Fully discrete polynomial de Rham sequences of arbitrary degree on polygons and polyhedra
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Abstract

In this work, merging ideas from compatible discretisations and polyhedral methods, we construct novel fully discrete polynomial de Rham sequences of arbitrary degree on polygons and polyhedra. The spaces and operators that appear in these sequences are directly amenable to computer implementation. Besides proving exactness, we show that the usual sequence of Finite Element spaces forms, through appropriate interpolation operators, a commutative diagram with our sequence, which ensures suitable approximation properties. A discussion on reconstructions of potentials and discrete $L^2$-products completes the exposition.

Key words. Fully discrete de Rham sequences, compatible discretisations, polyhedral methods, mixed methods

MSC2010. 65N08, 65N30, 65N99

1 Introduction

In this paper we construct novel fully discrete polynomial de Rham sequences of arbitrary degree on polygons and polyhedra. By fully discrete, we mean that both the spaces and vector operators that appear in the sequence are directly amenable to computer implementation.

The ideas underlying this work result from the confluence of two streams of research that have gathered an enormous amount of attention in the numerical community over the last years: compatible discretisations and polytopal methods.

Compatible discretisations aim at preserving structural features of the continuous model at the discrete level. Such features are instrumental to obtaining the stability and consistency properties required for convergence when non-trivial operators and domains are considered, as is the case in computational electromagnetism. The origins of compatible discretisations can be tracked back to, e.g., [33, 50, 51] for the mathematical community and [17, 37, 47, 49, 52] for the electromagnetic one. The importance of concepts from differential geometry and algebraic topology in the formulation of compatible discretisations is nowadays widely recognised; see, e.g., [2, 3, 10, 16, 26, 38, 42, 46]. As a matter of fact, by reformulating the continuous problems in terms of differential forms, one gets some indications on the design of suitable Finite Element (FE) discretisations. Specifically:

(i) the choice of the degrees of freedom should reflect the nature (and global regularity properties) of the fields they represent;
(ii) the discrete spaces and operators should form an exact sequence;
(iii) the maps that reconstruct a form from a set of degrees of freedom should be such that the continuous
and discrete de Rham sequences form commutative diagrams.

An important issue when generating high-order discrete de Rham sequences in the FE spirit lies in the
choice of the bases and of the degrees of freedom. A reformulation of classical moments that underlines
their geometrical aspects has been recently proposed in [11]. This reformulation has been made possible
by the precursor works [22, 44], where new degrees of freedom in terms of weights of forms on small
chains have been proposed. These weights have shed new light on the high-order approximations
originally proposed by Nédélec [43] confirming, also for the high-order version, the tight relation with
Whitney forms [51]; see [12, 15, 17] for the low-order case. A generalisation to the high-order case of
the relations between the moments of a field and those of its potential has been proposed in [1].

The extension of the FE approach to more general meshes is, however, not straightforward. The
main reason is that, in order to construct a conforming FE discretisation, one has to devise discrete
spaces that, through the single-valuedness of degrees of freedom at element boundaries, satisfy suitable
global continuity requirements. Recent efforts in this direction have been made in, e.g., [21, 39] (see
also references therein), focusing mainly on the lowest-order case and with some limitations on the
element shapes in three dimensions.

The problem of devising discretisation methods that support more general meshes than classical
FE (including, e.g., polytopal elements and nonmatching interfaces) has been recently tackled with
great impetus by the numerical community. Supporting general meshes paves the way to computational
strategies that are typically not accessible to traditional conforming FE (nonconforming mesh refinement,
mesh coarsening, seamless handling of fractures and microstructures, etc.). We will focus here only on
those developments that bear relations to the approach proposed in this work, and refer the reader to the
preface of [29] for a literature review of broader scope.

Let us start with lowest-order methods that fall within the category of compatible discretisations.
Mimetic Finite Differences (MFD) are derived by mimicking the Stokes theorem to formulate discrete
counterparts of differential operators and $L^2$-products [9]. Their extension to polytopal meshes has been
first carried out in [40, 41], then analysed in [18, 19]; see also [35] for a link with the Mixed Hybrid Finite
Volume (MHFV) methods of [34, 36] and [32, Section 2.5] along with [31, Section 3.5] for links with
Hybrid High-Order (HOO) methods. In the Discrete Geometric Approach (DGA), originally introduced
in [25] and extended to polyhedral meshes in [23, 24], as well as in Compatible Discrete Operators
[12, 13], formal links with the continuous operators are expressed in terms of Tonti diagrams [47, 48].
Similarly to the approach pursued here, MFD, DGA, and CDO methods work on discrete unknowns
and rely on discrete counterparts of the vector operators. Contrary to the present work, however, they
are typically limited to the lowest-order and their analysis often relies on an interplay of functional and
algebraic arguments that is not required in our presentation.

The development of high-order schemes is more recent. A high-order approach with structure-
preserving features is provided by the Virtual Element Method (VEM); see [4]. VEM can be described
as FE methods where explicit expressions for the basis functions are not available at each point; hence
the term “virtual” in reference to the function space they span. The degrees of freedom are selected so
as to enable the computation of polynomial projections of virtual functions and vector operators, which are
used in turn to formulate local contributions involving consistency and stabilisation terms. An exact de
Rham sequence of virtual spaces on polyhedra has been recently proposed in [8], with polynomial degrees
decreasing by one at each application of the exterior derivative (other virtual sequences are presently
under investigation [20], see also the related works [6, 7] concerning applications to magnetostatics).

Owing to the variational crime committed when taking projections on polynomial spaces, the
exactness of the virtual sequence cannot be directly exploited to obtain stable numerical approximations.
The approach proposed in this work, inspired by the HHO literature [29, 31], aims at establishing the
exactness property for fully discrete de Rham sequences, i.e., sequences involving spaces of (polynomial) discrete unknowns and discrete counterparts of vector operators acting thereon. The starting point is to identify arbitrary-order reconstructions of vector operators in full polynomial spaces. These reconstructions allow one to identify appropriate sets of discrete unknowns, which play the role of degrees of freedom in standard FE (or VEM). To ensure the compatibility with the choice of unknowns, each discrete vector operator is attached to the appropriate geometric entities: in three space dimensions, the discrete gradient has components on the edges, faces, and inside the polyhedron; the discrete curl has components at faces and inside the polyhedron; the discrete divergence has only one component inside the polyhedron. The full reconstructions of vector operators cannot be directly used to form an exact sequence, but their study permits to identify the modifications required to recover exactness. Specifically, an exact sequence is obtained by restricting the domains/co-domains of the operators in the middle of the sequence and taking the $L^2$-orthogonal projections of the full vector operator reconstructions on these spaces. Crucially, the proof of exactness relies on purely discrete arguments, that do not involve spaces of non-polynomial functions. The sequence we focus on is constructed so that all the spaces involved have the same polynomial degree and so that, through appropriate interpolation operators, it forms commutative diagrams with the usual sequence of FE spaces, which warrants suitable approximation properties. To complete the exposition, we also show how to reconstruct consistent potentials in each space and write discrete and consistent counterparts of $L^2$-products based on the latter. The focus of this paper is on the development of the exact discrete sequence; applications are postponed to future works.

The rest of this work is organised as follows: in Section 2 we introduce the basic tools and notations; in Sections 3 and 4 we construct fully discrete arbitrary-order exact sequences in two and three space dimensions, respectively.

## 2 Basic tools and notation

### 2.1 Polyhedra and polygons

A polytope of $\mathbb{R}^d$, $d \geq 1$, is a connected set that is the interior of a finite union of simplices. Our focus will be here on polytopes in dimension $d = 2$ (polygons) and $d = 3$ (polyhedra). Given a polyhedron $T \subset \mathbb{R}^3$, we denote by $\mathcal{F}_T$ the set of planar polygonal faces that lie on the boundary of $T$. For all $F \in \mathcal{F}_T$, an orientation is set by prescribing a unit normal vector $n_F$, and we denote by $\omega_T \in \{-1, 1\}$ the orientation of $F$ relative to $T$, that is, $\omega_T = 1$ if $n_F$ points out of $T$, $-1$ otherwise. With this choice, $\omega_T n_F$ is the unit normal vector to $F$ that points out of $T$. Similarly, for a polygon $F$, we denote by $\mathcal{E}_F$ the set of edges that lie on the boundary $\partial F$ of $F$. Notice that, throughout the paper, polygons are tacitly regarded as immersed in $\mathbb{R}^3$ whenever needed. For all $E \in \mathcal{E}_F$, an orientation is set by prescribing the unit tangent vector $t_E$. The boundary of $F$ is oriented counter-clockwise with respect to $n_F$, and we denote by $\omega_F \in \{-1, 1\}$ the orientation of $t_E$ opposite to $\partial F$: $\omega_F = 1$ if $t_E$ points on $E$ in the opposite orientation to $\partial F$, $\omega_F = -1$ otherwise. The vertices $V_1, V_2$ of the edge $E$ have coordinates $x_{E,1}, x_{E,2}$ and are numbered so that $|E|t_E = x_{E,2} - x_{E,1}$, where $|E|$ denotes the length of $E$. For any polygon $F$ and any edge $E \in \mathcal{E}_F$, we also denote by $n_{FE}$ the unit normal vector to $E$ lying in the plane of $F$ such that $(t_E, n_{FE})$ form a system of right-handed coordinates in the plane of $F$, which means that the system of coordinates $(t_E, n_{FE}, n_F)$ is right-handed. It can be checked that $\omega_{FE} n_{FE}$ is the normal to $E$, in the plane where $F$ lies, pointing out of $F$, and that, if $F_1, F_2$ are two faces of $T$ that share an edge $E$, it holds

$$\omega_{TF_1} \omega_{F_1E} + \omega_{TF_2} \omega_{F_2E} = 0. \quad (2.1)$$

In what follows, we will also need the sets of edges and vertices of a polyhedron $T \subset \mathbb{R}^3$, which we denote by $\mathcal{E}_T$ and $\mathcal{V}_T$, respectively, as well as the set $\partial^2 T := \bigcup_{E \in \mathcal{E}_T} \bar{E}$. The set of vertices of a polygon
2.2 Polynomial spaces and vector operators

For given integers \( \ell \geq 0 \) and \( n \geq 0 \), we denote by \( \mathbb{P}_n^\ell \) the space of \( n \)-variate polynomials of total degree \( \leq \ell \), with the convention that \( \mathbb{P}_0^\ell = \mathbb{R} \) for any \( \ell \) and \( \mathbb{P}_n^{-1} = \{0\} \) for any \( n \). For \( X \) polyhedron, polygon (immersed in \( \mathbb{R}^3 \)), or segment (again immersed in \( \mathbb{R}^3 \)), we denote by \( \mathcal{P}^\ell(X) \) the space spanned by the restriction to \( X \) of functions in \( \mathbb{P}_n^\ell \). Denoting by 0 \( \leq n \leq 3 \) the dimension of \( X \), \( \mathcal{P}^\ell(X) \) is isomorphic to \( \mathbb{P}_n^\ell \) (the proof, quite simple, follows the ideas of [29, Proposition 1.23]). With a little abuse of notation, we denote both spaces with \( \mathcal{P}^\ell(X) \), and the exact meaning of this symbol should be inferred from the context. We will also need the subspace \( \mathcal{P}^{0,\ell}(X) := \{ q \in \mathcal{P}^\ell(X) : \int_X q = 0 \} \). For any \( X \) polygon or polyhedron, the \( L^2 \)-orthogonal projector \( \pi^\ell_{\mathcal{P},X} : L^1(X) \to \mathcal{P}^\ell(X) \) is such that, for any \( q \in L^1(X) \),

\[
\int_X (\pi^\ell_{\mathcal{P},X} q - q) r = 0 \quad \forall r \in \mathcal{P}^\ell(X). \tag{2.2}
\]

As a projector, \( \pi^\ell_{\mathcal{P},X} \) is polynomially consistent, that is, it maps any \( r \in \mathcal{P}^\ell(X) \) onto itself. Optimal approximation properties for this projector have been proved in [28]; see also [27] for more general results on projectors on local polynomial spaces. Denoting by \( n \) the dimension of \( X \), we also denote by \( \pi^\ell_{\mathcal{P},X} : L^1(X)^n \to \mathcal{P}^\ell(X)^n \) the vector version defined applying the projector component-wise.

Let \( T \) be a polyhedron in \( \mathbb{R}^3 \). We denote by \( \mathcal{P}^\ell(\mathcal{T}_F) \) the space of functions \( q : \partial T \to \mathbb{R} \) such that \( q|_F \in \mathcal{P}^\ell(F) \) for all \( F \in \mathcal{T}_T \); an element \( q \in \mathcal{P}^\ell(\mathcal{T}_F) \) is identified with the family \( (q|_F)_{F \in \mathcal{T}_T} \) of its restrictions to the faces. The space \( \mathcal{P}^\ell(\mathcal{E}_T) \) is defined similarly, replacing faces by edges of \( T \). If \( F \) is a polygon immersed in \( \mathbb{R}^3 \), we define in the same way the space \( \mathcal{P}^\ell(\mathcal{E}_F) \). We also define the spaces \( \mathcal{P}^\ell_c(\partial^2 T) := \mathcal{P}^\ell(\mathcal{E}_T) \cap C^0(\partial^2 T) \) and \( \mathcal{P}^\ell_c(\partial F) := \mathcal{P}^\ell(\mathcal{E}_F) \cap C^0(\partial F) \) of functions that are continuous on the corresponding boundary and polynomial of degree \( \leq \ell \) on each edge of this boundary. It is easily checked that the following mapping is an isomorphism:

\[
\mathcal{P}^\ell_c(\partial F) \ni q \mapsto (\pi^\ell_{\mathcal{P},E} q|_E)_{E \in \mathcal{E}_F}, (q(x_V)|_{V \in \mathcal{V}_F}) \in \times_{E \in \mathcal{E}_F} \mathcal{P}^{\ell-1}(E) \times \mathbb{R}^{\mathcal{V}_F}. \tag{2.3}
\]

A similar isomorphism can be constructed for \( \mathcal{P}^\ell_c(\partial^2 T) \).

We respectively denote by \( \text{grad}_F \) and \( \text{div}_F \) the tangent gradient and divergence operators acting on smooth enough functions defined on \( F \in \mathcal{T}_T \). Moreover, for any \( r : F \to \mathbb{R} \) smooth enough, we define the two-dimensional vector curl operator such that

\[
\text{rot}_F r := \nabla - \nabla (\text{grad}_F r), \tag{2.4}
\]

where \( \nabla - \nabla \) is the rotation, in the oriented tangent space to \( F \), of angle \( -\frac{\pi}{2} \). We will also need the two-dimensional scalar curl operator such that, for any \( v : F \to \mathbb{R}^2 \) smooth enough,

\[
\text{rot}_F v := \text{div}_F (\nabla - \nabla v). \tag{2.5}
\]

We note for future use the following formulas linking volume and surface operators, which can be established selecting an orthonormal basis of \( \mathbb{R}^3 \) in which \( \mathbf{n}_F \) is the third vector: For any polyhedron \( T \subset \mathbb{R}^3 \), any face \( F \in \mathcal{T}_T \) and any sufficiently smooth functions \( v : T \to \mathbb{R}^3 \) and \( r : T \to \mathbb{R} \),

\[
(\text{grad} r)|_F \times \mathbf{n}_F = \text{rot}_F (r|_F), \tag{2.6}
\]

\[
(\text{curl} v)|_F \cdot \mathbf{n}_F = \text{div}_F (v|_F \times \mathbf{n}_F) = \text{rot}_F (\mathbf{n}_F \times (v|_F \times \mathbf{n}_F)). \tag{2.7}
\]
Above, \( n_F \times (v|_F \times n_F) \) is the orthogonal projection of \( v|_F \) on the plane that contains \( F \). Notice that, here and in what follows, with a little abuse of notation, this and similar quantities are regarded as functions \( F \to \mathbb{R}^2 \) whenever necessary.

For any integer \( \ell \geq -1 \), we define the following relevant subspaces of \( \mathcal{P}^\ell(F)^2 \):

\[
\mathcal{G}^\ell(F) := \text{grad}_F \mathcal{P}^{\ell+1}(F), \quad \mathcal{G}^\ell(F)^\perp := L^2\text{-orthogonal complement of } \mathcal{G}^\ell(F) \text{ in } \mathcal{P}^\ell(F)^2, \\
\mathcal{R}^\ell(F) := \text{rot}_F \mathcal{P}^{\ell+1}(F), \quad \mathcal{R}^\ell(F)^\perp := L^2\text{-orthogonal complement of } \mathcal{R}^\ell(F) \text{ in } \mathcal{P}^\ell(F)^2.
\]

The corresponding \( L^2 \)-orthogonal projectors are, with obvious notation, \( \pi_{\mathcal{G},F}^\ell, \pi_{\mathcal{G},F}^{\ell,\perp}, \pi_{\mathcal{R},F}^\ell, \) and \( \pi_{\mathcal{R},F}^{\ell,\perp} \).

Similarly, given a polyhedron \( T \subset \mathbb{R}^3 \), for any integer \( \ell \geq -1 \) we introduce the following subspaces of \( \mathcal{P}^\ell(T)^3 \):

\[
\mathcal{G}^\ell(T) := \text{grad} \mathcal{P}^{\ell+1}(T), \quad \mathcal{G}^\ell(T)^\perp := L^2\text{-orthogonal complement of } \mathcal{G}^\ell(T) \text{ in } \mathcal{P}^\ell(T)^3, \\
\mathcal{R}^\ell(T) := \text{curl} \mathcal{P}^{\ell+1}(T), \quad \mathcal{R}^\ell(T)^\perp := L^2\text{-orthogonal complement of } \mathcal{R}^\ell(T) \text{ in } \mathcal{P}^\ell(T)^3.
\]

The corresponding \( L^2 \)-orthogonal projectors are \( \pi_{\mathcal{G},T}^\ell, \pi_{\mathcal{G},T}^{\ell,\perp}, \pi_{\mathcal{R},T}^\ell, \) and \( \pi_{\mathcal{R},T}^{\ell,\perp} \).

For any polygon \( F \), polyhedron \( T \), and polynomial degree \( \ell \geq 0 \), the following mappings are isomorphisms:

\[
\text{rot}_F : \mathcal{P}^{0,\ell}(F) \xrightarrow{\cong} \mathcal{R}^{\ell-1}(F), \quad \text{grad} : \mathcal{P}^{0,\ell}(T) \xrightarrow{\cong} \mathcal{G}^{\ell-1}(T), \\
\text{div}_F : \mathcal{R}^{\ell}(F)^\perp \xrightarrow{\cong} \mathcal{P}^{\ell-1}(F), \quad \text{div} : \mathcal{R}^{\ell}(T)^\perp \xrightarrow{\cong} \mathcal{P}^{\ell-1}(T), \\
\text{curl} : \mathcal{G}^{\ell}(T)^\perp \xrightarrow{\cong} \mathcal{R}^{\ell-1}(T).
\]

The isomorphisms in (2.8) are trivial using (2.4) and the definitions of the respective co-domains. The other isomorphisms follow from [2, Corollary 7.3], except in the following situations that can easily be verified by hand: \( \ell = 0 \) in (2.9), and \( \ell = 0 \) or \( 1 \) in (2.10).

### 2.3 Integration by parts formulas

We recall a few inspiring integration by parts formulas, starting with those relevant for the design of the discrete gradient and divergence operators. Given a polyhedron \( T \in \mathbb{R}^3 \) and two functions \( v_T : T \to \mathbb{R}^3 \) and \( q_T : T \to \mathbb{R} \) smooth enough, we have

\[
\int_T \text{grad} q_T \cdot v_T = - \int_T q_T \text{div} v_T + \sum_{F \in T_T} \omega_{TF} \int_F q_T (v_T \cdot n_F). \tag{2.11}
\]

Similarly, for any polygon \( F \) and functions \( v_F : F \to \mathbb{R}^2 \) and \( q_F : F \to \mathbb{R} \) smooth enough, we have

\[
\int_F \text{grad}_F q_F \cdot v_F = - \int_F q_F \text{div}_F v_F + \sum_{E \in E_F} \omega_{FE} \int_E q_F (v_F \cdot n_{FE}). \tag{2.12}
\]

while, for any edge \( E \) and functions \( v_E : E \to \mathbb{R} \) and \( q_E : E \to \mathbb{R} \) smooth enough,

\[
\int_E q'_E v_E = - \int_E q_E v'_E + (q_E v_E)(x_{E,2}) - (q_E v_E)(x_{E,1}), \tag{2.13}
\]

where the derivatives are taken along the direction \( t_E \).
Let us now move to the formulas used in the design of the discrete curl operators. Given a polyhedron $T \in \mathbb{R}^3$ and two smooth enough functions $v_T : T \to \mathbb{R}^3$ and $w_T : T \to \mathbb{R}^3$, we have that

$$\int_T \text{curl } v_T \cdot w_T = \int_T v_T \cdot \text{curl } w_T + \sum_{F \in \mathcal{T}} \omega_{TF} \int_F v_T \cdot (w_T \times n_F)$$

$$= \int_T v_T \cdot \text{curl } w_T + \sum_{F \in \mathcal{T}} \omega_{TF} \int_F (n_F \times (v_T \times n_F)) \cdot (w_T \times n_F),$$

the second equality being justified recalling that $n_F \times (v_T \times n_F)$ is the projection of $v_T$ on the plane spanned by $F$, and noting that $w_T \times n_F$ belongs to that plane. Similarly, for any polygon $F$ and smooth enough functions $v_F : F \to \mathbb{R}^2$ and $r_F : F \to \mathbb{R}$

$$\int_F \text{rot } v_F \cdot r_F = \int_F v_F \cdot \text{rot } r_F - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E (v_F \cdot t_E) r_E.$$

(2.15)

3 An exact two-dimensional sequence

In this section we define a discrete counterpart of the following exact two-dimensional sequence on a polygon $F$ (which may be thought of as a mesh face):

$$\mathbb{R} \xrightarrow{i_F} H^1(F) \xrightarrow{\text{grad}_F} H(\text{rot}; F) \xrightarrow{\text{rot}_F} L^2(F) \xrightarrow{0} \{0\},$$

where $i_F$ is the operator that maps a real value to a constant function over $F$ and, with usual notation, $H^1(F)$ denotes the space of functions that are square integrable along with their (tangential) derivatives on $F$, while $H(\text{rot}; F) := \{ v \in L^2(F)^2 : \text{rot}_F v \in L^2(F) \}$. The starting point is, in Section 3.1, the design of reconstructions of the two-dimensional gradient and curl operators in full polynomial spaces, which drive the choice of the discrete unknowns. These operators cannot be directly used to form an exact discrete sequence, as we show in Section 3.2. Their properties, however, point out to the modifications required to obtain exactness, as detailed in Section 3.3. The discrete counterparts of $L^2$-products along with their stability and consistency properties are discussed in Section 3.5. In the rest of this section, we fix a polynomial degree $k \geq 0$.

3.1 Two-dimensional full vector operators reconstructions

We start by defining reconstructions of the vector operators in full polynomial spaces. As a general convention of notation, we use underlines to denote vectors made of components in different polynomial spaces, and bold fonts for vector-valued polynomials or vectors that have at least one vector-valued polynomial component. Thus, $\mathbf{a} = (b, c, d)$ denotes the vector whose components are the vector-valued polynomials $b$ and $c$ and the scalar-valued polynomial $d$, while $\mathbf{a} = (b, c, d)$ is the vector whose components are the scalar-valued polynomials $b, c,$ and $d$. Also, the full vector operators that will only enter in the sequence through $L^2$-projections or restrictions of their domain will be denoted using sans serif font to facilitate their identification.

3.1.1 Gradient

From the polytopal methods literature, it is well known that a consistent gradient can be reconstructed in $P^k(F)^2$ using polynomials of degree $(k - 1)$ inside $F$ and boundary polynomials in $P^k(\mathcal{E}_F)$ (see, e.g., [29, Section 2.1]). However, in order for the discrete gradient operator to map on the domain of the
discrete curl operator, we also aim here at reconstructing a gradient of degree $k$ on $\partial F$. For this reason, we will rather consider boundary polynomials in $\mathcal{P}^{k+1}(\partial F)$. We therefore define

$$
G^k_F := (G^k_F, G^k_{\partial F}^{k+1}) : \mathcal{P}^{k-1}(F) \times \mathcal{P}^{k+1}(\partial F) \to \mathcal{P}^k(F)^2 \times \mathcal{P}^k(\mathcal{E} F)
$$
such that, for all $q_F = (q_F, q_{\partial F}) \in \mathcal{P}^{k-1}(F) \times \mathcal{P}^{k+1}(\partial F)$,

$$
\int_F G^k_F q_F \cdot w_F = - \int_F q_F \, \text{div}_F w_F + \sum_{E \in \mathcal{F}_F} \omega_{FE} \int_E q_{\partial F}(w_F \cdot n_{FE}) \quad \forall w_F \in \mathcal{P}^k(F)^2
$$

and

$$
(G^k_F q_F)_{|E} = G^k_{\partial F} q_F := (q_{\partial F})_{|E} \quad \forall E \in \mathcal{E} F,
$$

where the derivative on $E$ is taken along the direction $t_E$. Since $G^k_{\partial F}$ only depends on the boundary values of $q_F$, by a slight abuse of notation we also write $G^k_{\partial F} q_{\partial F}$ instead of $G^k_{\partial F} q_F$ when needed.

We next state two results that will be useful in what follows: the consistency of both components of $G^k_F$, and the surjectivity of $G^k_{\partial F}$. Let us first introduce the interpolator $I^k_{\text{grad,f}} : \mathcal{C}^0(\overline{F}) \to \mathcal{P}^{k-1}(F) \times \mathcal{P}^{k+1}(\partial F)$ such that, for all $q \in \mathcal{C}^0(\overline{F})$,

$$
I^k_{\text{grad,f}} q := (q_F, q_{\partial F}) \in \mathcal{P}^{k-1}(F) \times \mathcal{P}^{k+1}(\partial F) \quad \text{with} \quad q_F = \pi^{k-1}_F q,
$$

$$
\pi^{k-1}_{\partial F}(q_{\partial F})_{|E} = \pi^{k-1}_{\partial F} q_{|E} \quad \forall E \in \mathcal{E} F, \quad \text{and} \quad q_{\partial F}(x_V) = q(x_V) \quad \forall V \in \mathcal{V}_F.
$$

The isomorphism (2.3) shows that the last two relations define $q_{\partial F}$ uniquely.

**Proposition 1** (Polynomial consistency of the reconstructed gradient). It holds

$$
G^k_F(I^k_{\text{grad,f}} q) = \text{grad}_F q \quad \forall q \in \mathcal{P}^{k+1}(F),
$$

$$
G^k_{\partial F}(I^k_{\text{grad,f}} q) = (q_{\partial F})' \quad \forall q \in \mathcal{P}^{k+1}(F), \quad \forall E \in \mathcal{E} F.
$$

**Proof.** Let $q \in \mathcal{P}^{k+1}(F)$ and set $q_F := I^k_{\text{grad,f}} q$. The restriction $q_{\partial F}$ of $q$ to $\partial F$ obviously satisfies the conditions imposed on $q_{\partial F}$ in (3.4), and thus $q_{\partial F} = q_{|\partial F}$. This establishes (3.6). The definition (3.2) of $G^k_F$ then yields, for all $w_F \in \mathcal{P}^k(F)^2$,

$$
\int_F G^k_F(I^k_{\text{grad,f}} q) \cdot w_F = - \int_F (\pi^{k-1}_{\partial F} q) \, \text{div}_F w_F + \sum_{E \in \mathcal{F}_F} \omega_{FE} \int_E q_{\partial F}(w_F \cdot n_{FE}) = - \int_F q \, \text{div}_F w_F + \sum_{E \in \mathcal{F}_F} \omega_{FE} \int_E q_{\partial F}(w_F \cdot n_{FE}) = \int_F \text{grad}_F q \cdot w_F,
$$

where the removal of $\pi^{k-1}_{\partial F}$ in the second line is justified by its definition (2.2) along with the fact that $\text{div}_F w_F \in \mathcal{P}^{k-1}(F)$, and the conclusion follows from the integration by parts formula (2.12). Since both $G^k_F(I^k_{\text{grad,f}} q)$ and $\text{grad}_F q$ belong to $\mathcal{P}^k(F)^2$, (3.7) proves (3.5). \qed

**Proposition 2** (Surjectivity of $G^k_{\partial F}$). For all $r_{\partial F} \in \mathcal{P}^k(\mathcal{E} F)$ such that $\sum_{E \in \mathcal{E} F} \omega_{FE} \int_E r_{\partial F} = 0$, there exists $q_{\partial F} \in \mathcal{P}^{k+1}(\partial F)$ such that $G^k_{\partial F} q_{\partial F} = r_{\partial F}$.

**Remark 3** (Bijection of $G^k_{\partial F}$). It is not difficult to check that the condition $\sum_{E \in \mathcal{E} F} \omega_{FE} \int_E r_{\partial F} = 0$ is also necessary for $r_{\partial F}$ to be in the image of $G^k_{\partial F}$, which is therefore an isomorphism between $\mathcal{P}^{k+1}(\partial F)$ and $\{r_{\partial F} \in \mathcal{P}^k(\mathcal{E} F) : \sum_{E \in \mathcal{E} F} \omega_{FE} \int_E r_{\partial F} = 0\}$. 

7
Proof of Proposition 2. Define the function \( \tilde{\partial F} : \partial F \to \mathbb{R} \) by setting \( (\tilde{\partial F})|_E := \omega_{FE}(\tilde{\partial F})|_E \) for all \( E \in \mathcal{E}_F \). Then,
\[
\int_{\partial F} \tilde{\partial F} = 0. \tag{3.8}
\]
Fix an arbitrary \( V \in \mathcal{V}_F \) and, for a given \( x \in \partial F \), let \( \Gamma_{xV} \to x \) be the path in \( \partial F \) that goes from \( x \) to \( x \) in a clockwise direction. Define then \( q_{\tilde{\partial F}}(x) \) to be the integral of \( \tilde{\partial F} \) along \( \Gamma_{xV} \to x \). The condition (3.8) ensures the continuity of \( q_{\tilde{\partial F}} \) at \( V \) after a complete loop around \( \partial F \). By construction, the derivative of \( q_{\tilde{\partial F}} \) in the clockwise direction along \( \partial F \) is equal to \( \tilde{\partial F} \). This means that, on any \( E \in \mathcal{E}_F \) with orientation \( t_E \), we have \( \omega_{FE}(q_{\tilde{\partial F}})|_E = (\tilde{\partial F})|_E = \omega_{FE}(\tilde{\partial F})|_E \), which precisely establishes \( G_{\tilde{\partial F}}^{k} q_{\tilde{\partial F}} = r_{\tilde{\partial F}} \).

\[ \square \]

3.2 An almost-exact two-dimensional sequence

The two-dimensional full gradient and curl reconstructions define the following sequence:

\[
\mathbb{R} \xrightarrow{I_F^{\text{grad},F}} \mathcal{P}^{k-1}(F) \times \mathcal{P}^{k+1}_c(\partial F) \xrightarrow{G_F^{k}} \mathcal{P}^k(F)^2 \times \mathcal{P}^k(\mathcal{E}_F) \xrightarrow{C_F^{k}} \mathcal{P}^k(F) \xrightarrow{0} \{0\}. \tag{3.13}
\]

This sequence satisfies some exactness properties, but is not completely exact. Analysing these properties, as done in the following proposition, will guide us to define an exact two-dimensional sequence of spaces and operators.
**Proposition 6** (Properties of the sequence \( (3.13) \)). It holds:

\[
P_{k\text{-grad},F} \mathbb{R} = \ker \mathbf{G}^k_F, \tag{3.14}
\]

\[
\im \mathbf{G}^k_F \subset \ker \mathbf{C}^k_F, \tag{3.15}
\]

For all \( \nu_F = (v_F, \nu_{\partial F}) \in \ker \mathbf{C}^k_F \), there exists \( q_F = (q_F, \nu_{\partial F}) \in \mathcal{P}^{k-1}(F) \times \mathcal{P}^{k+1}(\partial F) \) such that \( \nu_{\partial F} = G^k_{\partial F} q_{\partial F} \), \( \pi^{k-1}_{\partial F} v_F = \pi^{k-1}_{\partial F} (\mathbf{G}^k_F q_F) \), and \( \pi^{k+1}_{\partial F} v_F = \pi^{k+1}_{\partial F} (\mathbf{G}^k_F q_F) \), and

\[
\im \mathbf{C}^k_F = \mathcal{P}^k(F). \tag{3.17}
\]

**Proof.** 1. **Proof of \( (3.14) \).** If \( C \in \mathbb{R} \) and \( q_F = p_{k\text{-grad},F} C \), then the consistency properties \( (3.5) \) and \( (3.6) \) give, respectively, \( \mathbf{G}^k_F q_F = 0 \) and \( G^k_{\partial F} q_F = 0 \). These two relations establish the inclusion \( C \subset \mathfrak{C}^k_F \).

To prove the converse inclusion, we first notice that, if \( q_F = (q_F, \nu_{\partial F}) \in \mathcal{P}^{k-1}(F) \times \mathcal{P}^{k+1}(\partial F) \) is such that \( \mathbf{G}^k_F q_F = 0 \), then \( G^k_{\partial F} q_{\partial F} = 0 \) and hence \( q_{\partial F} = C \) for some \( C \in \mathbb{R} \) (since \( q_{\partial F} \) is continuous and its derivative vanishes on each \( E \in \mathcal{E}_F \)). Plugging this result into the definition \( (3.2) \) of \( \mathfrak{C}^k_F \) and using the integration by parts formula \( (2.12) \) with \( C \) instead of \( q_F \), we infer, for all \( w_F \in \mathcal{P}^k(F)^2 \),

\[
\int_F \mathbf{G}^k_F q_F \cdot w_F = - \int_F q_F \text{div}_F w_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \mathbf{C}(w_F \cdot n_{FE}) = \int_F (C - q_F) \text{div}_F w_F. \tag{3.18}
\]

Using \( \mathbf{G}^k_F q_F = 0 \), we see that the left-hand side vanishes and, since \( \text{div}_F : \mathfrak{P}^k(F)^2 \rightarrow \mathfrak{P}^{k-1}(F) \) is surjective (consequence of \( (2.9) \) and \( q_F \in \mathfrak{P}^{k-1}(F) \), this yields \( \pi^{k-1}_{\partial F} C = \pi^{k-1}_{\partial F} q_F = q_F \). Together with \( q_{\partial F} = C \), this establishes that \( p_{k\text{-grad},F} C = q_F \), which concludes the proof of the inclusion \( C \subset \mathfrak{C}^k_F \).

2. **Proof of \( (3.15) \).** Let \( q_F \in \mathfrak{P}^{k-1}(F) \times \mathfrak{P}^{k+1}(\partial F) \) and write, using the definition \( (3.9) \) of \( \mathfrak{C}^k_F \), for all \( r_F \in \mathfrak{P}^k(F) \),

\[
\int_F \mathbf{C}^k_F (\mathbf{G}^k_F q_F) r_F = \int_F \mathbf{C}^k_F q_F \cdot \text{rot}_F r_F - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E G^k_{\partial F} q_F r_F = \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \left[ q_{\partial F} \text{rot}_F r_F \cdot n_{FE} - (q_{\partial F})'_{\partial F} \right],
\]

where the second line follows using the definitions \( (3.2) \) of \( \mathbf{G}^k_F \) with \( w_F = \text{rot}_F r_F \) (additionally noticing that \( \text{div}_F (\text{rot}_F r_F) = 0 \)) and \( (3.3) \) of \( G^k_{\partial F} \). We then use the integration by parts formula \( (2.13) \) on each \( E \in \mathcal{E}_F \) to obtain

\[
\int_F \mathbf{C}^k_F (\mathbf{G}^k_F q_F) r_F = \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_{\partial F} \left[ \text{rot}_F r_F \cdot n_{FE} + (r_F)'_{\partial F} \right] - \sum_{E \in \mathcal{E}_F} \omega_{FE} \left( q_{\partial F} (x_{E,2}) r_F (x_{E,2}) - q_{\partial F} (x_{E,1}) r_F (x_{E,1}) \right) = 0, \tag{3.19}
\]

where the cancellation comes from

\[
\text{rot}_F r_F \cdot n_{FE} = \text{grad}_F r_F \cdot (q_{\partial F})'_{\partial F} n_{FE} = \text{grad}_F r_F \cdot (q_{\partial F})'_{\partial F} n_{FE} = - \text{grad}_F r_F \cdot t_E = -(r_F)'_{\partial F} \tag{3.20}
\]

(since \( (t_E, n_{FE}) \) is right-handed in \( F \)), and we have concluded using the fact that, for all \( V \in \mathcal{V}_F \), the term \( q_{\partial F}(x_V) r_F(x_V) \) appears exactly twice in the last sum, with opposite signs. This proves \( (3.15) \).
3. **Proof of (3.16)**. Suppose that \( \underline{\pi}_F \in \mathcal{P}^k(F)^2 \times \mathcal{P}^k(\mathcal{E}_F) \) is such that \( C_F^k \underline{\pi}_F = 0 \). Then, \( (3.9) \) with \( r_F = 1 \) shows that \( \sum_{E \in \mathcal{E}_F} q_{\partial E} \int_{E} v_{\partial E} = 0 \) and thus, by Proposition 2, there exists \( q_{\partial F} \in \mathcal{P}^{k+1}(\partial F) \) such that \( G^k_{\partial F} q_{\partial F} = v_{\partial F} \). This establishes the first conclusion in (3.16).

Using \( C_F^k \underline{\pi}_F = 0 \) and again the definition \( (3.9) \) of \( C_F^k \), for all \( r_F \in \mathcal{P}^k(F) \) we then have

\[
\int_F v_F \cdot \text{rot}_F r_F = \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E G^k_F q_{\partial F} r_F = - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_{\partial F}(r_F)'_E = \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_{\partial F}(\text{rot}_F r_F \cdot n_{FE}) = \int_F G^k_F(q_F, q_{\partial F}) \cdot \text{rot}_F r_F,
\]

where the second line follows integrating by parts on each edge and cancelling out the vertex values in a similar way as in (3.19), the third line from (3.20), and the conclusion is obtained applying the definition \( (3.2) \) of \( G^k_F \) to \( w_F = \text{rot}_F r_F \) (which satisfies \( \text{div}_F w_F = 0 \)) and \( q_{\partial F} = (q_F, q_{\partial F}) \) for an arbitrary \( q_F \in \mathcal{P}^{k+1}(F) \). Since \( (3.21) \) is valid for any \( r_F \in \mathcal{P}^k(F) \), this proves the second conclusion in (3.16).

To conclude the proof of (3.16), we need to identify a specific \( q_F \in \mathcal{P}^{k-1}(F) \) such that \( \pi_{\mathcal{R},F} v_F = \pi_{\mathcal{R},F}^k G^k_F q_F \), that is, for any \( w_F \in \mathcal{R}^k(F)^\perp \),

\[
\int_F v_F \cdot w_F = \int_F G^k_F q_F \cdot w_F = - \int_F q_F \cdot \text{div}_F w_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_{\partial F}(w_F \cdot n_{FE}),
\]

where we have used the definition \( (3.2) \) of \( G^k_F \) in the second passage. Since \( q_{\partial F} \) is already given, we simply have to take \( q_F \in \mathcal{P}^{k-1}(F) \) such that:

\[
\int_F q_F \cdot \text{div}_F w_F = - \int_F v_F \cdot w_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_{\partial F}(w_F \cdot n_{FE}) \quad \forall w_F \in \mathcal{R}^k(F)^\perp.
\]

By \( (2.9) \), this relation defines \( q_F \) uniquely.

4. **Proof of (3.17)**. We only have to prove \( \mathcal{P}^k(F) \subset \text{Im} C_F^k \). Let \( q_F \in \mathcal{P}^k(F) \). Since \( \text{rot}_F : \mathcal{P}^{k+1}(F)^2 \to \mathcal{P}^k(F) \) is surjective (this is a consequence of its definition \( (2.5) \) along with the surjectivity of \( \text{div}_F : \mathcal{P}^{k+1}(F)^2 \to \mathcal{P}^k(F) \), which follows from \( (2.9) \)), there is \( v \in \mathcal{P}^{k+1}(F)^2 \) such that \( \text{rot}_F v = q_F \). Hence, using the polynomial consistency of \( \pi_{\mathcal{P},F} \) followed by the commutation property \( (3.11) \), we have \( q_F = \text{rot}_F v = \pi_{\mathcal{P},F}^k(\text{rot}_F v) = C_F^k(q_{\mathcal{R}^1,F}^k v) \in \text{Im} C_F^k \), which is the desired result.

\[ \square \]

3.3 An exact two-dimensional sequence

Proposition 3 shows that the defect of exactness of the sequence \( (3.12) \) lies in the domain of \( C_F^k \)/co-domain of \( G_F^k \), which is too large. Specifically, the space \( \mathcal{P}^k(F)^2 \times \mathcal{P}^k(\mathcal{E}_F) \) in this sequence must be restricted to its subspace \( (\mathcal{R}^{k-1}(F) \oplus \mathcal{R}^k(F)^\perp) \times \mathcal{P}^k(\mathcal{E}_F) \), which still contains sufficient information to reconstruct a discrete curl satisfying the commutation property \( (3.11) \) (cf. Remark 5). Obviously, this restriction requires to project \( G_F^k \) onto this space in order for the sequence to be well-defined.
The discrete curl operator that it is \( q \) and a generic vector \( v \in X^{k}_{\text{grad},F} \) is denoted by \( (q_{F}, q_{\partial F}) \) with \( q_{F} \in \mathcal{P}_{c}^{k-1}(F) \) and \( q_{\partial F} \in \mathcal{P}_{c}^{k+1}(\partial F) \). The interpolator on \( X^{k}_{\text{grad},F} \) does not change either:

\[
I^{k}_{\text{grad},F} q := (q_{F}, q_{\partial F}) \in \mathcal{P}_{c}^{k-1}(F) \times \mathcal{P}_{c}^{k+1}(\partial F) \text{ with } q_{F} = \pi^{k-1}_{\mathcal{P}_{c},F} q, \quad \pi^{k+1}_{\mathcal{P}_{c},F}(q_{\partial F})_{E} = \pi^{k+1}_{\mathcal{P}_{c},F} q_{E} \text{ for all } E \in F_{E}, \text{ and } q_{\partial F}(x_{V}) = q(x_{V}) \text{ for all } V \in V_{F}.
\]

The domain of the reconstructed curl, which plays the role of the space \( H^{1}(F) \) in the discrete level, is now

\[
X^{k}_{\text{rot},F} := \left( \mathcal{R}^{k-1}(F) \oplus \mathcal{R}^{k}(F) \right) \times \mathcal{R}^{k}(\mathcal{E}_{F}),
\]

and a generic vector \( \mathbf{v} \in X^{k}_{\text{rot},F} \) is decomposed into \( (v_{F}, v_{\partial F}) \) with \( v_{F} \in \mathcal{R}^{k+1}(F) \), \( v_{\partial F} \in \mathcal{R}^{k}(\mathcal{E}_{F}) \), and \( v_{\partial F} \in \mathcal{P}^{k}(\mathcal{E}_{F}) \).

The discrete gradient operator \( C^{k}_{F} : X^{k}_{\text{grad},F} \to X^{k}_{\text{rot},F} \) is defined by projecting \( C^{k}_{F} \) onto \( X^{k}_{\text{rot},F} \): For all \( q_{F} \in X^{k}_{\text{grad},F} \),

\[
G^{k}_{\mathbf{v}} q_{F} := \left( G^{k}_{R,F} q_{F} + G^{k}_{R,F} q_{F} \right) \quad \text{with } G^{k}_{R,F} := \pi^{k}_{R,F} G^{k}_{F} \text{ and } G^{k}_{R,F} := \pi^{k}_{R,F} G^{k}_{F} \text{ for all } F \in \mathcal{F}_{F}.
\]

It can easily be checked from (3.2) that the following two relations characterise \( G^{k}_{R,F} \) and \( G^{k}_{R,F} \). For all \( q_{F} \in X^{k}_{\text{grad},F} \),

\[
\int_{F} G^{k-1}_{R,F} q_{F} \cdot w_{F} = \sum_{E \in \mathcal{E}_{F}} \omega_{FE} \int_{E} q_{\partial F}(w_{F} \cdot n_{FE}) \quad \forall w_{F} \in \mathcal{R}^{k-1}(F), \quad (3.23a)
\]

\[
\int_{F} G^{k}_{R,F} q_{F} \cdot w_{F} = -\int_{F} q_{F} \text{ div}_{F} w_{F} + \sum_{E \in \mathcal{E}_{F}} \omega_{FE} \int_{E} q_{\partial F}(w_{F} \cdot n_{FE}) \quad \forall w_{F} \in \mathcal{R}^{k}(F). \quad (3.23b)
\]

The discrete curl operator \( C^{k}_{F} : X^{k}_{\text{rot},F} \to \mathcal{P}^{k}(F) \) is given by the restriction of \( C^{k}_{F} \) to \( X^{k}_{\text{rot},F} \), that is: For all \( \mathbf{v}_{F} = (v_{R,F} + v_{\partial F}, v_{\partial F}) \in X^{k}_{\text{rot},F} \),

\[
\int_{F} C^{k}_{F} \mathbf{v}_{F} \cdot r_{F} = \int_{F} v_{R,F} \cdot \mathbf{r}_{F} - \sum_{E \in \mathcal{E}_{F}} \omega_{FE} \int_{E} v_{\partial F}(r_{F} \cdot n_{FE}) \quad \forall r_{F} \in \mathcal{P}^{k}(F). \quad (3.24)
\]

Notice that, in the integral over \( F \), we have removed the component \( v_{\partial F} \) of \( v_{F} \) accounting for the fact that it is \( L^{2} \)-orthogonal to \( \mathbf{r}_{F} \), \( r_{F} \in \mathcal{R}^{k-1}(F) \subset \mathcal{R}^{k}(F) \).

Letting \( I^{k}_{\text{rot},F} : H^{1}(F)^{2} \to X^{k}_{\text{rot},F} \) be the interpolator obtained projecting \( \mathbf{v} \in X^{k}_{\text{rot},F} \), that is, for all \( \mathbf{v} \in H^{1}(F)^{2} \),

\[
I^{k}_{\text{rot},F} \mathbf{v} := (\pi^{k-1}_{R,F} \mathbf{v} + \pi^{k}_{R,F} \mathbf{v},(\pi^{k}_{R,F}(v_{\cdot} r_{E}))_{E \in \mathcal{E}_{F}}),
\]

we have, following Remark 5, the commutation property

\[
C^{k}_{F}(I^{k}_{\text{rot},F} \mathbf{v}) = \pi^{k}_{\mathcal{P}_{c},F}(\mathbf{r}_{F} \mathbf{v}) \quad \forall \mathbf{v} \in H^{1}(F)^{2}. \quad (3.26)
\]
Table 1: Number of discrete unknowns attached to each geometric entity for the two-dimensional sequence \((3.27)\) on a triangle \(F\) for \(k \in \{0, \ldots, 3\}\). For comparison, we also report in parentheses the number of degrees of freedom of the corresponding spaces in the FE sequence \((P_{k+1}(F), \varrho_{\pi/2}RT_k(F), P_k(F))\).

$$\begin{array}{c|c|c|c|c|c}
 k & V & E & F & \text{Total} \\
\hline
 X_{\text{grad},F}^k (P_{k+1}(F)) & 0 & 1 (1) & 0 (0) & 0 (0) & 3 (3) \\
 & 1 & 1 (1) & 1 (1) & 1 (0) & 7 (6) \\
 & 2 & 1 (1) & 2 (2) & 3 (1) & 12 (10) \\
 & 3 & 1 (1) & 3 (3) & 6 (3) & 18 (15) \\
 X_{\text{rot},F}^k (\varrho_{\pi/2}RT^k(F)) & 0 & 1 (1) & 0 (0) & 3 (3) & 1 (1) \\
 & 1 & 2 (2) & 3 (2) & 9 (8) & 2 (2) \\
 & 2 & 3 (3) & 8 (6) & 17 (15) & 3 (3) \\
 & 3 & 4 (4) & 15 (12) & 27 (24) & 4 (4) \\
 P_k(F) (P^k(F)) & 0 & 1 (1) & 1 (1) & 0 (0) & 1 (1) \\
 & 1 & 3 (3) & 3 (3) & 10 (10) & 2 (2) \\
 & 2 & 6 (6) & 6 (6) & 0 (0) & 3 (3) \\
 & 3 & 10 (10) & 10 (10) & 10 (10) & 6 (6) \\
\end{array}$$

Theorem 7 (Exact two-dimensional sequence). The following sequence is exact:

$$R \overset{I_k}{\rightarrow} X_{\text{grad},F}^k \overset{G_k}{\rightarrow} X_{\text{rot},F}^k \overset{C_k}{\rightarrow} P^k(F) \overset{0}{\rightarrow} \{0\}.$$ (3.27)

Remark 8 (Comparison with Finite Elements). When \(F\) is a triangle, the sequence \((3.27)\) can be compared with the usual FE sequence \((P_{k+1}(F), \varrho_{\pi/2}RT^k(F), P^k(F))\) (with \(\varrho_{\pi/2}RT^k(F)\) denoting the rotated two-dimensional Raviart–Thomas space, cf. [45]). The number of discrete unknowns in each case is reported in Table 1. We notice that, for each space in the sequence, the number of discrete unknowns attached to the lowest-dimensional geometric support is the same in both cases. On the other hand, our sequence has more internal unknowns. This phenomenon is known from the Virtual Element literature and can be countered using serendipity spaces; see, e.g., [5]. Also, when writing a scheme, internal unknowns can usually be locally eliminated by static condensation; see, e.g., [29, Section B.3.2].

Proof of Theorem 7 We have to prove that

$$\begin{align*}
I_k & \subset \text{Ker} G_k^F, \\
\text{Im} G_k^F & = \text{Ker} C_k^F, \\
\text{Im} C_k^F & = P^k(F). 
\end{align*}$$ (3.28) (3.29) (3.30)

1. Proof of \((3.28)\). Since \(G_k^F\) is a projection of \(G_k^F\), we have \(\text{Ker} G_k^F \subset \text{Ker} G_k^F\) and, thus, \((3.14)\) gives

$$I_k \subset \text{Ker} G_k^F.$$ 

Assume now that \(q_F \in X_{\text{grad},F}^k\) is such that \(G_k^F q_F = 0\). As in the proof of \((3.14)\), since the boundary components of \(G_k^F\) and \(G_k^F\) are the same, this shows that \(q_{\partial F} = C\) for some constant \(C \in \mathbb{R}\). Moreover, equation \((3.18)\) still holds, and can be applied to a generic \(w_F \in R^k(F)^\perp\) to infer

$$\int_F (C - q_F) \text{div}_F w_F = \int_F G_k^F q_F \cdot w_F = \int_F G_{\text{rot},F}^k q_F \cdot w_F = 0,$$
where the second equality follows from the definition of \( \pi_{R,F}^\perp \), recalling that \( G_{R,F}^{\perp,k} = \pi_{R,F}^{\perp,k} G_F^k \) (see \( \text{(3.22)} \)). Together with the surjectivity of \( \text{div} v_F : \mathcal{R}^k(F)^\perp \rightarrow \mathcal{P}^{k-1}(F) \), this ensures as before that \( q_F = \pi_{R,F}^{k-1} C \), which concludes the proof that \( q_F = l_{\text{grad},F}^k C \). Hence, we have

\[
\text{Ker} G_F^k \subset l_{\text{grad},F}^k \mathbb{R}.
\]

concluding the proof of \( \text{(3.28)} \).

2. Proof of \( \text{(3.29)} \). Let \( q_F \in X_{\text{grad},F}^k \) and set \( v_F := G_F^k q_F \), that is, by definition \( \text{(3.22)} \) of \( G_F^k \), \( v_F = (v_F = \pi_{R,F}^{k-1} w_F + \pi_{R,F}^{\perp,k} w_F, w_F) \). By \( \text{(3.15)} \), we have \( C_F^k w_F = 0 \). Using the definitions \( \text{(3.9)} \) and \( \text{(3.24)} \) of \( C_F^k w_F \) and \( C_F^k v_F \) and the relation

\[
\int_F w_F \cdot \text{rot}_F r_F = \int_F \pi_{R,F}^{k-1} w_F \cdot \text{rot}_F r_F = \int_F v_F \cdot \text{rot}_F r_F \quad \forall r_F \in \mathcal{P}^k(F),
\]

which follows from the definitions of \( \pi_{R,F}^{k-1} \) and \( v_{R,F} \), we see that \( 0 = C_F^k v_F = C_F^k v_{R,F} \). Hence, \( C_F^k(G_F^k q_F) = 0 \) and

\[
\text{Im} G_F^k \subset \text{Ker} C_F^k \quad \text{(3.31)}
\]

Let now \( v_F = (v_F = v_{R,F} + v_{R,F}^{\perp,k}) \in \text{Ker} C_F^k \subset X_{\text{grad},F}^k \cap \text{Ker} C_F^k \). By \( \text{(3.16)} \), there is \( q_F \in \mathcal{P}^{k-1}(F) \times \mathcal{P}^{k-1}(\partial F) = \mathbb{R} \times \mathbb{R} \) such that \( v_F = G_{q,F}^k q_F \), \( \pi_{R,F}^{k-1} v_F = \pi_{R,F}^{k-1}(G_{q,F}^k q_F) = G_{R,F}^{k-1} q_F \) and \( \pi_{R,F}^{\perp,k} v_F = \pi_{R,F}^{\perp,k}(G_{q,F}^k q_F) = G_{R,F}^{\perp,k} q_F \). Since \( \mathcal{R}^{k-1}(F) \subset \mathcal{P}^k(F) \) is orthogonal to \( \mathcal{R}^k(F)^\perp \), we have \( \pi_{R,F}^{k-1} v_F = v_{R,F} \) and \( \pi_{R,F}^{\perp,k} v_F = v_{R,F}^{\perp,k} \), which proves that \( v_F = G_{q,F}^k q_F \). Hence,

\[
\text{Ker} C_F^k \subset \text{Im} G_F^k
\]

and the second exactness property \( \text{(3.29)} \) is proved.

3. Proof of \( \text{(3.30)} \). Consequence of the commutation property \( \text{(3.26)} \) proceeding as in the proof of \( \text{(3.17)} \). \( \square \)

3.4 Two-dimensional potentials

We next exhibit reconstructions of the potential for each space in the sequence.

3.4.1 Scalar potential (scalar trace)

We start with a scalar potential reconstruction \( \gamma_{F}^{k+1} : X_{\text{grad},F}^k \rightarrow \mathcal{P}^{k+1}(F) \) which, when \( F \) represents a face of a polyhedron \( T \), plays the role of a reconstruction of the scalar trace on \( F \). The required properties on \( \gamma_{F}^{k+1} \) are the following:

\[
\gamma_{F}^{k+1}(l_{\text{grad},F}^k q) = q \quad \forall q \in \mathcal{P}^{k+1}(F), \tag{3.32a}
\]

\[
\pi_{R,F}^{k+1} \gamma_{F}^{k+1}(q_{F}^k) = q_F \quad \forall q_F \in (q_F, q_{\partial F}) \in X_{\text{grad},F}^k. \tag{3.32b}
\]

The first property expresses the polynomial consistency of the reconstruction, while the second enforces its projection on \( \mathcal{P}^{k-1}(F) \) and shows that \( \gamma_{F}^{k+1} \) has to be a higher-order correction of the projection \( X_{\text{grad},F}^k \ni q_F = (q_F, q_{\partial F}) \mapsto q_F \in \mathcal{P}^{k-1}(F) \). It is easily checked that, if \( \gamma_{F}^{k+1} : X_{\text{grad},F}^k \rightarrow \mathcal{P}^{k+1}(F) \) is a reconstruction that satisfies the consistency property \( \text{(3.32a)} \), then a reconstruction \( \gamma_{F}^{k+1} \) satisfying \( \text{(3.32a)–(3.32b)} \) is obtained setting

\[
\gamma_{F}^{k+1} q_F = q_F + \gamma_{F}^{k+1} q_{F}^k - \pi_{R,F}^{k-1}(\gamma_{F}^{k+1} q_{F}^k) \quad \forall q_F \in X_{\text{grad},F}^k.
\]
Remark 9 (A consistent potential reconstruction). There are several ways to devise a reconstruction \( \hat{\gamma}_F^{k+1} : X_{\text{grad}, F}^k \to \mathcal{P}^{k+1}(F) \) that satisfies (3.32a). One of them is to define, for all \( q_F \in X_{\text{grad}, F}^k \), \( \hat{\gamma}_F^{k+1} q_F \in \mathcal{P}^{k+1}(F) \) such that

\[
\int_F \hat{\gamma}_F^{k+1} q_F \, \text{div}_F \, v_F = - \int_F G_F^k q_F \cdot v_F + \sum_{E \in \partial F} \omega_{FE} \int_E q_F (v_F \cdot n_{FE}) \quad \forall v_F \in \mathcal{R}^{k+2}(F)\,.
\]

This relation defines \( \hat{\gamma}_F^{k+1} \) uniquely since \( \text{div}_F : \mathcal{R}^{k+2}(F) \to \mathcal{P}^{k+1}(F) \) is an isomorphism (see (2.9)). The consistency property (3.32a) for this reconstruction can be checked setting \( q_F = I_{\text{grad}, F}^k q \) for \( q \in \mathcal{P}^{k+1}(F) \), invoking the polynomial consistency property (3.5) to infer \( G_F^k I_{\text{grad}, F}^k q = \text{grad}_F q \), using the fact that \( q_{\partial F} = q_{\partial F} \) as a consequence of the definition (3.4) of \( I_{\text{grad}, F}^k \) together with the isomorphism (2.3) and \( q_{\partial F} \in \mathcal{P}_c^{k+1}(\partial F) \), and applying the integration by parts formula (2.12).

3.4.2 Vector potential (tangential vector trace)

We next define the two-dimensional vector potential \( \gamma_{t,F}^k : X_{\text{rot}, F}^k \to \mathcal{P}(F)^2 \) such that, for all \( v_F \in X_{\text{rot}, F}^k \),

\[
\int_F \gamma_{t,F}^k v_F \cdot \text{rot}_F r_F = \int_F C_F^k v_F \cdot r_F + \sum_{E \in \partial F} \omega_{FE} \int_E v_F \cdot n_{FE} \quad \forall r_F \in \mathcal{P}^{0,k+1}(F), \tag{3.33a}
\]

\[
\int_F \gamma_{t,F}^k v_F \cdot w_F = \int_F v_F \cdot w_F \quad \forall w_F \in \mathcal{R}^k(F)^\perp. \tag{3.33b}
\]

To check that, given \( v_F \in X_{\text{rot}, F}^k \), (3.33) defines a unique polynomial \( \gamma_{t,F}^k v_F \in \mathcal{P}(F)^2 \), observe that (3.33a) and (3.33b) prescribe, respectively, \( \pi_{\mathcal{R},F}^k(\gamma_{t,F}^k v_F) \) (use (2.8)) and \( \pi_{\mathcal{R},F}^k(\gamma_{t,F}^k v_F) \), and recall the orthogonal decomposition \( \mathcal{P}(F)^2 = \mathcal{R}^k(F) \oplus \mathcal{R}^k(F)^\perp \). When \( F \) represents a face of a polyhedron \( T \), \( \gamma_{t,F}^k \) corresponds to a reconstruction of the tangential trace.

Remark 10 (Validity of (3.33a)). Observing that both sides of (3.33a) vanish for \( r_F = 1 \) (use the definition of \( C_F^k \) for the right-hand side), we deduce that (3.33a) holds in fact for any \( r_F \in \mathcal{P}^{k+1}(F) \).

We now state and prove two propositions on the commutation properties of the vector potential reconstruction.

Proposition 11 (Commutation property for the two-dimensional vector potential reconstruction). For all \( v \in H^1(F)^2 \) such that \( \text{rot}_F v \in \mathcal{P}^{k}(F) \) and \( v|_E \cdot n_E \in \mathcal{P}^{k}(E) \) for all \( E \in \mathcal{E}_F \), it holds

\[
\gamma_{t,F}^k(I_{\text{rot},F}^k v) = \pi_{\mathcal{P},F}^k v. \tag{3.34}
\]

Proof. Take \( r_F \in \mathcal{P}^{k+1}(F) \), write (3.33a) (using Remark 10) for \( v_F = I_{\text{rot},F}^k v \), and apply the commutation property (3.26) of \( C_F^k \) to get

\[
\int_F \gamma_{t,F}^k(I_{\text{rot},F}^k v) \cdot \text{rot}_F r_F = \int_F \pi_{\mathcal{P},F}^k(\text{rot}_F v) \cdot r_F + \sum_{E \in \partial F} \omega_{FE} \int_E \pi_{\mathcal{P},E}^k(v|_E \cdot n_E) \cdot r_F.
\]

Using the assumptions on \( v \) along with their definition (2.2), the projectors \( \pi_{\mathcal{P},F}^k \) and \( \pi_{\mathcal{P},E}^k \) can be removed from the equation above, and the integration by parts formula (2.15) then leads to

\[
\int_F \gamma_{t,F}^k(I_{\text{rot},F}^k v) \cdot \text{rot}_F r_F = \int_F v \cdot \text{rot}_F r_F.
\]
The polynomial \( r_F \) being arbitrary in \( P^{k+1}(F) \), this relation implies \( \pi_{R,F}^k(\gamma_{l,F}(I_{\text{rot},F}^k v)) = \pi_{R,F}^k v \). On the other hand, \((3.33b)\) with \( v_F = I_{\text{rot},F}^k v \) and the definition of \( I_{\text{rot},F}^k \) yield \( \gamma_{l,F}^k(\gamma_{l,F}^k(I_{\text{rot},F}^k v)) = v_{R,F}^k = \pi_{R,F}^k v \). The relation \((3.34)\) then follows using the decomposition \( P^k(F)^2 = R^k(F) \oplus R^k(F)^\perp \) to write
\[
\gamma_{l,F}^k(I_{\text{rot},F}^k v) = \pi_{R,F}^k(\gamma_{l,F}^k(I_{\text{rot},F}^k v)) + \pi_{R,F}^k(\pi_{R,F}^k(v)) = \pi_{R,F}^k v + \pi_{R,F}^k v = \pi_{R,F}^k v.
\]

Proposition 12 (Two-dimensional discrete vector potential reconstruction and gradient). It holds
\[
\gamma_{l,F}^k(G_{F}^k q_F) = G_{F}^k q_F, \quad \forall q_F \in X_{\text{grad},F}^k.
\]

Proof. For all \( r_F \in P^{k+1}(F) \), writing (3.33a) for \( v_F = G_{F}^k q_F \), it is inferred that
\[
\int_F \gamma_{l,F}^k(G_{F}^k q_F) \cdot \text{rot}_F r_F = \int_F C_F^k(G_{F}^k q_F) r_F + \sum_{E \in E_F} \omega_{FE} \int_E G_{E}^k q_F r_F = \int_F G_{F}^k q_F \cdot \text{rot}_F r_F,
\]
where we have used the inclusion (3.31) in the cancellation, while the conclusion follows proceeding as in (3.21). This implies \( \pi_{R,F}^k(\gamma_{l,F}^k(G_{F}^k q_F)) = \pi_{R,F}^k(G_{F}^k q_F) \). On the other hand, \((3.33b)\) also applied to \( v_F = G_{F}^k q_F \), implies \( \pi_{R,F}^k(\pi_{R,F}^k(G_{F}^k q_F)) = G_{R,F}^k q_F = \pi_{R,F}^k(G_{F}^k q_F) \). Combining these relations with the orthogonal decomposition \( P^k(F)^2 = R^k(F) \oplus R^k(F)^\perp \), \((3.35)\) follows.

3.5 Two-dimensional discrete \( L^2 \)-products

We next define discrete counterparts of the \( L^2 \)-products in \( H^1(F) \) and \( H(\text{rot}; F) \). The discrete \( L^2 \)-products are composed of consistent and stabilising terms. The former correspond to the \( L^2 \)-product of the full potential reconstructions, whereas the latter penalise in a least square sense high-order differences between the potential reconstruction and the discrete unknowns. The design of these high-order differences is inspired by the stabilisation terms in HHO methods, see [29 Section 2.1.4]. Specifically, we define

- \((\cdot, \cdot)_{\text{grad},F} : X^k_{\text{grad},F} \times X^k_{\text{grad},F} \to \mathbb{R}\) such that, for all \( q_F, r_F \in X^k_{\text{grad},F} \),
  \[
  (q_F, r_F)_{\text{grad},F} := \int_F \gamma_{F}^{k+1} q_F \gamma_{F}^{k+1} r_F + \int_F \delta_{\text{grad},F}^{k+1} q_F \delta_{\text{grad},F}^{k+1} r_F + \sum_{E \in E_F} h_F \int_E \delta_{\text{grad},\partial F}^{k+1} q_F \delta_{\text{grad},\partial F}^{k+1} r_F,
  \]
  where \( h_F \) denotes the diameter of \( F \) and we have set, for any \( q_F \in X^k_{\text{grad},F} \),
  \[
  (\delta_{\text{grad},F}^{k+1} q_F, \delta_{\text{grad},\partial F}^{k+1} q_F) := \int_F \delta_{\text{grad},F}^{k+1} q_F - q_F.
  \]

- \((\cdot, \cdot)_{\text{rot},F} : X^k_{\text{rot},F} \times X^k_{\text{rot},F} \to \mathbb{R}\) such that, for all \( v_F, w_F \in X^k_{\text{rot},F} \),
  \[
  (v_F, w_F)_{\text{rot},F} := \int_F \gamma_{F}^{k+1} v_F \gamma_{F}^{k+1} w_F + \int_F \delta_{\text{rot},F}^{k+1} v_F \delta_{\text{rot},F}^{k+1} w_F + \int_F \delta_{\text{rot},\partial F}^{k+1} v_F \delta_{\text{rot},\partial F}^{k+1} w_F + \sum_{E \in E_F} h_F \int_E \delta_{\text{rot},E}^{k+1} v_F \delta_{\text{rot},E}^{k+1} w_F,
  \]
  where we have set, for any \( v_F \in X^k_{\text{rot},F} \),
  \[
  (\delta_{\text{rot},F}^{k+1} v_F + \delta_{\text{rot},\partial F}^{k+1} v_F, (\delta_{\text{rot},E}^{k+1} v_F)_{E \in E_F}) := I_{\text{rot},F}^k(\gamma_{l,F}^k v_F) - v_F.
  \]
The bilinear forms \((\cdot, \cdot)_{\text{grad}, F}\) and \((\cdot, \cdot)_{\text{rot}, F}\) are obviously symmetric and positive semi-definite. Using arguments similar to the ones deployed in the three-dimensional case (cf. Lemma 28 below), it can be proved that they are actually positive definite, hence they define proper inner products on \(X^k_{\text{grad}, F}\) and \(X^k_{\text{rot}, F}\), respectively. By (3.32a) and (3.34), they also enjoy the following consistency properties:

\[
\begin{align*}
(I_{\text{grad}, F}^k q, I_{\text{grad}, F}^k r)_{\text{grad}, F} &= (q, r)_{L^2(F)} & \forall q, r \in \mathcal{P}^{k+1}(F), \\
(I_{\text{rot}, F}^k v, I_{\text{rot}, F}^k w)_{\text{rot}, F} &= (v, w)_{L^2(F)^2} & \forall v, w \in \mathcal{P}^k(F)^2.
\end{align*}
\]

4 An exact three-dimensional sequence

In this section we define a discrete counterpart of the following exact three-dimensional sequence on a polyhedron \(T\) (which may be thought of as a mesh element):

\[
\mathbb{R} \xrightarrow{\text{tr}} H^1(T) \xrightarrow{\text{grad}} H(\text{curl}; T) \xrightarrow{\text{curl}} H(\text{div}; T) \xrightarrow{\text{div}} L^2(T) \xrightarrow{0} \{0\},
\]

where \(\text{tr}\) is the operator that maps a real value to a constant function over \(T\), \(H^1(T)\) denotes the space of functions that are square integrable over \(T\) along with their derivatives, \(H(\text{curl}; T) := \{v \in L^2(T)^3 : \text{curl } v \in L^2(T)\}\), and \(H(\text{div}; T) := \{v \in L^2(T)^3 : \text{div } v \in L^2(T)\}\). The principle is, as in two dimensions, to start from reconstructions of vector operators in full polynomial spaces, and to project them on restricted domains/co-domains to form an exact sequence. For the sake of conciseness, and since it is similar to the two-dimensional case, we do not detail the initial analysis (i.e., the derivation of an almost-exact sequence and the equivalent of Proposition 0), but directly provide appropriate choices of spaces and discrete operators.

4.1 Three-dimensional discrete spaces and interpolators

4.1.1 Discrete \(H^1(T)\) space

The discrete counterpart of \(H^1(T)\) is

\[
X^k_{\text{grad}, T} := \mathcal{P}^{k-1}(T) \times \mathcal{P}^{k-1}(\mathcal{F}_T) \times \mathcal{P}^{k+1}(\partial^2 T).
\]

A generic vector \(q_T \in X^k_{\text{grad}, T}\) is written

\[
q_T = (q_T, q_T\partial T, q_T\partial^2 T), \quad \text{with } q_T \in \mathcal{P}^{k-1}(T), \quad q_T\partial T \in \mathcal{P}^{k-1}(\mathcal{F}_T) \quad \text{and} \quad q_T\partial^2 T \in \mathcal{P}^{k+1}(\partial^2 T).
\]

For any \(F \in \mathcal{F}_T\) we let \(q_F := (q_F)_{\partial F}\) and, for any \(E \in \mathcal{E}_T\), \(q_E := (q_E)_{\partial E}\). The interpolator associated with this space is \(I_{\text{grad}, T}^k : C^0(\overline{T}) \to X^k_{\text{grad}, T}\) such that, for all \(q \in C^0(\overline{T})\),

\[
\begin{align*}
I_{\text{grad}, T}^k q &:= q_T, \\
I_{\text{grad}, T}^k q &= (q_T, q_T\partial T, q_T\partial^2 T) \in X^k_{\text{grad}, T} \quad \text{with } q_T = \pi_T^k q, \quad q_F = \pi_F^k q, \quad q_E = \pi_E^k q \quad \text{for all } F \in \mathcal{F}_T, \\
\pi_T^k q &\in \mathcal{P}^{k-1}(T), \\
\pi_F^k q &\in \mathcal{P}^{k-1}(\partial T), \\
\pi_E^k q &\in \mathcal{P}^{k-1}(\partial E).
\end{align*}
\]

Arguments similar to the two-dimensional case show that the component \(q_T\partial^2 T\) is well-defined by the conditions in the second line of (4.2).

We denote by \(X^k_{\text{grad}, \partial T}\) the restriction of \(X^k_{\text{grad}, T}\) to \(\partial T\), and the corresponding interpolator, with obvious definition, is denoted by \(I_{\text{grad}, \partial T}^k\). Similarly, \(X^k_{\text{rot}, \partial T}\) is the restriction of \(X^k_{\text{rot}, T}\) to \(\partial^2 T\). The restriction of \((q_T, q_T\partial T, q_T\partial^2 T) \in X^k_{\text{grad}, T}\) to \(X^k_{\text{grad}, \partial T}\) is \((q_T, q_T\partial T)\).

Remark 13 (Relation with two-dimensional spaces). The restriction of an element \(q_T \in X^k_{\text{grad}, T}\) to a face \(F \in \mathcal{F}_T\) defines an element \(q_F := (q_F, q_F\partial T)\) of \(X^k_{\text{grad}, F}\). Conversely, gluing together a family \((q_F)_{F \in \mathcal{F}_T}\) with \(q_F = (q_F, q_F\partial F) \in X^k_{\text{grad}, F}\) for all \(F \in \mathcal{F}_T\) defines an element of \(X^k_{\text{grad}, \partial T}\) provided that the edge values coincide: For any \(F_1, F_2\) faces of \(T\) sharing an edge \(E\), \((q_{\partial F_1})_E = (q_{\partial F_2})_E\).
4.1.2 Discrete $H(\text{curl}; T)$ space

The role of the space $H(\text{curl}; T)$ is played, at the discrete level, by

$$X^k_{\text{curl}, T} := \left( \mathcal{R}^{k-1}(T) \oplus \mathcal{R}^k(T) \right) \times \left( \bigotimes_{F \in \mathcal{T}_T} \left[ \mathcal{R}^{k-1}(F) \oplus \mathcal{R}^k(F) \right] \right) \times \mathcal{P}^k(\mathcal{E}_T).$$

(4.3)

A generic vector $v_T \in X^k_{\text{curl}, T}$ is denoted by

$$v_T = (v_T = v_{R,T} + v^\perp_{R,T}, (v_F = v_{R,F} + v^\perp_{R,F})_{F \in \mathcal{T}_T}, v_{\partial T})$$

with $(v_{R,T}, v^\perp_{R,T}) \in \mathcal{R}^{k-1}(T) \times \mathcal{R}^k(T)^\perp$, $(v_{R,F}, v^\perp_{R,F}) \in \mathcal{R}^{k-1}(F) \times \mathcal{R}^k(F)^\perp$ for all $F \in \mathcal{T}_T$, and $v_{\partial T} \in \mathcal{P}^k(\mathcal{E}_T)$. The interpolator $I^k_{\text{curl}, T} : C^0(\overline{T})^3 \to X^k_{\text{curl}, T}$ is such that, for all $v \in C^0(\overline{T})^3$,

$$I^k_{\text{curl}, T} v := \left( \pi_{\text{curl}, T}^k v + \pi_{\text{curl}, T}^\perp v, (\pi_{\text{curl}, F}^k(n_F \times (v_{F} \times n_F)) + \pi_{\text{curl}, F}^\perp(n_F \times (v_{F} \times n_F)))_{F \in \mathcal{T}_T}, (\pi_{\text{curl}, E}^k v \cdot t_E)_{E \in \mathcal{E}_T} \right).$$

(4.4)

We remind the reader that $n_F \times (v_{F} \times n_F)$ is the orthogonal projection of $v$ on the plane spanned by $F$. The restriction of $X^k_{\text{curl}, T}$ to the boundary of $T$ is

$$X^k_{\text{curl}, \partial T} := \left\{ v_{\partial T} := ((v_F = v_{R,F} + v^\perp_{R,F})_{F \in \mathcal{T}_T}, v_{\partial T}) : (v_{R,F}, v^\perp_{R,F}) \in \mathcal{R}^{k-1}(F) \times \mathcal{R}^k(F)^\perp \text{ for all } F \in \mathcal{T}_T \text{ and } v_{\partial T} \in \mathcal{P}^k(\mathcal{E}_T) \right\}.$$

Remark 14 (Relation with two-dimensional spaces). The restriction of an element $v_T \in X^k_{\text{curl}, T}$ to a face $F \in \mathcal{T}_T$ defines an element $v_F := (v_F, (v_{\partial F})_{|\partial F}) \in X^k_{\text{curl}, F}$. Conversely, gluing together a family $(v_F)_{F \in \mathcal{T}_T}$ with $v_F = (v_{F}, v_{\partial F}) \in X^k_{\text{curl}, F}$ for all $F \in \mathcal{T}_T$ defines an element of $X^k_{\text{curl}, \partial T}$ provided that the edge values coincide: For any $F_1, F_2$ faces of $T$ sharing an edge $E$, $(v_{\partial F_1})_E = (v_{\partial F_2})_E$.

4.1.3 Discrete $H(\text{div}; T)$ space

Finally, the discrete counterpart of the space $H(\text{div}; T)$ is

$$X^k_{\text{div}, T} := \left( \mathcal{G}^{k-1}(T) \oplus \mathcal{G}^k(T)^\perp \right) \times \mathcal{P}^k(\mathcal{T}_T),$$

(4.5)

with a generic vector $v_T \in X^k_{\text{div}, T}$ decomposed as

$$v_T := (v_T = v_{G,T} + v^\perp_{G,T}, v_{\partial T})$$

with $(v_{G,T}, v^\perp_{G,T}) \in \mathcal{G}^{k-1}(T) \times \mathcal{G}^k(T)^\perp$ and $v_{\partial T} \in \mathcal{P}^k(\mathcal{T}_T)$. The interpolator $I^k_{\text{div}, T} : C^0(\overline{T})^3 \to X^k_{\text{div}, T}$ is such that, for all $v \in C^0(\overline{T})^3$,

$$I^k_{\text{div}, T} v := \left( \pi_{\text{div}, T}^k v + \pi_{\text{div}, T}^\perp v, (\pi_{\text{div}, F}^k(v \cdot n_F))_{F \in \mathcal{T}_T} \right).$$

(4.6)
4.2 Three-dimensional vector operators reconstructions

4.2.1 Gradient

The three-dimensional full gradient operator $G_T^k : X_{\text{grad},T}^k \rightarrow \mathcal{P}^k(T)^3$ is defined such that, for all $q_T \in X_{\text{grad},T}^k$:

$$
\int_T G_T^k q_T \cdot v_T = - \int_T q_T \, \text{div} \, v_T + \sum_{F \in \mathcal{T}_T} \omega_{TF} \int_F \gamma_{F}^{k+1} (q_T \cdot n_F) \quad \forall v_T \in \mathcal{P}^k(T)^3,
$$

(4.7)

where $(\gamma_F^{k+1} : X_{\text{grad},\partial T}^k \rightarrow \mathcal{P}^{k+1}(F))_{F \in \mathcal{T}_T}$ is a family of trace reconstruction operators such that, for all $F \in \mathcal{T}_T$, $\gamma_F^{k+1}$ only depends on the unknowns on $F$ and satisfies the polynomial consistency and projection properties (3.32).

The discrete gradient operator $G_T^k : X_{\text{grad},T}^k \rightarrow X_{\text{grad},T}^k$ is defined by projecting $G_T^k$ onto $R^{k-1}(T) \oplus R^k(T)^+$. The discrete and full divergence operators are both equal to $D_T^k : X_{\text{grad},T}^k \rightarrow \mathcal{P}^k(T)$ defined such that, for all $v_T \in X_{\text{grad},T}^k$:

$$
\int_T G_T^k v_T \cdot w_T = \int_T v_T \cdot \text{curl} \, w_T + \sum_{F \in \mathcal{T}_T} \omega_{TF} \int_F \gamma_{F}^{k} v_T \cdot \omega_T \cdot (w_T \times n_F) \quad \forall w_T \in \mathcal{P}^k(T)^3,
$$

(4.9)

where $(\gamma_F^{k} : X_{\text{grad},\partial T}^k \rightarrow \mathcal{P}^k(F)^2)_{F \in \mathcal{T}_T}$ is the family of tangential trace reconstruction operators such that, for all $F \in \mathcal{T}_T$, $\gamma_F^{k}$ is formally defined by (3.33). The discrete curl operator $C_T^k : X_{\text{curl},T}^k \rightarrow X_{\text{div},T}^k$ is obtained projecting $C_T^k$ onto $G^{k-1}(T) \oplus G^k(T)^+$ and completing using the face curl operators: For all $v_T \in X_{\text{curl},T}^k$,

$$
C_T^k v_T := (C_{G,T}^{k-1} v_T + C_{G,T}^{\perp,k} v_T)_{F \in \mathcal{T}_T},
$$

(4.10)

where $C_{G,T}^{k-1} : X_{G,T}^k \rightarrow \mathcal{P}^k(T)$ is formally defined by (3.24).

4.2.2 Curl

The full curl operator $C_T^k : X_{\text{curl},T}^k \rightarrow \mathcal{P}^k(T)^3$ is defined such that, for all $v_T \in X_{\text{curl},T}^k$:

$$
\int_T C_T^k v_T \cdot w_T = \int_T v_T \cdot \text{curl} \, w_T + \sum_{F \in \mathcal{T}_T} \omega_{TF} \int_F \gamma_{F}^{k} (w_T \times v_T) - (w_T \times n_F) \quad \forall w_T \in \mathcal{P}^k(T)^3,
$$

(4.11)

where $\gamma_F^{k} : X_{\text{curl},\partial T}^k \rightarrow \mathcal{P}^k(F)^2$.

4.2.3 Divergence

The discrete and full divergence operators are both equal to $D_T^k : X_{\text{div},T}^k \rightarrow \mathcal{P}^k(T)$ defined such that, for all $v_T \in X_{\text{div},T}^k$:

$$
\int_T D_T^k v_T \cdot q_T = - \int_T v_T \cdot \text{grad} \, q_T + \sum_{F \in \mathcal{T}_T} \omega_{TF} \int_F v_T \cdot q_T \quad \forall q_T \in \mathcal{P}^k(T).
$$

(4.11)
Table 2: Number of discrete unknowns attached to each geometric entity for the three-dimensional sequence \((4.12)\) on a tetrahedron \(T\) for \(k \in \{0, \ldots, 3\}\). For comparison, we also report in parentheses the number of degrees of freedom of the corresponding spaces in the FE sequence \((P^{k+1}(T), N^k(T), RT^k(T), P^k(T))\).

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<th>(k)</th>
<th>(V)</th>
<th>(E)</th>
<th>(F)</th>
<th>(T)</th>
<th>Total</th>
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4.3 Exactness of the three-dimensional sequence

The goal of this section is to prove the following exactness result.

**Theorem 15** (Exact three-dimensional sequence). The following sequence is exact:

\[ \mathbb{R} \xrightarrow{\nabla^k_{\text{grad},T}} X^k_{\text{grad},T} \xrightarrow{\nabla^k_{\text{curl},T}} X^k_{\text{curl},T} \xrightarrow{\nabla^k_{\text{div},T}} X^k_{\text{div},T} \xrightarrow{\nabla^k_{\text{div},T}} P^k(T) \xrightarrow{0} \{0\}. \]  

**Remark 16** (Comparison with Finite Elements). When \(T\) is a tetrahedron, the sequence \((4.12)\) can be compared with the usual FE sequence \((P^{k+1}(T), N^k(T), RT^k(T), P^k(T))\) (with \(N^k(T)\) and \(RT^k(T)\) denoting the usual three-dimensional edge and face spaces of \([43]\), whose definitions are respectively recalled in \((4.18)\) and \((4.40)\) below). The corresponding number of discrete unknowns is reported in Table 2. Similar considerations as for the two-dimensional case hold; see Remark 8.

4.3.1 Preliminary results

We establish first a few properties on the discrete operators that will be useful during the proof of Theorem 15. We start by noticing the following relations:

\[ \sum_{F \in T} \sum_{E \in F} \omega_{TE} \omega_{FE} a_E = 0 \quad \forall (a_E)_{E \in E_T} \in \mathbb{R}^{E_T}, \]  

\[ (z \times n_F) \cdot n_{FE} = z \cdot t_E \quad \forall z \in \mathbb{R}^3, \forall F \in T_T, \forall E \in E_F. \]

Relation \((4.13)\) follows from the fact that, after rearranging the sum over the edges, each \(a_E\) appears in factor of the quantity in the left-hand side of \((2.1)\). Equation \((4.14)\) is a direct consequence of \((z \times n_F) \cdot n_{FE} = -z \cdot (n_F \times n_{FE})\) together with the fact that \((t_E, n_{FE}, n_F)\) is a right-handed system of coordinates.
Lemma 17 (Properties of the gradient operator). It holds:

\[ \mathbf{G}^k_T(I^k_{\text{grad},T} \mathbf{q}) = \text{grad } \mathbf{q} \quad \forall \mathbf{q} \in P^{k+1}(T), \]  
and

\[ \int_T \mathbf{G}^{k-1}_{R,T} \mathbf{q}_T \cdot \text{curl } \mathbf{w}_T = \int_T \mathbf{G}^k_T \mathbf{q}_T \cdot \text{curl } \mathbf{w}_T = - \sum_{F \in T_T} \omega_T F \int_F \mathbf{G}^k_T \mathbf{q}_F \cdot (\mathbf{w}_T \times \mathbf{n}_F) \quad \forall \mathbf{q}_T \in X^{k}_{\text{grad},T}; \forall \mathbf{w}_T \in P^k(T)^3. \]  

Remark 18 (Properties (4.15)-(4.16)). The relation (4.15) states a consistency property on the full gradient reconstruction, while (4.16) provides a link between volume and face gradients.

Proof of Lemma 17. 1. Proof of (4.15). Let \( \mathbf{q} \in P^{k+1}(T) \) and apply the definition (4.7) of \( \mathbf{G}^k_T \) to \( q_T := I^k_{\text{grad},T} \mathbf{q} \). Using the definition (4.2) of \( I^k_{\text{grad},T} \) together with the consistency (3.32a) of each \( \gamma_f^{k+1} \), this gives, for all \( \mathbf{v}_T \in P^k(T)^3 \),

\[ \int_T \mathbf{G}^k_T(I^k_{\text{grad},T} \mathbf{q}) \cdot \mathbf{v}_T = - \int_T \pi_{P,f,T}^{-1} \mathbf{q} \text{ div } \mathbf{v}_T + \sum_{F \in T_T} \omega_T F \int_F q_T \cdot \mathbf{n}_F = \int_T \text{grad } q \cdot \mathbf{v}_T, \]

where the second equality is obtained removing the projector \( \pi_{P,f,T}^{-1} \) (since \( \text{div } \mathbf{v}_T \in P^{k-1}(T) \)) and using the integration by parts formula (2.11). Since both \( \mathbf{G}^k_T(I^k_{\text{grad},T} \mathbf{q}) \) and \( \text{grad } q \) belong to \( P^k(T)^3 \), this relation establishes (4.15).

2. Proof of (4.16). The first equality is a straightforward consequence of \( \mathbf{G}^{k-1}_{R,T} = \pi_{P,f,T}^{-1} \mathbf{G}^k_T \) (see (4.8)) and of \( \text{curl } \mathbf{w}_T \in R^{k-1}(T) \). To prove the second equality, apply the definition (4.7) of \( \mathbf{G}^k_T \) to \( \mathbf{v}_T = \text{curl } \mathbf{w}_T \in P^{k-1}(T)^3 \) and introduce, using its definition, the projector \( \pi_{P,f}^{-1} \) in each boundary integral to get

\[ \int_T \mathbf{G}^k_T(I^k_{\text{grad},T} \mathbf{q}) \cdot \text{curl } \mathbf{w}_T = - \int_T q_T \text{ div } \text{curl } \mathbf{w}_T + \sum_{F \in T_T} \omega_T F \int_F \pi_{P,f}^{-1} \mathbf{q}_F \cdot \gamma_f^{k+1} \mathbf{q}_F \cdot \mathbf{curl } \mathbf{w}_T \cdot \mathbf{n}_F. \]

Using the projection property (3.32b) of \( \gamma_f^{k+1} \) and the identity (2.7), we infer

\[ \int_T \mathbf{G}^k_T \mathbf{q}_T \cdot \text{curl } \mathbf{w}_T = \sum_{F \in T_T} \omega_T F \int_F q_F \text{ div}_F (\mathbf{w}_T \times \mathbf{n}_F) \]

\[ = - \sum_{F \in T_T} \omega_T F \int_F \mathbf{G}^k_T \mathbf{q}_F \cdot (\mathbf{w}_T \times \mathbf{n}_F) + \sum_{F \in T_T} \sum_{E \in E_F} \omega_T F \omega_{E_F} \int_E q_E (\mathbf{w}_T \times \mathbf{n}_F) \cdot \mathbf{n}_{FE}, \]  

where the second equality follows applying the definition (3.12) of \( \mathbf{G}^k_T \) to \( \mathbf{w}_F := \mathbf{w}_T \times \mathbf{n}_F \). Using (4.14) we have \( (\mathbf{w}_T \times \mathbf{n}_F) \cdot \mathbf{n}_{FE} = \mathbf{w}_T \cdot \mathbf{t}_E \) on each \( E \in E_T \), and the double sum in (4.17) therefore vanishes owing to (4.13) with \( \mathbf{a}_E = \int_E q_E (\mathbf{w}_T \cdot \mathbf{t}_E) \). The proof of the second equality in (4.16) is complete. \( \square \)

The following Nédélec space, in which \( \bar{x}_T := \frac{1}{|T|} \int_T x \, dx \) denotes the centroid of \( T \), will be useful to formulate a commutation property for \( \mathbf{C}^k_T \):

\[ \mathcal{N}^k(T) := P^k(T)^3 + (x - \bar{x}_T) \times P^k(T)^3. \]  

(4.18)
Lemma 19 (Properties of the curl operator). It holds:

\[
\mathbf{C}_T^k (I_{\text{curl}, T} v) = \text{curl } v \quad \forall v \in \mathcal{N}^k(T), \tag{4.19}
\]

\[
\int_T \mathbf{C}_T^{k-1} \mathbf{y}_T^* \cdot w_T = \sum_{F \in \mathcal{T}_T} \omega_T \int_F v_T \cdot (w_T \times n_F) \quad \forall v_T \in X_{\text{curl}, T}, \forall w_T \in G^{k-1}(T), \tag{4.20}
\]

\[
\int_T \mathbf{C}_T^k \mathbf{y}_T^* \cdot \text{grad } r_T = \sum_{F \in \mathcal{T}_T} \omega_T \int_F \mathbf{C}_T^k \mathbf{y}_{\partial T} \cdot r_T \quad \forall v_T \in X_{\text{curl}, T}, \forall r_T \in \mathcal{P}^{k+1}(T), \tag{4.21}
\]

\[
\int_T \mathbf{C}_T^{k-1} \mathbf{y}_T^* \cdot \text{grad } r_T = \sum_{F \in \mathcal{T}_T} \omega_T \int_F \mathbf{C}_T^{k-1} \mathbf{y}_{\partial T} \cdot r_T \quad \forall v_T \in X_{\text{curl}, T}, \forall r_T \in \mathcal{P}^k(T). \tag{4.22}
\]

Remark 20 (Properties \[4.19\]–\[4.22\]). Equation \[4.19\] states a polynomial consistency property for \( \mathbf{C}_T^k \), \( \mathbf{C}_T^{k-1} \) is given on each face \( F \), \( \omega_T \), and the cancellation in the second line being justified by the fact that \( x - \bar{x}_E \parallel t_E \). Hence, applying the definition \( \pi_{R,F}^k (n_F \times (v_F \times n_F)) \), we infer that

\[
\gamma_{1,F}^k \mathbf{y}_{\partial T} = \pi_{R,F}^k (n_F \times (v_F \times n_F)) \quad \forall F \in \mathcal{T}_T. \tag{4.23}
\]

By definition of \( I_{\text{curl}, T} \), we have \( v_T = \pi_{R,F}^{k-1} v + \pi_{R,F}^{1,k} v \), and thus, for any \( w_T \in \mathcal{P}^k(T)^3 \),

\[
\int_T v_T \cdot \text{curl } w_T = \int_T (\pi_{R,F}^{k-1} v + \pi_{R,F}^{1,k} v) \cdot \text{curl } w_T = \int_T v \cdot \text{curl } w_T,
\]

where we have removed \( \pi_{R,F}^{k-1} \) using its definition, and \( \pi_{R,F}^{1,k} \) using its \( L^2 \)-orthogonality to \( \text{curl } w_T \in \mathcal{P}^{k-1}(T) \subset \mathcal{P}^k(T) \). Hence, applying the definition \( \mathbf{C}_T^k \) and using \( \mathbf{C}_T^{k-1} \), we obtain, for all \( w_T \in \mathcal{P}^k(T)^3 \),

\[
\int_T \mathbf{C}_T^k (I_{\text{curl}, T} v) \cdot w_T = \int_T v \cdot \text{curl } w_T + \sum_{F \in \mathcal{T}_T} \omega_T \int_F \pi_{R,F}^k (n_F \times (v_F \times n_F)) \cdot (w_T \times n_F)
\]

\[
= \int_T v \cdot \text{curl } w_T + \sum_{F \in \mathcal{T}_T} \omega_T \int_F (n_F \times (v_F \times n_F)) \cdot (w_T \times n_F)
\]

\[
= \int_T v \cdot \text{curl } w_T,
\]

where the second line follows from \( w_T|_F \times n_F \in \mathcal{P}^k(F)^3 \), and the conclusion is a consequence of the integration by parts formula \[2.14\]. Since both \( \mathbf{C}_T^k (I_{\text{curl}, T} v) \) and \( \text{curl } v \) belong to \( \mathcal{P}^k(T)^3 \), this concludes
2. Proof of (4.20). Recalling that \( C^{k-1}_{G,T} = \pi^{k-1}_{G,T} C^k_T \) (see (4.10)) and using the definition (4.9) of \( C^k_T \), we infer, for all \( w_T \in G^{k-1}(T) \),

\[
\int_T C^{k-1}_{G,T} w_T \cdot w_T = \int_F C^k_T \nu_T \cdot w_T = \int_T \nu_T \cdot \text{curl } w_T + \sum_{F \in \mathcal{T}} \omega_{TF} \int_F \gamma^k_{\nu_T} \nu_{\partial T} \cdot (w_T \times n_F),
\]

the cancellation coming from the vector calculus identity \( \text{curl } \text{grad} = 0 \). Let now \( r_T \in \mathcal{P}^k(T) \) be such that \( w_T = \text{grad } r_T \). Then the identity (2.6) yields \( w_T \times n_F = \text{rot}_F r_T|_F \), and thus

\[
\int_T C^{k-1}_{G,T} w_T \cdot w_T = \sum_{F \in \mathcal{T}} \omega_{TF} \int_F \gamma^k_{\nu_T} \nu_{\partial T} \cdot \text{rot}_F r_T|_F
\]

where the second line follows from the definition (3.33a) of \( \gamma^k_{\nu_T} \) applied to \( r_T = r_T|_F \in \mathcal{P}^k(F) \subset \mathcal{P}^{k+1}(F) \) (see also Remark 10) and the third line is a consequence of the definition (3.24) of \( C^k_T \) with the same \( r_T \), together with the definition \( v_F = v_{R,F} + v_{\perp R,F} \) and the \( L^2 \)-orthogonality of \( v_{\perp R,F} \) and \( \text{rot}_F r_T \). The relation (4.20) follows by recalling that \( \text{rot}_F r_T|_F = w_T \times n_F \).

3. Proof of (4.21)–(4.22). Let \( v_T \in X^k_{\text{curl},T} \) and \( r_T \in \mathcal{P}^{k+1}(T) \). Writing the definition (4.9) of \( C^k_T \) with \( w_T = \text{grad } r_T \), observing that \( \text{curl } \text{grad} = 0 \), and using the identity (2.6), we infer

\[
\int_T C^k_T v_T \cdot \text{grad } r_T = \sum_{F \in \mathcal{T}} \omega_{TF} \int_F \gamma^k_{\nu_T} \nu_{\partial T} \cdot \text{rot}_F r_T|_F.
\]

We continue using, for all \( F \in \mathcal{T} \), the definition (3.33a) of the tangential trace reconstruction with \( r_F = r_T|_F \in \mathcal{P}^{k+1}(F) \) (see also Remark 10) to write

\[
\int_F C^k_T v_T \cdot \text{grad } r_T = \sum_{F \in \mathcal{T}} \omega_{TF} \int_F C^k_T \nu_{\partial T} r_T + \sum_{F \in \mathcal{T}} \omega_{TF} \sum_{E \in \mathcal{E}_F} \omega_{TE} \int_E v_{E,F},
\]

where, to cancel the rightmost term, we have used (4.13) with \( \omega_E = \int_E v_{E,F} \). This proves (4.21). The relation (4.22) is obtained applying (4.21) with \( r_T \in \mathcal{P}^k(T) \) and using \( C^{k-1}_{G,T} = \pi^{k-1}_{G,T} C^k_T \). \( \square \)

**Lemma 21** (Surjectivity of the boundary curl). Let \( v_F = (v_F)_F \in \mathcal{P}^k(\mathcal{T}) \) be such that

\[
\sum_{F \in \mathcal{T}} \omega_{TF} \int_F v_F = 0.
\]

Then, there exists \( z_{\partial T} \in X^k_{\text{curl},\partial T} \) such that \( C^k_T z_{\partial T} = v_F \) for all \( F \in \mathcal{T} \).

**Remark 22.** The condition (4.24) is also necessary for the conclusion of the lemma to hold.

**Proof of Lemma 22**. By the exactness in two dimensions (cf. Theorem 7), for each \( F \in \mathcal{T} \) there is \( w_F \in X^k_{\text{curl},F} \) such that \( C^k_T w_F = v_F \). Following Remark 14, a vector \( z_{\partial T} \in X^k_{\text{curl},\partial T} \) can be defined by gluing together the vectors \( (w_F)_F \) if their edge values coincide. The idea of the proof is to add to
each \( \mathbf{w}_F \) a vector \( \mathbf{y}_F \in X_{\text{rot},F}^k \) such that \( C_F^k(\mathbf{w}_F + \mathbf{y}_F) = \mathbf{v}_F \) and the edge values of \( \mathbf{w}_F + \mathbf{y}_F \) coincide; by gluing these vectors together, we obtain \( \mathbf{z}_{\partial T} \in X_{\text{curl,}\partial T}^k \) such that \( C_F^k \mathbf{z}_{\partial T} = \mathbf{v}_F \) for all \( F \in \mathcal{T}_T \).

To ensure the relation \( C_F^k(\mathbf{w}_F + \mathbf{y}_F) = \mathbf{v}_F \), we have to look for \( \mathbf{y}_F \) in \( \text{Ker} \, C_F^k \), that is, owing to Theorem 7, we have to look for \( \mathbf{y}_F = G_F^k \mathbf{q}_F \) for some \( \mathbf{q}_F \in \mathcal{X}_{\text{grad},F}^k \). The other condition on \( \mathbf{y}_F \) only concerns its boundary values \( \mathbf{y}_{\partial F} = G_{\partial F}^k \mathbf{q}_{\partial F} \) (which means that the component \( q_F \) of \( \mathbf{q}_F \) is irrelevant to our purpose and can be set to 0), and is written:

\[
(\mathbf{w}_F)_E + (\mathbf{y}_{\partial F})_E = (\mathbf{w}_F')_E + (\mathbf{y}_{\partial F})_E \quad \forall E \in \mathcal{E}_T \text{ with } \mathcal{T}_E = \{F_1, F_2\},
\]

where \( \mathcal{T}_E \) denotes the set containing the two faces of \( T \) that share \( E \). By Proposition 2, in order for \( \mathbf{y}_{\partial F} \in \mathcal{P}^k(\mathcal{E}_F) \) to be represented as \( G_{\partial F}^k \mathbf{q}_{\partial F} \), it needs to verify

\[
\sum_{E \in \mathcal{E}_F} \omega_{EF} \int_E \mathbf{y}_{\partial F} = 0.
\]

We are therefore reduced to finding, for each \( F \in \mathcal{T}_T \), \( \mathbf{y}_{\partial F} \in \mathcal{P}^k(\mathcal{E}_F) \) such that (4.25) and (4.26) hold.

Let us set, for \( E \in \mathcal{E}_F \) with \( \mathcal{T}_E = \{F_1, F_2\}, r_E := \frac{1}{2} ((\mathbf{y}_{\partial F})_E + (\mathbf{y}_{\partial F})_E) \). We also set, for \( F \in \mathcal{T}_T \) and \( E \in \mathcal{E}_F \), \( \mathbf{W}_{FE} := \frac{1}{2} (\mathbf{w}_F)_E - (\mathbf{w}_F)_E \), where \( F' \) is the other face of \( T \) that shares \( E \) with \( F \). Then, (4.25) is equivalent to

\[
(\mathbf{y}_{\partial F})_E = r_E + \mathbf{W}_{FE} \quad \forall F \in \mathcal{T}_T, \forall E \in \mathcal{E}_F.
\]

Using this expression, we obtain the following equivalent reformulation of (4.26):

\[
\sum_{E \in \mathcal{E}_F} \omega_{EF} \int_E r_E = - \sum_{E \in \mathcal{E}_F} \omega_{EF} \int_E \mathbf{W}_{FE}.
\]

We thus have to find \( (r_E)_E \in \mathcal{E}_F \) such that each \( r_E \) belongs to \( \mathcal{P}^k(E) \) and (4.28) holds for all \( F \in \mathcal{T}_T \). Defining then \( (\mathbf{y}_{\partial F})_E \in \mathcal{E}_F \) by (4.27), we obtain a family that satisfies (4.25) and (4.26) for all \( F \in \mathcal{T}_T \). The relation (4.28) only involves the integral of \( r_E \) over the constant polynomials \( r_E \in \mathbb{R} \), in which case (4.28) is recast, after multiplying by \( \omega_{TF} \), as

\[
\omega_{TF} \sum_{E \in \mathcal{E}_F} \omega_{EF} |E|r_E = - \omega_{TF} \sum_{E \in \mathcal{E}_F} \omega_{EF} \int_E \mathbf{W}_{FE} \quad \forall F \in \mathcal{T}_T.
\]

This is a linear system of size \( \text{card}(\mathcal{T}_T) \times \text{card}(\mathcal{E}_T) \) in the unknowns \( (r_E)_E \in \mathcal{E}_F \in \mathbb{R}^\mathcal{E}_T \). Denoting by \( A \) its matrix, this system has a solution if and only if its right-hand side belongs to \( \text{Im} \, A = (\ker A')^\perp \), with \( A' \) denoting the transpose of \( A \) through the standard dot products of \( \mathbb{R}^\mathcal{T}_T \) and \( \mathbb{R}^\mathcal{E}_T \). It is easy to check that \( A' \) corresponds to the mapping

\[
\mathbb{R}^\mathcal{T}_T \ni (\xi_F)_{F \in \mathcal{T}_T} \mapsto \left( \sum_{F \in \mathcal{T}_E} \omega_{TF} \xi_F \right)_{E \in \mathcal{E}_T} \in \mathbb{R}^\mathcal{E}_T.
\]

Invoking (2.1) we see that \( (\xi_F)_{F \in \mathcal{T}_T} \in \ker A' \) if and only if \( \xi_{F_1} = \xi_{F_2} \) whenever \( F_1, F_2 \in \mathcal{T}_T \) share a common edge. Working from neighbouring face to neighbouring face, this shows that the vectors in \( \ker A' \) are those with all components equal. Hence, the right-hand side of (4.29) belongs to \( (\ker A')^\perp \) if and only if it is orthogonal to the vector with all components equal to 1, which translates into

\[
0 = - \sum_{F \in \mathcal{T}_T} \sum_{E \in \mathcal{E}_F} \omega_{TF} \omega_{EF} \int_E \mathbf{W}_{FE}.
\]
Proof of Theorem 15. \[ q \] all where the conclusion follows from the integration by parts formula (2.11). Since this relation holds for \( v \), we see that \( \int_T v \cdot \text{grad} q_T = \int_T v \cdot \text{grad} q_T \). Since \( v_F = \pi_{T,F}^k (v \cdot n_F) \) and \((q_T)_F \in \mathcal{P}^k(F)\) for all \( F \in \mathcal{T}_T \), the definition (4.11) of \( D^k_T \) therefore shows that
\[
\int_T D^k_T(I^k_{\text{div}, T} v) q_T = - \int_T v \cdot \text{grad} q_T + \sum_{F \in \mathcal{T}_T} \omega_{TF} \int_F (v \cdot n_F) q_T = \int_T \text{div} v q_T,
\]
where the conclusion follows from the integration by parts formula (2.11). Since this relation holds for all \( q_T \in \mathcal{P}^k(T) \), it proves (4.31).

**Lemma 23** (Commutation property for \( D^k_T \)). It holds
\[
D^k_T(I^k_{\text{div}, T} v) = \pi_{\text{grad}, T}^k (\text{div} v) \quad \forall v \in H^1(T)^3.
\] (4.31)

**Proof.** Setting \( v_F := I_{\text{div}, T}^k v \), we have \( v_F = \pi_{1,T}^{k-1} v + \pi_{0,T}^{1,k} v \) and thus, for all \( q_T \in \mathcal{P}^k(T) \), by definition of these projectors and since \( \text{grad} q_T \in \mathcal{G}^{k-1}(T) \),
\[
\int_T v_T \cdot \text{grad} q_T = \int_T v \cdot \text{grad} q_T.
\]
Since \( v_F = \pi_{T,F}^k (v \cdot n_F) \) and \((q_T)_F \in \mathcal{P}^k(F)\) for all \( F \in \mathcal{T}_T \), the definition (4.11) of \( D^k_T \) therefore shows that
\[
\int_T D^k_T(I^k_{\text{div}, T} v) q_T = - \int_T v \cdot \text{grad} q_T + \sum_{F \in \mathcal{T}_T} \omega_{TF} \int_F (v \cdot n_F) q_T = \int_T \text{div} v q_T,
\]
where the conclusion follows from the integration by parts formula (2.11). Since this relation holds for all \( q_T \in \mathcal{P}^k(T) \), it proves (4.31).

**4.3.2 Proof of the exactness of the three-dimensional sequence**

**Proof of Theorem 15** We have to prove that
\[
\text{Ker} G^k_T = \text{Im} G^k_T, \quad (4.32)
\]
\[
\text{Ker} C^k_T = \text{Im} G^k_T, \quad (4.33)
\]
\[
\text{Ker} D^k_T = \text{Im} C^k_T, \quad (4.34)
\]
\[
\text{Im} D^k_T = \mathcal{P}^k(T). \quad (4.35)
\]
1. Proof of (4.32). By the consistency properties of the boundary and volume gradients (see (3.6), (3.5) and (4.15)), it holds that \( G^k_T(\nabla_{\text{grad},\partial T} C) = 0 \), \( G^k_F(\nabla_{\text{grad},\partial T} C) = 0 \), and \( G^k_T(\nabla_{\text{grad},T} C) = 0 \) for all \( C \in \mathbb{R} \), proving by definition (4.8) of \( G^k_T \) that

\[
I^k_{\text{grad},T} \mathbb{R} \subset \text{Ker} G^k_T.
\]

Let us prove the converse inclusion, i.e.,

\[
\text{Ker} G^k_T \subset I^k_{\text{grad},T} \mathbb{R}. \tag{4.36}
\]

Let \( q_T \in X^k_{\text{grad},T} \) be such that \( G^k_T q_T = 0 \). Then, \( G^k_F q_{\partial T} = 0 \) for all \( F \in \mathcal{T}_T \) and thus, recalling the two-dimensional exactness proved in Theorem 7 and accounting for the single-valuedness of vertex and edge unknowns, there exists \( C \in \mathbb{R} \) such that \( q_{\partial T} = I^k_{\text{grad},\partial T} C \). Therefore, it only remains to prove that \( q_T = C \). Enforcing \( G^k_T q_T = 0 \) and using (4.8) together with the definition (4.7) of \( G^k_T \) and the fact that \( \gamma^k_F q_{\partial T} = \gamma^k_F(I^k_{\text{grad},\partial T} C) = C \) by the polynomial consistency (3.32a) of this trace reconstruction operator, we infer, for all \( w_T \in \mathcal{R}^k(T) \),

\[
0 = \int_T G^k_T q_T \cdot w_T = \int_T G^k_F q_{\partial T} \cdot w_T = - \int_T q_T \text{div} w_T + \sum_{F \in T} \omega_T F \int_T C(w_T \cdot n_F) = \int_T (q_T - C) \text{div} w_T,
\]

where we have used the integration by parts formula (2.11) with \( C \) instead of \( q_T \) to conclude. Since \( \text{div} : \mathcal{R}^k(T) \longrightarrow \mathcal{P}^{k-1}(T) \) is surjective by (2.9) and \( q_T \in \mathcal{P}^{k-1}(T) \), this implies \( q_T = \pi^k_{\mathcal{P},T} C \), thus proving (4.36).

2. Proof of (4.33). We start by proving that

\[
\text{Im} G^k_T \subset \text{Ker} C^k_{G,T}, \tag{4.37}
\]

that is \( C^k_{G,T}(G^k_T q_T) = 0 \) for all \( q_T \in X^k_{\text{grad},T} \). Theorem 7 implies \( C^k_F(G^k_T q_T) = 0 \) for all \( F \in \mathcal{T}_T \). Let us prove that \( C^{k-1}_{G,T}(G^k_T q_T) = 0 \). From the characterisation (4.20) of \( C^{k-1}_{G,T} \) we infer, for all \( w_T \in \mathcal{G}^{k-1}(T) \),

\[
\int_T C^{k-1}_{G,T}(G^k_T q_T) \cdot w_T = \sum_{F \in T} \omega_T F \int_F (G^{k-1}_{\mathcal{R},F} q_T + G^{1,k}_{\mathcal{R},F} q_T) \cdot (w_T \times n_F).
\]

Since \( w_T \in \mathcal{G}^k(T) \), the relation (2.6) shows that \( w_T|F \times n_F \in \mathcal{R}^k(F) \subset \mathcal{R}^k(F) \) and thus, using \( G^{1,k}_{\mathcal{R},F} q_T \in \mathcal{R}^k(F) \) and the relation (3.23a) with \( w_F = w_T|F \times n_F \), we continue with

\[
\int_T C^{k-1}_{G,T}(G^k_T q_T) \cdot w_T = \sum_{F \in T} \omega_T F \int_F G^{k-1}_{\mathcal{R},F} q_T \cdot (w_T \times n_F)
\]

\[
= \sum_{F \in T} \omega_T F \sum_{E \in \mathcal{E}_T} \omega_{FE} \int_E q_E(w_T \times n_F) \cdot n_{FE}
\]

\[
= \sum_{F \in T} \omega_T F \sum_{E \in \mathcal{E}_T} \omega_{FE} \int_E q_E(w_T \cdot t_E) = 0,
\]

where we have used (4.14) to pass to the third line and (4.13) with \( a_E = \int_E q_E(w_T \cdot t_E) \) to conclude. This implies \( C^{k-1}_{G,T}(G^k_T q_T) = 0 \).
We next notice that it holds, for all $w_T \in G^k(T)^+$,

$$
\int_T C_{G,T}^k (G_{R,T}^k q_T^k) \cdot w_T = \int_T (G_{R,T}^{k-1} q_T^k + G_{R,T}^{k} q_T^k) \cdot \text{curl} \ w_T + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \gamma_{F,T}^k (G_{F}^k q_{\partial F}) \cdot (w_T \times n_F) = \int_T G_{R,T}^{k-1} q_T^k \cdot \text{curl} \ w_T + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F G_{F}^k q_{\partial F} \cdot (w_T \times n_F) = 0,
$$

where we have used the definition (4.39) of $C_{G}^k$ together with $C_{G,T}^k = \pi_{R,T}^{1,k} C_{G}^k$ (cf. (4.10)) in the first line, the relation $G_{R,T}^{k} = \pi_{R,T}^{1,k} G_{G,T}^k$ (cf. (4.8)) together with $\text{curl} \ w_T \in \mathcal{R}^{k-1}(T)$ and the property (3.35) of the tangential trace reconstruction in the second line, and the link (4.16) between volume and face gradients to conclude. This proves (4.37).

We next prove the converse inclusion, that is,

$$\text{Ker} \ C_{G}^k \subset \text{Im} \ G_{R,T}^{k}. \quad (4.38)$$

This requires to show that, for all $v_T \in X_{\text{curl},T}$ such that $C_{G}^k v_T = 0$, there exists $q_T \in X_{\text{grad},T}$ such that $v_T = C_{G}^k q_T$. For all $F \in \mathcal{F}_T$, enforcing $C_{G}^k v_T = 0$ in (3.24) and taking $r_F = 1$, we see that $\sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E v_E = 0$. Hence, Proposition 2 provides $q_{\partial F} \in P_c^{k+1}(\partial F)$ such that $v_E = G_{E}^k q_{\partial F}$ for all $E \in \mathcal{E}_F$. Each function $q_{\partial F}$ for $F \in \mathcal{F}_T$ is defined up to an additive constant which, by the single-valuedness of $(v_E)_{E \in \mathcal{E}_T}$ across the faces, can be selected so as to form a continuous function $q_{\partial T} \in P_c^{k+1}(\partial T)$ defined on the whole $\partial T$, and such that $v_E = G_{E}^k q_{\partial F}$ for all $E \in \mathcal{E}_T$.

We next proceed as in Point 3 of the proof of Proposition 6, first, to infer that, for any choice of $q_{\partial T} = (q_F)_{F \in \mathcal{F}_T} \in P^{k-1}(\partial T)$, letting $q_{\partial T} := (q_T, q_{\partial T})$, it holds $v_{R,F} = G_{R,F}^{k-1} q_{\partial T}$; then, to select a proper $q_{\partial T}$ such that $v_{R,F} = G_{R,F}^{k-1} q_{\partial T}$ for all $F \in \mathcal{F}_T$. This proves that $v_T = G_{R,F}^{k-1} q_{\partial T}$ for all $F \in \mathcal{F}_T$.

Let us now show that, for any $q_T \in P^{k-1}(T)$, setting $q_{\partial T} := (q_T, q_{\partial T}) \in X_{\text{grad},T}$, it holds $v_{R,T} = G_{R,T}^{k-1} q_{\partial T}$. Applying the definition (4.9) of $C_{G}^k$ to an arbitrary $w_T \in G^k(T)^+$ and enforcing $\theta = C_{G,T}^k v_T = \pi_{G,T}^{1,k} (G_{G,T}^k q_T)$, we get

$$
\int_T v_T \cdot \text{curl} \ w_T = - \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \gamma_{F,T}^k v_{\partial T} \cdot (w_T \times n_F) = - \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \gamma_{F,T}^k G_{F}^k q_{\partial F} \cdot (w_T \times n_F) = - \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F G_{F}^k q_{\partial F} \cdot (w_T \times n_F) = \int_T G_{R,T}^{k-1} q_T^k \cdot \text{curl} \ w_T,
$$

where we have used the property (3.35) of the tangential trace reconstruction to pass to the third line and the link (4.16) between face and volume gradients to conclude. By (2.10), $\text{curl} \ w_T$ spans $\mathcal{R}^{k-1}(T)$ when $w_T$ spans $G^k(T)^+$, and we therefore deduce that $G_{R,T}^{k-1} q_T^k = \pi_{R,T}^{k-1} v_T = v_{R,T}$ as desired.

It only remains to prove the existence of $q_T \in P^{k-1}(T)$ such that $v_{R,T} = G_{R,T}^{k-1} q_T^k$. Recalling that $G_{R,T}^{k-1} q_T^k = \pi_{R,T}^{1,k} (G_{G,T}^k q_T^k)$ (see (4.8)) and applying the definition (4.7) of $G_{G,T}^k q_T^k$ to an arbitrary test function $w_T \in \mathcal{R}^{k}(T)^+$, this requires the following condition to hold:

$$
\int_T q_T \text{ div} \ w_T = - \int_T v_{R,T}^k \cdot w_T + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \gamma_{F,T}^{k+1} q_{\partial T} \cdot (w_T \times n_F) \quad \forall w_T \in \mathcal{R}^{k}(T)^+,
$$

26
which appropriately defines \( q_T \) since \( \text{div} : \mathcal{R}^k(T)^\perp \to \mathcal{P}^{k-1}(T) \) is an isomorphism by (2.9). This concludes the proof of (4.38).

3. Proof of (4.34). Let us start by proving that \( D^k_T(C^k_T \nu_T) = 0 \) for all \( \nu_T \in X^k_{\text{curl}, T} \), which implies \( \text{Im} \ C^k_T \subset \text{Ker}(D^k_T) \).

For all \( q_T \in \mathcal{P}^k(T) \), we have

\[
\int_T D^k_T(C^k_T \nu_T) q_T = - \int_T (C^{k-1}_{G, T} \eta_T + C^k_{G, T} \nu_T) \cdot \text{grad} q_T + \sum_{F \in \mathcal{T}_T} \omega_T F \int_F C^k_T \nu_T q_T = 0,
\]

where we have used the definition (4.11) of \( D^k_T \) in the first equality, the \( L^2 \)-orthogonality of \( C^k_{G, T} q_T \) and \( \text{grad} q_T \in (\mathcal{P}^{k-1}(T) \in \text{the cancellation, and we have concluded using the link (4.22) between volume and face curls. Since } q_T \text{ is arbitrary in } \mathcal{P}^k(T), \text{this shows that } D^k_T(C^k_T \nu_T) = 0.\)

Let us now prove the inclusion

\[
\text{Ker}(D^k_T) \subset \text{Im} \ C^k_T. \tag{4.39}
\]

We fix an element \( \nu_T \in X^k_{\text{div}, T} \) such that \( D^k_T \nu_T = 0 \) and prove the existence of \( z_T \in X^k_{\text{curl}, T} \) such that \( \nu_T = C^k_T z_T \). Enforcing \( D^k_T \nu_T = 0 \) in (4.11) with \( q_T = 1 \), we infer that \( \sum_{F \in \mathcal{T}_T} \omega_T F \int_F v_T = 0 \). Lemma 21 then provides \( z_{\partial T} \in X^k_{\text{curl}, \partial T} \) such that \( v_T = C^k_T z_{\partial T} \) for all \( F \in \mathcal{T}_T \).

Enforcing again \( D^k_T \nu_T = 0 \) in (4.11), this time for a generic test function \( q_T \in \mathcal{P}^k(T) \), and accounting for the previous result, we can write, for all \( z_T \in X^k_{\text{curl}, T} \) whose boundary values are given by \( z_{\partial T} \).

\[
\int_T v_T \cdot \text{grad} q_T = \sum_{F \in \mathcal{T}_T} \omega_T F \int_F C^k_F z_{\partial T} q_T = \int_T C^{k-1}_{G, T} z_T \cdot \text{grad} q_T,
\]

where the conclusion follows from the relation (4.22) linking volume and face curls. Since \( \text{grad} q_T \) spans \( \mathcal{G}^{k-1}(T) \) as \( q_T \) spans \( \mathcal{P}^k(T) \), this proves that \( C^{k-1}_{G, T} z_T = \pi^{k-1}_{G, T} v_T = v_{G, T} \).

Finally, we show that, for some \( z_T \in \mathcal{R}^{k-1}(T) \), the vector \( z_T = (z_T, z_{\partial T}) \in X^k_{\text{curl}, T} \) satisfies \( v_{G, T} = C^k_{G, T} z_T \). Recalling the definition (4.9) of the full curl reconstruction and (4.10) to write \( C^k_{G, T} = \pi^k_{G, T} G^k_T \), this amounts to enforcing the following condition: For all \( w_T \in \mathcal{G}^k(T)^\perp ,
\]

\[
\int_T z_T \cdot \text{curl} w_T = \int_T v_{G, T} \cdot w_T - \sum_{F \in \mathcal{T}_T} \omega_T F \int_F \gamma_{U, F}^k z_{\partial T} \cdot (w_T \times n_F). 
\]

By (2.10), \( \text{curl} : \mathcal{G}^k(T)^\perp \to \mathcal{R}^{k-1}(T) \) is an isomorphism and this relation therefore defines a unique \( z_T \in \mathcal{R}^{k-1}(T) \). This concludes the proof of (4.39).

4. Proof of (4.35). Let \( q_T \in \mathcal{P}^k(T) \) and let us show the existence of \( \nu_T \in X^k_{\text{div}, T} \) such that \( q_T = D^k_T \nu_T \).

By (2.9), there exists \( v \in \mathcal{R}^{k+1}(T)^\perp \) such that \( \text{div} v = q_T \). Using the polynomial consistency of \( \pi^k_{G, T} \) followed by the commutation property (4.31), we have \( q_T = \text{div} v = \pi^k_{G, T} (\text{div} v) = D^k_T (L^k_{\text{div}, T} v) \), which is the desired result with \( \nu_T = L^k_{\text{div}, T} v \).
4.4 Commutative diagrams

In this section we prove commutative diagram properties for the discrete three-dimensional sequence. These commutative diagrams express, in a synthetic manner, crucial compatibility properties of the discrete three-dimensional sequence (4.12). To this end, we recall the definition (4.18) of the Nédélec space and we introduce the Raviart–Thomas–Nédélec space

\[ \mathcal{R}T^k(T) := \mathcal{P}^k(T)^3 + (x - \bar{x}_T)\mathcal{P}^k(T). \] 

(4.40)

**Theorem 24** (Commutative diagrams). Denoting by \( i_T : \mathcal{P}^k(T) \to \mathcal{P}^k(T) \) the identity operator, the following diagrams commute:

\[
\begin{array}{c}
\mathcal{P}^{k+1}(T) \xrightarrow{\text{grad}} \mathcal{N}^k(T) \xrightarrow{\text{curl}} \mathcal{R}T^k(T) \xrightarrow{\text{div}} \mathcal{P}^k(T) \\
X_{\text{grad},T}^k \xrightarrow{G_{\text{T},T}^k} X_{\text{curl},T}^k \xrightarrow{C_{\text{T},T}^k} X_{\text{div},T}^k \xrightarrow{D_{\text{T},T}^k} \mathcal{P}^k(T)
\end{array}
\]

(4.41)

**Proof.** Using the polynomial consistency properties (4.15) of \( G_{T}^k \), (3.5) of \( G_{F}^k \), and (3.6) of \( G_{E}^k \) we infer, for all \( q \in \mathcal{P}^{k+1}(T) \),

\[
G_{T}^k(I_{\text{grad},T}^k q) = \text{grad } q, \quad G_{F}^k(I_{\text{grad},T}^k q) = \text{grad } q |_F \quad \forall F \in \mathcal{T}_T,
\]

\[
G_{E}^k(I_{\text{grad},T}^k q) = (q |_E)' \quad \forall E \in \mathcal{E}_T.
\]

Plugging these relations into the definition (4.8) of \( G_{T}^k \) and recalling the definition (4.4) of \( I_{\text{grad},T}^k \) proves the leftmost commutative diagram in (4.41).

The commutation properties (4.19) of \( C_{T}^k \) and (3.26) of \( C_{F}^k \), together with \( (I_{\text{curl},T}^k v)_F = I_{\text{rot},F}(n_F \times (v |_F \times n_F)) \) give, for all \( v \in \mathcal{N}^k(T) \),

\[
C_{T}^k(I_{\text{curl},T}^k v) = \text{curl } v, \quad C_{F}^k(I_{\text{curl},T}^k v) = \text{rot}_F(n_F \times (v |_F \times n_F)) = (\text{curl } v)|_F \cdot n_F \quad \forall F \in \mathcal{T}_T,
\]

where we have additionally used the fact that \( \text{rot}_F(n_F \times (v |_F \times n_F)) \in \mathcal{P}^k(F) \) and the identity (2.7).

Plugging these relations into the definition (4.10) of \( C_{T}^k \) and recalling the definition (4.6) of \( I_{\text{div},T}^k \) concludes the proof of the middle commutative diagram in (4.41).

Finally, the rightmost commutative diagram follows combining (4.31) and \( iT = \pi_{\mathcal{P},T}^k \) on \( \mathcal{P}^k(T) \). \( \square \)

4.5 Three-dimensional potentials

4.5.1 Scalar potential

Starting from the full gradient (4.7) and scalar trace reconstructions \( (\gamma_F^k F \in \mathcal{T}_T \) satisfying the properties (3.32), we define a scalar potential reconstruction \( p_{k+1}^{\text{grad},T} : X_{\text{grad},T}^k \to \mathcal{P}^{k+1}(T) \) as follows: For all \( q_T \in X_{\text{grad},T}^k \),

\[
\int_T P_{\text{grad},T}^{k+1} q_T \text{ div } v_T = - \int_T G_{T}^k q_T \cdot v_T + \sum_{F \in \mathcal{T}_T} \omega_T \int_F \gamma_F^{k+1} q_{\text{rot}}^k (v_T \cdot n_F) \quad \forall v_T \in \mathcal{R}^{k+2}(T)'. \]

(4.42)

This relation defines a unique \( P_{\text{grad},T}^{k+1} q_T \) since \( \text{div} : \mathcal{R}^{k+2}(T) \to \mathcal{P}^{k+1}(T) \) is an isomorphism by (2.9). Combining the polynomial consistencies (4.15) of the full gradient and (3.32) of the scalar trace reconstructions with the integration by parts formula (2.11), it is inferred that

\[
P_{\text{grad},T}^{k+1} (I_{\text{grad},T}^k q) = q \quad \forall q \in \mathcal{P}^{k+1}(T).
\]

(4.43)

Notice that other choices are possible for a scalar potential reconstruction satisfying (4.43).
4.5.2 Vector potential on $X_{\text{curl},T}^k$

A vector potential reconstruction $P_{\text{curl},T}^k : X_{\text{curl},T}^k \rightarrow \mathcal{P}^k(T)^3$ is obtained as follows: For all $v_T \in X_{\text{curl},T}^k$,

\[
\int_T P_{\text{curl},T}^k v_T \cdot \text{curl} w_T = \int_T C_T^k v_T \cdot w_T - \sum_{F \in \partial T} \omega_T F \int_F \gamma_{k,F} \gamma_{k,F} \cdot (w_T \times n_F) \quad \forall w_T \in \mathcal{G}^{k+1}(T)^\perp, \quad (4.44a)
\]

\[
\int_T P_{\text{curl},T}^k v_T \cdot w_T = \int_T v_T \cdot w_T \quad \forall w_T \in \mathcal{R}_k(T)^\perp. \quad (4.44b)
\]

To check that these equations define a unique $P_{\text{curl},T}^k v_T \in \mathcal{P}^k(T)^3$, observe that (4.44a) and (4.44b) prescribe, respectively, $\pi_{R,T}^k(P_{\text{curl},T}^k v_T)$ (since $\text{curl} : \mathcal{G}^{k+1}(T)^\perp \rightarrow \mathcal{R}_k(T)$ is an isomorphism, see (2.10)) and $\pi_{R,T}^\perp(P_{\text{curl},T}^k v_T)$, and recall the orthogonal decomposition $\mathcal{P}_k(T)^3 = \mathcal{R}_k(T) \oplus \mathcal{R}_k(T)^\perp$.

**Proposition 25** (Consistency of $P_{\text{curl},T}^k$). It holds

\[
P_{\text{curl},T}^k(I_{\text{curl},T}^k v) = v \quad \forall v \in \mathcal{P}^k(T)^3. \quad (4.45)
\]

**Proof.** Let $v \in \mathcal{P}^k(T)^3$. Applying (4.44a) to $v_T = I_{\text{curl},T}^k v$ and using the consistency properties (4.19) of $C_T^k$ and (3.34) of $\gamma_{k,F}$ we obtain, for all $w_T \in \mathcal{G}^{k+1}(T)^\perp$,

\[
\int_T P_{\text{curl},T}^k (I_{\text{curl},T}^k v) \cdot \text{curl} w_T = \int_T v_T \cdot w_T - \sum_{F \in \partial T} \omega_T F \int_F \pi_{R,F}^k (n_F \times (v_T \times n_F)) \cdot (w_T \times n_F).
\]

Since $n_F \times (v_T \times n_F) \in \mathcal{P}^k(F)^2$, the projector $\pi_{R,F}^k$ above can be removed and, invoking the integration by parts formula (2.14), it is inferred that $\pi_{R,T}^k(P_{\text{curl},T}^k(I_{\text{curl},T}^k v)) = \pi_{R,T}^k v$. On the other hand, (4.44b) and the definition (4.4) of $I_{\text{curl},T}^k$ readily imply $\pi_{R,T}^\perp(P_{\text{curl},T}^k(I_{\text{curl},T}^k v)) = \pi_{R,T}^\perp v$. Recalling the orthogonal decomposition $\mathcal{P}_k(T)^3 = \mathcal{R}_k(T) \oplus \mathcal{R}_k(T)^\perp$, the conclusion follows. \(\square\)

4.5.3 Vector potential on $X_{\text{div},T}^k$

The vector potential in $X_{\text{div},T}^k$ is $P_{\text{div},T}^k : X_{\text{div},T}^k \rightarrow \mathcal{P}^k(T)^3$ such that, for all $v_T \in X_{\text{div},T}^k$,

\[
\int_T P_{\text{div},T}^k v_T \cdot \text{grad} q_T = -\int_T D_T^k v_T q_T + \sum_{F \in \partial T} \omega_T F \int_F v_T q_T \quad \forall q_T \in \mathcal{P}^{0,k+1}(T), \quad (4.46a)
\]

\[
\int_T P_{\text{div},T}^k v_T \cdot w_T = \int_T v_T \cdot w_T \quad \forall w_T \in \mathcal{G}^k(T)^\perp. \quad (4.46b)
\]

These equations prescribe, respectively, $\pi_{\mathcal{R},T}^k(P_{\text{div},T}^k v_T)$ and $\pi_{\mathcal{R},T}^\perp(P_{\text{div},T}^k v_T)$, hence $P_{\text{div},T}^k v_T$ by virtue of the orthogonal decomposition $\mathcal{P}_k(T)^3 = \mathcal{G}_k(T) \oplus \mathcal{G}_k^k(T)^\perp$.

**Proposition 26** (Consistency of $P_{\text{div},T}^k$). It holds:

\[
P_{\text{div},T}^k(I_{\text{div},T}^k v) = \pi_{\mathcal{R},T}^k v \quad \forall v \in \mathcal{R}^k(T). \quad (4.47)
\]

**Proof.** Let $v \in \mathcal{R}^k(T)$ and set $v_T = I_{\text{div},T}^k v$. Recalling the commutation property (4.31) of $D_T^k$ and the definition (4.6) of $I_{\text{div},T}^k$, (4.46a) gives: For all $q_T \in \mathcal{P}^{0,k+1}(T)$,

\[
\int_T P_{\text{div},T}^k(I_{\text{div},T}^k v) \cdot \text{grad} q_T = -\int_T \pi_{\mathcal{R},T}^k(\text{div} v) q_T + \sum_{F \in \partial T} \omega_T F \int_F \pi_{\mathcal{R},T}^k(v \cdot n_F) q_T.
\]

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Since \( v \in RT^k(T) \), we have \( \text{div}\ v \in \mathcal{P}^k(T) \) and \( v_{|F} \cdot n_F \in \mathcal{P}^k(F) \) for all \( F \in \mathcal{T}_T \). This is a consequence of the definition \( (4.40) \) of the Raviart–Thomas space observing that, for all \( F \in \mathcal{T}_T \), the mapping \( F \ni x \mapsto (x - \overline{x}_F) \cdot n_F \) is constant; hence, the projectors can be removed from the right-hand side of the above expression. Invoking then the integration by parts formula \( (2.11) \), it is inferred that \( \pi_{G,T}^k(\mathcal{P}^k_{\text{div},T}(L^k_{\text{div},T} v)) = \pi_{G,T}^k v \). Equation \( (4.46b) \), on the other hand, readily implies \( \pi_{G,T}^k(\mathcal{P}^k_{\text{div},T}(L^k_{\text{div},T} v)) = \pi_{G,T}^k v \). Combining these facts with the orthogonal decomposition \( \mathcal{P}^k(T)^3 = G^k(T) \oplus \mathcal{G}^k(T)^k \), \( (4.47) \) follows.

### 4.6 Three-dimensional discrete \( L^2 \)-products

We next define discrete counterparts of the \( L^2 \)-products in \( H^1(T) \), \( H(\text{curl}; T) \), and \( H(\text{div}; T) \):

- \((\cdot, \cdot)_{\text{grad}, T} : X^k_{\text{grad}, T} \times X^k_{\text{grad}, T} \to \mathbb{R} \) such that, for all \( q_T, r_T \in X^k_{\text{grad}, T} \),

\[
(q_T, r_T)_{\text{grad}, T} := \int_F P^k_{\text{grad}, T} q_T \cdot P^k_{\text{grad}, T} r_T + \int_T \delta^k_{\text{grad},T} q_T \cdot \delta^k_{\text{grad},T} r_T + \sum_{F \in \mathcal{T}_T} h_T \int_F \delta^k_{\text{grad}, F} q_T \cdot \delta^k_{\text{grad}, F} r_T + \sum_{E \in \mathcal{E}_T} h_T^2 \int_E \delta^k_{\text{grad}, E} q_T \cdot \delta^k_{\text{grad}, E} r_T,
\]

where \( h_T \) is the diameter of \( T \) and we have set, for any \( q_T \in X^k_{\text{grad}, T} \),

\[
(\delta^k_{\text{grad},T} q_T, (\delta^k_{\text{grad},F} q_T)_{F \in \mathcal{T}_T}, (\delta^k_{\text{grad},E} q_T)_{E \in \mathcal{E}_T}) := L^k_{\text{grad},T}(P^k_{\text{grad},T} q_T) - q_T.
\]

- \((\cdot, \cdot)_{\text{curl}, T} : X^k_{\text{curl}, T} \times X^k_{\text{curl}, T} \to \mathbb{R} \) such that, for all \( v_T, w_T \in X^k_{\text{curl}, T} \),

\[
(v_T, w_T)_{\text{curl}, T} := \int_T P^k_{\text{curl}, T} v_T \cdot P^k_{\text{curl}, T} w_T + \int_T \delta^k_{\text{curl}, T} v_T \cdot \delta^k_{\text{curl}, T} w_T + \int_T \delta^k_{\text{curl}, T} v_T \cdot \delta^k_{\text{curl}, T} w_T + \sum_{F \in \mathcal{T}_T} h_T \int_F \delta^k_{\text{curl}, F} v_T \cdot \delta^k_{\text{curl}, F} w_T + \sum_{E \in \mathcal{E}_T} h_T^2 \int_E \delta^k_{\text{curl}, E} v_T \cdot \delta^k_{\text{curl}, E} w_T,
\]

where we have set, for all \( v_T \in X^k_{\text{curl}, T} \), with obvious notations,

\[
(\delta^k_{\text{curl},T} v_T, (\delta^k_{\text{curl},F} v_T)_{F \in \mathcal{T}_T}, (\delta^k_{\text{curl},E} v_T)_{E \in \mathcal{E}_T}) := L^k_{\text{curl},T}(P^k_{\text{curl},T} v_T) - v_T.
\]

- \((\cdot, \cdot)_{\text{div}, T} : X^k_{\text{div}, T} \times X^k_{\text{div}, T} \to \mathbb{R} \) such that, for all \( v_T, w_T \in X^k_{\text{div}, T} \),

\[
(v_T, w_T)_{\text{div}, T} := \int_T P^k_{\text{div}, T} v_T \cdot P^k_{\text{div}, T} w_T + \int_T \delta^k_{\text{div}, T} v_T \cdot \delta^k_{\text{div}, T} w_T + \sum_{F \in \mathcal{T}_T} h_T \int_F \delta^k_{\text{div}, F} v_T \cdot \delta^k_{\text{div}, F} w_T + \sum_{E \in \mathcal{E}_T} h_T^2 \int_E \delta^k_{\text{div}, E} v_T \cdot \delta^k_{\text{div}, E} w_T,
\]

where we have set, for all \( v_T \in X^k_{\text{div}, T} \),

\[
\delta^k_{\text{div}, T} v_T := n^k_F (P^k_{\text{div}, T} v_T \cdot n_F) - v_T \quad \forall F \in \mathcal{T}_T.
\]
Remark 27 (Discrete \(L^2\)-product in \(X^k_{\text{div},T}\)). Also for \(X^k_{\text{div},T}\) it is possible to define a discrete \(L^2\)-product where all the components of \(P^k_{\text{div},T}(X^k_{\text{div},T}Y^k_T) - Y^k_T\) are penalised. It turns out, however, that penalising the volume differences is not required to prove definiteness; cf. the proof of Lemma 28 below.

Lemma 28 (Discrete \(L^2\)-products). The bilinear forms \((\cdot, \cdot)_T\), with \(\bullet \in \{\text{grad, curl, div}\}\), are positive definite. Additionally, they satisfy the following consistency properties:

\[
\begin{align*}
(I^{\text{grad},T}_q, I^{\text{grad},T}_r)_{\text{grad},T} &= (q, r)_{L^2(T)} & \forall q, r \in P^{k+1}(T), \\
(I^{\text{curl},T}_v, I^{\text{curl},T}_w)_{\text{curl},T} &= (v, w)_{L^2(T)^3} & \forall v, w \in P^k(T)^3, \\
(I^{\text{div},T}_v, I^{\text{div},T}_w)_{\text{div},T} &= (v, w)_{L^2(T)^3} & \forall v, w \in P^k(T)^3.
\end{align*}
\]

Proof. Let us first prove the positive definiteness of the bilinear forms \((\cdot, \cdot)_T\). By inspection, they are positive semi-definite, and it only remains to prove that they are definite.

Consider first the case of \((\cdot, \cdot)_{\text{grad},T}\). Let \(q_T \in X^k_{\text{grad},T}\) be such that \((q_T, q_T)_{\text{grad},T} = 0\). Then, obviously from (4.48), we have \(P^{k+1}_{\text{grad},T}q_T = 0\), \(H^{k+1}_{\text{grad},T}q_T = 0\), and \(q_T = 0\) for all \(F \in T\) and \(E \in T\). This gives \(0 = (I^{\text{grad},T}_q, I^{\text{grad},T}_q)_{\text{grad},T} = q_T\), and thus \(q_T = 0\) as required.

The definiteness of \((\cdot, \cdot)_{\text{curl},T}\) is obtained exactly the same way, so let us turn to \((\cdot, \cdot)_{\text{div},T}\). If \(v_T \in X^k_{\text{div},T}\) is such that \((v_T, v_T)_{\text{div},T} = 0\) then \(P^k_{\text{div},T}v_T = 0\) and \(I^k_{\text{div},T}v_T = 0\) for all \(F \in T\). This shows that \(v_T = \pi^k_{\text{div},T}(P^k_{\text{div},T}v_T, v_T)_{\text{div},T} = 0\) for all \(F \in T\). Using then (4.46a), we infer that

\[
\int_T D^k_Tv_T = 0 \quad \forall v_T \in P^0(T).
\]

The definition (4.11) of \(D^k_T\) together with the fact that \(\text{grad} : P^0(T) \to G^{k-1}(T)\) is surjective then shows that \(v_{\text{grad},T} = \pi^k_{\text{grad},T}v_T = 0\). Since \(P^k_{\text{div},T}v_T = 0\), the relation (4.46c) obviously yields \(v_{\text{grad},T} = 0\), which concludes the proof of \(v_T = 0\).

The consistency properties (4.51)–(4.53) follow easily from the consistency properties (4.43), (4.45) and (4.47) of the potential reconstructions. 

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