $\mathcal{C}_{S5}$-unifier of ϕ.*

Proof: Suppose σ is a $\mathcal{C}_{S5}$-unifier of ϕ. By Lemma 11,

$\mathcal{C}_{S5} \models \Box\phi \rightarrow (\tau_\phi^\sigma(\phi) \leftrightarrow \phi)$. Hence,
$\mathcal{C}_{S5} \models \Box\phi \rightarrow (\phi \rightarrow \tau_\phi^\sigma(\phi))$. Since
$\mathcal{C}_{S5} \models \Box\phi \rightarrow \phi$, therefore
$\mathcal{C}_{S5} \models \Box\phi \rightarrow \tau_\phi^\sigma(\phi)$. By Lemma 11,
$\mathcal{C}_{S5} \models \Diamond\neg\phi \rightarrow (\tau_\phi^\sigma(\phi) \leftrightarrow \sigma(\phi))$. Thus,
$\mathcal{C}_{S5} \models \Diamond\neg\phi \rightarrow (\sigma(\phi) \rightarrow \tau_\phi^\sigma(\phi))$. Since σ is a $\mathcal{C}_{S5}$-unifier of ϕ, therefore
$\mathcal{C}_{S5} \models \Diamond\neg\phi \rightarrow \tau_\phi^\sigma(\phi)$. Since
$\mathcal{C}_{S5} \models \Box\phi \rightarrow \tau_\phi^\sigma(\phi)$, therefore
$\mathcal{C}_{S5} \models \tau_\phi^\sigma(\phi)$. Consequently, τ_ϕ^σ is a $\mathcal{C}_{S5}$-unifier of ϕ. $\square$

Lemma 13: *If μ is a $\mathcal{C}_{S5}$-unifier of ϕ then $\tau_\phi^\sigma \leq_{\mathcal{C}_{S5}} \mu$.*

Proof: Suppose μ is a $\mathcal{C}_{S5}$-unifier of ϕ. Hence, $\mathcal{C}_{S5} \models \Box\mu(\phi)$. Let x be an arbitrary variable. By Lemma 11, $\mathcal{C}_{S5} \models \Box\phi \rightarrow (\tau_\phi^\sigma(x) \leftrightarrow x)$. Thus, $\mathcal{C}_{S5} \models \Box\mu(\phi) \rightarrow (\mu(\tau_\phi^\sigma(x)) \leftrightarrow \mu(x))$. Since

$\mathcal{C}_{S5} \models \Box\mu(\phi)$, therefore $\mathcal{C}_{S5} \models \mu(\tau_\phi^\sigma(x)) \leftrightarrow \mu(x)$. As x is an arbitrary variable, $\tau_\phi^\sigma \circ \mu \simeq_{\mathcal{C}_{S5}} \mu$. Consequently, $\tau_\phi^\sigma \preceq_{\mathcal{C}_{S5}} \mu$. $\square$

Proposition 6: *$\mathcal{C}_{S5}$ is unitary.*

Proof: Let ϕ be a $\mathcal{C}_{S5}$-unifiable formula. Let σ be a $\mathcal{C}_{S5}$-unifier of ϕ. By Lemmas 12 and 13, τ_ϕ^σ is a most $\mathcal{C}_{S5}$-general unifier of ϕ. Hence, ϕ is of type unitary. As ϕ is an arbitrary $\mathcal{C}_{S5}$-unifiable formula, $\mathcal{C}_{S5}$ is unitary. $\square$

Example: Consider again the formula $\phi = \Box x \vee \Box\neg x$. The substitution σ such that $\sigma(x) = \Box x$ is one of its $\mathcal{C}_{S5}$-unifiers. Let τ_ϕ^σ be the substitution defined by $\tau_\phi^\sigma(x) = (\Box\phi \wedge x) \vee (\Diamond\neg\phi \wedge \sigma(x))$. In $\mathcal{C}_{S5}$, $\tau_\phi^\sigma(x)$ is equivalent to $\Box x$. By Lemmas 12 and 13, this means that σ is a most $\mathcal{C}_{S5}$-general unifier of ϕ. Remark that the substitution σ' such that $\sigma'(x) = \Diamond x$ is a $\mathcal{C}_{S5}$-unifier of ϕ too. Let $\tau_\phi^{\sigma'}$ be the substitution defined by $\tau_\phi^{\sigma'}(x) = (\Box\phi \wedge x) \vee (\Diamond\neg\phi \wedge \sigma'(x))$. In $\mathcal{C}_{S5}$, $\tau_\phi^{\sigma'}(x)$ is equivalent to $\Diamond x$. By Lemmas 12 and 13, this means that σ' is a most $\mathcal{C}_{S5}$-general unifier of ϕ too. In other respect, the reader may easily verify that $\sigma \preceq_{\mathcal{C}_{S5}} \sigma'$ and $\sigma' \preceq_{\mathcal{C}_{S5}} \sigma$ by showing that $\sigma \circ \sigma' \simeq_{\mathcal{C}_{S5}} \sigma'$ and $\sigma' \circ \sigma \simeq_{\mathcal{C}_{S5}} \sigma$.

Example: Consider again the formula $\phi = \Diamond x \vee \Diamond\neg y$. The substitution σ such that $\sigma(x) = x$ and $\sigma(y) = \Box x$ is one of its $\mathcal{C}_{S5}$-unifiers. Let τ_ϕ^σ be the substitution defined by $\tau_\phi^\sigma(x) = (\Box\phi \wedge x) \vee (\Diamond\neg\phi \wedge \sigma(x))$ and $\tau_\phi^\sigma(y) = (\Box\phi \wedge y) \vee (\Diamond\neg\phi \wedge \sigma(y))$. In $\mathcal{C}_{S5}$, $\tau_\phi^\sigma(x)$ is equivalent to x and $\tau_\phi^\sigma(y)$ is equivalent to $(\Diamond x \vee \Diamond\neg y) \wedge y$. Remark that the substitution σ' such that $\sigma'(x) = x$ and $\sigma'(y) = \Diamond x$ is a $\mathcal{C}_{S5}$-unifier of ϕ too. Let $\tau_\phi^{\sigma'}$ be the substitution defined by $\tau_\phi^{\sigma'}(x) = (\Box\phi \wedge x) \vee (\Diamond\neg\phi \wedge \sigma'(x))$ and $\tau_\phi^{\sigma'}(y) = (\Box\phi \wedge y) \vee (\Diamond\neg\phi \wedge \sigma'(y))$. In $\mathcal{C}_{S5}$, $\tau_\phi^{\sigma'}(x)$ is equivalent to x and $\tau_\phi^{\sigma'}(y)$ is equivalent to $(\Diamond x \vee \Diamond\neg y) \wedge y$.

The results contained in Lemmas 11–13 and Proposition 6 have previously been discussed in Baader and Ghilardi (2011); Dzik (2003); Ghilardi (2000), sometimes with no proof. The proofs that we have given above allow the reader to exactly understand where the specific properties of $\mathcal{C}_{S5}$-frames (reflexivity, transitivity and Euclideanity) are used. In this respect, the proof of Lemma 12 uses the fact, corresponding to the reflexivity of $\mathcal{C}_{S5}$-frames, that $\mathcal{C}_{S5} \models \Box\phi \to \phi$. Hence, it cannot be repeated in the case of $\mathcal{C}_{K45}$ and $\mathcal{C}_{KD45}$. The main drawback with $\mathcal{C}_{K45}$ and $\mathcal{C}_{KD45}$ is that the universal modality (interpreted in models by the universal relation) is not definable in our language for these classes of frames. Nevertheless, it can be proved that unification is directed both in $\mathcal{C}_{K45}$ and in $\mathcal{C}_{KD45}$. The directedness of unification in $\mathcal{C}_{K45}$ and in $\mathcal{C}_{KD45}$ is a consequence of the characterisation by Ghilardi and Sacchetti (2004) of the normal extensions of $K4$ with a directed unification problem. More precisely, Ghilardi and Sacchetti demonstrate in their Theorem 8.4 that a normal extension L of $K4$ has a directed unification problem iff $\Diamond^+\Box^+x \to \Box^+\Diamond^+x$ is in L. Since $\Diamond^+\Box^+x \to \Box^+\Diamond^+x$ is both $\mathcal{C}_{K45}$-valid and $\mathcal{C}_{KD45}$-valid, therefore, by Ghilardi and Sacchetti (2004, Theorem 8.4), unification is directed both in $\mathcal{C}_{K45}$ and in $\mathcal{C}_{KD45}$. The proof presented by Ghilardi and Sacchetti uses advanced notions from algebraic and relational semantics of normal modal logics. In the remaining part of this Section, we give an explicit and simpler proof of the directedness of unification in $\mathcal{C}_{K45}$ and in $\mathcal{C}_{KD45}$. Consider substitutions σ, τ. Suppose for all variables x, the variable y occurs neither in $\sigma(x)$, nor in $\tau(x)$. Let $\alpha_{KD45}^{\sigma,\tau}$ be the substitution defined by $\alpha_{KD45}^{\sigma,\tau}(x) = (\Box y \wedge \sigma(x)) \vee (\Diamond\neg y \wedge \tau(x))$.

Lemma 14:

(1) $\alpha_{KD45}^{\sigma,\tau} \preceq_{\mathcal{C}_{KD45}} \sigma$.
(2) $\alpha_{KD45}^{\sigma,\tau} \preceq_{\mathcal{C}_{KD45}} \tau$.

Proof:

(1) Let β be the substitution defined by $\beta(y) = \top$. Since the variable y occurs neither in $\sigma(x)$, nor in $\tau(x)$, the reader may easily verify that $\alpha^{\sigma,\tau}_{KD45} \circ \beta \simeq_{\mathcal{C}_{KD45}} \sigma$. Hence, $\alpha^{\sigma,\tau}_{KD45} \preceq_{\mathcal{C}_{KD45}} \sigma$.

(2) Similar to the proof of (1), this time using the substitution γ defined by $\gamma(y) = \bot$.

$\square$

Lemma 15: *Let ψ be a formula.*

(1) $\mathcal{C}_{KD45} \models \Box y \to (\alpha^{\sigma,\tau}_{KD45}(\psi) \leftrightarrow \sigma(\psi))$.

(2) $\mathcal{C}_{KD45} \models \Diamond\neg y \to (\alpha^{\sigma,\tau}_{KD45}(\psi) \leftrightarrow \tau(\psi))$.

Proof: (1) The proof is done by induction on ψ. The case when $\psi = x$ is easy whereas the Boolean cases are left to the reader. Thus, we only give the proof of the case $\psi = \Box\psi'$. By induction hypothesis, we know that

$\mathcal{C}_{KD45} \models \Box y \to (\alpha^{\sigma,\tau}_{KD45}(\psi') \leftrightarrow \sigma(\psi'))$. Then
$\mathcal{C}_{KD45} \models \Box\Box y \to \Box(\alpha^{\sigma,\tau}_{KD45}(\psi') \leftrightarrow \sigma(\psi'))$. Since
$\mathcal{C}_{KD45} \models \Box y \to \Box\Box y$, therefore
$\mathcal{C}_{KD45} \models \Box y \to \Box(\alpha^{\sigma,\tau}_{KD45}(\psi') \leftrightarrow \sigma(\psi'))$. Since
$\mathcal{C}_{KD45} \models \Box(\alpha^{\sigma,\tau}_{KD45}(\psi') \leftrightarrow \sigma(\psi')) \to (\Box\alpha^{\sigma,\tau}_{KD45}(\psi') \leftrightarrow \Box\sigma(\psi'))$, therefore
$\mathcal{C}_{KD45} \models \Box y \to (\Box\alpha^{\sigma,\tau}_{KD45}(\psi') \leftrightarrow \Box\sigma(\psi'))$. Consequently,
$\mathcal{C}_{KD45} \models \Box y \to (\alpha^{\sigma,\tau}_{KD45}(\Box\psi') \leftrightarrow \sigma(\Box\psi'))$.

(2) Similar to the proof of (1), this time using the fact that $\mathcal{C}_{KD45} \models \Diamond\neg y \to \Box\Diamond\neg y$. $\square$

Lemma 16: *Let ϕ be a formula. If σ and τ are $\mathcal{C}_{KD45}$-unifiers of ϕ then $\alpha^{\sigma,\tau}_{KD45}$ is a $\mathcal{C}_{KD45}$-unifier of ϕ.*

Proof: Suppose σ and τ are $\mathcal{C}_{KD45}$-unifiers of ϕ. By Lemma 15,

$\mathcal{C}_{KD45} \models \Box y \to (\alpha^{\sigma,\tau}_{KD45}(\phi) \leftrightarrow \sigma(\phi))$ and
$\mathcal{C}_{KD45} \models \Diamond\neg y \to (\alpha^{\sigma,\tau}_{KD45}(\phi) \leftrightarrow \tau(\phi))$. Hence,
$\mathcal{C}_{KD45} \models \Box y \to (\sigma(\phi) \to \alpha^{\sigma,\tau}_{KD45}(\phi))$ and
$\mathcal{C}_{KD45} \models \Diamond\neg y \to (\tau(\phi) \to \alpha^{\sigma,\tau}_{KD45}(\phi))$. Since σ and τ are $\mathcal{C}_{KD45}$-unifiers of ϕ, therefore
$\mathcal{C}_{KD45} \models \Box y \to \alpha^{\sigma,\tau}_{KD45}(\phi)$ and
$\mathcal{C}_{KD45} \models \Diamond\neg y \to \alpha^{\sigma,\tau}_{KD45}(\phi)$. Thus,
$\mathcal{C}_{KD45} \models \alpha^{\sigma,\tau}_{KD45}(\phi)$. Consequently, $\alpha^{\sigma,\tau}_{KD45}$ is a $\mathcal{C}_{KD45}$-unifier of ϕ. $\square$

Proposition 7: *Unification in $\mathcal{C}_{KD45}$ is directed.*

Proof: Let ϕ be a $\mathcal{C}_{KD45}$-unifiable formula. Let σ, τ be $\mathcal{C}_{KD45}$-unifiers of ϕ. By Lemmas 14 and 16, $\alpha^{\sigma,\tau}_{KD45}$ is a $\mathcal{C}_{KD45}$-unifier of ϕ such that $\alpha^{\sigma,\tau}_{KD45} \preceq_{\mathcal{C}_{KD45}} \sigma$ and $\alpha^{\sigma,\tau}_{KD45} \preceq_{\mathcal{C}_{KD45}} \tau$. As ϕ is an arbitrary $\mathcal{C}_{KD45}$-unifiable formula, $\mathcal{C}_{KD45}$ is directed. $\square$

Consider substitutions σ, τ. Suppose for all variables x, the variable y occurs neither in $\sigma(x)$, nor in $\tau(x)$. Let $\alpha^{\sigma,\tau}_{K45}$ be the substitution defined by $\alpha^{\sigma,\tau}_{K45}(x) = ((\Box y \wedge (y \vee \Diamond\top)) \wedge \sigma(x)) \vee ((\Diamond\neg y \vee (\neg y \wedge \Box\bot)) \wedge \tau(x))$.

Lemma 17:

(1) $\alpha^{\sigma,\tau}_{K45} \preceq_{\mathcal{C}_{K45}} \sigma$.

(2) $\alpha^{\sigma,\tau}_{K45} \preceq_{\mathcal{C}_{K45}} \tau$.

Proof: Similar to the proof of Lemma 14. $\square$

Lemma 18: *Let ψ be a formula.*

(1) $\mathcal{C}_{K45} \models \Box y \wedge (y \vee \Diamond\top) \rightarrow (\alpha^{\sigma,\tau}_{K45}(\psi) \leftrightarrow \sigma(\psi))$.
(2) $\mathcal{C}_{K45} \models \Diamond\neg y \vee (\neg y \wedge \Box\bot) \rightarrow (\alpha^{\sigma,\tau}_{K45}(\psi) \leftrightarrow \tau(\psi))$.

Proof: Similar to the proof of Lemma 15, this time using the fact that $\mathcal{C}_{K45} \models \Box y \wedge (y \vee \Diamond\top) \rightarrow \Box(\Box y \wedge (y \vee \Diamond\top))$ and $\mathcal{C}_{K45} \models \Diamond\neg y \vee (\neg y \wedge \Box\bot) \rightarrow \Box(\Diamond\neg y \vee (\neg y \wedge \Box\bot))$. $\Box$

Lemma 19: *Let ϕ be a formula. If σ and τ are $\mathcal{C}_{K45}$-unifiers of ϕ then $\alpha^{\sigma,\tau}_{K45}$ is a $\mathcal{C}_{K45}$-unifier of ϕ.*

Proof: Similar to the proof of Lemma 16. $\Box$

Proposition 8: *Unification in $\mathcal{C}_{K45}$ is directed.*

Proof: Similar to the proof of Proposition 7. $\Box$

Proposition 9:

(1) *Either $\mathcal{C}_{K45}$ is unitary, or $\mathcal{C}_{K45}$ is nullary.*
(2) *Either $\mathcal{C}_{KD45}$ is unitary, or $\mathcal{C}_{KD45}$ is nullary.*

Proof: By Lemma 10 and Propositions 7 and 8. $\Box$

We conjecture that $\mathcal{C}_{K45}$ is unitary and $\mathcal{C}_{KD45}$ is unitary.

6. Unifiability with parameters

In Sections 4 and 5, we have considered that a formula ϕ is unifiable if there exists a substitution σ such that $\sigma(\phi)$ is valid. But it rarely happens that we accept all variables to be possibly replaced by formulas. This leads us to a new definition of the syntax. Let *PAR* be a countable set of new atomic formulas called parameters (with typical members denoted p, q, etc.). The formulas are now defined by the rule:

- $\phi ::= p \mid x \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid \Box\phi$.

We write $\phi(p_1, \ldots, p_m)$ to denote a formula whose parameters form a subset of $\{p_1, \ldots, p_m\}$, $\phi(x_1, \ldots, x_n)$ to denote a formula whose variables form a subset of $\{x_1, \ldots, x_n\}$ and $\phi(p_1, \ldots, p_m, x_1, \ldots, x_n)$ to denote a formula whose parameters form a subset of $\{p_1, \ldots, p_m\}$ and whose variables form a subset of $\{x_1, \ldots, x_n\}$. Like in Section 2, the result of the replacement of $x_1, \ldots, x_n$ in their places in $\phi(x_1, \ldots, x_n)$ with formulas $\psi_1, \ldots, \psi_n$ will be denoted $\phi(\psi_1, \ldots, \psi_n)$. A substitution is still a function σ associating to each variable x a formula $\sigma(x)$. And again, we shall say that a substitution σ is closed if for all variables x, $\sigma(x)$ is a variable-free formula. Nevertheless, when σ is a closed substitution, for some variable x, the formula $\sigma(x)$ may contain parameters. We shall say that a substitution σ is a closed substitution with parameters in $\{p_1, \ldots, p_m\}$ if for all variables x, $\sigma(x)$ is a closed formula whose parameters form a subset of $\{p_1, \ldots, p_m\}$. As before, for all formulas $\phi(x_1, \ldots, x_n)$, we define $\sigma(\phi)$ to be the formula $\phi(\sigma(x_1), \ldots, \sigma(x_n))$. And about the composition $\sigma \circ \tau$ of the substitutions σ and τ, it still associates to each variable x the formula $\tau(\sigma(x))$. Now, the semantics. Concerning the frames, there is no change: frames are still structures of the form $\mathcal{F} = (W, R)$ where W is a nonempty set of states and R is a binary relation on W. The change in the semantics is concerning the models. More precisely, in a model $\mathcal{M} = (W, R, V)$, the valuation V is not only a function associating a subset $V(x)$ of W to each $x \in VAR$, it is also a function associating a subset $V(p)$ of W to each $p \in PAR$. And of course, the truth conditions now include the following line:

- $\mathcal{M}, s \models p$ iff $s \in V(p)$.

Let $\mathcal{C}$ be a class of frames. A formula ϕ (possibly containing parameters) is $\mathcal{C}$-unifiable if there exists a substitution σ such that $\mathcal{C} \models \sigma(\phi)$. In this case, σ is a $\mathcal{C}$-unifier of ϕ. The unification problem is still defined to be the following decision problem:

input: a formula ϕ (possibly containing parameters),
output: determine whether ϕ is $\mathcal{C}$-unifiable.

Example: The formula $\phi = \Box p \vee \Box\neg x$ is $\mathcal{C}_{K45}$-unifiable, the substitution σ such that $\sigma(x) = \Box p$ being one of its $\mathcal{C}_{K45}$-unifiers.

Example: The formula $\phi = \Diamond p \vee \Diamond\neg x$ is $\mathcal{C}_{KD45}$-unifiable, the substitution σ such that $\sigma(x) = \Box p$ being one of its $\mathcal{C}_{KD45}$-unifiers.

In this variant with parameters, what happens to the unifiability problem? If either $\mathcal{C}$ is $\mathcal{C}_{K45}$, or $\mathcal{C}$ is $\mathcal{C}_{KD45}$, or $\mathcal{C}$ is $\mathcal{C}_{S5}$, is it still computable whether a given formula is unifiable? When the answer is 'yes', how complex is the problem? When a formula is unifiable, has it a minimal complete set of unifiers? When the answer is 'yes', how large is this set?

Lemma 20: *Let $\phi(p_1, \ldots, p_m, x_1, \ldots, x_n)$ be a formula. If ϕ possesses a $\mathcal{C}$-unifier, then ϕ possesses a closed $\mathcal{C}$-unifier with parameters in $\{p_1, \ldots, p_m\}$.*

Proof: This follows from the fact that for all $\mathcal{C}$-unifiers σ of ϕ and for all closed substitutions τ, $\sigma \circ \tau$ is a closed $\mathcal{C}$-unifier of ϕ and the fact that for all parameter-free variable-free formulas $\psi_1, \ldots, \psi_n$, if a closed formula $\phi(p_1, \ldots, p_m, q_1, \ldots, q_n)$ is $\mathcal{C}$-valid then the closed formula $\phi(p_1, \ldots, p_m, \psi_1, \ldots, \psi_n)$ obtained from $\phi(p_1, \ldots, p_m, q_1, \ldots, q_n)$ as the result of the replacement of $q_1, \ldots, q_n$ in their places in $\phi(p_1, \ldots, p_m, q_1, \ldots, q_n)$ with formulas $\psi_1, \ldots, \psi_n$ is $\mathcal{C}$-valid too. $\Box$

Proposition 10: *If either $\mathcal{C}$ is $\mathcal{C}_{K45}$, or $\mathcal{C}$ is $\mathcal{C}_{KD45}$, or $\mathcal{C}$ is $\mathcal{C}_{S5}$ then the $\mathcal{C}$-unifiability problem with parameters is decidable.*

Proof: Let $\phi(p_1, \ldots, p_m, x_1, \ldots, x_n)$ be a formula. By Lemma 20, to determine if $\phi(p_1, \ldots, p_m, x_1, \ldots, x_n)$ is $\mathcal{C}$-unifiable, it suffices to guess variable-free formulas $\psi_1, \ldots, \psi_n$ based on the parameters $p_1, \ldots, p_m$ such that $\mathcal{C} \models \phi(p_1, \ldots, p_m, \psi_1, \ldots, \psi_n)$. Suppose either $\mathcal{C}$ is $\mathcal{C}_{K45}$, or $\mathcal{C}$ is $\mathcal{C}_{KD45}$, or $\mathcal{C}$ is $\mathcal{C}_{S5}$. As is well-known, there exists finitely many pairwise non-$\mathcal{C}$-equivalent variable-free formulas based on the parameters $p_1, \ldots, p_m$. Moreover, these formulas can be enumerated. Since $\mathcal{C}$-validity is decidable, therefore the $\mathcal{C}$-unifiability problem with parameters is decidable. $\Box$

The exact complexity of the unifiability problem with parameters in $\mathcal{C}_{K45}$, $\mathcal{C}_{KD45}$ or $\mathcal{C}_{S5}$ is still unknown. Now, with parameters, we want to determine the $\mathcal{C}$-unification type when $\mathcal{C}$ is one of the classes $\mathcal{C}_{K45}$, $\mathcal{C}_{KD45}$ and $\mathcal{C}_{S5}$. It happens that the proofs of Lemmas 10–19 can be repeated. As a result,

Proposition 11: *Unification with parameters in $\mathcal{C}_{S5}$ is unitary.*

Proposition 12: *Unification with parameters in $\mathcal{C}_{KD45}$ is directed.*

Proposition 13: *Unification with parameters in $\mathcal{C}_{K45}$ is directed.*

Proposition 14:

(1) *Unification with parameters in $\mathcal{C}_{K45}$ is either unitary, or nullary.*
(2) *Unification with parameters in $\mathcal{C}_{KD45}$ is either unitary, or nullary.*

7. Multi-agent setting

In Sections 4–6, we have considered languages with only one modal connective. But it rarely happens that we are only interested in one agent. This leads us to the following new syntax. Let AGT be a finite set of agents (with typical members denoted a, b, etc.) and $n = Card(AGT)$. We assume $n \geq 2$. The formulas are now defined by the rule

- $\phi ::= x \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid \Box_a\phi$.

Let

- $\Diamond_a\phi ::= \neg\Box_a\neg\phi$.

Concerning substitutions, we will use the definitions introduced in Section 2. Now, the semantics. In a frame $\mathcal{F} = (W, R)$, R is now a function associating a binary relation $R(a)$ on W to each $a \in AGT$. In this multi-agent setting, the truth conditions in a model $\mathcal{M} = (W, R, V)$ now include the following line:

- $\mathcal{M}, s \models \Box_a\phi$ iff for all $t \in W$, if $sR(a)t$ then $\mathcal{M}, t \models \phi$.

Let $\mathcal{C}^n_{K45}$ be the class of all transitive and Euclidean frames, $\mathcal{C}^n_{KD45}$ be the class of all serial, transitive and Euclidean frames and $\mathcal{C}^n_{S5}$ be the class of all reflexive, transitive and Euclidean frames. Now, defining unifiability, unifiers and unification types as in Sections 4 and 5, what happens to the unifiability problem?

Example: The formula $\phi = \Box_a x \vee \Box_b \neg x$ is $\mathcal{C}^n_{K45}$-unifiable, the substitutions $\sigma_\bot$ such that $\sigma_\bot(x) = \bot$ and $\sigma_\top$ such that $\sigma_\top(x) = \top$ being two of its $\mathcal{C}^n_{K45}$-unifiers.

Suppose either $\mathcal{C}$ is $\mathcal{C}^n_{KD45}$, or $\mathcal{C}^n_{S5}$. Arguments similar to the ones considered in Section 4 about $\mathcal{C}^n_{KD45}$ and $\mathcal{C}^n_{S5}$ can be repeated here. In fact, every variable-free formula is $\mathcal{C}$-equivalent to $\bot$ or $\top$. Hence, to determine if $\phi(x_1, \ldots, x_n)$ is $\mathcal{C}$-unifiable, it suffices to guess formulas $\psi_1, \ldots, \psi_n$ in $\{\bot, \top\}$ such that $\mathcal{C} \models \phi(\psi_1, \ldots, \psi_n)$. Since, given a variable-free formula ϕ, determining whether $\mathcal{C} \models \phi$ can be done in polynomial time, therefore this proves membership in NP of the $\mathcal{C}$-unifiability problem. As for its NP-hardness, an argument similar to the one considered in the second part of the proof of Proposition 1 can be easily repeated. Thus,

Proposition 15: *If either $\mathcal{C}$ is $\mathcal{C}^n_{KD45}$, or $\mathcal{C}$ is $\mathcal{C}^n_{S5}$ then the $\mathcal{C}$-unifiability problem is NP-complete.*

As for the computability of the $\mathcal{C}^n_{K45}$-unifiability problem, it is still an open question. This issue seems to be a difficult one, similar to the computability of the unifiability problem in ordinary normal modal logic K. Now, what about the $\mathcal{C}$-unification type when $\mathcal{C}$ is one of the classes $\mathcal{C}^n_{K45}$, $\mathcal{C}^n_{KD45}$ and $\mathcal{C}^n_{S5}$? About the type of unification in $\mathcal{C}^n_{S5}$, it is still unknown. In fact, the proof of Proposition 6 cannot be repeated: in the definition of the substitution τ^σ_ϕ associated there to the formula ϕ and the substitution σ, which modal connective in $\{\Box_a : a \in AGT\}$ to use instead of the modal connective $\Box$? As for the types of unification in $\mathcal{C}^n_{KD45}$ and $\mathcal{C}^n_{K45}$, they are still unknown too. In fact, the proof of Proposition 9 cannot be repeated: in the definition of the substitution $\mu^{\sigma,\tau}$ associated there to the substitutions σ and τ, again, which modal connective in $\{\Box_a : a \in AGT\}$ to use instead of the modal connective $\Box$?

Table 1. Cl: classes of frames; MuAg: Multi-agent; Pa: parameters; CoUn: computability of unifiability; UnTy: unification types.

♯	Cl	MuAg	Pa	CoUn	UnTy
1	$\mathcal{C}_{K45}$	No	No	*NP*-complete (Proposition 2)	1 or 0 (Proposition 9)
2	$\mathcal{C}_{KD45}$	No	No	*NP*-complete (Proposition 1)	1 or 0 (Proposition 9)
3	$\mathcal{C}_{S5}$	No	No	*NP*-complete (Proposition 1)	1 (Proposition 6)
4	$\mathcal{C}_{K45}$	No	Yes	Decidable (Proposition 10)	1 or 0 (Proposition 14)
5	$\mathcal{C}_{KD45}$	No	Yes	Decidable (Proposition 10)	1 or 0 (Proposition 14)
6	$\mathcal{C}_{S5}$	No	Yes	Decidable (Proposition 10)	1 (Proposition 11)
7	$\mathcal{C}^n_{K45}$	Yes	No	?	?
8	$\mathcal{C}^n_{KD45}$	Yes	No	*NP*-complete (Proposition 15)	?
9	$\mathcal{C}^n_{S5}$	Yes	No	*NP*-complete (Proposition 15)	?
10	$\mathcal{C}^n_{K45}$	Yes	Yes	?	?
11	$\mathcal{C}^n_{KD45}$	Yes	Yes	?	?
12	$\mathcal{C}^n_{S5}$	Yes	Yes	?	?

8. Conclusion

In the context of epistemic logics, classes of frames such as the ones underlying $K45$, $KD45$ and $S5$ give rise to quite similar sets of valid formulas for what concerns axiomatisation and decidability but have different properties for what concerns unifiability and unification types. For instance, unifiability is *NP*-complete in $\mathcal{C}^n_{KD45}$ and $\mathcal{C}^n_{S5}$ whereas the computability of the unifiability problem in $\mathcal{C}^n_{K45}$ seems to be a difficult issue. Putting known results adapted from Baader and Ghilardi (2011); Dzik (2003, 2007); Ghilardi (2000); Jeřábek (2015) together with new ones enables us to establish basic facts and outline open problems. See the lines 1–9 of Table 1 which present results that have been proved in this paper (the lines 10–12 concern unifiability and unification types in multi-epistemic logics with parameters). While the study of $K45$, $KD45$ and $S5$ has limited logical value, considering unifiability and unification types in epistemic logics is justified from applied perspectives: methods for deciding the unifiability of formulas can be used to improve the efficiency of automated theorem provers as in Babenyshev et al. (2010); deciding the unifiability of formulas like $\phi \leftrightarrow \psi$ helps us to understand what is the overlap between the properties ϕ and ψ correspond to as in Baader and Ghilardi (2011); in description logics, unification algorithms are used to detect redundancies in knowledge-based systems as in Baader, Borgwardt, and Morawska (2012). Depending on the level of abstractness and precision (with one or several agents, with or without common knowledge, etc.), one readily observes that, while attacking the above-mentioned open problems, little, if anything, from the standard tools in epistemic logics (canonical models, filtrations, etc.) is helpful. In order to successfully solve them, new techniques in epistemic logics must be developed. The study of unifiability and unification types in epistemic logics has still many secrets to reveal.

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