REPORT

Sampled-data tracking under model predictive control and multi-rate planning: Further simulations and remarks

Mohamed Elobaid[1,2], Mattia Mattioni[1], Salvatore Monaco[1,], Dorothée Normand-Cyrot[2]

[1]Dipartimento di Ingegneria Informatica, Automatica e Gestionale (La Sapienza University of Rome) Rome, 00185 Italy (e-mail: {mohamed.elobaid, mattia.mattioni, salvatore.monaco}@uniroma1.it),
[2]Laboratoire de Signaux et Systèmes (L2S, CNRS); 3, Rue Joliot Curie, 91192, Gif-sur-Yvette, France (e-mail: {mohamed.elobaid, dorothee.normand-cyrot}@centralesupelec.fr)

ABSTRACT
This technical report summarizes all the simulations performed for the PVTOL case study testing the control scheme proposed in [3], combining single-rate MPC with a multi-rate trajectory planner (MR MPC). The proposed scheme is compared to both stand-alone multi-rate control (MR) and a standard MPC controller with an cascade FIR based filters for trajectory generation (FIR MPC).

1. Case study: Planar Vertical Take Off and Landing aircraft PVTOL

In this section we introduce the case study, and highlight the validity of the proposed approach in terms of following a given trajectory, namely a straight line reference on the position. This scheme is compared to the stand-alone MR control (see for example [1] and the references therein), thus specifying the result referenced in Remark 4.3 and Theorem 3.1 in [3] to this particular example, as well as a typical SR MPC with a given trajectory planner. Let the model of the PVTOL take the form:

$$\begin{aligned}\ddot{x} &= -sin(\theta)v_1 + \epsilon cos(\theta)v_2\\ \ddot{z} &= cos(\theta)v_1 - 1 + \epsilon sin(\theta)v_2\\ \ddot{\theta} &= v_2\end{aligned} \tag{1}$$

for which the output $y = h(x, \dot{x}, z, \dot{z}, \theta, \dot{\theta}) = (x, z)^\top$.

1.1. *Construction of the MR planner model*

To use our control scheme, we first define the multi-rate sampled data model of the PVTOL. To this end, it is known (e.g. [1]) that (1) is feedback equivalent to a finitely discretizable system, by setting

$$v = \begin{pmatrix} \frac{1}{cos\theta} + \varepsilon\dot{\theta}^2 \\ -2\dot{\theta}^2 \tan\theta \end{pmatrix} + \begin{pmatrix} \frac{1}{cos\theta} & 0 \\ 0 & \cos^2\theta \end{pmatrix} u$$

plus together with the coordinates change

$$\zeta = \varphi(x, \dot{x}, z, \dot{z}, \theta, \dot{\theta}) = \epsilon \begin{pmatrix} \cos\theta \\ -\dot{\theta}\sin\theta \\ 0 \\ -\sin\theta \\ 0 \\ -\dot{\theta}\cos\theta \end{pmatrix} + \begin{pmatrix} z \\ \dot{z} \\ \tan\theta \\ x \\ \frac{\dot{\theta}}{\cos^2\theta} \\ \dot{x} \end{pmatrix}.$$

Thus obtaining

$$\begin{aligned} \dot{\zeta} &= \tilde{f}(\zeta) + \tilde{g}_1(\zeta)u_1 + \tilde{g}_2(\zeta)u_2 \\ \tilde{f}(\zeta) &= \begin{pmatrix}\zeta_2 & 0 & \zeta_5 & \zeta_6 & 0 & -\zeta_3\end{pmatrix}^\top, \tilde{g}_1(\zeta) = \begin{pmatrix}0 & 1 & 0 & 0 & 0 & -\zeta_3\end{pmatrix}^\top \\ \tilde{g}_2(\zeta) &= \begin{pmatrix}0 & 0 & 0 & 0 & 1 & 0\end{pmatrix}^\top \end{aligned} \tag{2}$$

The above model (2) admits the closed-form sampled-data equivalent model of the form ([3], eqn. 2) that can be computed. We then select our multi-rate order $m = 4$ and, in particular

$$\begin{aligned} u_1(t) &= u_1^j(k),\ t \in [(k + \frac{j-1}{2})T, (k + \frac{j}{2})T[,\ \ j = 1, 2 \\ u_2(t) &= u_2^j(k),\ t \in [(k + \frac{j-1}{4})T, (k + \frac{j}{4})T[,\ \ j = 1, \dots, 4. \end{aligned}$$

so getting the MR planner dynamics(dropping the subscript k for the control):

$$\begin{aligned} \zeta_{k+1} = {} & (A^\delta + B_1^\delta(u_1^2))^2(A^\delta + B_1^\delta(u_1^1))^2\zeta_k \\ & + (A^\delta + B_1^\delta(u_1^2))^2(I + B_1^\delta(u_1^1))B_0^\delta(u_1^1, u_2^1) \\ & + (A^\delta + B_1^\delta(u_1^2))^2 B_0^\delta(u_1^1, u_2^2) \\ & + (I + B_1^\delta(u_1^1))B_0^\delta(u_1^2, u_2^3) + B_0^\delta(u_1^2, u_2^4) \end{aligned} \tag{3}$$

with $\zeta_k = \varphi(x_k, \dot{x}_k, z_k, \dot{z}_k, \theta_k, \dot{\theta}_k)$ for all $k \geq 0$ with

$$A^\delta = \begin{pmatrix} 1 & \delta & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \delta & 0 \\ 0 & 0 & -\frac{\delta^2}{2} & 1 & -\frac{\delta^3}{6} & \delta \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\delta & 0 & -\frac{\delta^2}{2} & 1 \end{pmatrix}, B_1^\delta = \begin{pmatrix} & \multicolumn{4}{c}{\mathbf{0}_{3\times 6}} \\ \mathbf{0}_{3\times 2} & -\frac{\delta^2}{2}u_1 & 0 & -\frac{\delta^3}{6}u_1 & 0 \\ & 0 & 0 & 0 & 0 \\ & -\delta u_1 & 0 & -\frac{\delta^2}{2}u_1 & 0 \end{pmatrix}$$

$$B_0^\delta = \begin{pmatrix} \frac{\delta^2 u_1}{2} & \delta u_1 & \frac{\delta^2 u_2}{2} & \frac{-\delta^4(1+u_1)u_2}{24} & \delta u_2 & \frac{-\delta^3(1+u_1)u_2}{6} \end{pmatrix}^\top$$

with $\delta \geq 0$ being the sampling period and $T = 4\delta$.

1.2. *Planning and control*

For all $t = kT$ with $k \geq 0$ planning is made on the basis of the simplified equivalent model (3) so getting, for the original system (1) a sequence of admissible outputs $\{(\hat{x}^i_{k+j}, \hat{z}^i_{k+j}),\quad i = 1,\dots,4$ and $j = 0,1\}$ when setting $(\hat{x}^i_{k+j}, \dot{\hat{x}}^i_{k+j}, \hat{z}^i_{k+j}, \dot{\hat{z}}^i_{k+j}, \hat{\theta}^i_{k+j}, \dot{\hat{\theta}}^i_{k+j})^\top = \varphi^{-1}(\hat{\zeta}^i_k)$.

Consequently, for all $t = kT + i\delta$, the MPC computes the feedback u^i_k (for $i = 0,\dots,3$) with the sampled data SR model of the PVTOL used for prediction. This feedback is then applied to the simulation model of system (1) while recomputing the reference for all $t = kT$.

1.3. *Simulations and Remarks*

In the following we will compare the proposed control algorithm (MR MPC) to both the stand-alone MR control and MPC with the trajectory planner proposed by [2] (FIR MPC). In all simulations $\delta = 1$ seconds while $n_p = n_c = 4$. In the exact steering scenarios, The PVTOL is required to perform the classical lateral maneuver of 10 m, namely a reference on the normalized position of $(x\ \ z)^\top = (1\ \ 0)^\top$. While in the time varying references, a linear path is fixed on both the lateral and vertical displacements (that is x, z respectively) as a ramp signal with velocity $v_0 = 1$m/s to be tracked at $t = kT, T = 4\delta$.

1.3.1. Exact steering with Perturbation

Here we perform the manuever of a lateral displacement, assuming the system is perturbed by a disturbance $w(t)$ where w is a randomly generated actuation white noise (that is $u(t) = u_{mpc}(t) + w(t)$). In that case, the following comparisons are made:

- Figure 1 compare the performance of this control scheme to that obtained in [1]. Notice that in the figure the MR MPC is able to stabilize the vertical displacement to zero despite the disturbance, while the MR alone fails at doing so. On the other hand, the rotation is kept bounded by the MR MPC below 0.1 rads after 10 seconds.
- Figure 2 compare the performance of this control scheme to that obtained with an FIR MPC trajectory planner with a rest to rest motion. In this case, both control schemes perform similarly in the steady state keeping the desired position while also ensuring the stability of the θ dynamics (the PVTOL doesn't perform flips around its axis). It's worth mentioning that the FIR MPC does better in the transient compared to our scheme and requires less vertical movement to recover the required lateral manuever. Once stabilization is achieved, both control schemes stabilize the PVTOL.

1.3.2. Exact steering with ϵ variation

Here we perform the same manuever above, without an actuation perturbation, and we assume that the value of $\epsilon = 1.1$ in the actual model, while it's the nominal value for defining the MR planner simplified model and the MPC prediction models, we perform the following comparisons

- Figure 3 compare the performance of MR MPC to that obtained with an FIR MPC. Similar performances are recovered, although MR MPC performs slightly worse than FIR MPC in the transient as expected due to the fact that the feedback and coordinate change bringing (1) to (2) are not defined for $\epsilon \neq 0.8$

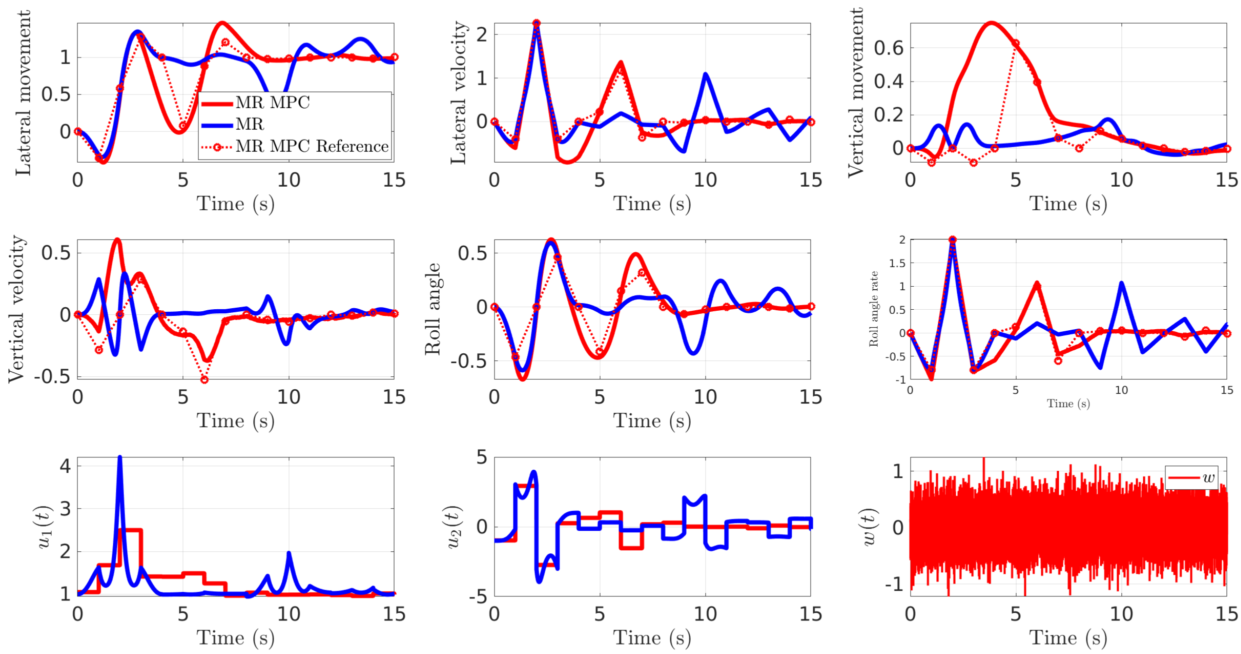

Figure 1. perturbed steering to $(x, y) = (1, 0)$ MR VS MR MPC with $R = 0, Q = \mathrm{I}$

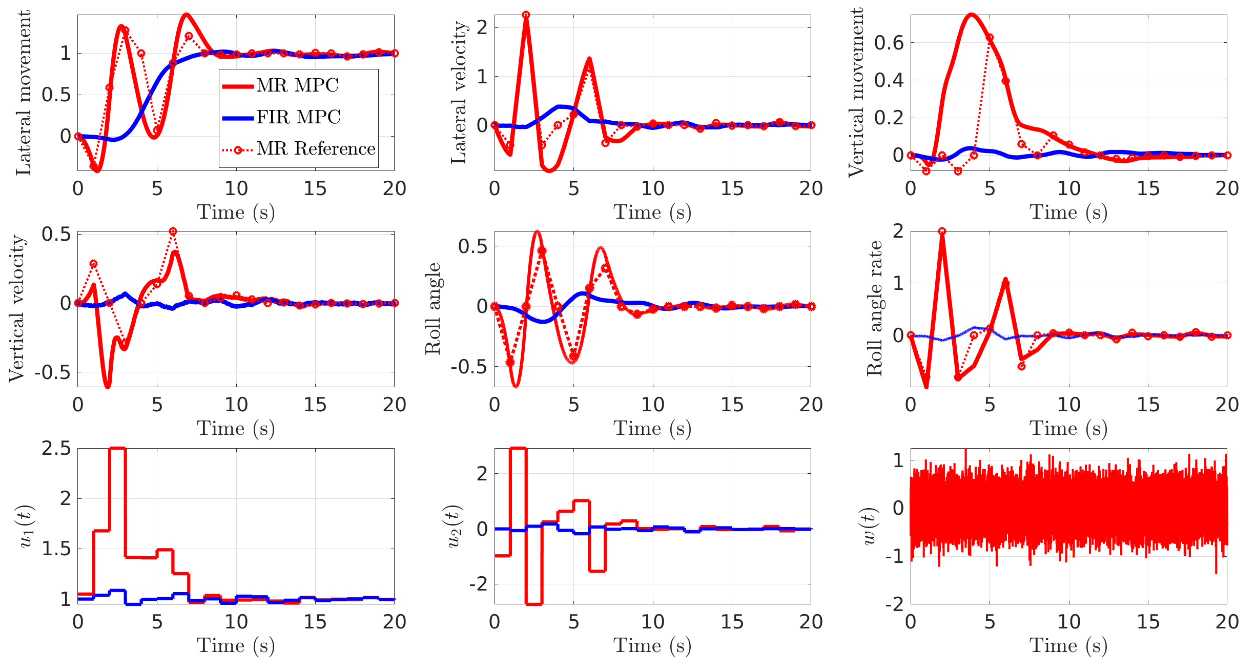

Figure 2. perturbed steering to $(x, y) = (1, 0)$ FIR MPC VS MR MPC with $R = 0, Q = \mathrm{I}$

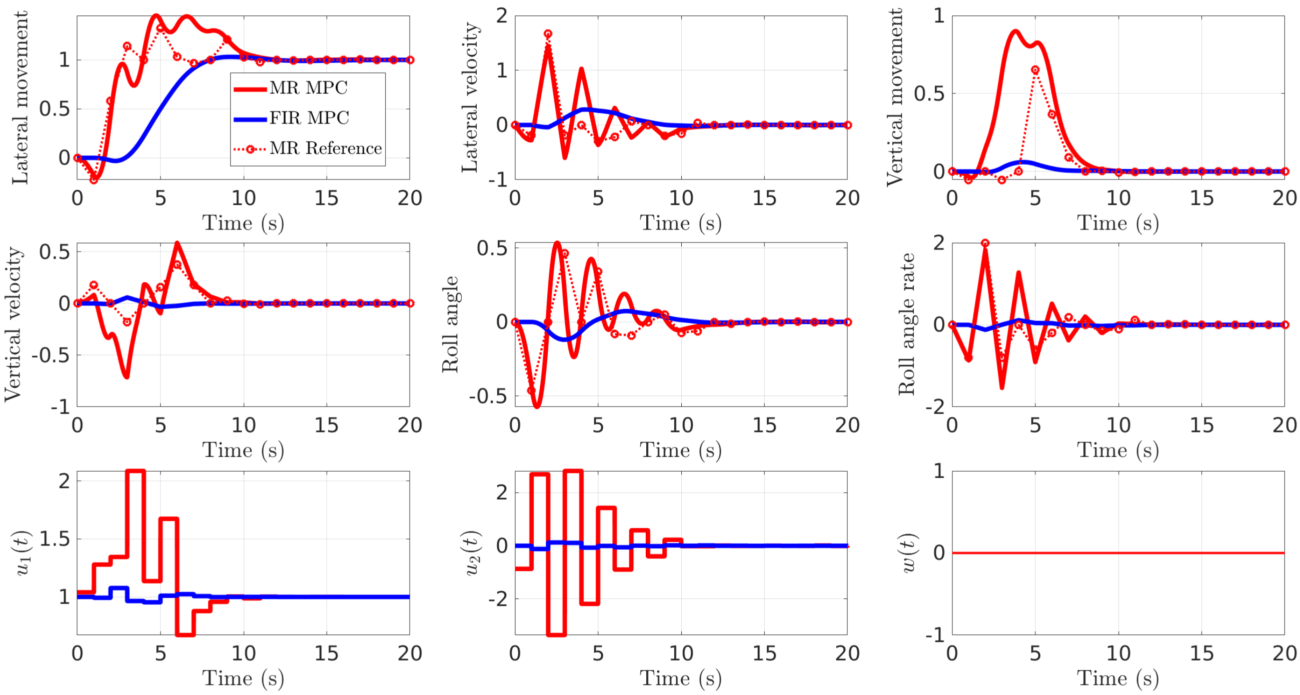

Figure 3. perturbed steering to $(x, y) = (1, 0)$ FIR MPC VS MR MPC with $\epsilon = 1.1, R = 0, Q = \mathrm{I}$

1.3.3. Time-varying references tracking

We require both the lateral x and vertical displacement z to follow a given reference, in this case, a ramp signal with velocity $v_0 = 0.1$ (i.e without normalization $v_0 = 1m/s$). We show that the proposed scheme outperforms both stand-alone MR control and FIR MPC one, and to this end, the following comparisons are made:

- In the nominal case depicted by Figure 4, we set $R = 0$. Notice that, contrarily to the MR MPC scheme, the FIR MPC is unable to follow the reference over the larger steps T, and for the same choices of Q, R, n_p, n_c an off-set is evident.
- Figure 5 on the other hand, compare the performance of this control scheme to that obtained in [1], which in the nominal case turns out to be similar as expected. In fact, one can interpret it as that this scheme is a way of implementing sampled data multi-rate controllers via optimization.
- Figure 6 shows that, even when there is actuation perturbation, and parameter variation, our proposed scheme outperforms the FIR MPC one, not just on the big sampling instants kT, but also during the smaller sub-intervals $k\delta$.
- Figure 7 compares our scheme tracking a straight line, under the action of a perturbation, against the stand-alone MR control, and the benefit in using our proposed scheme is evident as expected, both in terms of tracking at the small sampling instants, and maintaining the variation in the θ dynamics small. This reflects the idea that MR MPC is a robust way to implement MR controls.
- Figure 8 shows the effect of weighting the output $R > 0$ and verifying that the MR MPC scheme performs better compared to FIR MPC in this case, under both the action of a perturbation and the parameter ϵ change.

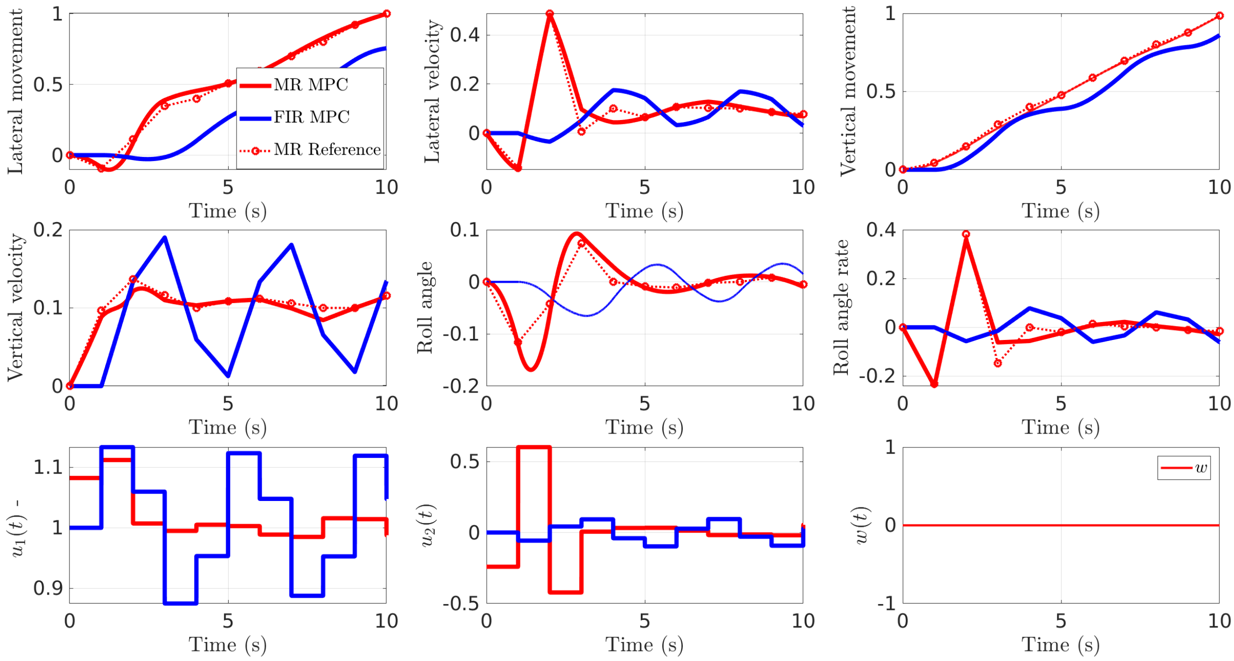

Figure 4. Nominal tracking of a straight line FIR MPC VS MR MPC with $R = 0, Q = \mathrm{I}$

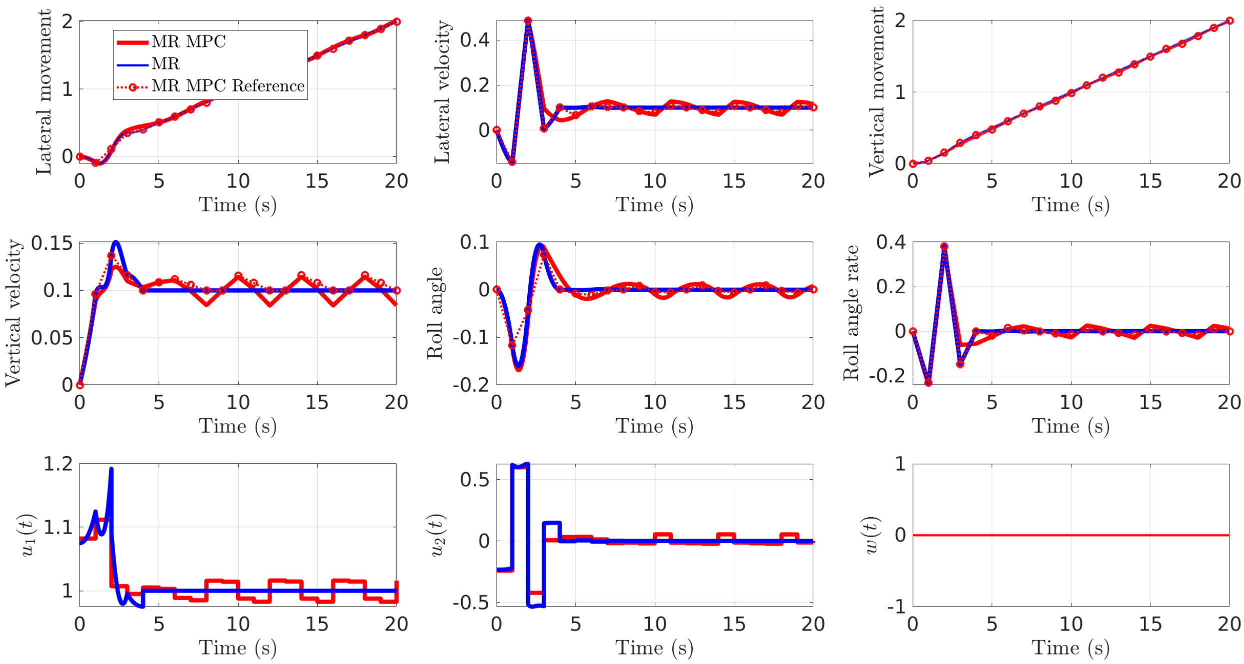

Figure 5. Nominal tracking of a stright line MR VS MR MPC with $R = 0, Q = \mathrm{I}$

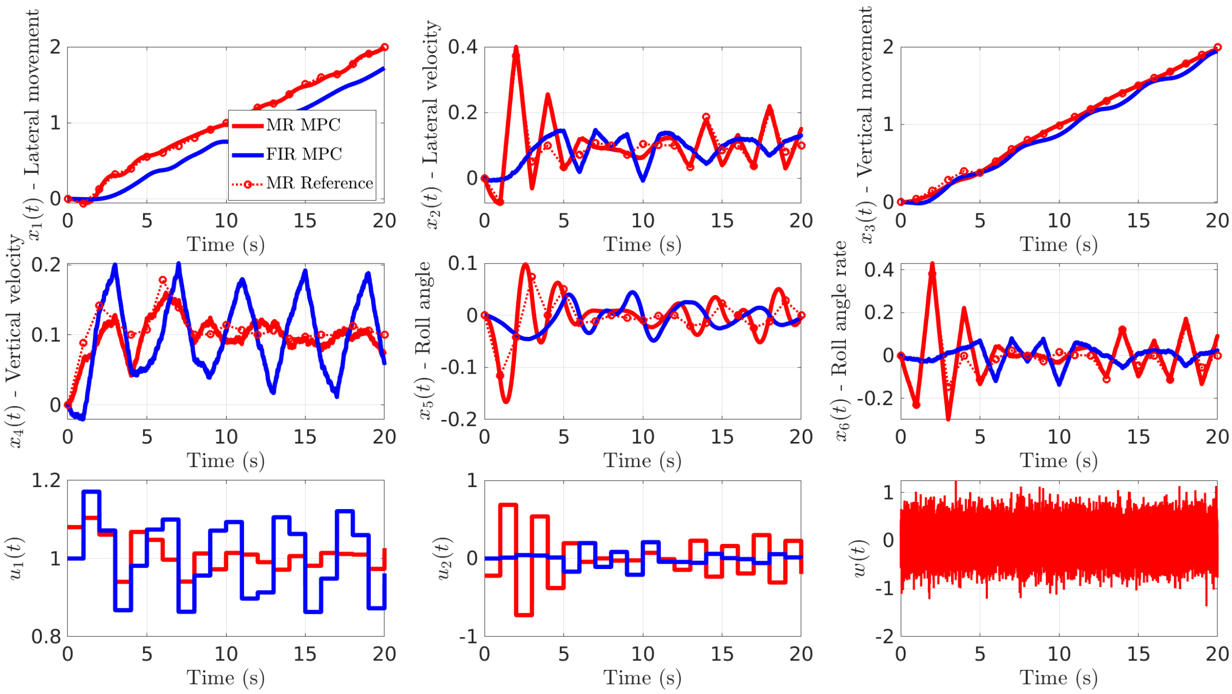

Figure 6. perturbed tracking of a straight line FIR MPC VS MR MPC with $\epsilon = 0.5, R = 0, Q = \mathrm{I}$

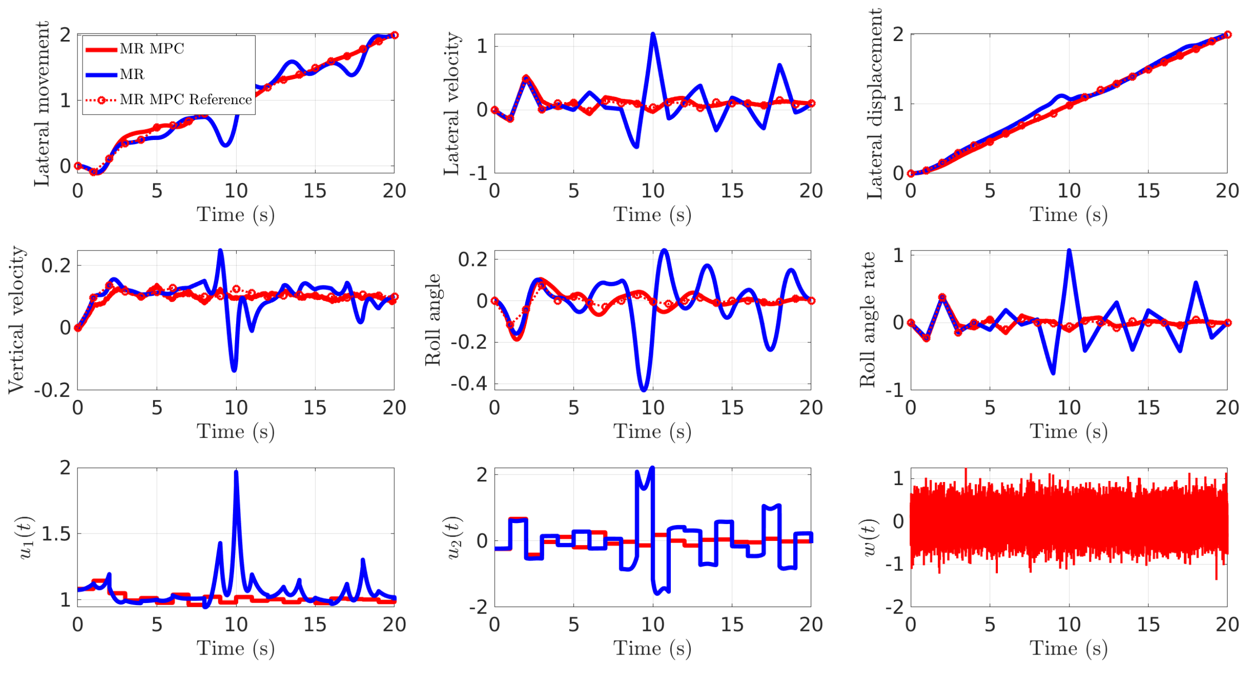

Figure 7. perturbed tracking of a straight line MR VS MR MPC with $\epsilon = 0.8, R = 0, Q = \mathrm{I}$

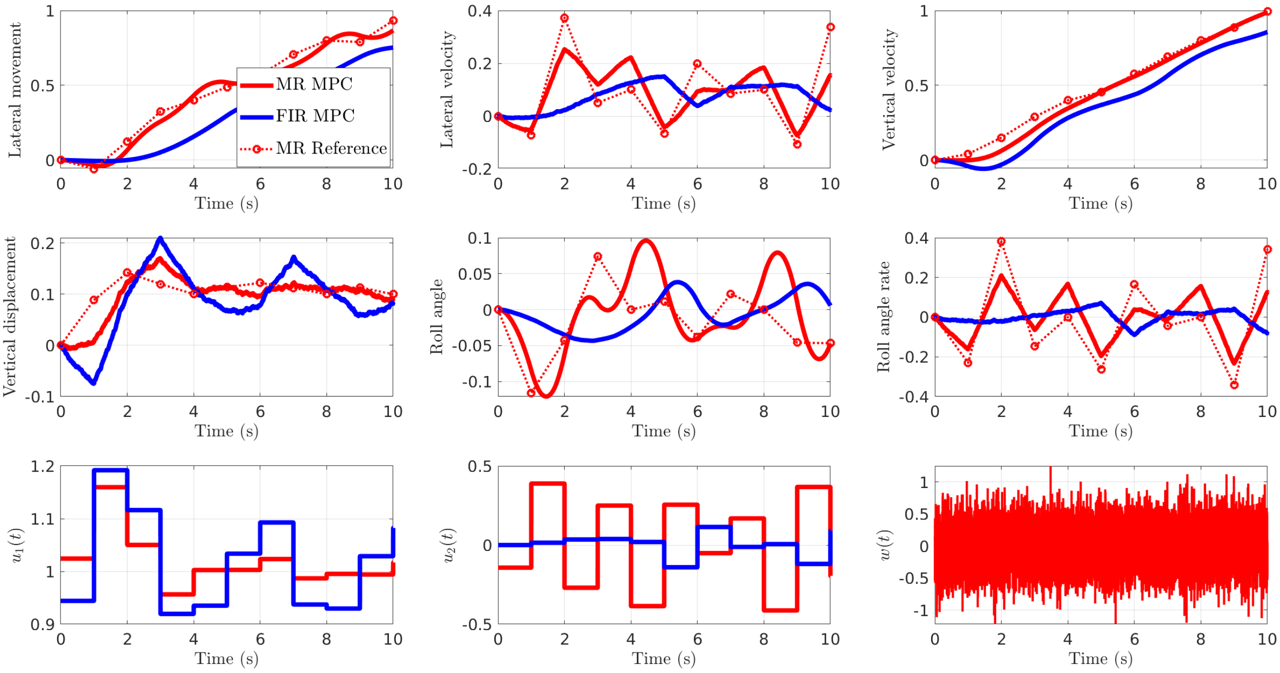

Figure 8. perturbed tracking of a straight line FIR MPC VS MR MPC with $\epsilon = 0.5, R = 0.5\mathrm{I}, Q = \mathrm{I}$

2. Concluding remarks

In this brief manuscript, we report further simulation results that support the intuition discussed in the attached paper, further motivating the use of the proposed scheme of combining multi-rate planning with single-rate model predictive control.

References

[1] P. Di Giamberardino and M. Djemai, *Multirate digital control of a PVTOL*, in *Proc. 33th CDC.* 1994, pp. 3844–3849.

[2] C.M. Luigi Biagiotti, *Trajectory generation via fir filters: A procedure for time-optimization under kinematic and frequency constraints*, Control Engineering Practice 87 (2019), pp. 43 – 58.

[3] M. Elobaid, M. Mattioni, S. Monaco and D. Normand-Cyrot, *sampled-data tracking under model predictive control and multi-rate planning*, submitted for review to the IFAC World congress 2020.