FIBERED COHOMOLOGY CLASSES IN DIMENSION THREE, TWISTED ALEXANDER POLYNOMIALS AND NOVIKOV HOMOLOGY

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Abstract. We prove that for "most" closed 3-dimensional manifolds N, the existence of a closed non singular one-form in a given cohomology class $u \in H^1(M, \mathbb{R})$ is equivalent to the non-vanishing modulo p of all twisted Alexander polynomials associated to finite Galois coverings of N. When $u \in H^1(M, \mathbb{Z})$, a stronger version of this had been proved by S. Friedl and S. Vidussi in 2013, asking only the non-vanishing of the Alexander polynomials.

1. Introduction and statement of the main result

We consider N a closed connected 3-manifold, $G := \pi_1(M)$ and $u \in \mathrm{Hom}(G, \mathbb{R}) \setminus \{0\} = H_1(M, \mathbb{R}) \setminus \{0\}$. Denote $\mathrm{rk}(u)$ the rank of u, ie the number of free generators of $u(G)$. We are interested in the existence of a nonsingular closed 1-form ω in the class u. If such a form exist, we say that u is *fibered*. The reason is that if $\mathrm{rk}(u) = 1$ so that $au(G) \subset \mathbb{Z}$ for a suitable $a \neq 0$, such a form is f^*dt for f a fibration to $S^1 = \mathbb{R}/\mathbb{Z}$.

More generally, by [Tischler 1970], if u fibers then N fibers over S^1: perturb ω to $\omega' = \omega + \varepsilon$ such that $\mathrm{rk}([\omega']) = 1$ and ε is C^0-small. Then ω' is still nonsingular, thus N fibers.

An answer to this question was given in rank 1 by [Stallings 1962]: *if* $\mathrm{rk}(u) = 1$*, u fibers if and only if* $\ker u$ *is finitely generated*. In any rank, the famous paper [Thurston 1986] introducing the Thurston (semi-)norm in $H^1(M;\mathbb{R})$ proved the following results: 1) the unit ball of the norm is a "integer polyhedron", ie it is defined by a finite number of inequalities $u(g) \leq n$, $g \in G$, $n \in \mathbb{N}^*$; 2) *the set of fibered* $u \in H^1(M;\mathbb{R}) \setminus \{0\}$ *is a cone over the union of some maximal open faces of the unit sphere of the Thurston norm.* Note that thanks to Stallings, to know if a given face is "fibered", it suffices to test one element u of rank one and see if $\ker u$ is finitely generated.

In the 2000s and beginning of 2010s, S. Friedl and S. Vidussi studied again this question mostly in rank one, in connection with what was then a conjecture of Taubes: u fibers if and only if $u \wedge [dt] + a \in H^1(M \times S^1)$ is represented by a symplectic form, where $a \in H^2(M;\mathbb{R})$ satisfies $a \wedge u \neq 0$. The starting point was the relation of Seiberg-Witten invariants of $N \times S^1$ and *twisted Alexander polynomials*, see below and Section 3. They ultimately solved that conjecture in [Friedl-Vidussi 2013], and obtained as a byproduct a new answer for the characterization of fibered classes in the case of rank 1: *if* $\mathrm{rk}(u) = 1$*, u fibers if and only if all twisted Alexander polynomials* $\Delta^H(G, u)$ *are nonzero.*

Let us describe briefly what are these twisted Alexander polynomials. In fact, we do it only for a special case, which is already sufficient: those associated to finite covers, see [Friedl-Vidussi 2008], section 3.2. Let H be a normal subgroup of G with finite index (denoted $H \triangleleft_{f.i.} G$). By definition, $\Delta^H(G, u)$ is an element of $\mathbb{Z}[G/\ker u] \approx \mathbb{Z}[t_1^{\pm 1}, \cdots, t_r^{\pm 1}]$ defined up to multiplication by a unit: it is the *order* of

$$H_1(G; \mathbb{Z}[G/\ker u])^{G/H} \approx H_1(H; \mathbb{Z}[G/\ker u])$$

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viewed as as a $\mathbb{Z}[G/\ker u]$-module, ie the greatest common divisor of the m-minors of A for a presentation

$$\mathbb{Z}[\mathbb{Z}^r]^{(I)} \xrightarrow{\times A} Z[\mathbb{Z}^m] \to H_1(G;\mathbb{Z}[G/\ker u])^{G/H}.$$

It is not difficult to prove that, if u is fibered, $\Delta^H(G,u)$ is always nonzero and even "bi-monic", ie its u-maximal and u-minimal terms have a coefficient ± 1.

We can now dstate our main result.

Theorem 1.1. *Assume that $\widetilde{M}$ is contractible and $G := \pi_1(M)$ is virtually residually torsion-free nilpotent (VRTFN). If there exists a prime p such that for every $H \triangleleft_{f.i.} G$, $\Delta^H(G,u) \not\equiv 0 \bmod p$, then u is fibered, ie represented by a nonsingular closed 1-form.*

Comments. 1) In rank one, our result is weaker than [Friedl-Vidussi 2013]. However, even in that case we believe that our proof, whose only "hard" ingredient is [Thurston 1986], may be of interest.

2) It is easy to prove that if N fibers over S^1, then $\pi_1(M)$ is residually torsion-free nilpotent (RTFN) (see [Koberda 2013]). Building on [Agol 2014], [Koberda 2013] proves that $\pi_1(M)$ is VTRFN for all geometric manifolds which are not Sol. Presumably this is always true.

The hypothesis that $\widetilde{M}$ is not contractible can be dispensed with: if it does not hold, then (since $b_1(M) > 0$) we are in one of the two following cases: either N is nonprime thus nonfibered and the twisted Alexander polynomials always vanish; or N fibers over S^1 with fiber S^2 or $\mathbb{RP}^2$.

3) Of course, one expects that the hypothesis can be weakened to $\Delta^H(G,u) \neq 0$.

2. Sketch of the proof and content of the paper

The main idea is to express the fibering condition on u by the vanishing of some *Novikov homology* $H_1(G,u;\mathbb{Z}/p\mathbb{Z})$, and the nonvanishing of $\Delta^H(G,u)$ by the vanishing of some *Abelianized relative Novikov homology* $H_1^{ab}(G,H,u;\mathbb{Z}/p\mathbb{Z})$. In turn, these vanishings are expressed by the invertibility of some matrix in the Novikov rings $\mathbb{Z}/p\mathbb{Z}[G]_u$ and of its image in $\mathbb{Z}/p\mathbb{Z}[G/H\cap\ker u]_{\overline{u}}$.

Then the theorem is reduced to a result about "finite detectability of invertible matrices". We prove it for a VRTFN group, using results of [Hall 1959], the bi-orderability of a RTFN group and the associated Mal'cev-Neumann completion of $\mathbb{Z}/p\mathbb{Z}[G]$, which contains the Novikov ring by a remark of [Kielak 2018].

We now describe the content of the paper.

In Section 3, we define the twisted Alexander polynomials $\Delta^H(G,u)$, and compare them to their version modulo p.

In Section 4, we define the Novikov ring $R[G]_u$ (R a commutative unital ring), and the Novikov homology $H_*(G,u;R)$. We quote the result of [Bieri-Neumann-Strebel 1987], building upon previous results of Stallings and Thurston: *if $G = \pi_1(M)$ with N a closed 3-manifold, u fibers if and only if $H_1(G,u;\mathbb{Z}) = 0$, $G = \pi_1(M)$.*

In Section 5, we compute $H_1(G,u;R)$ via a presentation of G and Fox differential calculus, and give a topological characterization of the property ($H_1(G,u;R) = 0$) which implies that it is independent of R: this is a remark of [Kielak 2018] (it was implicit in [Bieri-Neumann-Strebel 1987] and [Sikorav 1987], but never explicitly stated). This will be used with $R = \mathbb{Z}/p\mathbb{Z}$.

In Section 6, we explain the relations between twisted Alexander polynomials and an"Abelian relative" version of Novikov homology.

In Section 7, we precise the computation of $H_1(G,u;R)$ for $G = \pi_1(M^3)$ with $\widetilde{M}$ contractible, thanks to the form of a presentation of G given by a handle decomposition and Poincaré duality. We deduce that ($H_1(G,u;R) = 0$) is equivalent to the invertibility in $\mathrm{M}_{m-1}(R[G]_u)$ of some matrix $A \in \mathrm{M}_{m-1}(R[G])$ where m is the genus of the decomposition.

Similarly, the vanishing of $H_1^{ab}(G/(H\cap\ker u),\overline{u};R)$ is equivalent to the invertibility of the image of A in $\mathrm{M}_{m-1}(R[G/H\cap\ker u]_{\overline{u}})$. Thus we have reduced Theorem 1.1 to Theorem 7.2: *if G is VRTFN, a matrix $A\in\mathrm{M}_n(\mathbb{Z}/p\mathbb{Z}[G])$ whose image in $\mathrm{M}_n(\mathbb{Z}/p\mathbb{Z}[G/H]_u)$ is invertible for every $H\triangleleft_{f.i.}G$, is invertible in $\mathrm{M}_n(\mathbb{Z}/p\mathbb{Z}[G]_u)$.*

Theorem 7.2 is proven in Section 11. There are two main ingredients:

- (Sections 8 and 9) the case when G is nilpotent (Section 9), where we use ideas from [Hall 1959] about simple $\mathbb{Z}/p\mathbb{Z}[G]$-modules to prove a stronger result (finite detectability of full ideals);
- (Section 10) the fact that when G is RTFN, it admits a bi-invariant order, thus one can embed $\mathbb{Z}/p\mathbb{Z}[G]$ in the Mal'cev-Neumann completion $\mathbb{Z}/p\mathbb{Z}\langle G\rangle$, which is a skew-field; moreover, by a remark of [Kielak 2018] the order can be chosen so that $\mathbb{Z}/p\mathbb{Z}\langle G\rangle$ contains $\mathbb{Z}/p\mathbb{Z}[G]_u$. Actually, we work mostly with the division closure of $\mathbb{Z}/p\mathbb{Z}[G]$ in $\mathbb{Z}/p\mathbb{Z}\langle G\rangle$, whose elements have "controlled" support.

Notations. In the following text, G is a finitely generated group and $u:G\to\mathbb{R}$ a nonzero homomorphism. Thus $G/\ker u\approx\mathbb{Z}^r$, $r=\mathrm{rk}(u)$, and $\mathbb{Z}[G/\ker u]\approx\mathbb{Z}[t_1^{\pm},\cdots,t_r^{\pm1}]$ is a UFD (unique factorization domain). More generally, if R is a UFD, $R[G/\ker u]\approx R[t_1^{\pm},\cdots,t_r^{\pm1}]$ is a UFD.

3. Twisted Alexander polynomials

3.1. **Order of a finitely generated module over a UFD.** Let M be finitely generated R-module where R is a UFD. [usually, one requires R to be Noetherian, but actually it is not necessary]. One defines its order $\mathrm{ord}_R(M)$ as the greatest common divisor (gcd) of the m-minors of a matrix $A\in\mathrm{M}_{(I),m}(R)$ where

$$R^{(I)}\xrightarrow{r}R^m\xrightarrow{p}M$$

is a presentation of M with r represented by the right multiplication by A (we use right instead of left since later we will have mostly noncommutative rings, and we prefer to work with left modules). It is easy to prove that this does not depend on the presentation: if $R^{(I_1)}\xrightarrow{\times A_1}R^{m_1}\xrightarrow{p_1}M$ and $R^{(I_2)}\xrightarrow{\times A_1}R^{m_2}\xrightarrow{p_2}M$ are two presentations, one can lift p_1 to $\times B:R^{m_1}\to R^{m_2}$ and obtain a presentation

$$R^{m_1}\oplus R^{(I_2)}\xrightarrow{\times A_1}R^{m_1}\oplus R^{m_2}\xrightarrow{p_1+p_2}M\ ,\ A=\begin{pmatrix}\mathrm{I}_{m_1}&B\\0&A_2.\end{pmatrix}.$$

Similarly, there is a presentation

$$R^{m_1}\oplus R^{(I_2)}\xrightarrow{\times C}R^{m_1}\oplus R^{m_2}\xrightarrow{p_1+p_2}M\ ,\ C=\begin{pmatrix}A_1&0\\D&\mathrm{I}_{m_2}.\end{pmatrix}.$$

Thus $\ker(p_1+p_2)$ is generated by the lines of $\begin{pmatrix}\mathrm{I}_{m_1}&B\\0&A_2.\end{pmatrix}$, and also by the lines of $\begin{pmatrix}A_1&0\\D&\mathrm{I}_{m_2}\end{pmatrix}$: this implies (for any ring) that the ideals generated by the m-minors of A_1 and A_2 coincide. When R is a UFD, this implies that they have tyhe same gcd.

The main property of this order is the

Proposition 3.1. *If M is a finitely generated R-module where R is a UFD , and $\mathrm{ann}_R(M)$ is its annihilator, $\mathrm{ord}_R(M)$ and $\mathrm{ann}_R(M)$ have the same prime divisors. More precisely, if M is generated by m elements, one has*

$$\mathrm{ann}_R(M)^m\subset\langle\mathrm{ord}_R(M)\rangle\subset\langle\gcd(\mathrm{ann}_R(M))\rangle.$$

Thus

$$\mathrm{ord}_R(M)\neq0\Leftrightarrow M\text{ is a torsion module over }R.$$

Proof. Let $R^{(I)} \xrightarrow{\times A} R^m \xrightarrow{p}$ be a presentation of M. Then

$$a \in \operatorname{ann}(M) \Leftrightarrow aM = 0 \Leftrightarrow aR^m \supset \ker p \Leftrightarrow aR^m \supset R^{(I)}A.$$

By Cramer's formulas, $R^{(I)}A$ contains μR^m for every minor μ of A, thus $\operatorname{ann}_R(M)$ contains all m-minors of A. Thus the gcd of the m-minors divides the gcd of $\operatorname{ann}(M)$, which proves the right inclusion.

For the left inclusion: $a \in \operatorname{ann}(M) \Leftrightarrow aR^n \supset R^{(I)}A \Leftrightarrow$ (there exists $X \in \mathrm{M}_{m,I}(R)$ such that $XA = a\mathrm{I}_m$). This implies that $a^m = \sum \mu_{X,i}\mu_{A,i}$ where $\mu_{X,i}$ and $\mu_{A,i}$ are m-minors of A and X, thus $\operatorname{ord}_R(M)$ divides a^m, qed.

3.2. **Twisted Alexander polynomials and version mod p.** As said in the Introduction, we define only those of [Friedl-Vidussi 2008], section 3.2 (cf. also [Friedl-Vidussi 2011b], 3.3.5). We define

$$\Delta^H(G,u) : \operatorname{ord}_{\mathbb{Z}[G/\ker u]}(H_1(G;\mathbb{Z}[G/\ker u]^{[G/H]}) \in \mathbb{Z}[G/\ker u].$$

This corresponds to their $\Delta^\alpha_{N,\varphi,1}$, where $\varphi : H \to F$ is the natural map $H_1(G)/\mathrm{Tors} \to G/\ker u$ and $\alpha : \pi_1(N) \to G$ corresponds to $G \to G/H$ for us.

Note that

$$H_1(G;\mathbb{Z}[G/\ker u]^{[G/H]} \approx H_1(H;\mathbb{Z}[G/\ker u]),$$

thus $\Delta^H(G,u)$ can be viewed as a kind of untwisted Alexander polynomial for $(H,u_{|H})$, depending however on G. Moreover, on $\mathbb{Z}[H/H\cap\ker u]$, we have

$$H_1(H;\mathbb{Z}[G/\ker u]) \approx H_1(\mathbb{Z}[H/H\cap\ker u])^{[G:H]}.$$

Thus $\Delta^H(G,u)$ begins to resemble to $\Delta(H,u_{|H})$. In particuler, we have the property:

$$\Delta^H(G,u) \Leftrightarrow \Delta(H,u_{|H}).$$

This follows from Proposition 3.1 and the fact that if M is a $\mathbb{Z}[\mathbb{Z}^r]$-module and $L \subset \mathbb{Z}^r$ a lattice, (M torsion over $\mathbb{Z}[\mathbb{Z}^r] \Rightarrow M$ torsion over $\mathbb{Z}[L]$: if $P(t_1,\cdots,t_r)$ annihilates M, and N is such that $L \supset N\mathbb{Z}^r$, the product $\prod P(\omega_1 t_1,\cdots,\omega_r t_r)$ over all r-tuples of m-th roots of unity belongs to $R[L]$ and annihilates M.

3.3. **Twisted Alexander polynomials mod p.** We define

$$\Delta^H_p(G,u) := \operatorname{ord}_{\mathbb{F}_p[G/\ker u]}(H_1(G;\mathbb{F}_p[G/\ker u]^{[G/H]}) \in \mathbb{F}_p[G/\ker u].$$

The comparison with $\Delta^H(G,u)$ mod p is not obvious, since R is not principal thus there is no theorem of universal coefficients. We shall need the following proposition, whose proof lasts until the end of the section and uses two lemmas. (presumably, there should be a simpler argument).

Proposition 3.2. $\Delta^H_p(G,u) \neq 0 \Leftrightarrow \Delta^H(G,u) \not\equiv 0$ *mod p.*

Proof. We express $H_1(G,\mathbb{Z}[G/H\cap\ker u])$ and $H_1(G,\mathbb{Z}/p\mathbb{Z}[G/H\cap\ker u])$ via Fox differential calculus. Let $\langle x_1,\cdots,x_m \mid (r_i)_{i\in I}\rangle$ be a presentation of G. We have an exact complex

$$\mathbb{Z}[G]^{(I)} \xrightarrow{\times D_2} \mathbb{Z}[G]^m \xrightarrow{\times D_1} \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z} \to 0,$$

where $D_2 \in \mathrm{M}_{m,I}(\mathbb{Z}[G])$ is equal to $\left(\dfrac{\partial r_i}{\partial x_j}\right)$, D_1 is the column $(x_i-1)_{1\le i\le m}$ and ε is the augmentation. If M is a left $R[G]$-module, this enables us to compute

$$H_1(G,M) \approx \ker(\times D_1 \otimes_{R[G]} M)/\mathrm{im}(\times D_2 \otimes_{R[G]} M).$$

In particular,

$$H_1(G,R[G/\ker u]^{G/H}) \approx \ker(\times\widehat{D}_1)/\mathrm{im}(\times\widehat{D}_2)$$

where $\widehat{D}_1, \widehat{D}_2$ are obtained from D_1, D_2 by changing the coefficients from $\mathbb{Z}[G]$ to $\mathbb{Z}[G/\ker u]^{G/H}$.

Lemma 3.3. *Let $R = \mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z}$. We have an isomorphism of left $R[G]$-modules*

$$R[G/\ker u]^{G/H} \approx R[G/H \cap \ker u]^{[G:H \ker u]}.$$

Note that we write $[G:H \cap \ker u]$ and not $G/H \cap \ker u$, since G acts trivially.

Proof. We prove it for $\ker u, H$ replaced by any two normal subgroups H_1, H_2. The group $G/H \cap H_2$ acts freely on the left on $G/H_1 \times G/H_2$ (diagonally), and the quotient is in bijection with G/H_1H_2 via

$$\psi_0(x \bmod H_1, y \bmod H_2) = x^{-1}y \bmod H_1H_2.$$

Since the map $\pi(x \bmod H_1, y \bmod H_2) = x^{-1}y \bmod H_1H_2$ induces a bijection from $G\backslash(G/H_1 \times G/H_2)$ to G/H_1H_2, we have a bijection

$$\psi = \psi_0 \times \pi : G/H_1 \times G/H_2 \to G/H_1 \cap H_2 \times G/H_1H_2,$$

which is G-equivariant with $G/H_1 \cap H_2$ acting naturally on all quotients except on G/H_1H_2 where it acts trivially. This gives the desired isomorphism $R[G/H_1]^{G/H_2} \approx R[G/H_1 \cap H_2]^{[G:H_1H_2]}$, qed.

Lemma 3.3 implies $H_1(G, R[G/\ker u]^{G/H}) = \ker(\times\widetilde{D}_1)/\mathrm{im}(\times\widetilde{D}_2)$, where now

$$\widetilde{D}_1 \in \mathrm{M}_{m[G:H],[G:H]}(R[G/\ker u]) \ , \ \widetilde{D}_2 \in \mathrm{M}_{I\times[G:H],m[G:H]}(R[G/\ker u]).$$

Moreover,

$$\widetilde{D}_1 = \begin{pmatrix} T_1 - \mathrm{I}_{[G:H]} \\ \cdots \\ T_m - \mathrm{I}_{[G:H]} \end{pmatrix}$$

with each $T_i = \overline{x}_i\sigma_i$, $\overline{x}_i$ the image of x_i in $G/\ker u$, and σ_i a permutation matrix of size $[G:H]$ associated to the action of $\overline{x}_i$ on G/H. We have $\det(T_i - \mathrm{I}_{[G:H]}) = \prod_{j=1}^{r}(\overline{x}_i^{c_j} - 1)$, where $c_1, \cdots, c_r$ are the lengths of the cycles of the permutation σ_i. Thus $(T_i - \mathrm{I}_k)^{-1} = d_iA_i$ where $d_i = (\overline{x}_i^{[G:H]} - 1)^{[G:H]}$ and $A_i \in \mathrm{M}_{[G:H]}(R[G/\ker u])$. Thus

$$\ker(\times\widetilde{D}_2) = \{(X_1 \mid \cdots \mid X_m) \mid X_i = -d_i^{-1}\sum_{j\neq i}^{m} X_j(T_j - \mathrm{I}_k)A_i\}.$$

Note that since the $\overline{x}_i$ generate $G/\ker u$, the gcd of $d_1, \cdots, d_m$ is 1. We want to deduce that $\mathrm{ord}_{R[G/\ker u]}(H_1(G, R[G/\ker u]))$ and the gcd of m-minors of $\widetilde{D}_2$ have the same prime divisors. The projection

$$p_i : (X_1 \mid \cdots \mid X_m) \mapsto (X_1 \mid \cdots \widehat{X}_i \cdots \mid X_m)$$

is injective on $\ker(\times\widetilde{D}_2)$, with image containing $d_iR[G/\ker u]^{(m-1)[G:H]}$. Thus, writing

$$\widetilde{D}_2 = (L_1 \mid \cdots \mid L_m) \ , \ \widetilde{D}_2^{(i)} = (L_1 \mid \cdots \widehat{L}_i \cdots \mid L_m),$$

we have $H_1(G, R[G/\ker u]) \approx p_i(\ker(\times\widetilde{D}_1))/\mathrm{im}(\times\widetilde{D}_2^{(i)})$, which is sandwiched between $\mathrm{coker}(\times\widetilde{D}_2^{(i)})$ and $d_i\mathrm{coker}(\times\widetilde{D}_2^{(i)})$.

Lemma 3.4. *Let $\widetilde{R}$ be a Noetherian UFD and M a $\widetilde{R}$-module such that (up to isomorphism) one has*

$$d \ \mathrm{coker}(\times A) \subset M \subset \mathrm{coker}(\times A)$$

for some $d \in \widetilde{R}^*$ and $A \in \mathrm{M}_{I,m}$. Let μ be the gcd of m-minors of A. Then $\mathrm{ord}_{\widetilde{R}}(M)d$ and μd have the same prime divisors.

Proof. By definition, $\mathrm{ord}_{\widetilde{R}}(\mathrm{coker}(\times A)) = \mu$ and $\mathrm{ord}_{\widetilde{R}}(d\ \mathrm{coker}(\times A)) = d^m\mu$. By hypothesis,

$$\mathrm{ann}(d\ \mathrm{coker}(\times A)) \supset \mathrm{ann}(M) \supset \mathrm{ann}(\mathrm{coker}(\times A)),$$

thus $\mathrm{ann}_{\widetilde{R}}(d\ \mathrm{coker}(\times A))$ and $d\ \mathrm{ann}_{\widetilde{R}}(M)$ have the same prime divisors. By Proposition 3.1, this means that $\mathrm{ord}_{\widetilde{R}}(d\,\mathrm{coker}(\times A))$ and $d\,\mathrm{ord}_{\widetilde{R}}(M)$ have the same divisors. Since by definition, $\mathrm{ord}_{\widetilde{R}}(d\,\mathrm{coker}(\times A))$ $= \mathrm{ord}_{\widetilde{R}}(\mathrm{coker}(\times dA)) = d^m\mu$, this proves Lemma 3.4.

End of the proof of Proposition 3.2. We apply Lemma 3.4 with $\widetilde{R} = R[G/\ker u]$, $R = \mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z}$, $M = H_1(G, R[G \cap \ker u])$, $A = \widetilde{D}_2^{(i)}$ and $d = d_i$. Let μ, μ_i be the gcd of m-minors of D_2, $D_2^{(i)}$ for $R = \mathbb{Z}$ (resp. $R = \mathbb{Z}/p\mathbb{Z}$). Then μ mod p, μ_i mod $p$are the gcd of m-minors of D_2 mod p, $D_2^{(i)}$ mod p, thus

- $\Delta^H(G,u)d_i$ and $\mu_i d_i$ have the same prime divisors in $\mathbb{Z}[G/\ker u]$
- $\Delta_p^H(G,u)d_i$ and $(\mu_i \bmod p)d_i$ have the same prime divisors in $\mathbb{Z}/p\mathbb{Z}[G/\ker u]$.

Moreover, $\gcd(d_1, \cdots, d_m) = 1$ in $\mathbb{Z}[G/\ker u]$ and in $\mathbb{Z}/p\mathbb{Z}[G/\ker u]$. Thus $\Delta^H(G,u)$ and μ have the same prime divisors in $\mathbb{Z}[G/\ker u]$, and similarly $\Delta_p^H(G,u)$ and μ mod p have the same prime divisors in $\mathbb{Z}/p\mathbb{Z}[G/\ker u]$. Thus $\Delta_p^H(G,u)$ has the same prime divisors as $\Delta^H(G,u)$ mod p. Thus it is a unit if and only if $\Delta^H(G,u) \not\equiv 0 \bmod p$. Since $\Delta_u^H(G,u)$ is a unit if and only $H_1(G, \mathbb{Z}/p\mathbb{Z}[G/\ker u]) = 0$, this finishes the proof of proposition 3.2.

4. Novikov homology

4.1. **Novikov ring.** Let R a commutative unital ring (not necessarily a UFD for the moment). We define the *Novikov ring* $R[G]_u$ as the formal series over G with coefficients in R having finite support in each sublevel of u:

$$R[G]_u := \{\lambda \in R[[G]] \mid (\forall C \in \mathbb{R})\ \mathrm{supp}(\lambda) \cap (u \le C) \text{ is finite}\}.$$

It is easy to see that the multiplication

$$\sum_{g_1 \in G} a_{g_1} g_1 . \sum_{g_2 \in G} b_{g_2} g_2 = \sum_{g \in G} \Big(\sum_{g_1 g_2 = g} a_{g_1} b_{g_2} \Big) g$$

is well defined and makes $R[G]_u$ a ring containing $R[G]$ as a subring.

Property. Maybe the most important property of $R[G]_u$ is the following one: if $x = 1 + a \in R[G]_u$ and $\mathrm{supp}(a) \subset \{u > 0\}$, x is invertible, with inverse $x^{-1} = \sum_{n=0}^{\infty} (-a)^n$. More generally, if $A = \mathrm{I}_n + B \in \mathrm{M}_n(R[G]_u)$ and $\mathrm{supp}(B) \subset \{u > 0\}$, A is invertible, with inverse $A^{-1} = \sum_{n=0}^{\infty} (-B)^n$.

Corollary. *If $u : \mathbb{Z}^r \to \mathbb{R}$ is an injective homomorphism and R is a field, $R[\mathbb{Z}^r]_u$ is a field.*

4.2. **Novikov homology, relation with fibering.** The Novikov homology $H_*(G,u;R)$ is defined as the homology of G with coefficients in the left $R[G]$-module $R[G]_u$:

$$H_*(G,u;R) := H_*(G, R[G]_u).$$

Although we shall not use it, let us quote an important about the vanishing of $H_1(G,u)$ in rank one.

Theorem 4.1. [Bieri-Neumann-Strebel 1987], [Sikorav 1987] *If* $\mathrm{rk}(u) = 1$, $\ker u$ *is finitely generated if and only if $H_1(G,u) = 0 = H_1(G,-u)$.*

The proof is quite easy. Harder is the generalization by [Bieri-Neumann-Strebel 1987] to all ranks: $\ker u$ is finitely generated if and only if $H_1(G, v) = 0 =$ for every v vanishing on $\ker u$. But the rank one is of special interest for us because of the quoted result of [Stallings 1962].

For this paper, the interest of Novikov homology lies in the following

Theorem 4.2. [Bieri-Neumann-Strebel 1987]**.** *Let $G = \pi_1(M)$, where N is a closed and connected three-manifold. The following are equivalent:*

- *u is fibered.*
- *$H_1(G, u; \mathbb{Z}) = 0$.*

Remarks. 1) This is their Theorem E, reinterpreted in terms of Novikov homology, cf. p.456 of the paper.

2) At the time, one needed N to contain no fake cells, and also the hypothesis $\pi_1(M) \neq \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ (to avoid a possible fake $\mathbb{RP}^2 \times S^1$), restrictions removed later thanks to Perelman.

3) In [Sikorav 1987], I could only prove the equivalence between (i) and the vanishing of both $(H_1(G, u; \mathbb{Z})$ and $H_1(G, -u; \mathbb{Z})$: this is an immediate consequence of [Stallings 1962], [Thurston 1986] and Theorem 4.1.This would suffice for the proof of our main result, but it seemed better to quote the stronger result.

5. Computation of $H_1(G, u; R)$, independence on R of its vanishing

Let $\langle x_1, \cdots, x_m \mid (r_i)_{i \in I} \rangle$ be a presentation of G, and let D_1, D_2 be defined as in Section 3. Then, denoting $(D_i)_{R[G]_u} \in \mathrm{M}_{m,I}(R[G]_u)$ the matrix obtained by the base change $\mathbb{Z}[G] \to R[G]_u$, we have

$$H_1(G, u; R) = \ker(\times D_1)_{R[G]_u} / \mathrm{im}(\times D_2)_{R[G]_u}.$$

Since $u \neq 0$, there exists $i \in \{1, \cdots, m\}$, such that $u(x_i) \neq 0$. Denoting $D_2^{(i)}$ the matrix obtained by deleting the i-eth column of D_2, we have $H_1(G, u; R) \approx \mathrm{coker}(\times D_2^{(i)})_{R[G]_u}$.

Corollary 5.1. *We have*

$$H(G, u; R) = 0 \Leftrightarrow \times (D_2^{(i)})_{R[G]_u} : R[G]_u^{(I)} \to R[G]_u^{(I \setminus \{i\})} \text{ is onto.}$$

Equivalently, there exists $\widetilde{X} \in \mathrm{M}_{m-1,I}(R[G]_u)$ such that $\widetilde{X} D_2^{(i)} = \mathrm{I}_{m-1}$. Truncating $\widetilde{X}$ below a sufficiently high level of u, we obtain $X \in \mathrm{M}_{m-1,I}(R[G])$ such that $X D_2^{(i)} = \mathrm{I}_{m-1} + A$ with $u > 0$ on $\mathrm{supp}(A)$. Since such a matrix is invertible over $R[G]_u$, we obtain the following

Corollary 5.2. *$(H_1(G, u; R) = 0)$ is equivalent to the existence of a matrix $X \in \mathrm{M}_{m-1,I}(R[G])$ such that $X D_2^{(i)} = \mathrm{I}_{m-1} + A$ with $u > 0$ on $\mathrm{supp}(A)$.*

The following property is proved for $\mathbb{Z}$-coefficients by [Bieri-Neumann-Strebel 1987] and (in a different form) by [Sikorav 1987]. Actually, the proof works exactly the same for R-coefficients. We shall use the corollary, which was observed by [Kielak 2018], with $R = \mathbb{Z}/p\mathbb{Z}$ (Kielak used it with $R = \mathbb{Q}$ or $\mathbb{C}$).

Proposition 5.3. *Let $\Gamma = \Gamma(G, S)$ be a Cayley graph for some finite generating system S. Denote $\Gamma_{u>0}$ the subgraph of Γ generated by the vertices with $u(g) > 0$. The following are equivalent:*

(1) $H_1(G, u; R) = 0$
(2) *$\Gamma_{u>0}$ is connected.*

Corollary 5.4. *If G is finitely generated, the property $(H_1(G, u; R) = 0)$ does not depend on R.*

Proof. We consider X the 2-complex associated to some presentation $\langle S \mid (r_i)_{i\in I}\rangle$. Then the 1-skeleton $\widetilde{X}^{(1)} = \Gamma$. We extend u affinely to $\widetilde{X}^{(1)}$ and then to $\widetilde{X}$, keeping the equivariance $u \circ g = u + u(g)$ and with $u(\sigma) \subset u(\partial\sigma)$ for every 2-cell. Then

$$H_1(G,u;R) = H_1(C_*(\widetilde{X}) \otimes_{\mathbb{Z}[G]} R[G]_u) = H_1(C_*(\widetilde{X};R) \otimes_{R[G]} R[G]_u).$$

Here $C_*(\widetilde{X})$ is the chain complex with $\mathbb{Z}$-coefficients, viewed as a $\mathbb{Z}[G]$-left module, and

$$C_*(\widetilde{X};R) = C_*(\widetilde{X}) \otimes_{\mathbb{Z}} R \approx C_*(\widetilde{X}) \otimes_{\mathbb{Z}[G]} R[G]$$

the one with R-coefficients, viewed as a left $R[G]$-module. Without loss of generality we can assume that there exists $s \in S$ such that $u(s) > 0$. Then $\partial e_s = s - 1$ is invertible in $R[G]_u$, with $\mathrm{supp}((s-1)^{-1} - 1) \subset \{u > 0\}$.

(1) $\Rightarrow$ (2). The key point is that a cell complex Y is connected if and only if $\widetilde{H}_0(Y;R) = 0$ for one or every R. Thus it suffices to prove that for every $g \in G$ such that $u(g) > 0$, $g - 1$ bounds a chain in $C_1(\widetilde{X}_{u>0};R)$. Choose $\gamma \in C_1(\widetilde{X})$ such that $\partial_1\gamma = g - 1$. Let s be an element of S such that $u(s) \neq 0$. Define

$$\widetilde{\gamma} = \gamma + (1-g)(s-1)^{-1}e_s \in C_1(\widetilde{X},u;R).$$

Then $\partial_1\widetilde{\gamma} = 0$, thus by hypothesis there exists $\widetilde{c} \in C_2(\widetilde{X}) \otimes_{\mathbb{Z}[G]} R[G]_u)$ such that $\partial_2\widetilde{c} = \widetilde{\gamma}$. By truncating $\widetilde{c}$ below a sufficiently high level of u, we find $c \in C_2(\widetilde{X};R)$ such that $\widetilde{\gamma} - \partial_2 c \in C_1(\widetilde{X}_{u>0};R)$. Moreover, since $\mathrm{supp}((s-1)^{-1} - 1) \subset \{u > 0\})$ and $u(g) > 0$, we have $\widetilde{\gamma} - \gamma \in C_1(\widetilde{X}_{u>0};R)$, thus

$$\gamma - \partial_2 c = (\widetilde{\gamma} - \gamma) + (\widetilde{\gamma} - \partial_2 c) \in C_1(\widetilde{X}_{u>0};R).$$

Thus $\gamma - \partial_2 c$ is a ∂_1-primitive of $g - 1$ in $C_1(\widetilde{X}_{u>0};R)$, qed.

(2) $\Rightarrow$ (1). It suffices to treat the case $R = \mathbb{Z}$. Let $D_2^{(s)}$ be the matrix obtained from D_2 by deleting the column corresponding to s. ByCorollary 5.2, it suffices to find $M \in \mathrm{M}_{S\setminus\{s\},I}(\mathbb{Z}[G])$ such that $D_2^{(s)}M - \mathrm{I}_{S\setminus\{s\}}$ has support in $\{u > 0\}$. Equivalently to find, for every $t \in S \setminus \{s\}$, elements $c \in C_2(\widetilde{X})$ and $\lambda \in \mathbb{Z}[G]$ such that $\partial_2 c - e_t - \lambda e_s \in C_1(\widetilde{X}_{u>0})$. For n large enough, $u(ts^n) = u(s^n t) > 0$. By hypothesis, there exists $\gamma \in C_1(\widetilde{X}_{u>0})$ such that $s^n t - ts^n = \partial_1\gamma$. Thus

$$\begin{aligned}\partial_1\big((1-s^n)e_t + (t-1)(1+s+\cdots+s^{n-1})e_s\big) &= (1-s^n)(t-1) + (t-1)(s^n-1)\\ &= ts^n - s^n t = \partial_1\gamma.\end{aligned}$$

Since $\ker\partial_1 = \mathrm{im}\ \partial_2$, there exists $c \in C_2(\widetilde{X})$ such that

$$(1-s^n)e_t + (t-1)(1+s+\cdots+s^{n-1})e_s = \gamma + \partial_2 c.$$

Thus $\partial_2 c - e_t - (t-1)(1+s+\cdots+s^{n-1})e_s \in C_1(\widetilde{X}_{u>0})$. Setting $\lambda = (t-1)(1+s+\cdots+s^{n-1})$, this finishes the proof of the Proposition.

6. Abelianized relative Novikov homology and twisted Alexander polynomials

Consider the induced morphism $\overline{u} : G/\cap\ker u \to \mathbb{R}$ and the associated Novikov ring $R[G\cap\ker u]_{\overline{u}}$, which is a left $R[G]$-module, and define the *Abelianized Novikov homology*

$$H_1^{ab}(G,u;R) := H_1(G;R[G/\ker u]_{\overline{u}}).$$

It is in fact the original homology defined in [Novikov 1981]. Since $G/\ker u$ is free Abelian of rank $r = \mathrm{rk}(u)$, $R[G/\ker u]_{\overline{u}})$ is Abelian. Moreover, when R is a field, $R[G/\ker u]_{\overline{u}})$ is also a field.

If $H \triangleleft_{f.i} G$ be a normal subgroup with finite index, we generalize the above definition. Consider the induced morphism $\overline{u} : G/H \cap \ker u \to \mathbb{R}$ and the associated Novikov ring $R[G/H \cap \ker u]_{\overline{u}}$, and we define the *Abelianized relative Novikov homology*

$$H_1^{ab}(G, H, u; R) := H_1(G; R[G \cap H \ker u]_{\overline{u}}).$$

To relate this with Alexander polynomials, we need a lemma.

Lemma 6.1. *Let R be a field and $u : \mathbb{Z}^r \to \mathbb{R}$ an injective morphism. Consider a complex of $R[\mathbb{Z}^r]$-linear maps*

$$R[\mathbb{Z}^r]^{(I)} \xrightarrow{f} R[\mathbb{Z}^r]^m \xrightarrow{g} R[\mathbb{Z}^r]^n.$$

The following are equivalent:

(1) $\dfrac{\ker g}{\operatorname{im} f}$ *is torsion over* $R[\mathbb{Z}^r]$*. Or:* $\operatorname{ord}(\dfrac{\ker g}{\operatorname{im} f}) \neq 0$.

(2) *The induced complex* $R[\mathbb{Z}^r]_u^{(I)} \xrightarrow{f_u} R[\mathbb{Z}^r]_u^m \xrightarrow{g_u} R[\mathbb{Z}^r]_u^n$ *is exact.*

Proof. More generally, we prove it for $R[\mathbb{Z}^r]$ replaced by any Noetherian domain A and $R[\mathbb{Z}^r]$ by a field F containing it. It suffices to do it when F is the fraction field of A. (i) is equivalent to: $(\exists a \in A^* \mid a \ker g \subset \operatorname{im} f)$, ie $(\ker g \subset (A^*)^{-1} \operatorname{im} f)$. On the other hand, $\ker g_u = (A^*)^{-1} \ker g \cap A$, thus (ii) is equivalent to

$$(A^*)^{-1} \ker g \cap A \subset (A^*)^{-1} \operatorname{im} f.$$

Since A is a domain, this is also equivalent to $(\ker g \subset (A^*)^{-1} \operatorname{im} f)$.

Proposition 6.2. *The following are equivalent:*

(1) $\Delta^H(G, u) \not\equiv 0$ *mod* p.
(2) $H_1(G; \mathbb{Z}/p\mathbb{Z}[G/\ker u]^{G/H})$ *is torsion over* $\mathbb{Z}/p\mathbb{Z}[G/\ker u]$.
(3) $H_1^{ab}(G, H, u; \mathbb{Z}/p\mathbb{Z}) = 0$*, ie* $H_1(G, \mathbb{Z}/p\mathbb{Z}[G/H \cap \ker u]_{\overline{u}}) = 0$.

Proof. In Section 3, we saw that (1) $\Leftrightarrow$ $\Delta_p^H(G, u) \neq 0$. Since $\Delta_p^H(G, u)$ is the order of $H_1(G; \mathbb{Z}/p\mathbb{Z}[G/\ker u]^{G/H})$, ((1) $\Leftrightarrow$ (2)) follows from Proposition 3.1. By Lemma 6.1, (2) is equivalent to $H_1(G; \mathbb{Z}/p\mathbb{Z}[G/\ker u]_{\overline{u}}^{G/H}) = 0$.

Finally, the isomorphism $\mathbb{Z}/p\mathbb{Z}[G/\ker u]^{G/H} \approx \mathbb{Z}/p\mathbb{Z}[G/H \cap \ker u]^{[G:H \ker u]}$ from Lemma 3.3 extends, via a tensorisation $\otimes_{\mathbb{Z}/p\mathbb{Z}[G]} \mathbb{Z}/p\mathbb{Z}[G]_u$, to an isomorphism of $\mathbb{Z}/p\mathbb{Z}[G]_u$-modules

$$\mathbb{Z}/p\mathbb{Z}[G/\ker u]_{\overline{u}}^{G/H} \approx \mathbb{Z}/p\mathbb{Z}[G/H \cap \ker u]_{\overline{u}}^{[G:H \ker u]}.$$

Thus

$$H_1(G; \mathbb{Z}/p\mathbb{Z}[G/\ker u]_{\overline{u}}^{G/H}) \approx H_1(\mathbb{Z}/p\mathbb{Z}[G/H \cap \ker u]_{\overline{u}})^{[G:H \ker u]}.$$

Thus $(H_1(G; \mathbb{Z}/p\mathbb{Z}[G/\ker u]_{\overline{u}}^{G/H}) = 0)$ is equivalent to (3), qed.

7. Computations in dimension three and reduction of the main result

In this section we consider the case where $G = \pi_1(M)$ where N is a closed and connected three-manifold with a contractible universal covering $\widetilde{M}$.

7.1. **Complex of a special form for computing the homology of G.** Using a handle decomposition of genus m [or a self-indexing Morse function with one minimum and one maximum], one can obtain $H_*(\widetilde{M};\mathbb{Z})$ by a complex of left modules over $\mathbb{Z}[G]$, of the form

$$\mathbb{Z}[G] \xrightarrow{\times D_3} \mathbb{Z}[G]^m \xrightarrow{\times D_2} Z[G]^m \xrightarrow{\times D_1} \mathbb{Z}[G]$$

with

$$D_1 = \begin{pmatrix} x_1 - 1 \\ \cdots \\ x_m - 1 \end{pmatrix} \ , \ D_3 = (y_1 - 1 \mid \cdots \mid y_m - 1).$$

$(x_1, \cdots, x_m)$ and $(y_1, \cdots, y_m)$ are generating systems for G. Since $u \neq 0$, we can reorder them so that $u(x_m)$ and $u(y_m)$ are nonzero, thus $x_m - 1$ and $y_m - 1$ are invertible in $\mathbb{Z}[G]_u$. Then we denote c the column $(x_i - 1)_{i<m}$ and ℓ the line $(y_j - 1)_{j<m}$, so that

$$D_1 = \begin{pmatrix} c \\ x_m - 1 \end{pmatrix} \ , \ D_3 = (\ell \mid y_m - 1).$$

Even if $\widetilde{M}$ is not contractible, this complex gives a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[G]$ up to degree 2, thus $H_1(G, u; R) \approx \ker(\partial_1)_{R[G]_u}/\mathrm{im}(\partial_2)_{R[G]_u}$ where we have changed the coefficients from $\mathbb{Z}[G]$ to $R[G]_u$. By the contractibility of $\widetilde{M}$, it is a complete free resolution, and G is a group with 3-dimensional Poincaré duality, which can be expressed as follows. Denote w the orientation morphism $G \to \{1, -1\}$, and define modified adjoint isomorphisms

$$\lambda = \sum_g a_g g \mapsto \lambda^* = \sum_g a_g \varepsilon(g) g^{-1} \ , \ A = (a_{i,j})^* = (a^*_{j,i}).$$

Then Poincaré duality can be expressed by the fact that C_* is quasi-isomorphic to the complex $(C^*_i = C_{3-i}, \times D^*_{4-i})$. Note that (D^*_1, D^*_2, D^*_3) has the same properties as (D_1, D_2, D_3). Let us write $D_2 = \begin{pmatrix} A & C \\ L & a \end{pmatrix}$ where $A \in \mathrm{M}_{m-1}(\mathbb{Z}[G])$, $L \in M_{1,m-1}(\mathbb{Z}[G])$, $C \in \mathrm{M}_{m-1,1}(\mathbb{Z}[G])$ and $a \in \mathbb{Z}[G]$. Note that $D_2^{(m)} = (A \mid C)$. Since $\partial_1 \circ \partial_2 = 0$ and $\partial_2 \circ \partial_3 = 0$, we have $D_2 D_1 = 0$ and $D_3 D_2 = 0$. Working over $\mathbb{Z}[G]_u$, we obtain

$$\begin{aligned} C &= Ac(1 - x_m)^{-1} \ , \ a = Lc(1 - x_m)^{-1} \\ L &= (1 - y_m)^{-1} \ell A \\ a &= (1 - y_m)^{-1} \ell C = (1 - y_m)^{-1} \ell A c (1 - x_m)^{-1}. \end{aligned}$$

These computations have the following consequence.

Proposition 7.1. *Let $u \in H^1(M;\mathbb{R}) \setminus \{0\}$, and let R be a commutative unital ring. The two following properties are equivalent:*

1. *u is fibered.*
2. *The matrix $A \in \mathrm{M}_{m-1}(\mathbb{Z}[G])$ becomes invertible in $\mathrm{M}_{m-1}(R[G]_u)$.*

Proof. By [Bieri-Neumann-Strebel 1987], (1) is equivalent to $(H_1(G, u; R) = 0)$. By Corollary 5.1, this is equivalent to the left invertibility of $(A \mid L)$ with $R[G]_u$-coefficients). Since $L = (1 - y_m)^{-1} \ell A$, (1) is equivalent to the left invertibility of A in $\mathrm{M}_{m-1}(R[G]_u)$.

Since (D^*_1, D^*_2, D^*_3) has the same properties as (D_1, D_2, D_3), this is also equivalent to the left invertibility of A^* in $\mathrm{M}_{m-1}(R[G]_{-u})$, i.e. the right-invertibility of A in $\mathrm{M}_{m-1}(R[G]_u)$. This proves Proposition 7.1.

Remarks. In [Sikorav 2017] I proved that $\mathbb{Z}[G]_u$ is always *stably finite*: an matrix $A \in \mathrm{M}_n(\mathbb{Z}[G]_u)$ is invertible if and only if it is left invertible. This is a well-known result of Kaplansky for $\mathbb{Z}[G]$, which was proved by [Kochloukova 2006] for $\mathbb{Z}[G]_u$ when $\mathrm{rk}(u) = 1$. With Poincaré duality, this allows to prove $(H_1(G,u;\mathbb{Z}) = 0 \Rightarrow H_1(G,-u;\mathbb{Z}) = 0)$ without using the results of Stallings and Thurston.

On the other hand, note that the stable finiteness of $\mathbb{Z}/p\mathbb{Z}[G]$ (and of $\mathbb{Z}/p\mathbb{Z}[G]_u$) is still an open question.

Proposition 7.1 reduces the proof of the main result to the following result of "finite detectability of invertible matrices".

Theorem 7.2. *Assume that G is VRTFN. Let p be a prime and $n \in \mathbb{N}$. Let $A \in \mathrm{M}_n(\mathbb{Z}/p\mathbb{Z}[G]_u)$ be* finitely invertible *as an element of $\mathrm{M}_n(\mathbb{Z}/p\mathbb{Z}[G]_u)$, i.e. for every $H \lhd_{f.i.} G$ the image of A in $\mathrm{M}_n(\mathbb{Z}/p\mathbb{Z}[\mathbb{F}_p/H \cap \ker u]_{\overline{u}})$ is invertible. Then A is invertible in $\mathrm{M}_n(\mathbb{Z}/p\mathbb{Z}[G]_u)$.*

Remarks. 1) The validity of Theorem 7.2 for a finite index subgroup $G_0 \subset G$ implies its validity for G: this follows from the fact that an n-matrix over $\mathbb{Z}/p\mathbb{Z}[G]_u$ can be represented by a $(n.[G\!:\!G_0])$-matrix over $\mathbb{Z}/p\mathbb{Z}[G_0]_{u_{|G_0}}$. Thus it suffices to prove Theorem 7.2 when G is RTFN.

2) Theorem 7.2 is also true for $\mathrm{M}_n(\mathbb{Z}[G]_u)$, but not interesting for us and we do not prove it here. If it were true for $\mathrm{M}_n(\mathbb{Q}[G]_u)$, we would obtain our main result with the weaker hypothesis $\Delta^H(G,u) \neq 0$. However, this fails miserably: if $G = \mathbb{Z}^2 = \langle x, y\rangle$, $u(x) = 1$, $u(y) = 0$, then $2 - y$ is invertible in all $\mathbb{Q}[\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}]_{\overline{u}}$, but is not invertible in $\mathbb{Q}[\mathbb{Z}^2]_{\overline{u}}$.

8. Finitely detectable full left ideals

8.1. **Definitions.** A left $R[G]_u$-submodule $M \subset (R[G]_u)^n$ is *finitely full* if its image in every quotient $(R[G/(H \cap \ker u)]_{\overline{u}})^n$, $H \lhd_{f.i.} G$, is equal to $(R[G]_u)^n$. In particular, for $n = 1$: a left ideal $I \subset R[G]_u$ is finitely full if its image in every quotient $R[G/H \cap \ker u]_u$, $H \lhd_{f.i.} G$, is equal to $R[G]_u$. The group ring $R[G]_u$ has *finitely detectable full left ideals* if every left ideal $I \subset R[G]_u$ which is finitely full is equal to $R[G]_u$.

Proposition 8.1. *Assume that $R[G]_u$ has finitely detectable full left ideals. Then every left $R[G]_u$-submodule $M \subset (R[G]_u)^n$ which is finitely full is equal to $(R[G]_u)^n$.*

Proof. For $n = 1$, it is the hypothesis. Assume that $n > 1$ and the result is true for $n - 1$. Consider the set I of $\lambda \in R[G]_u$ such that there exists $\lambda_1, \cdots, \lambda_{n-1} \in R[G]_u$ with $(\lambda_1, \cdots, \lambda_{n-1}, \lambda) \in M$. It is a left ideal, and its image is full in every quotient $R[G/H \cap \ker u]_u$ with $H \lhd_{f.i.} G$. Thus $I = R[G]_u$, i.e. M contains an element $= (\lambda_1, \cdots, \lambda_{n-1}, 1)$. One has a direct sum decomposition $(R[G]_u)^n = (R[G]_u)^{n-1} \oplus R[G]_u x$. Substracting $\mu_n x$ from every element $(\mu_1, \cdots, \mu_n) \in M$, one sees that $M = (M \cap (R[G]_u)^{n-1}) \oplus R[G]_u x$. It suffices to prove that $M \cap (R[G]_u)^{n-1} = (R[G]_u)^{n-1}$. Clearly, $M \cap (R[G]_u)^{n-1}$ is a left submodule which maps onto every $(R[G/H \cap \ker u])^{n-1}$, $H \lhd_{f.i.} G$. By the induction hypothesis, $M \cap (R[G]_u)^{n-1} = (R[G]_u)^{n-1}$, qed.

Corollary 8.2. *Assume that $R[G]_u$ has finitely detectable full left ideals. Then for every $n \in \mathbb{N}^*$, $\mathrm{M}_n(R[G]_u)$ has finitely detectable units.*

Proof. If $A \in \mathrm{M}_n(R[G]_u)$ is finitely invertible, $M = (R[G]_u)^n A$ is finitely full. Thus $M = (R[G]_u)^n$, ie A is right invertible. Similarly, the adjoint A^*, defined here via $(\sum_{g \in G} a_g g)^* = \sum_{g \in G} a_g g^{-1}$, is right invertible. Thus A is left invertible, thus it is a unit, qed.

Proposition 8.3. *Let $\Gamma \subset G$ be a finite index subgroup such that $R[\Gamma]_{u_{|\Gamma}}$ has finitely detectable full left ideals. Then $R[G]_u$ has finitely detectable full left ideals.*

Proof. We can assume that Γ is normal. Using a section $\sigma : G/\Gamma \to G$, we have an isomorphism of $R[\Gamma]_{u_{|\Gamma}}$-modules

$$\overline{\sigma} : (R[\Gamma]_{u_{|\Gamma}})^{G/\Gamma} \approx R[G]_u, (a_q)_{q\in G/\Gamma} \mapsto \sum_{q\in G/\Gamma} a_q\sigma(q).$$

Let $I \subset R[G]_u$ be a residually full left ideal. By the above isomorphism, we associate to I a submodule $M \subset (R[\Gamma]_{u_{|\Gamma}})^{G/\Gamma}$. Let $H \triangleleft_{f.i} G$, and set $H_1 = H \cap \Gamma$. Then the isomorphism of $R[\Gamma]_{u_{|\Gamma}}$-modules $R[G]_u \to (R[\Gamma]_{u_{|\Gamma}})^{G/\Gamma}$ induces an isomorphism of $R[\Gamma/H_1]_{\overline{u}}$-modules

$$R[G/H \cap \ker u]_{\overline{u}} \to (R[\Gamma/H_1 \cap \ker u]_{\overline{u}})^{G/\Gamma}.$$

By hypothesis, the image of I in $R[G/H\cap\ker u]_{\overline{u}}$ is full, thus the image of M in $(R[\Gamma/H_1\cap\ker u])^{G/\Gamma}$ is full. Since $R[\Gamma]_{u_{|\Gamma}}$ has finitely detectable full left ideals, $M = (R[\Gamma]_{u_{|\Gamma}})^{G/\Gamma}$, thus $I = R[G]_u$, qed.

9. The case of nilpotent groups

In this section we prove the following result.

Proposition 9.1. *If G is nilpotent, $\mathbb{Z}/p\mathbb{Z}[G]_u$ has finitely detectable full left ideals.*

Corollary 9.2. *If G is nilpotent, every matrix in $\mathrm{M}_n(\mathbb{Z}/p\mathbb{Z}[G]_u)$ [not only in $\mathrm{M}_n(\mathbb{Z}/p\mathbb{Z}[G])$ which has an invertible image in all $\mathrm{M}_n(\mathbb{Z}/p\mathbb{Z}[G/H/\cap\ker u]_{\overline{u}})$, $H \triangleleft_{f.i.} G$, is invertible.*

Proof. We argue by contradiction, thus we assume that I is a finitely full left ideal in $\mathbb{Z}/p\mathbb{Z}[G]_u$ which is not full. I is contained in a maximal ideal I_1, without the axiom of choice since it is not difficult to prove that $R[G]_u$ is left Noetherian if R is Noetherian and G polycyclic (in particular, finitely generated and nilpotent): one follows the proof by [Hall 1954] that $R[G]$ is left Noetherian, which itself it close to the proof that $R[t]$ is Noetherian if R is Noetherian.

Then I_1 is again a finitely full left ideal in $\mathbb{Z}/p\mathbb{Z}[G]_u$ which is not full, thus we can assume that I is maximal. Thus $M := \mathbb{Z}/p\mathbb{Z}[G]_u/I$ is a simple $\mathbb{Z}/p\mathbb{Z}[G]_u$-module.

We shall need the following lemma, which uses several arguments from [Hall 1959].

Lemma 9.3. *Let p a prime and M a simple (or irreducible) left $\mathbb{Z}/p\mathbb{Z}[G]_u$-module. Denote Z the center of G. Then $Z \cap \ker u$ has a finite image in $\mathrm{Aut}(M)$.*

Proof. (cf. [Hall 1959], proof of Theorem 3.1, p. 610). Since M is simple, $M = \mathbb{Z}/p\mathbb{Z}[G]_u/I$ where I is a maximal left ideal. Set $M_0 = \mathbb{Z}/p\mathbb{Z}[\ker u]/I \cap \mathbb{Z}/p\mathbb{Z}[\ker u]$, which is a sub-$\mathbb{Z}/p\mathbb{Z}[\ker u]$-module of M, and define

$$D := \mathrm{End}_{\mathbb{Z}/p\mathbb{Z}[G]_u}(M)\ ,\ D_0 := \{d \in D \mid d(M_0) = M_0\}.$$

Since M is simple, by Schur's lemma D is a skew field. Clearly, its prime field is $\mathbb{Z}/p\mathbb{Z}$ and D_0 is a subfield. Also, the natural map $D_0 \to \mathrm{End}_{\mho_p}(M_0)$ is an embedding: if $d \in D_0$, $dx = \lambda x$ for some $\lambda \in \mathbb{Z}/p\mathbb{Z}[\ker u]$ such that $\lambda\mathbb{Z}/p\mathbb{Z}[\ker u] \subset \mathbb{Z}/p\mathbb{Z}[\ker u] + I$, and $d_{|M_0} = 0$ iff $\lambda\mathbb{Z}/p\mathbb{Z}[\ker u] \subset I$. In particular, $\lambda \in I$, thus $d = 0$.

Arguing by contradiction, we assume that $Z\cap\ker u$ contains an element z whose image $\overline{z} \in \mathrm{Aut}(M)$ is of infinite order. We have $\overline{z} \in D$ since $z \in Z$, and $\overline{z}M_0 = M_0$ since $u(z) = 0$. Thus $\overline{z} \in D_0$, and since $\overline{z}$ has infinite order it is transcendent over $\mathbb{Z}/p\mathbb{Z}$ (here we use the fact that $\mathbb{Z}/p\mathbb{Z}$ is finite). Thus $\overline{z}$ generates a field $\mathbb{Z}/p\mathbb{Z}(\overline{z}) \subset D_0$ which is isomorphic to $\mathbb{Z}/p\mathbb{Z}(t)$.

Also, $\overline{z}$ defines on M_0 a structure of $K[t,t^{-1}]$module by $t^m.x = \overline{z}^m x$. If $x_0 \in M \setminus \{0\}$, $\mathbb{Z}/p\mathbb{Z}(z)x_0$ is a sub-$K[t,t^{-1}]$-module isomorphic to $\mathbb{Z}/p\mathbb{Z}(t)$. Since $\ker u$ is again nilpotent and finitely generated, this contradicts Hall's Lemma 3, p.599: *If K is a field and M is a simple $K[G]$-module with G polycyclic, M does not contain a submodule $K(z)x$ isomorphic as a $K[t,t^{-1}]$-module to $K(t)$ where t is sent to some central element $z \in G$.*

End of the proof of Proposition 9.2. Let G_0 be a normal finite index subgroup such that G_0 and torsion-free, and let $\ell(G) = \min \sum \mathrm{rk}(G_i/G_{i+1})$ where (G_i) is a finite descending sequence of normal subgroups of G such that G_i/G_{i+1} is free Abelian. We make an induction on $\ell(G)$, distinguishing two cases:

- $Z \cap \ker u$ is infinite. Since $Z \cap \ker u$ has a finite image in $\mathrm{Aut}(M)$, it contains a nontrivial torsion-free subgroup F such that F acts trivially on M, thus M is a $\mathbb{Z}/p\mathbb{Z}[G/F]$-module. Over $\mathbb{Z}/p\mathbb{Z}[G/F]_{\overline{u}}$, M is also simple, finitely full and not full. Since $\ell(G/F) < \ell(G)$, this contradicts the induction hypothesis.
- $Z \cap \ker u$ is finite. Up to passing to a subgroup of finite index, we have $Z \cap \ker u = \{1\}$. Then $Z = G$, otherwise Z contains a nonzero commutator, on which u vanishes. Thus u is injective, so that $H \cap \ker u = \{1\}$ for all $H \triangleleft G$, thus finitely full is the same as full, which gives a contradiction.

10. Bi-invariant order on a residually torsion-free nilpotent group

Here we assume that G is residually torsion-free nilpotent (RTFN). Equivalently, if $(\gamma_n(G))$ is the lower central series,

$$G_n := \sqrt{\gamma_n(G)} := \{x \in G \mid (\exists m > 0)\ x^m \in \gamma_n(G)\}$$

is a subgroup and one has $\bigcap_{n\in\mathbb{N}} G_n = \{1\}$. Note that G_n is normal, G/G_n is torsion-free nilpotent and G_n/G_{n+1} is torsion-free Abelian. Since G is finitely generated, $\gamma_n(G)$ is finitely generated. Moreover, $G_n/\gamma_n(G)$ is the torsion of the nilpotent group $G/\gamma_n(G)$, thus it is also finitely generated. Thus G_n/G_{n+1} is finitely generated, and thus is free Abelian and injects in $\mathbb{R}$.

10.1. **Bi-invariant order on G.** Recall that a *bi-invariant order* on a group H is a total order relation on H which is left and right-invariant, ie

$$x \leq y \Rightarrow (\forall z)\ zx \leq zy\ ,\ xz \leq yz.$$

Such a bi-invariant order is defined by giving its positive cone $H^+ = \{x \in H \mid x > 1\}$, which is any subset $P \subset G$ satisfying

$$PP \subset P\ ,\ G = P \amalg P^{-1} \amalg \{1\}\ ,\ (\forall x \in G)\ xPx^{-1} = P.$$

The relation $x > y$ is then defined by $xy^{-1} \in P$, equivalently $y^{-1}x \in P$.

Following [Eizenbud-Lichtman 1987], we define a bi-invariant order on G as follows.

1) One first constructs by induction on n a bi-invariant order on G/G_n, using the following facts:

(1) A finitely generated free Abelian group can be embedded in $\mathbb{R}$, inducing a bi-invariant order.
(2) If $H_1 \to H \to H_2$ is an extension with H_1, H_2 having a bi-invariant order and $H_1 \subset \sqrt{Z(H)}$, one orders H lexicographically via a section $s : H_2 \to H$: an element $x \in H$ can be uniquely written $x = h_1 s(h_2)$, and we say that $x > 1$ if $h_2 > 1$ or $h_2 = 1$ and $h_1 > 1$. Using $H_1 \subset \sqrt{Z(H)}$, it is easy to see this is a bi-invariant order: the key point is that if $h_1 > 1$ and $h \in H$, then for some $m > 0$ we have $hh_1^m h^{-1} = h_1^m > 1$, thus $h_1 > 1$.

For $n = 0$, the result is obvious, and for $n = 1$ it is the first fact. If it holds for $n-1$, G/G_{n-1} and G_{n-1}/G_n admit bi-invariant orders. Since we have an extension $G_{n-1}/G_n \to G/G_n \to G_{n-1}$ and $G_{n-1}/G_n \subset \sqrt{Z(G/G_n)}$, the second fact implies the result for n.

2) Let $(G/G_n)^+$ be the cone of positive elements for a bi-invariant order on G/G_n, and π_n be the projection $G_{n-1} \to G_{n-1}/G_n$. We define a bi-invariant order on G with positive cone

$$G_n^+ := \pi_n^{-1}((G/G_n)^+ \cap (G_{n-1}/G_n)) \subset G_{n-1} \setminus G_n.$$

Then we define a "lexicographical" order on G with positive cone $G^+ := \coprod_{n=1}^{\infty} G_n^+$. In other words, an element $x \in G$ is > 1 if and only if its first nontrivial image in a subquotient G_{n-1}/G_n is > 1. It is clear that G^+ is indeed the positive cone of a bi-invariant order on G.

We shall use the following properties, immediate by construction.

Properties. *Let $x, y \in G^+$.*

- *If $x \in G_n$ and $y \notin G_n$, x is negligible with respect to y i.e. $x^N < y$ for all $N > 0$.*
- *If $x, y \in G_{n-1} \setminus G_n$, x and y are comparable i.e. there exist $N_1, N_2 > 0$ such that $y < x^{N_1}$ and $x < y^{N_2}$.*

10.2. **Mal'cev-Neumann completion and subfield with controlled support.** We recall a celebrated result of A.I. Mal'cev and B.H. Neumann: if G is a bi-invariantly ordered group and K a field, the formal series

$$K\langle G\rangle := \{\lambda \in K[[G]] \mid \mathrm{supp}(\lambda) \text{ is well-ordered}\}$$

form a skew-field (or division ring) for the natural operations, containing $K[G]$ as a subring.

Actually, we shall work mostly in $\mathcal{D}_K(G)$, the division closure of $K[G]$ in $K\langle G\rangle$. This is because of the following proposition.

Proposition 10.1. *Every element of $\mathcal{D}_K(G)$ has a support in some subset of the form $g\overline{\langle F\rangle^+} = g(\langle F\rangle^+ \cup \{1\})$, the submonoid generated by a finite subset $F \subset G^+$.*

Proof. Let $\widehat{K[G]} \subset K\langle G\rangle$ be the set of elements having such a support. Clearly, $\widehat{K[G]}$ contains $K[G]$, thus it suffices to prove that $\widehat{K[G]}$ is a subfield.

The stability by sum and product is easy.: assuming that $\mathrm{supp}(\lambda) \subset g\overline{\langle F_1\rangle^+}$ and $\mathrm{supp}(\lambda') \subset g'\overline{\langle F'\rangle^+}$, with $g \leq g'$, then

- $\mathrm{supp}(\lambda + \lambda') \subset g\overline{\langle F\rangle^+} \cup g'\overline{\langle F'\rangle^+}$. If $g = g'$, it is contained in $g\overline{\langle F \cup F'\rangle^+}$. If $g < g'$, in $g\langle F \cup F' \cup \{g^{-1}g'\}\rangle^+$.
- $\mathrm{supp}(\lambda\lambda') \subset g\overline{\langle F\rangle^+}g'\overline{\langle F'\rangle^+} \subset gg'\overline{\langle g'^{-1}Fg' \cup F'\rangle^+}$.

The stability by inverse is harder. Assuming that $\mathrm{supp}(\lambda) \subset g\overline{\langle F\rangle^+}$ and $\lambda \neq 0$, we want to prove $\mathrm{supp}(\lambda^{-1})$ has the same type of property. We have $\lambda = \mu g$ with $\mathrm{supp}(\mu) \subset \overline{\langle gFg^{-1}\rangle^+}$, thus $\lambda^{-1} = g^{-1}\mu^{-1}$, thus we can assume that $g = 1$, $\mathrm{supp}(\lambda) \subset \overline{\langle F\rangle^+}$.

Let $m = \min \mathrm{supp}(\lambda)$. There are two cases.

1) If $m = 1$, we have $\lambda = \alpha(1+\mu)$ with $\alpha \in K^*$ and $\mathrm{supp}(\mu) \sup\langle F\rangle^+$. Thus $\lambda^{-1} = \alpha^{-1}\sum_{n=0}^{\infty}(-1)^n\mu^n$ and $\mathrm{supp}(\lambda) \subset \alpha^{-1}\langle F\rangle^+$.

2) Now we assume $m > 1$. let N_F be the smallest n such that F injects in G/G_n. Then $m \in G_{N_F-r-1} \setminus G_{N_F-r}$ for some $r \geq 0$. We prove by induction on r that there exists $F_1 \subset G^+$ finite and containing F such that

$$m^{-1}\langle F\rangle^+ \cap G^+ \subset \langle F_1\rangle^+.$$

Then $\text{supp}(m^{-1}\lambda) \subset m^{-1} \subset \langle F_1\rangle^+$ and $\min\text{supp}(m^{-1}\lambda) = 1$, thus by 1) $\text{supp}(\lambda^{-1})$ will have the desired form.

• Assume that $r = 0$. Let $x \in \langle F\rangle^+$ such that $m < x$. Write $x = f_1 \cdots f_k f_{k+1} \cdots f_n$ with $f_1 \cdots f_{k-1} \leq m < f_1 \cdots f_k$. Then $f_1, \cdots, f_{k-1}$ are in $G_{N-1} \setminus G_N$ and comparable to m, thus

$$\min(F)^k = \min(G_{N-1} \setminus G_N)^k \leq f_1 \cdots f_k \leq m f_k \leq m \max(F),$$

thus $k \leq k_0$ depending only on m, F. Let

$$F_1 = F \cup \{m^{-1} f_1 \cdots f_k \mid k \leq k_0 \text{ and } f_i \in F\},$$

then we have $m^{-1}\langle F\rangle^+ \cap G^+ \subset \langle F_1\rangle^+$, as desired.

• Assume that $r > 0$ and the result is known for $r-1$. Let $x \in \langle F\rangle^+$ such that $m < x$. Write $x = f_1 \cdots f_k f_{k+1} \cdots f_n$ as above. Necessarily, $f_1, \cdots, f_{k-1}$ are in G_{N-r}. Let $i_1 < \cdots < i_\ell \leq k$ be the indices such that $f_j \notin G_{N_F-r+1}$, thus $m \sim f_j$ or $m \ll f_j$, and $j_1 < \cdots < j_{\ell'}$ the indices such that $f_j \in G_{N_F-r+1}$, thus $f_j < m$. Then

$$\min(F \setminus G_{N_F-r+1})^\ell = \min(G_{N-r} \setminus G_{N_F-r+1})^\ell \leq f_{i_1} \cdots f_{i_\ell} \leq f_1 \cdots f_k \leq m \max(F),$$

thus $\ell \leq \ell_0$ depending only on m, F. Setting $B = \{1\} \cup F \cup F^2 \cdots \cup F^{\ell_0}$, we have

$$f_1 \cdots f_k = f'_{j_1} \cdots f'_{j_{\ell'}} f_{i_1} \cdots f_{i_\ell},$$

where $f'_{j_s} = b f_{j_s} b^{-1}$ and $b \in \{1\} \cup F \cup \cdots \cup F^\ell$. Then set

$$F_1 := F \cup \{b f b^{-1} \mid b \in B\ ,\ f \in F\} \cup \{m^{-1} b \mid b \in B\ ,\ m^{-1} b > 1\}.$$

We distinguish two cases.

(i) Assume that $f_{i_1} \cdots f_{i_\ell} > m$. Then

$$m^{-1} f_1 \cdots f_k = (m^{-1} f_{i_1} \cdots f_{i_\ell}) \prod_{r=1}^{\ell'} (f_{i_1} \cdots f_{i_\ell})^{-1} f'_{j_r} (f_{i_1} \cdots f_{i_\ell}).$$

Thus

$$m^{-1} x = (m^{-1} f_1 \cdots f_k) f_{k+1} \cdots f_n \in \langle F_1\rangle^+,$$

thus $m^{-1}\langle F\rangle^+ \cap G^+ \subset \langle F_1\rangle^+$.

(ii) Assume that $f_{i_1} \cdots f_{i_\ell} < m$. Then $1 < (f_{i_1} \cdots f_{i_\ell})^{-1} m < f'_{j_1} \cdots f'_{j_{\ell'}}$, thus $(f_{i_1} \cdots f_{i_\ell})^{-1} m \in G_{N_F-r+1}$. Set $m_{i_1,\cdots,i_\ell} = (f_{i_1} \cdots f_{i_\ell})^{-1} m$. Then

$$m^{-1} x = m^{-1}_{i_1,\cdots,i_\ell} f'_{j_1} \cdots f'_{j_{\ell'}} \in m^{-1}_{i_1,\cdots,i_\ell} \langle F_1\rangle^+,$$

thus

$$m^{-1}\langle F\rangle^+ \cap G^+ \subset \bigcup_{\ell \leq \ell_0} \bigcup_{(i_1,\cdots,i_\ell)} m^{-1}_{i_1,\cdots,i_\ell} \langle F_1\rangle^+.$$

Moreover, either $N_{F_1} = N_F$ and $m_{i_1,\cdots,i_\ell} \in G_{N_F-r+1}$, or $N_{F_1} > N_F$ and $m_{i_1,\cdots,i_\ell} \in G_{N_{F_1}}$. In both cases, by the induction hypothesis there exists $F_{i_1,\cdots,i_\ell} \subset G^+$ finite and containing F_1 such that

$$m^{-1}_{i_1,\cdots,i_\ell} \langle F_1\rangle^+ \cap G^+ \subset \langle F_{i_1,\cdots,i_\ell}\rangle^+,$$

thus

$$m^{-1}\langle F\rangle^+ \cap G^+ \subset \Big\langle \bigcup_{\ell\leq \ell_0} \bigcup_{(i_1,\cdots,i_\ell)} F_{i_1,\cdots,i_\ell}\rangle\Big\rangle^+$$

which finishes the proof.

10.3. **Property of finite fibers.** The interest of the subfield $\mathcal{D}_K(G)$ lies in the following proposition and its corollaries.

Proposition 10.2. *If $F \subset G^+$ is finite and injects in G/G_N (which is true for N large enough), the projection $\pi_N : G \to G/G_N$ has finite fibers on $\langle F\rangle^+$ for N large enough.*

Corollary 10.3. *Every element $\lambda \in \mathcal{D}_K(G)$ has a well-defined image $\bar\lambda$ in $K\widehat{[G/G_N]}$ for N large enough. Furthermore, if $\mu \in K[G]$, the image of $\lambda\mu$ is $\bar\lambda\bar\mu$.*

Corollary 10.4. *If $\lambda \in \mathcal{D}_K(G)$ has a well-defined image $\bar\lambda$ and $\overline{\lambda} \in K[G/G_N]_{\overline{u}}$, then $\lambda \in K[G]_u$.*

Proof. This follows from [Passman 1977], Lemma 2.10 p.599-601, an ingredient of the proof of the well-orderedness of $\langle F\rangle^+$ in the construction of Mal'cev-Neumann. Let us give a simpler proof.

Let N be the smallest integer such that F injects in G/G_N. We fix $g \in \langle F\rangle^+$, and let r be such that $g \in G_{N-r-1} \setminus G_N$. We want to show the slightly stronger property that the natural map $p_N : W(F^+) \to \langle F\rangle^+ \to G/G_N$ has finite fibers. We prove it by induction on r, with an argument partially similar to 10.2, but simpler:

- Assume that $r = 0$. If $p_N(w) = g$, we have $w = f_1 \cdots f_\ell$, where every f_i is in $F\cap(G_{N-1}\setminus G_{k-1})$. Thus $\min(F \setminus G_k)^\ell \leq g$, thus $\ell \leq \ell_0$ depending only on F, g, and $\mathrm{card}(p_N^{-1}(\{g\}) \leq (\mathrm{card}(F))^{\ell_0}$.
- Assume that $r > 0$ and the result is known for $r - 1$. Assume to the contrary that we have an infinite sequence (w_n) of words in $W(F)$ such that $p_N(w_n) = g$. We write

  $$w_n = f_{n,1}w_{n,1} \cdots f_{n,i_n}w_{n,i_n},$$

  where each $f_{n,i}$ is in $F\setminus G_{N-r+1}$ and each $w_{n,i}$ is in $W(F\cap G_{N-r+1})$. Necessarily, $f_{n,i} \in G_{N-r}$, thus $\min(F \setminus G_{N-r})^{i_n} = \min(F \cap (G_{N-r} \setminus G_{N-r+1}))^{i_n} \leq g$, thus $i \leq i_0$ depending only on F, g. Thus at least one of the sequences $(w_{i,n})_n$ is infinite.

  By the induction hypothesis, every such infinite sequence has a subsequence on which p_N is injective, and since $p_N(\langle F\rangle^+)$ is well ordered, another subsequence is increasing. By the diagonal process, we can assume that every sequence $(w_{i,n})_n$ is increasing or constant, and that one of them is infinite. But then $(p_N(w_n))_n$ is increasing, contradiction.

Remark. If we did not know already that $p_N(\langle F\rangle^+)$ (or $\langle F\rangle^+$) is well-ordered, we would make it part of the induction on r, and we would get the bonus that its ordinal is at most ω^N.

10.4. **Compatibility with u.** Inspired by [Kielak 2018], for the proof of Theorem 7.2 we shall require the order to be compatible with u in the sense that $(u(x) > 0 \Rightarrow x > 1)$. This is possible by changing the definition of (G_n), setting

- $G_1^{new} = \ker u$
- $G_2^{new} = G_{n-1}$ if $n \geq 2$.

and defining the order on G/G_1 by embedding it in $\mathbb{R}$ via $\overline{u}$ induced by u. This implies

$$\mathbb{Z}/p\mathbb{Z}[G]_u \subset \mathbb{Z}/p\mathbb{Z}\langle G\rangle.$$

11. Proof of Theorem 7.2

By Remark 1 after the statement of Theorem 7.2, it suffices to treat the case when G is RTFN. We define G_n, G_n^+, G^+ and the order on G as in the previous section, with the property $(u(x) > 0 \Rightarrow x > 1)$, thus $\mathbb{Z}/p\mathbb{Z}[G]_u \subset \mathbb{Z}/p\mathbb{Z}\langle G\rangle$.

Let $A \in \mathrm{M}_n(\mathbb{Z}/p\mathbb{Z}[G])$ with every image in $\mathrm{M}_n(\mathbb{Z}/p\mathbb{Z}[G/(H \cap \ker u]_{\overline{u}})$ invertible. We want to prove that A is invertible in $\mathrm{M}_n(\mathbb{Z}/p\mathbb{Z}[G]_u)$.

We first prove that A is invertible in $\mathrm{M}_n(\widehat{\mathbb{Z}/p\mathbb{Z}[G]})$. Assume the contrary, then since $\widehat{\mathbb{Z}/p\mathbb{Z}[G]}$ is a skew field, there exists $L = (\lambda_1|\cdots|\lambda_n) \in \mathrm{M}_{1,n}(\widehat{\mathbb{Z}/p\mathbb{Z}[G]}) \setminus \{0\}$ such that $LA = 0$. By Corollary 10.3, for N large enough the image $\bar{L} \in \mathbb{Z}/p\widehat{\mathbb{Z}[G/}G_N]$ is well-defined, and we have $\bar{L}\bar{A} = 0$, where $\bar{A}$ is the image of A in $\mathrm{M}_n(\mathbb{Z}/p\mathbb{Z}[G/G_N]) \subset \mathbb{Z}/p\widehat{\mathbb{Z}[G/}G_N]$. Thus $\bar{A}$ is not invertible in $\mathrm{M}_n(\mathbb{Z}/p\widehat{\mathbb{Z}[G/}G_N]$. Since $\mathbb{Z}/p\mathbb{Z}\langle G/G_N\rangle$ is a field containing $\mathbb{Z}/p\widehat{\mathbb{Z}[G/}G_N]$, it is not invertible in $\mathrm{M}_n(\mathbb{Z}/p\mathbb{Z}\langle G/G_N\rangle)$. A fortiori it is not invertible in $\mathrm{M}_n(\mathbb{Z}/p\mathbb{Z}[G/G_N]_{\overline{u}})$. On the other hand, since G/G_N is nilpotent, Corollary 9.2 implies that $\bar{A}$ is not finitely invertible, thus A is not finitely invertible, contradiction.

Let B be the inverse of A in $\mathrm{M}_n(\widehat{\mathbb{Z}/p\mathbb{Z}[G]})$, there remains to prove $B \in \mathrm{M}_n(\mathbb{Z}/p\mathbb{Z}[G]_u)$. Again by Corollary 10.3, B has a well-defined image $\bar{B} \in \mathrm{M}_n(\mathbb{Z}/p\widehat{\mathbb{Z}[G/}G_N])$, and $\bar{A}\bar{B} = \mathrm{I}_n = \bar{B}\bar{A}$. Thus $\bar{B}$ is the inverse of $\bar{A} \in \mathrm{M}_n(\mathbb{Z}/p\mathbb{Z}\langle G/G_N\rangle)$ thus also in $\mathrm{M}_n(\mathbb{Z}/p\mathbb{Z}\langle G/G_N\rangle)$. Since $\bar{A}$ is invertible already in $\mathrm{M}_n(\mathbb{Z}/p\mathbb{Z}[G/G_N]_u)$, $\bar{B}$ is in $\mathrm{M}_n(\mathbb{Z}/p\mathbb{Z}[G/G_N]_u)$. By Corollary 10.4, $B \in \mathrm{M}_n(\mathbb{Z}/p\mathbb{Z}[G]_u)$, qed.

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