Nonlinear free Lévy-Khinchine formula and conformal mapping

Philippe Biane

NONLINEAR FREE LÉVY-KHINCHINE FORMULA AND CONFORMAL MAPPING

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Abstract. There are two natural notions of Lévy processes in free probability: the first one has free increments with homogeneous distributions and the other has homogeneous transition probabilities [6]. In the two cases one can associate a Nevanlinna function to a free Lévy process. The Nevanlinna functions appearing in the first notion were characterized by Bercovici and Voiculescu [3]. I give an explicit parametrization for the Nevanlinna functions associated with the second kind of free Lévy processes. This gives a nonlinear free Lévy-Khinchine formula.

1. Introduction

The convolution of two probability measures on the real line, λ and μ, is characterized by

$$\int_{\mathbf{R}} f(x)\lambda * \mu(dx) = E[f(X+Y)] \tag{1}$$

for bounded continuous functions f, where X and Y are independent random variables, distributed as λ and μ. A probability distribution μ is called infinitely divisible if, for every integer $n > 0$, it can be written as a convolution power $\mu = (\mu_{1/n})^{*n}$, for some probability distribution $\mu_{1/n}$. The Lévy-Khinchine formula gives an integral representation of the logarithm of the Fourier transform of an infinitely divisible distribution i.e.

$$\int_{\mathbf{R}} e^{ivx}\mu(dx) = e^{\theta_\mu(v)}$$

where

$$\theta_\mu(v) = imv + \int_{\mathbf{R}} \frac{e^{ivy} - 1 - ivy}{y^2}(1+y^2)\nu(dy) \tag{2}$$

for some real number m and a positive finite measure ν on $\mathbf{R}$ (the function under the integral being extended by continuity to $y = 0$). As a consequence, there exists a convolution semigroup of measures $(\mu_t)_{t\geq 0}$ satisfying $\mu_t * \mu_s = \mu_{s+t}$ and $\int e^{ivx}\mu_t(dx) = e^{-t\theta_\mu(v)}$. The formula (2) is an instance of Choquet's integral representation theorem for convex cones and the extreme cases correspond to Dirac measures (for $\nu = 0$), Gaussian measures ($\nu = \delta_0$) or Poisson distributions ($\nu = \delta_t, t \neq 0$).

The free convolution of two probability measures on the real line, λ and μ, defined by Voiculescu [10], [4], is characterized by

$$\int_{\mathbf{R}} f(x)\lambda \boxplus \mu(dx) = \tau(f(X+Y)) \tag{3}$$

for bounded continuous functions f, where X and Y are free elements in some non-commutative probability space $(\mathcal{A}, \tau)$, distributed as λ and μ, see section 2.2.1 below. The free convolution of measures can be computed using their Voiculescu transforms, which are analytic functions defined on a domain inside the complex upper halfplane. One can develop a theory of free convolution which parallels the classical theory of convolution of measures and sums of independent random variables on the real line. In particular there are analogues of the Gauss and Poisson distributions as well as a notion of free infinitely divisible distribution and a free analogue of the Lévy-Khinchine formula [3]. This last formula reduces to the integral representation formula for

Nevanlinna functions defined for z in the upper halfplane. Indeed the Voiculescu transform of a freely infinitely divisible measure can be expressed as:

$$\varphi(z) = \alpha + \int_{\mathbf{R}} \frac{1+xz}{z-x}\nu(dx), \tag{4}$$

as I recall below in section 2.2.3. This is the free analogue of the Lévy-Khinchine formula (2). Associated to a freely infinitely divisible distribution μ there is a free convolution semigroup of probability measures $(\mu_t)_{t\geq 0}$, indexed by real times t, satisfying

$$\mu = \mu_1; \quad \mu_s \boxplus \mu_t = \mu_{s+t} \text{ for } s,t \geq 0.$$

For X, Y as in (3) there exists a Markov kernel $p_{\lambda,\mu}(x, du)$ on $\mathbf{R}$ such that, for any bounded continuous functions f, g one has

$$\tau(f(X)g(X+Y)) = \int_{\mathbf{R}} \left(\int_{\mathbf{R}} g(u)p_{\lambda,\mu}(x,du)\right) f(x)\lambda(dx). \tag{5}$$

This is analogous to the classical situation where X and Y are independent random variables: in this case one has

$$E(f(X)g(X+Y)) = \int_{\mathbf{R}} \left(\int_{\mathbf{R}} g(u)q_{\lambda,\mu}(x,du)\right) f(x)\lambda(dx) \tag{6}$$

where the kernel is given by $q_{\lambda,\mu}(x, du) = (\mu * \delta_x)(du)$, the translate of μ by x. In particular this kernel does not depend on λ.

In the case of free convolution, the Markov kernel $p_{\lambda,\mu}$ can be computed in terms of the Cauchy transforms of the measures and this leads to a subordination property of these Cauchy transforms (cf [6]), which I recall in section 2.3 below. If $(\mu_t)_{t\geq 0}$ is a convolution semigroup of freely infinitely divisible distributions, then one can define accordingly Markov kernels $\mathcal{K}_{s,t}$, for $s < t$, corresponding to the convolution equations

$$\mu_s \boxplus \mu_{t-s} = \mu_t.$$

These kernels satisfy the Chapman-Kolmogorov equation:

$$\mathcal{K}_{s,t} \circ \mathcal{K}_{t,u} = \mathcal{K}_{s,u} \quad \text{for } s < t < u.$$

Contrary to the case of classical convolution, the homogeneity of the increments does not imply that the kernels are time-homogeneous, i.e. in general $\mathcal{K}_{s,t}$ does not depend only on $t - s$. The question therefore arises of finding continuous families of measures

$$\mu_t, \text{ for } t \geq 0, \quad \mu_{s,t}, \text{ for } s < t, \quad \text{such that} \quad \mu_s \boxplus \mu_{s,t} = \mu_t \tag{7}$$

and such that the corresponding kernels $\mathcal{K}_{s,t}$ depend only on $t - s$. In [6] I gave a necessary and sufficient condition for a family such as (7) to correspond to a time-homogeneous transition kernel. However these conditions are hard to check and I did not give an explicit description of all the solutions. The purpose of this paper is to give an answer to this question and in particular to give a parametrization of all solutions, which we will call the nonlinear free Lévy-Khinchine formula. This parametrization has a strong geometric flavour and uses in an essential way the theory of conformal mappings of the upper halfplane.

I will also consider the case of free multiplicative convolution, for which analogous results can be obtained.

This paper is organized as follows. In section 2, I recall the necessary facts from complex analysis: Cauchy transforms, Nevanlinna functions, and from free probability: free convolution, Voiculescu transform, free infinitely divisible distributions, subordination and the Markov property. I also state the main problem that is solved in the paper, which is to characterize free additive Lévy functions of the second kind. In section 3, I consider primitives of Nevanlinna functions and investigate their behaviour as conformal mapping. In particular, I give necessary and sufficient conditions for such functions to map the upper halfplane to a domain containing the upper halfplane, which is the crucial property needed later. In section 4, I solve the main problem by providing an explicit characterization of the free additive Lévy functions of the second kind. Finally, the case of multiplicative convolution is discussed in section 5.

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2. Preliminaries

2.1. **Some tools from complex analysis.**

2.1.1. *Cauchy and Voiculescu transforms.* The Cauchy transform of a probability measure μ on $\mathbf{R}$ is given by

$$G_\mu(\zeta) = \int_{\mathbf{R}} \frac{1}{\zeta - u}\mu(du), \quad \text{for } \zeta \in \mathbf{C} \setminus \mathbf{R}.$$

The function G_μ is analytic, it satisfies $G_\mu(\bar{\zeta}) = \overline{G_\mu(\zeta)}$ and $G_\mu(\mathbf{C}^+) \subset \mathbf{C}^-$, where $\mathbf{C}^\pm$ denote the upper and lower halfplanes i.e. $\mathbf{C}^+ = \{z \in \mathbf{C} | \Im(z) > 0\}, \mathbf{C}^- = -\mathbf{C}^+$. This function uniquely determines the measure μ. For $\alpha, \beta > 0$, let

$$\Theta_{\alpha,\beta} = \{z = x + iy \,|\, y < 0; \alpha y < x < -\alpha y; |z| \leq \beta\}.$$

For every $\alpha > 0$, there exists a real number $\beta > 0$ such that the function G_μ has a right inverse defined on the domain $\Theta_{\alpha,\beta}$, taking values in some domain of the form

$$\Gamma_{\gamma,\lambda} = \{z = x + iy \,|\, y > 0; -\gamma y < x < \gamma y; |z| \geq \lambda\}$$

with $\gamma, \lambda > 0$. Call K_μ this right inverse, and let $R_\mu(z) = K_\mu(z) - \frac{1}{z}$. We shall also need the notations

$$F_\mu(\zeta) = \frac{1}{G_\mu(\zeta)} \tag{8}$$

and

$$\varphi_\mu(z) = R_\mu(\frac{1}{z}) = F_\mu^{-1}(z) - z \tag{9}$$

where F_μ^{-1} is defined in some domain of the form $\Gamma_{\alpha,\beta}$. The function φ_μ is defined on the same domain as F_μ^{-1} and takes its values in $\mathbf{C}^- \cup \mathbf{R}$. It is called the Voiculescu transform of μ.

2.1.2. *Nevanlinna functions.* An analytic function φ, defined on $\mathbf{C}^+$, with values in $\mathbf{C}^- \cup \mathbf{R}$, is called a Nevanlinna function. The Nevanlinna representation gives real numbers $\alpha \leq 0$, β and a finite positive measure ν, on $\mathbf{R}$, such that

$$\varphi(z) = \alpha z + \beta + \int_{\mathbf{R}} \frac{1 + uz}{z - u}\nu(du). \tag{10}$$

The measure ν can be recovered from φ by

$$\nu(du) = \lim_{\varepsilon \to 0} \frac{-\Im(\varphi(u + i\varepsilon))}{2\pi(1 + u^2)} du \tag{11}$$

while

$$\alpha = \lim_{v \to +\infty} \frac{\varphi(iv)}{iv}. \tag{12}$$

Finally

$$\varphi(i) = \alpha i + \beta - i \int_{\mathbf{R}} \nu(du)$$

allows to recover all parameters.

Observe that, if φ takes a real value at some point, then it is constant, as follows from the maximum principle. The integral representation (10) exhibits the set of Nevanlinna functions as a Choquet simplex, whose extreme rays are generated by the maps $z \mapsto \frac{1+uz}{z-u}$, which are conformal mappings from the upper halfplane onto itself.

Finally we note that, if $\int |u|\nu(du) < +\infty$, then $\int_{-\infty}^{+\infty} \frac{1+uz}{z-u}\nu(du) \to \int u\,\nu(du)$ if $|z| \to \infty$ with z in some domain $\Gamma_{\gamma,\lambda}$.

2.2. **Free convolution and freely indivisible distributions.**

2.2.1. *Free convolution.* We recall the definition of the free convolution of measures and how it can be computed, see e.g. [4] for these results. Let λ and μ be probability measures on $\mathbf{R}$, then there exists a non-commutative probability space (A, τ) and self-adjoint elements X, Y affiliated to A, with respective distributions λ and μ, such that X and Y are free, i.e. the von Neumann algebras generated by their spectral projections are free. The distribution of $X + Y$ depends only on λ and μ, it is called the free additive convolution of λ and μ and is denoted by $\lambda \boxplus \mu$. This defines a symmetric and associative binary operation on the set of probability measures on $\mathbf{R}$. The free additive convolution is linearized by the Voiculescu transform (9), indeed one has

$$\varphi_{\lambda\boxplus\mu} = \varphi_\lambda + \varphi_\mu$$

on some domain of the form $\Gamma_{\alpha,\beta}$ where these three functions are defined. Since $\lambda\boxplus\mu$ is determined by the restriction of $\varphi_{\lambda\boxplus\mu}$ to one of these domains, this characterizes completely the measure $\lambda \boxplus \mu$.

2.2.2. *Processes with free increments.* Processes with free increments were studied in [6]. In short, a process with free increments is a family of non-commutative random variables $(X_t)_{t\geq 0}$, in a non-commutative probability space (A, τ), such that for any $s < t$ the increment $X_t - X_s$ is free from the von Neumann algebra generated by the $(X_u)_{u\leq s}$ (some care is needed when the operators are unbounded and one has to use affiliated subalgebras, see [6] for details). The laws of the increments $X_t - X_s$, for $s < t$, denoted $\mu_{s,t}$ satisfy the relations

$$\mu_{s,t} \boxplus \mu_{t,u} = \mu_{s,u} \quad \text{for} \quad s < t < u. \tag{13}$$

Conversely, given probability distributions $\mu_{s,t}$ satisfying relations (13), together with some continuity assumption and an initial distribution μ_0, there exists a non-commutative process with free increments distributed as $\mu_{s,t}$ [6].

2.2.3. *Free infinitely divisible distributions.* There is a notion of infinitely divisible measures for the free additive convolution: a measure μ is freely infinitely divisible if for all $n > 0$ there exists $\mu_{1/n}$ such that $\mu = (\mu_{1/n})^{\boxplus n}$. There is also an analogue of the Lévy-Khinchine formula, which was obtained in [3]. A probability measure μ on $\mathbf{R}$ is freely infinitely divisible if and only if its Voiculescu transform φ_μ has an analytic continuation to the whole of $\mathbf{C}^+$, with values in $\mathbf{C}^- \cup \mathbf{R}$ and one has

$$\lim_{v\to\infty, v\in\mathbf{R}} \frac{\varphi_\mu(iv)}{iv} = 0.$$

The Nevanlinna representation (10) implies that

$$\varphi_\mu(z) = \beta + \int_{\mathbf{R}} \frac{1+uz}{z-u}\nu(du) \tag{14}$$

for some positive finite measure ν, called the free Lévy measure of μ. The formula (14) is the free analogue of the Lévy-Khinchine formula. It expresses an arbitrary infinitely divisible distribution in terms of the Wigner semi-circle distribution (corresponding to $\nu = \delta_0$), which is the free analogue of the Gauss distribution and the Pastur-Marchenko distributions (for $\nu = \delta_x$ with $x \neq 0$), which are the free Poisson distributions. If μ is freely infinitely divisible then for all $t \geq 0$ there exists a probability measure on the real line μ_t such that $\varphi_{\mu_t} = t\varphi_\mu$ and these measures satisfy the relations

$$\mu_s \boxplus \mu_t = \mu_{s+t}.$$

The parallel between classical and free infinitely divisible distributions goes quite far, for example one can find free analogues of the classical theory of stable distributions and domains of attractions, see [2].

2.3. **Subordination and the Markov property.** Given probability distributions μ, ν on $\mathbf{R}$, there exists a subordination relation between the Cauchy transforms of μ (or ν) and of $\mu \boxplus \nu$. As shown in [6], this relation expresses the Markov property of the free convolution. We recall the main theorem of [6].

Theorem 2.1. *Let (A, τ) be a non-commutative probability space, B be a von Neumann subalgebra of A, let $Y \in A$ be a self-adjoint element which is free from B, and let $X \in B$ be self-adjoint. Denote by λ and μ the distributions of X and Y, then there exists a Feller Markov kernel $\mathcal{K} = k(x, du)$ on $\mathbf{R} \times \mathbf{R}$ and an analytic function F on $\mathbf{C} \setminus \mathbf{R}$ such that*

(1) For any Borel bounded function f on $\mathbf{R}$ one has $\tau(f(X+Y)|B) = \mathcal{K}f(X)$.
(2) $F(\bar{\zeta}) = \overline{F(\zeta)}$, $\quad F(\mathbf{C}^+) \subset \mathbf{C}^+$.
(3) $Im(F(\zeta)) \geq Im(\zeta)$ for $\zeta \in \mathbf{C}^+$.
(4) $\frac{F(iy)}{iy} \to 1$ as $y \to +\infty$, $y \in \mathbf{R}$
(5) for all $\zeta \in \mathbf{C} \setminus \mathbf{R}$ one has $\int_{\mathbf{R}} (\zeta - u)^{-1} k(x, du) = (F(\zeta) - x)^{-1}$.
(6) For all $\zeta \in \mathbf{C} \setminus \mathbf{R}$ one has $G_\lambda(F(\zeta)) = G_{\lambda \boxplus \mu}(\zeta)$

Here $\tau(.|B)$ denotes the conditional expectation, $\mathcal{K}f(x) = \int_{\mathbf{R}} f(u)k(x, du)$ and the map F is uniquely determined by properties (4) *and* (6)*.*

Property (1) above is the Markov property of free convolution while (6) is the subordination property relating the Cauchy transforms of λ and $\lambda \boxplus \mu$.

For each process with free increments Theorem 2.1 yields a family of Markov kernels $\mathcal{K}_{s,t}; s < t$ on the real line, satisfying the Chapman-Kolmogorov relations

$$\mathcal{K}_{s,t} \circ \mathcal{K}_{t,u} = \mathcal{K}_{s,u} \quad \text{for } s < t < u. \tag{15}$$

These kernels are determined, using (5) of Theorem 2.1, by analytic functions $F_{s,t}$ mapping the upper halfplane to itself and satisfying

$$F_{s,t} \circ F_{t,u} = F_{s,u} \quad \text{for } s < t < u. \tag{16}$$

We call such a family of kernels *time homogeneous* if $\mathcal{K}_{s,t} \equiv \mathcal{K}_{t-s}$ (or equivalently $F_{s,t} \equiv F_{t-s}$) depends only on $t - s$. If this is the case then the kernels $\mathcal{K}_t$ form a semigroup (and the maps F_t form a semigroup of analytic maps on $\mathbf{C}^+$).

As is easily seen on examples, see e.g. section 5 of [6], in general the kernels $\mathcal{K}_{s,t}$ are not time homogeneous, when the increments are i.e. when $\mu_{s,t} \equiv \mu_{t-s}$. It is therefore natural to ask whether there exists processes with non homogeneous free increments and with time homogeneous transition probabilities. In [6] a characterization was given in the following theorem.

Theorem 2.2. *Let $\mu_t, t \geq 0$ and $(\mu_{s,t})_{s<t \in \mathbf{R}_+}$ be families of probability measures satisfying*

$$\mu_s \boxplus \mu_{s,t} = \mu_t; \quad \mu_{s,t} \boxplus \mu_{t,u} = \mu_{s,u} \tag{17}$$

for all $s < t < u$. Let $(\mathcal{K}_{s,t})_{s<t \in \mathbf{R}_+}$ be the corresponding Markov transition functions on $\mathbf{R}$. Assume that the kernels are time homogeneous, then the kernels $\mathcal{L}_t \equiv \mathcal{K}_{0,t}$ for $t \geq 0$, form a Feller Markov semi-group. Let $F_{s,t}$ be the analytic functions associated to the kernels $\mathcal{K}_{s,t}$ by Theorem 2.1. The maps $F_t \equiv F_{0,t}$, where F_0 is the identity function, form a continuous semigroup, under composition, of analytic transformations of $\mathbf{C}^+$ and $F_{s,t} = F_{t-s}$, moreover there exists a Nevanlinna function φ such that

$$\frac{\varphi(\zeta)}{\zeta} \to_{\substack{\zeta \to \infty \\ \zeta \in \Gamma_{\alpha,\beta}}} 0 \tag{18}$$

in every domain of the form $\Gamma_{\alpha,\beta}$ and such that the maps F_t, for $t \geq 0$, satisfy the differential equation

$$\frac{\partial F_t}{\partial t} + \varphi(F_t) = 0, \qquad F_0(z) = z. \tag{19}$$

Conversely, let φ be a Nevanlinna function satisfying (18) in some domain of the form $\Gamma_{\alpha,\beta}$, and let $(F_t)_{t\in\mathbf{R}_+}$ be the semi-group of analytic maps of $\mathbf{C}^+$ obtained by solving (19) with initial condition $F_0(z) = z$, then there exists $(\mu_t)_{t\geq 0}$ and $(\mu_{s,t})_{s<t\in\mathbf{R}_+}$, families of probability measures satisfying (13) with associated semi-group of maps $(F_t)_{t\in\mathbf{R}_+}$, if and only if, for every $t > 0$ the function $\varphi \circ F_t^{-1} \circ F_{\mu_0}^{-1}$ has an analytic continuation to $\mathbf{C}^+$, with values in $\mathbf{C}^-$.

The Nevanlinna functions having the properties listed in Theorem 2.2 have been called *free additive Lévy functions of the second kind* (or FAL2) in [6]. One can easily check, by explicit computations [6], that the functions $z \mapsto -z^\rho$ with $0 < \rho < 1$ are FAL2 functions while the Nevanlinna functions $z \mapsto z^\theta$, for $-1 < \theta < 0$, are not.

The characterization of FAL2 functions in this theorem is rather indirect, it is not easy to check moreover it does not provide a nice parametrization of the set of FAL2 functions. In the following we shall show that one can give a more explicit parameterization these functions, at least in the case $\mu_0 = \delta_0$. For this we use properties of primitives of Nevalinna functions, as explained in the next section, as well as classical results on starlike domains in conformal mapping theory. We call this parametrization the *nonlinear free Lévy-Khinchine formula* since, as we shall see the set of FAL2 functions can be parametrized by a convex set, up to some non-linear transformation.

2.4. **Semigroups of analytic functions on the upper halfplane.** The semigroups of analytic functions $(\phi_t)_{t\geq 0}$, mapping the upper halfplane to itself, have been studied by Berkson and Porta in [5] where they give a characterization of infinitesimal generators of such semigroups (more precisely, they consider the right halfplane, but of course it is immediate to translate their result to the case of the upper halfplane). These fall into three classes according to the behaviour of the semi-group at infinity. The semigroups that we consider in the present paper belong to their first class, which is characterized by the fact that $\phi_t(z) \to \infty$ as $t \to \infty$.

3. Conformal mappings associated with Nevanlinna functions

3.1. **Primitives of Nevalinna functions.** Let ψ be a Nevalinna function and $\Psi = -\int \psi(z)dz$ be a primitive of $-\psi$, which is holomorphic on $\mathbf{C}^+$.

Lemma 3.1. *If $\psi \neq 0$ then the function Ψ is univalent on $\mathbf{C}^+$.*

Proof. If ψ is real, then it is constant and $\Psi(z) = az + b$ for some $a \neq 0$ therefore the claim is clear. If not then $\Im(\psi(z)) < 0$ for all $z \in \mathbf{C}^+$. For $z_1 \neq z_2$ in $\mathbf{C}^+$ one has

$$\frac{\Psi(z_2) - \Psi(z_1)}{z_2 - z_1} = -\int_0^1 \psi(z_1 + t(z_2 - z_1))dt$$

therefore $\Im(\frac{\Psi(z_2)-\Psi(z_1)}{z_2-z_1}) > 0$. ☐

It follows from Lemma 3.1 that Ψ maps conformally $\mathbf{C}^+$ onto some domain $\Omega \subset \mathbf{C}$. The class of domains which are obtained in this way is characterized by a geometric property recalled in section 3.2, which is the upper halfplane version of a classical result on univalent functions in the unit disk.

3.2. **Starlike domains.**

Definition 3.2. *A domain $\Omega \subset \mathbf{C}$ is called starlike at $-\infty$ if $\Omega \neq \emptyset, \mathbf{C}$ and, for any $t > 0$, one has $\Omega - t \subset \Omega$.*

A domain Ω, which is starlike at $-\infty$, is a union of open horizontal halflines

$$D_\Omega(q) = \Omega \cap \{p + iq \,|\, p \in \mathbf{R}\} = \{p + iq \,|\, p < d_\Omega(q)\}$$

where $d_\Omega : \mathbf{R} \to [-\infty, +\infty]$ is a lower semicontinuous function and $d_\Omega^{-1}(-\infty) =]-\infty, q_-] \cup [q_+, +\infty[$ whith $-\infty \leq q_- < q_+ \leq +\infty$.

Proposition 3.3. *Let ψ be a nonzero Nevanlinna function and Ψ be a primitive of $-\psi$ then the domain $\Psi(\mathbf{C}^+)$ is starlike at $-\infty$. Conversely, for any domain Ω, starlike at $-\infty$, there exists ψ, a nonzero Nevanlinna function and Ψ, a primitive of $-\psi$, such that $\Omega = \Psi(\mathbf{C}^+)$.*

The proof is similar to the case of univalent functions on the unit disk, cf Pommerenke [8] Chap. 2.2 .

It is instructive to consider the case of rational Nevanlinna functions. Let $\psi(z)$ be such a function, with partial fraction expansion

$$\psi(z) = az + b + \sum_{k=1}^{N} \frac{\alpha_k}{z - \xi_k}$$

where $a < 0$, b is real, the ξ_k are real (with $\xi_1 < \xi_2 < \ldots < \xi_N$) and the α_k are positive. We have

$$\Psi(z) = -\frac{1}{2}az^2 - bz - \sum_k \alpha_k \log(z - \xi_k)$$

where we take the determination of the logarithm on $\mathbf{C} \setminus \mathbf{R}_-$ such that $\log(t) > 0$ for $t > 0$. The map Ψ extends continuously (even analytically) to the boundary of $\mathbf{C}^+$ (i.e. to $\mathbf{R}$) except at the points ξ_k, moreover its imaginary part is constant on each interval $]\xi_k, \xi_{k+1}[$, while its real part is a stricly convex function on each of these intervals, whith limit $+\infty$ at each boundary point. It follows that the image of $\mathbf{C}^+$ by Ψ is the complement of a sequence of horizontal halflines $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_{N+1}$, each of the form $\mathcal{D}_j = \{p + iq_j \,|\, p \geq p_j\}$ hence

$$\Psi(\mathbf{C}^+) = \Omega = \mathbf{C} \setminus (\mathcal{D}_1 \cup \mathcal{D}_2 \cup \ldots \cup \mathcal{D}_{N+1}). \tag{20}$$

Conversely, it is not difficult to check that for any finite family of horizontal halflines, as above, the conformal map from $\mathbf{C}^+$ to $\mathbf{C} \setminus (\mathcal{D}_1 \cup \mathcal{D}_2 \cup \ldots \cup \mathcal{D}_{N+1})$, mapping ∞ to ∞, is the primitive of the opposite of a rational Nevalinna function.

As an example, the conformal mapping $\Psi(z) = z^2/2 - \log(z)$, corresponding to $\psi(z) = -z + \frac{1}{z}$, maps $\mathbf{C}^+$ to $\mathbf{C} \setminus (\mathcal{D}_1 \cup \mathcal{D}_2)$ where $\mathcal{D}_1 = \{p \,|\, p \geq 1/2\}$ and $\mathcal{D}_2 = \{p - i\pi \,|\, p \geq 1/2\}$ are two horizontal halflines. Figure 1 below shows some of the flow lines and equipotential lines, i.e. the images by Ψ of the lines $\Im(z), \Re(z) = cst$ in $\mathbf{C}^+$.

3.3. **Starlike domains containing an upper halfplane.** In this section, ψ denotes a Nevanlinna function with canonical representation

$$\psi(z) = \alpha z + \beta + \int_{-\infty}^{+\infty} \frac{1 + uz}{z - u} \nu(du)$$

and Ψ a primitive of $-\psi$. We look for conditions on ψ ensuring that the starlike domain $\Psi(\mathbf{C}^+)$ contains a translate of $\mathbf{C}^+$.

Lemma 3.4. *If $\Psi(\mathbf{C}^+)$ contains a translate of $\mathbf{C}^+$ then one has $\int_0^\infty u^2 \nu(du) < +\infty$.*

Proof. The region $\mathbf{C}^+_\varepsilon = \{z \,|\, \Im(z) > \varepsilon\}$ is mapped by Ψ to the region on the left of the curve $x \mapsto \Psi(x + i\varepsilon)$ where the function $\Psi(x + i\varepsilon)$ has strictly increasing imaginary part. The image of Ψ is a proper domain in $\mathbf{C}$ therefore there exists a point $w \in \mathbf{R}$ such that $\Im(\Psi(w + i\varepsilon))$ remains bounded as $\varepsilon \to 0$. One has

$$\begin{aligned}\Im(\Psi(A + i\varepsilon) - \Psi(w + i\varepsilon)) &= \int_w^A \Im(\psi(x + i\varepsilon))dx \\ &= -\varepsilon\alpha(A - w) + \int_w^A \left(\int_{-\infty}^{\infty} \frac{\varepsilon(1 + u^2)}{(x - u)^2 + \varepsilon^2} \nu(du)\right) dx\end{aligned}$$

One can easily see that

$$\int_w^A \left(\int_{-\infty}^{\infty} \frac{\varepsilon(1 + u^2)}{(x - u)^2 + \varepsilon^2} \nu(du)\right) dx \to_{\varepsilon \to 0} \frac{1 + w^2}{2} \nu(\{w\}) + \int_{]w,A[} (1 + u^2)\nu(du) + \frac{1 + A^2}{2} \nu(\{A\})$$

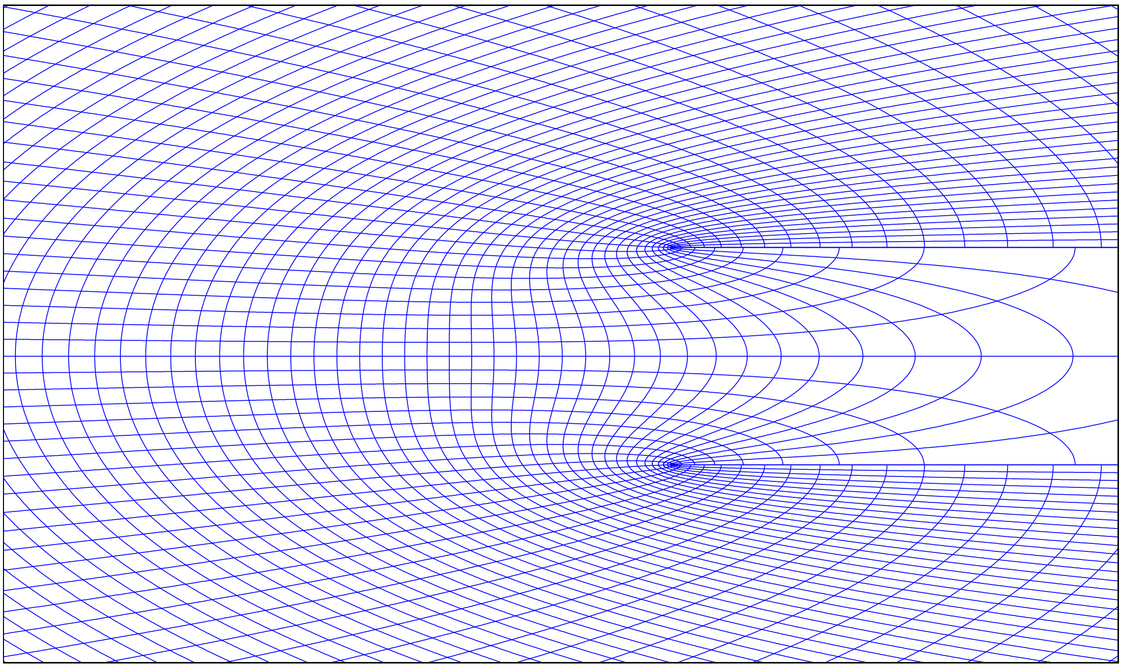

FIGURE 1. The image of $\Psi(z) = z^2/2 - \log(z)$

Since $\Psi(\mathbf{C}^+)$ contains a translate of $\mathbf{C}^+$ its boundary is contained in a halfplane $\Im(z) \le B$ for some real number B. It follows that $\Im(\Psi(A+i\varepsilon)) \le B+1$ for ε close to zero therefore $\int_{]w,A[}(1+u^2)\nu(du)$ is bounded above by a quantity independent of A and $\int_0^\infty u^2\,\nu(du) < +\infty$. □

From the description of starlike domains in section 3.2 we see that, if $\int_0^\infty u^2\nu(du) < +\infty$ then there exists some real number q_0 such that either $d_\Omega(q) = +\infty$ for $q \ge q_0$ or $d_\Omega(q) = -\infty$ for $q \ge q_0$. The domain $\Psi(\mathbf{C}^+)$ contains a translate of $\mathbf{C}^+$ if and only if the first case holds. It follows that, if $\Psi(\mathbf{C}^+)$ contains complex numbers of arbitrarily large imaginary part then $\Psi(\mathbf{C}^+)$ contains a translate of $\mathbf{C}^+$. Note that, by Cauchy-Schwarz inequality one has $\int_0^\infty u\,\nu(du) < +\infty$ and the quantity $\int_{\mathbf{R}} u\,\nu(du)$ is well defined in $[-\infty, +\infty[$.

Lemma 3.5. *Assume that $\int_0^\infty u^2\,\nu(du) < +\infty$.*

(1) If $\alpha < 0$ then $\Psi(\mathbf{C}^+)$ contains a translate of $\mathbf{C}^+$.
(2) If $\alpha = 0$ and $\beta + \int_{\mathbf{R}} u\,\nu(du) < 0$ then $\Psi(\mathbf{C}^+)$ contains a translate of $\mathbf{C}^+$.
(3) If $\alpha = 0$ and $\beta + \int_{\mathbf{R}} u\,\nu(du) \ge 0$ then $\Psi(\mathbf{C}^+)$ does not contain a translate of $\mathbf{C}^+$.

Proof.

(1) One has $\Psi(ye^{i\pi/4}) \sim \frac{-\alpha}{2}iy^2$ for large y therefore $\Psi(\mathbf{C}^+)$ contains complex numbers of arbitrarily large imaginary part and one concludes from the discussion before Lemma 3.5.
(2) If $\alpha = 0$ and $\beta + \int_{\mathbf{R}} u\,\nu(du) < 0$ then $\Re(\psi(iy)) < -\varepsilon$ for y large enough and some $\varepsilon > 0$. It follows that $\Im(\Psi(iy)) \to \infty$ as $y \to \infty$ and we conclude by the same argument.
(3) If $\alpha = 0$ and $\beta + \int_{\mathbf{R}} u\,\nu(du) = \gamma \ge 0$ then

$$\psi(z) = \gamma z + \int_{\mathbf{R}} \frac{1+u^2}{z-u}\nu(du)$$

and one has

$$\begin{aligned}\Im(\Psi(x+iy)-\Psi(x+i)) &= -\gamma(y-1)-\int_1^y\left[\int_{\mathbf{R}}\frac{(1+u^2)(x-u)}{(x-u)^2+w^2}\nu(du)\right]dw\\ &\leq -\gamma(y-1)+\int_1^y\left[\int_x^\infty\frac{(1+u^2)(u-x)}{(x-u)^2+w^2}\nu(du)\right]dw.\end{aligned}$$

One has, for $y \geq 0$,

$$\int_1^y \frac{v}{v^2+w^2}dw \leq \frac{\pi}{2} \tag{21}$$

therefore

$$\Im(\Psi(x+iy)-\Psi(x+i)) \leq \gamma(1-y)+\frac{\pi}{2}\int_0^\infty(1+u^2)\nu(du). \tag{22}$$

Since $\Im(\Psi(x+i))$ is uniformly bounded in x it follows that $\Im(\Psi(z))$ is uniformly bounded on $\mathbf{C}^+$.

□

Using Lemmas 3.4 and 3.5 we can now state necessary and sufficient conditions on ψ so that $\Psi(\mathbf{C}^+)$ contains a translate of $\mathbf{C}^+$.

Proposition 3.6. *Let ψ be a Nevanlinna function with canonical representation*

$$\psi(z)=\alpha z+\beta+\int_{-\infty}^{+\infty}\frac{1+uz}{z-u}\nu(du)$$

and Ψ be a primitive of $-\psi$, then $\Psi(\mathbf{C}^+)$ contains a translate of $\mathbf{C}^+$ if and only if one of the following exclusive conditions is fulfilled:

(1) $\int_0^\infty u^2\nu(du)<+\infty$ and $\alpha<0$.

(2) $\int_0^\infty u^2\nu(du)<+\infty$, $\alpha=0$ and $\beta+\int_{\mathbf{R}}u\,\nu(du)<0$.

Here are two examples illustrating the different situations.

Figure 2 shows $\psi(z)=-z^{1/2}$ where $\alpha=0$ and $\nu(du)=\frac{\sqrt{-u}du}{2\pi(1+u^2)}1_{u<0}$, with $\beta+\int u\,\nu(du)=-\infty$. One has $\Psi(z)=\frac{2}{3}z^{3/2}$ and the image $\Psi(\mathbf{C}^+)$ is a 3/4 plane, which contains the upper halfplane.

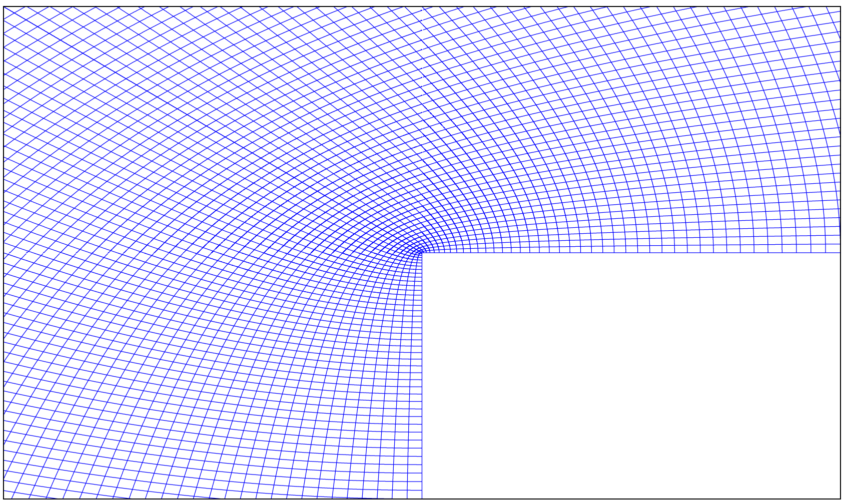

FIGURE 2. The image of $\Psi(z)=\frac{2}{3}z^{3/2}$

In Figure 3 one has $\psi(z) = z^{-1/2}$ where $\alpha = 0$ and $\nu(du) = \frac{du}{2\pi\sqrt{-u}(1+u^2)} 1_{u<0}$ while $\beta + \int u\, \nu(du) = 0$ and $\Psi(z) = -2z^{1/2}$. This time the image $\Psi(\mathbf{C}^+)$ is a quarterplane, it does not contain a translate of the upper halfplane.

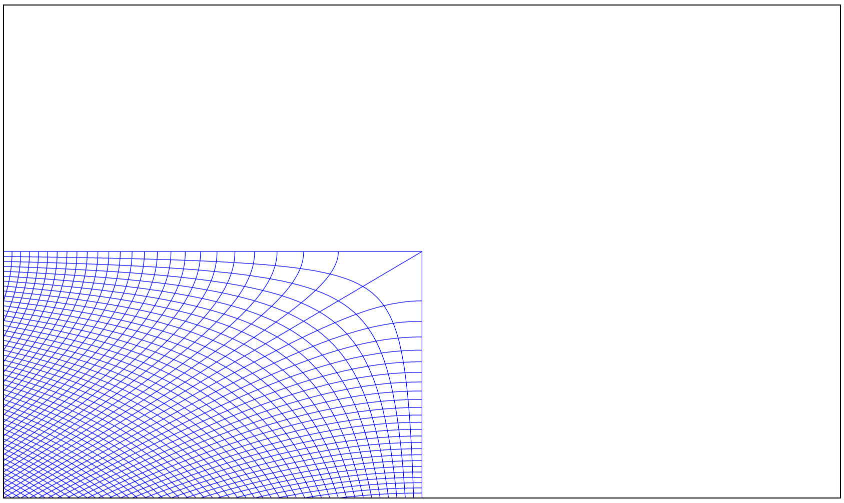

FIGURE 3. The image of $\Psi(z) = -2z^{1/2}$

4. Free Lévy processes with homogeneous transition probabilities

4.1. **Some preliminary computations.** I consider the case where $\mu_0 = \delta_0$ and will say a few words about the general case at the end of this section. By Theorem 2.2 we are trying to characterize Nevanlinna functions φ such that

(i) $\frac{\varphi(\zeta)}{\zeta} \to_{\substack{\zeta\to\infty \\ \zeta\in\Gamma_{\alpha,\beta}}} 0$ in every domain of the form $\Gamma_{\alpha,\beta}$

(ii) For any $t \geq 0$, $\varphi \circ F_t^{-1}$ has an analytic continuation to $\mathbf{C}^+$, with values in $\mathbf{C}^-$.

Here $F_t(z)$, for $t \geq 0$ is the semi-group of analytic maps of $\mathbf{C}^+$ obtained by solving

$$\frac{\partial F_t}{\partial t} + \varphi(F_t) = 0 \tag{23}$$

with initial condition $F_0(z) = z$,

In order to solve (23) it is natural to introduce a primitive of $-1/\varphi$, denoted Φ. Indeed Φ, being the primitive of a Nevanlinna function, is such that $\Phi(\mathbf{C}^+) + t \subset \Phi(\mathbf{C}^+)$ for $t \geq 0$ and one has $F_t(z) = \Psi(\Phi(z) + t)$ where Ψ is the inverse of the conformal mapping Φ. Note that one has $\Psi' = -\varphi \circ \Phi$.

4.2. The free nonlinear Lévy-Khinchine formula.

4.2.1. We denote by ψ a Nevanlinna function and Ψ a primitive of $-\psi$, such that $\Psi(\mathbf{C}^+)$ contains $\mathbf{C}^+$, as characterized in section 3.3. Let $\Phi = \Psi^{-1} : \Psi(\mathbf{C}^+) \to \mathbf{C}^+$, then the restriction of Φ to $\mathbf{C}^+$ maps conformally $\mathbf{C}^+$ to a domain inside $\mathbf{C}^+$. It follows that

$$F_t(z) := \Psi(\Phi(z) + t)$$

is well defined for all $z \in \mathbf{C}^+$ and all $t \in \mathbf{R}$. Moreover these functions satisfy the equation

$$\frac{d}{dt} F_t(z) + \varphi(F_t(z)) = 0, \qquad F_0(z) = z, \quad \text{for } z \in \mathbf{C}^+$$

where $\varphi = \psi \circ \Phi$ is a Nevanlinna function. One has $F_t(F_s(z)) = F_{t+s}(z)$ for all $s, t \in \mathbf{R}$ therefore $F_{-t} = F_t^{-1}$. The maps Ψ, Φ conjugate the flow F_t on $\Psi(\mathbf{C}^+)$ with the horizontal translation flow on $\mathbf{C}^+$. For every t the function $\varphi \circ F_t^{-1}(z) = \psi(\Phi(z) - t)$ is a Nevanlinna function. Since φ is a Nevanlinna function there exists $a = \lim_{y\to\infty} \frac{\varphi(iy)}{iy} \in]-\infty, 0]$. Assume that $a < 0$, then $\Phi(\mathbf{C}^+)$ is a domain included in a horizontal strip of height $-a\pi$ (this is easy to see if φ is rational and follows in the general case by approximation). However it follows from the proof of Lemma 3.5, parts (1) and (2), that $\Psi^{-1}(\mathbf{C}^+) = \Phi(\mathbf{C}^+)$ contains complex numbers with arbitrarily large imaginary parts. We conclude that $a = 0$ and the function φ satisfies all the conditions in Theorem 2.2.

4.2.2. For the converse, we use a technique of Cowen [7]. In Theorem 3.2 he shows that, under very general conditions, for an analytic function φ of the disk to itself there exists a linear fractional transformation Φ and an analytic function σ such that $\Phi \circ \sigma = \sigma \circ \varphi$ (see also [9], [1] for related results). Since we deal with a very particular case we obtain a more precise result and also we need less machinery, in particular we do not use the uniformization theorem.

Let φ be a Nevalinna function satisfying the hypothesis of Theorem 2.2 and let $(F_t)_{t\geq 0}$ be the solution to

$$\frac{d}{dt}F_t(z) + \varphi(F_t(z)) = 0, \quad F_0(z) = z.$$

Let Φ be a primitive of $-1/\varphi$ and $\Omega_0 = \Phi(\mathbf{C}^+)$. Since $-1/\varphi$ is a Nevanlinna function the function Φ is a conformal mapping $\Phi : \mathbf{C}^+ \to \Omega_0$ with inverse $\Psi : \Omega_0 \to \mathbf{C}^+$. The domain $-\Omega_0$ is starlike therefore $\Omega_0 + t \subset \Omega_0$ for all $t \geq 0$ moreover one has $F_t(z) = \Psi(\Phi(z) + t)$ for all $z \in \mathbf{C}^+, t \geq 0$. For each $t > 0$ the function F_t is univalent with inverse F_t^{-1} defined at least in a domain of the form $\Gamma_{\alpha,\beta}$ and one has

$$\frac{d}{dt}F_t^{-1}(z) - \varphi(F_t^{-1}(z)) = 0$$

in this domain, therefore putting $F_{-t} = F_t^{-1}$ the equation $\frac{d}{dt}F_t(z) + \varphi(F_t(z)) = 0$ also holds for $t < 0$ at least in some domain $\Gamma_{\alpha,\beta}$. The assumption is that, for all $t > 0$, $\varphi \circ F_t^{-1}$ extends analytically to a Nevalinna function. For every t one has

$$\frac{d}{dz}F_t(z) = \frac{\varphi(F_t(z))}{\varphi(z)}.$$

For $t < 0$ the function $z \mapsto \varphi(F_t(z))$ has an analytic continuation to $\mathbf{C}^+$, therefore the functions $\frac{d}{dz}F_t$ and F_t also have such an analytic continuation.

If $z \in \Omega_0$ and $t \geq 0$ one has $\Psi(z) = F_{-t}(\Psi(z + t))$. Let $\Omega = \cup_{t\geq 0}(\Omega_0 - t)$, then for every $z \in \Omega$ there exists $t \geq 0$ such that $z + t \in \Omega_0$ therefore $\Psi(z + t) \in \mathbf{C}^+$ and $F_{-t}(\Psi(z + t))$ is well defined, moreover it does not depend on t. This gives an analytic continuation of Ψ to Ω, such that Ψ' takes values with negative imaginary part. The domain Ω is stable under translation by real numbers. Since $\varphi(iy)/iy \to 0$ as $y \to \infty$ the function Φ takes values with arbitrarily high imaginary parts, therefore Ω is either the whole complex plane, or a translate of the upper halfplane. In the first case, since Ψ is univalent, it must be a polynomial of degree 1 and φ is a constant with negative imaginary part. In the second case, since Φ was defined up to an integration constant, we can assume that Ω is equal to $\mathbf{C}^+$ and Ψ is a univalent map whose derivative is $-\psi$ with ψ a Nevalinna function, moreover $\Psi(\mathbf{C}^+)$ contains $\mathbf{C}^+$ so that we are in the situation of section 4.2.1.

4.3. Finally we can summarize the preceding results and state the free nonlinear Lévy-Khinchine formula, which characterizes the Nevanlinna functions appearing in Theorem 2.2, for $\mu_0 = \delta_0$.

Theorem 4.1. *Let φ be a FAL2 function, then either φ is a constant, or there exists a univalent function Ψ, with inverse Φ and derivative $\Psi' = -\psi$, where ψ is a Nevanlina function, such that $\Psi(\mathbf{C}^+)$ contains $\mathbf{C}^+$ and such that $\varphi = \psi \circ \Phi$. Conversely, for any functions ψ, Ψ, Φ satisfying the above requirements, the Nevanlinna function $\varphi = \psi \circ \Phi$ is a FAL2 function.*

As we see from the above theorem, the FAL2 functions can be parametrized by a convex set, i.e. the functions Ψ, however going from Ψ to $\varphi = -\Psi' \circ \Psi^{-1}$ is a nonlinear map.

Finally we say a few words about the case of a general initial measure μ_0. In this case, using arguments as in 4.2.2 one can see that φ is a FAL2 function with initial measure μ_0 if and only if it is associated to a Nevalinna function ψ as in Theorem 4.1 above and $F_{\mu_0}^{-1}$ can be analytically continued to $\mathbf{C}^+$ with values in $\Psi(\mathbf{C}^+)$. Details are left to the reader.

5. The case of free multiplicative convolution

In this section we consider the case of free multiplicative convolutions of measures, on the unit circle and on the positive halfline, recalling Theorems 3.5, 3.6, 4.6.1 and 4.6.2 of [6] and giving the analogues of the nonlinear free Lévy-Khinchine formula, Theorem 4.1. Since this is very similar to the additive case, we only sketch the arguments.

5.1. **Free multiplicative convolution on the circle.**

5.1.1. Let μ and ν be probability measures on the unit circle $\mathbf{T}$ and let U and V be two unitary elements in some non-commutative probability space (A, τ), with respective distributions μ and ν, then the distribution of UV is called the free multiplicative convolution of μ and ν and is denoted by $\mu \boxtimes \nu$. Define

$$\eta_\mu(z) = \int_{\mathbf{T}} \frac{z\xi}{1 - z\xi} d\mu(\xi)$$

Let $\mathcal{M}_*$ be the set of probability measures on $\mathbf{T}$ such that $\int_{\mathbf{T}} \xi d\mu(\xi) \neq 0$. If $\mu \in \mathcal{M}_*$ then the function $\frac{\eta_\mu}{1+\eta_\mu}$ has a right inverse, called $\tilde{\chi}_\mu$, defined in a neighbourhood of 0, such that $\tilde{\chi}_\mu(0) = 0$, and we let $\Sigma_\mu(z) = \frac{1}{z}\tilde{\chi}_\mu(z)$ be the Σ-transform of μ. Then, for any measures $\mu, \nu \in \mathcal{M}_*$, one has $\mu \boxtimes \nu \in \mathcal{M}_*$ and

$$\Sigma_{\mu\boxtimes\nu}(z) = \Sigma_\mu(z)\Sigma_\nu(z)$$

in some neighbourhood of zero where these three functions are defined. If one of the measures has zero mean then $\mu \boxtimes \nu$ is the uniform measure on $\mathbf{T}$.

5.1.2. The analogue, for free multiplicative convolution on $\mathbf{T}$, of the Lévy-Khinchine formula, states that a probability measure on $\mathbf{T}$ is infinitely divisible, for the free multiplicative convolution, if and only if its Σ transform can be written as $\Sigma_\mu(z) = \exp(u(z))$ where u is an analytic function on the open unit disk $\mathbf{D}$, taking values with nonnegative real parts. Such a function has a representation of the form

$$u(z) = i\alpha + \int_{\mathbf{T}} \frac{1 + \zeta z}{1 - \zeta z} d\nu(\zeta)$$

for some finite positive measure ν on $\mathbf{T}$, and real number α.

5.1.3. The Markov and subordination property of the free multiplicative convolution is given by the following Theorem 3.5 from [6].

Theorem 5.1. *Let (A, τ) be a non commutative probability space, $B \subset A$ be a von Neumann subalgebra, and $U, V \in A$ such that U and V are unitary, with respective distributions μ and ν, one has $U \in B$, and V is free from B, then there exists a Feller Markov kernel $\mathcal{K} = k(\xi, d\omega)$ on $\mathbf{T} \times \mathbf{T}$ and an analytic function F, defined on $\mathbf{D}$, such that*

(1) For any bounded Borel function f on S, one has $\tau(f(UV)|B) = \mathcal{K}f(U)$.
(2) $|F(z)| \leq |z|$, for $z \in \mathbf{D}$.
(3) for all $z \in \mathbf{D}$ one has $\int_{\mathbf{T}} \frac{z\omega}{1-z\omega} k(\xi, d\omega) = \frac{F(z)\xi}{1-F(z)\xi}$
(4) for all $z \in \mathbf{D}$ one has $\eta_\mu(F(z)) = \eta_{\mu\boxtimes\nu}(z)$

If $\mu \in \mathcal{M}_*$, the map F is uniquely determined by the properties (2) *and* (4).

5.1.4. Processes with unitary multiplicative free increments are defined analogously to the additive case, and such processes with homogeneous transition probabilities were called FUL2 processes, for which we now recall the analogue of Theorem 2.2.

Theorem 5.2. *Let $\mu_t, t \geq 0$ and $(\mu_{s,t})_{s<t\in\mathbf{R}_+}$ be families of probability measures on $\mathbf{T}$ satisfying*

$$\mu_s \boxtimes \mu_{s,t} = \mu_t; \quad \mu_{s,t} \boxtimes \mu_{t,u} = \mu_{s,u} \tag{24}$$

for all $s < t < u$. Let $(\mathcal{K}_{s,t})_{s<t\in\mathbf{R}_+}$ be the corresponding Markov transition functions on $\mathbf{T}$. Assume that the kernels are time homogeneous, then the kernels $\mathcal{L}_t \equiv \mathcal{K}_{0,t}$ for $t \geq 0$, form a Feller Markov semi-group. Let $F_{s,t}$ be the analytic functions associated to the kernels $\mathcal{K}_{s,t}$ by Theorem 5.1. The maps $F_t \equiv F_{0,t}$, where F_0 is the identity function, form a continuous semigroup, under composition, of analytic transformations of $\mathbf{D}$, moreover there exists a function u on $\mathbf{D}$, taking values with nonnegative real part, such that the maps F_t, for $t \geq 0$, satisfy the differential equation

$$\frac{\partial F_t}{\partial t} + F_t u(F_t) = 0, \qquad F_0(z) = z. \tag{25}$$

Conversely, let u be an analytic function on $\mathbf{D}$, such that $\Re(u(z)) \geq 0$ for all $z \in \mathbf{D}$, and let F_t, for $t \geq 0$, be the solution of the differential equation $\frac{\partial F_t}{\partial t} + F_t u(F_t) = 0$, with $F_0(z) = z$, then there exists a free multiplicative Lévy process of the second kind, with initial distribution μ_0, with associated semi-group of maps $(F_t)_{t\in\mathbf{R}_+}$, if and only if, for every $t > 0$ the function $u \circ F_t^{-1} \circ \tilde{\chi}_{\mu_0}^{-1}$ has an analytic continuation to $\mathbf{D}$, taking values with nonnegative real part.

Functions like u in the above theorem are called FUL2 Lévy functions in [6]. In the following we consider the case where $\mu_0 = \delta_1$.

Let u be a FUL2 function. Let us change variables and put $z = e^{iw}$ with $\Im(w) > 0$. Then the function $\tilde{u}(w) = -iu(e^{iw})$ is a Nevanlinna function which is periodic of period 2π, and the differential equation (25) becomes $\frac{\partial \tilde{F}_t}{\partial t} + \tilde{F}_t\tilde{u}(\tilde{F}_t) = 0$, with $\tilde{F}_0(w) = w$. One has $\tilde{F}_t(w + 2\pi) = \tilde{F}_t(w) + 2\pi$. Introducing a primitive of $-1/\tilde{u}$ and reasoning as above we see that there must exist a 2π-periodic Nevanlinna function ψ, with Ψ a primitive of $-\psi$ such that $\Psi(\mathbf{C}^+)$ contains $\mathbf{C}^+$, however if ψ is 2π periodic then the measure $(1 + u^2)\nu(du)$ is also 2π periodic therefore the integral $\int_0^\infty (1 + u^2)\nu(du) = \infty$, unless ψ is constant. We conclude:

Theorem 5.3. *All FUL2 functions are constant.*

5.2. Multiplicative free convolution on the positive halfline.

5.2.1. Let μ be a probability measure on $\mathbf{R}_+$, different from δ_0, and define

$$\eta_\mu(z) = \int_{\mathbf{R}_+} \frac{z\xi}{1 - z\xi} d\mu(\xi)$$

This function is analytic on $\mathbf{C} \setminus \mathbf{R}_+$, and $\eta_\mu(\bar{z}) = \bar{\eta}_\mu(z)$ for $z \in \mathbf{C} \setminus \mathbf{R}_+$. The function $\frac{\eta_\mu}{1+\psi_\mu}$ is univalent on $i\mathbf{C}^+$, its image contains a neighbourhood of the interval $]\mu(\{0\}) - 1, 0[$ in $\mathbf{C}$. Let $\tilde{\chi}_\mu$ be the right inverse of this function on the image $\frac{\eta_\mu}{1+\eta_\mu}(i\mathbf{C}^+)$. We define the Σ-transform of μ as the function $\Sigma_\mu(z) = \frac{1}{z}\tilde{\chi}_\mu(z)$ defined on $\frac{\eta_\mu}{1+\eta_\mu}(i\mathbf{C}^+)$.

Let μ and ν be two probability measures on $\mathbf{R}_+$, different from δ_0 and let S and T be free random variables, in some non-commutative probability space, with respective distributions μ and ν, then the distribution of $S^{\frac{1}{2}}TS^{\frac{1}{2}}$ which is also the distribution of $T^{\frac{1}{2}}ST^{\frac{1}{2}}$, is the free multiplicative convolution of μ and ν, denoted by $\mu \boxtimes \nu$, and one has $\Sigma_{\mu\boxtimes\nu} = \Sigma_\mu\Sigma_\nu$ on some neighbourhood of the interval $] - \varepsilon, 0[$, for some $\varepsilon > 0$.

5.2.2. The Markov and subordination property of the free multiplicative convolution on the positive halfline is given by the following Theorem 3.6 from [6].

Theorem 5.4. *Let (A,τ) be a non commutative probability space, $B \subset A$ be a von Neumann subalgebra, and $S,T \in \tilde{A}_{sa}$ such that S and T are positive, with respective distributions μ and ν, different from δ_0, one has $S \in \tilde{B}_{sa}$ and T is free from B, then there exists a Feller Markov kernel $\mathcal{K} = k(u,dv)$ on $\mathbf{R}_+ \times \mathbf{R}_+$ and an analytic function F, defined on $\mathbf{C} \setminus \mathbf{R}_+$, such that*

(1) for any bounded Borel function f on S one has $\tau(f(S^{1/2}TS^{1/2})|B) = \mathcal{K}f(S)$.
(2) $F(\zeta) \in \mathbf{C}^+$, $F(\bar{\zeta}) = \bar{F}(\zeta)$ and $\mathrm{Arg}\,(F(\zeta)) \geq \mathrm{Arg}\,(\zeta)$ for $\zeta \in \mathbf{C}^+$.
(3) for all $\zeta \in \mathbf{C}^+$ one has $\int_{\mathbf{R}_+} \frac{\zeta v}{1-\zeta v} k(u,dv) = \frac{F(\zeta)u}{1-F(\zeta)u}$.
(4) For all $\zeta \in \mathbf{C}^+$ one has $\eta_\mu(F(\zeta)) = \eta_{\mu \boxtimes \nu}(\zeta)$.

The map F is uniquely determined by the properties (2) *and* (4).

5.2.3. Again one has a Lévy-Khinchine formula, where freely infinitely divisible probability measures on $\mathbf{R}_+$ are characterized as having Σ-transforms of the form $\Sigma_\mu(z) = \exp(v(z))$, where v is an analytic function on $\mathbf{C} \setminus \mathbf{R}_+$, with $v(\bar{z}) = \bar{v}(z)$, and $v(\mathbf{C}^+) \subset \mathbf{C}^- \cup \mathbf{R}$. Such functions have the representation

$$v(z) = az + b + \int_0^{+\infty} \frac{1+tz}{z-t} d\nu(t)$$

for some real numbers $a \leq 0$ and b, and ν a finite positive measure on $\mathbf{R}_+$. The analogue of Theorems 2.2 and 5.2 is the following.

Theorem 5.5. *Let $\mu_t, t \geq 0$ and $(\mu_{s,t})_{s<t\in\mathbf{R}_+}$ be families of probability measures on $\mathbf{R}_+$ satisfying*

$$\mu_s \boxtimes \mu_{s,t} = \mu_t; \quad \mu_{s,t} \boxtimes \mu_{t,u} = \mu_{s,u} \tag{26}$$

for all $s < t < u$. Let $(\mathcal{K}_{s,t})_{s<t\in\mathbf{R}_+}$ be the corresponding Markov transition functions on $\mathbf{R}_+$. Assume that the kernels are time homogeneous, then the kernels $\mathcal{L}_t \equiv \mathcal{K}_{0,t}$ for $t \geq 0$, form a Feller Markov semi-group. Let $F_{s,t}$ be the analytic functions associated to the kernels $\mathcal{K}_{s,t}$ by Theorem 5.4. The maps $F_t \equiv F_{0,t}$ for $t \geq 0$, form a semigroup of analytic maps on $\mathbf{C} \setminus \mathbf{R}_+$, such that $t \mapsto \mathrm{Arg}\, F_t(z)$ is an increasing map for $z \in \mathbf{C}^+$. There exists an analytic function v on $\mathbf{C} \setminus \mathbf{R}_+$, $\mathbf{C}^- \cup \mathbf{R}$, such that $v(\bar{z}) = \bar{v}(z)$ for $z \in \mathbf{C}^+$, $v(\mathbf{C}^+) \subset \mathbf{C}^- \cup \mathbf{R}$, and the maps F_t, for $t \geq 0$, satisfy the differential equation $\frac{\partial F_t}{\partial t} + F_t v(F_t) = 0$.
Let v be an analytic function on $\mathbf{C} \setminus \mathbf{R}_+$, such that $v(\mathbf{C}^+) \subset \mathbf{C}^- \cup \mathbf{R}$, and $v(\bar{z}) = \bar{v}(z)$ for all $z \in \mathbf{C}^+$, and let F_t, for $t \geq 0$, be the solution of the differential equation $\frac{\partial F_t}{\partial t} + F_t u(F_t) = 0$, with $F_0(z) = z$, then there exists a free multiplicative Lévy process of the second kind, with initial distribution μ_0, with associated semi-group of maps $(F_t)_{t\in\mathbf{R}_+}$, if and only if, for every $t > 0$ the function $v \circ F_t^{-1} \circ \tilde{\chi}_{\mu_0}^{-1}$ has an analytic continuation to $\mathbf{C} \setminus \mathbf{R}_+$, such that $v(\bar{z}) = \bar{v}(z)$, and $v(\mathbf{C}^+) \subset \mathbf{C}^- \cup \mathbf{R}$.

5.2.4. We now determine all FPL2 functions, in the case $\mu_0 = \delta_1$.

We change variables and put $z = -e^w$ where $z \in \mathbf{C} \setminus \mathbf{R}_+$ and $w \in \mathcal{S}$ where $\mathcal{S}$ is the symmetric horizontal strip $\mathcal{S} = \{w | \Im(w) \in]-\pi, \pi[\}$. Let v be a FPL2 function and define $\tilde{v}(w) = v(-e^w)$. Then v is analytic in the strip $\mathcal{S}$, satisfies $v(\bar{w}) = \bar{v}(w)$ and takes values with positive imaginary part on $\mathcal{S} \cap \mathbf{C}^+$. With $F_t(-e^w) := -\exp(\tilde{F}_t(w))$ the equation $\frac{\partial F_t(z)}{\partial t} + F_t v(F_t(z)) = 0$ becomes $\frac{\partial \tilde{F}_t(w)}{\partial t} + \tilde{v}(\tilde{F}_t(w)) = 0$. The function $\tilde{v}$ has at most one zero ω_0, on the real line. Let $\tilde{V}(w)$ be a primitive of $-1/\tilde{v}$ on $\mathcal{S} \setminus]-\infty, \omega_0]$ such that $\tilde{V}$ takes real values on $]\omega_0, +\infty]$. One has $\tilde{V}(\bar{w}) = \bar{\tilde{V}}(w)$, moreover $\tilde{V}$ is univalent on $\mathcal{S} \setminus]-\infty \omega_0]$ and the domain $\Omega = \tilde{V}(\mathcal{S})$ satisfies $\bar{\Omega} = \Omega$ and $\Omega + t \subset \Omega$ for $t \geq 0$. Let $\tilde{W}$ be the inverse of $\tilde{V}$ then one has $\tilde{F}_t(w) = \tilde{W}(\tilde{V}(w) + t)$. As in section 4.2.2 one can extend the map $\tilde{W}$ to a univalent function on the domain $\Omega_\infty = \cup_{t\in\mathbf{R}}(\Omega + t)$ which is either the whole complex plane or a horizontal strip, symmetric with respect to the real axis. The function $\tilde{W}$ satisfies $\bar{\tilde{W}}(z) = \tilde{W}(\bar{z})$ moreover $\Im(\tilde{W}'(w)) \leq 0$ for $\Im(w) > 0$. From these considerations we deduce the analogue of Theorem 4.1:

Theorem 5.6. *Let v be a FPL2 function, then either v is a constant, or there exists a univalent function U defined on a symmetric horizontal strip $\mathcal{T}$, such that $U(\mathcal{T})$ contains $\mathcal{S}$, one has*

$U(\bar{z}) = \bar{U}(z)$, and $U' = -u$, with $\Im(u(z)) > 0$ for $\Im(z) > 0$, and $v = u \circ U^{-1}$. Conversely, for any function U satisfying the above requirements, the function $v = u \circ U^{-1}$ is a FPL2 function.

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