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Raphaël Danchin, Jin Tan

ON THE WELL-POSEDNESS OF THE HALL-MAGNETOHYDRODYNAMICS SYSTEM IN CRITICAL SPACES

RAPHAËL DANCHIN AND JIN TAN

ABSTRACT. We investigate the existence and uniqueness issues of the 3D incompressible Hall-magnetohydrodynamic system supplemented with initial data in critical regularity spaces. First, we establish a global result for small initial data in the Besov spaces $\dot{B}^{\frac{3}{p}-1}_{p,1}$ with $1 \leq p < \infty$, and the conservation of higher regularity. Second, in the case where the viscosity is equal to the magnetic resistivity, we obtain the global well-posedness for (small) initial data in the *larger* critical Besov spaces of type $\dot{B}^{\frac{1}{2}}_{2,r}$ for any $r \geq 1$. In the particular case $r = 1$, we also establish the local existence for large data, and supplement our results with continuation criteria.

To the best of our knowledge, the present paper is the first one where well-posedness is proved for the Hall-MHD system, in a critical regularity setting.

1. Introduction

We are concerned with the following three dimensional incompressible resistive and viscous Hall-magnetohydrodynamics system (Hall-MHD):

$$\partial_t u + \operatorname{div}(u \otimes u) + \nabla P = (\nabla \times B) \times B + \mu \Delta u, \tag{1.1}$$

$$\operatorname{div} u = 0, \tag{1.2}$$

$$\partial_t B = \nabla \times ((u - \varepsilon \nabla \times B) \times B) + \nu \Delta B, \tag{1.3}$$

supplemented with the initial conditions

$$(u(0,x), B(0,x)) = (u_0(x), B_0(x)), \quad x \in \mathbb{R}^3. \tag{1.4}$$

The unknown vector-fields $u = u(t,x)$ and $B = B(t,x)$, and scalar function $P = P(t,x)$ with $t \geq 0$ and $x \in \mathbb{R}^3$ represent the velocity field, the magnetic field and the scalar pressure, respectively. The parameters μ and ν are the fluid viscosity and the magnetic resistivity, while the dimensionless number ε measures the magnitude of the Hall effect compared to the typical length scale of the fluid. In accordance with (1.2), we assume that $\operatorname{div} u_0 = 0$ and, for physical consistency, since a magnetic field has to be divergence free, we suppose that $\operatorname{div} B_0 = 0$, too, a property that is conserved through the evolution.

The above system is used to model the evolution of electrically conducting fluids such as plasmas or electrolytes (then, u represents the ion velocity), and takes into account the fact that in a moving conductive fluid, the magnetic field can induce currents which, in turn, polarize the fluid and change the magnetic field. That phenomenon which is neglected in the classical MHD equations, is represented

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by the *Hall electric field* $E_H := \varepsilon J \times B$ where the current J is defined by $J := \nabla \times B$. Hall term plays an important role in magnetic reconnection, as observed in e.g. plasmas, star formation, solar flares, neutron stars or geo-dynamo (for more explanation on the physical background of Hall-MHD system, one can refer to [2, 3, 12, 13, 15, 16]).

Despite its physical relevance, Hall-MHD system has been considered only recently in mathematics, following the work by Acheritogaray, Degond, Frouvelle and Liu in [1] where the Hall-MHD system was formally derived both from a two fluids system and from a kinetic model. Then, in [7], Chae, Degond and Liu showed the global existence of weak solutions as well as the local well-posedness for initial data u_0 and B_0 in sobolev spaces H^s with $s > 5/2$. Weak solutions have been further investigated by Dumas and Sueur in [11] both for the Maxwell-Landau-Lifshitz system and for the Hall-MHD system. In [8], blow-up criteria for smooth solutions and the global existence of smooth solutions emanating from small initial data have been obtained. More recently, [5], [17], [18] established the well-posedness of strong solutions with improved regularity conditions for initial data in sobolev or Besov spaces. Examples of smooth data with arbitrarily large L^∞ norms giving rise to global unique solutions have been exhibited very recently in [14].

Our main goal here is to establish the well-posedness of the Hall-MHD system with initial data in *critical spaces.* Since the system does not have any scaling invariance however (in contrast with the classical MHD system corresponding to $\varepsilon = 0$), one first has to explain what we mean by critical regularity. Observe that, on the one hand, if $B \equiv 0$, then u satisfies the incompressible Navier-Stokes equations:

$$(NS) \qquad \begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla P = \mu \Delta u, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0, \end{cases}$$

which are invariant for all $\lambda > 0$ by the rescaling

$$u(t,x) \rightsquigarrow \lambda u(\lambda^2 t, \lambda x) \quad \text{and} \quad P(t,x) \rightsquigarrow \lambda^3 P(\lambda^2 t, \lambda x) \tag{1.5}$$

provided the initial velocity u_0 is rescaled according to

$$u_0(x) \rightsquigarrow \lambda u_0(\lambda x). \tag{1.6}$$

On the other hand, if the fluid velocity in (1.3) is 0, then we get the following *Hall equation* for B:

$$(Hall) \qquad \begin{cases} \partial_t B + \nabla \times ((\nabla \times B) \times B) = \nu \Delta B, \\ B|_{t=0} = B_0, \end{cases}$$

which is invariant by the rescaling

$$B(t,x) \rightsquigarrow B(\lambda^2 t, \lambda x) \tag{1.7}$$

provided the data B_0 is rescaled according to

$$B_0(x) \rightsquigarrow B_0(\lambda x). \tag{1.8}$$

In other words, ∇B has the same scaling invariance as the fluid velocity u in (NS). Reverting to the whole Hall-MHD system however, we see that the term $(\nabla \times B) \times B$ in (1.1) is out of scaling.

Let us now look at the current function $J = \nabla \times B$ as an additional unknown. Owing to the vector identity

$$\nabla \times (\nabla \times v) + \Delta v = \nabla \mathrm{div}\, v \tag{1.9}$$

and since B is divergence free, we have $\Delta B = -\nabla \times J$, whence

$$B = \mathrm{curl}^{-1} J := (-\Delta)^{-1} \nabla \times J,$$

where the -1 order homogeneous Fourier multiplier curl^{-1} is defined on the Fourier side by

$$\mathcal{F}(\mathrm{curl}^{-1} J)(\xi) := \frac{i\xi \times \widehat{J}(\xi)}{|\xi|^2}\cdot \tag{1.10}$$

With that notation, one gets the following *extended Hall-MHD system*:

$$\begin{cases} \partial_t u + \mathrm{div}\,(u \otimes u) - \mu \Delta u + \nabla P = (\nabla \times B) \times B, \\ \mathrm{div}\, u = 0, \\ \partial_t B - \nabla \times ((u - \varepsilon J) \times B) - \nu \Delta B = 0, \\ \partial_t J - \nabla \times \big(\nabla \times ((u - \varepsilon J) \times \mathrm{curl}^{-1} J)\big) - \nu \Delta J = 0. \end{cases}$$

The advantage of that extended (and redundant) formulation is that it has a scaling invariance, which is actually the same as that of the incompressible Navier-Stokes equations. It is thus natural to study whether the Hall-MHD system written in terms of (u, B, J) is well-posed in the same functional spaces as the velocity in (NS), and if similar blow-up criteria and qualitative behavior may be established.

We end this introductory part presenting a few notations. As usual, we denote by C harmless positive 'constants' which may change from one line to the other, and we sometimes write $A \lesssim B$ instead of $A \leq CB$. Likewise, $A \sim B$ means that $C_1 B \leq A \leq C_2 B$ with absolute constants C_1, C_2. For X a Banach space, $p \in [1, \infty]$ and $T > 0$, the notation $L^p(0, T; X)$ or $L^p_T(X)$ designates the set of measurable functions $f : [0, T] \to X$ with $t \mapsto \|f(t)\|_X$ in $L^p(0, T)$, endowed with the norm $\|\cdot\|_{L^p_T(X)} := \|\|\cdot\|_X\|_{L^p(0,T)}$, and agree that $\mathcal{C}([0, T], X)$ denotes the set of continuous functions from $[0, T]$ to X. Slightly abusively, we keep the same notation for functions with several components.

2. Main results

Since the incompressible Navier-Stokes equations are well-posed in all homogeneous Besov spaces $\dot{B}^{\frac{3}{p}-1}_{p,r}$ with $1 \leq p < \infty$ and $1 \leq r \leq \infty$, it is tempting to study whether it is also the case for the Hall-MHD system written in its extended formulation. Here we shall address that issue in full generality if the last index of Besov spaces is $r = 1$, and under a smallness condition on the magnetic field. More results will be achieved if $\mu = \nu$ and $p = 2$ (an assumption that is usually made in mathematical papers devoted to the Hall-MHD system).

For the time being, let us consider data in the critical regularity space $\dot{B}^{\frac{3}{p}-1}_{p,1}$. Then, since (extended) Hall-MHD system has many similarity with the incompressible Navier-Stokes equations, one expects, after the work by J.-Y. Chemin in [9], to get a solution (u, B, J) in the space

$$E_p(T) := \Big\{ z \in \mathcal{C}([0, T], \dot{B}^{\frac{3}{p}-1}_{p,1}),\ \nabla^2_x z \in L^1(0, T; \dot{B}^{\frac{3}{p}-1}_{p,1}) \quad \text{and} \quad \mathrm{div}_x\, z = 0 \Big\}$$

or in its global version, denoted by E_p, if the data are small.

Our first result states the global well-posedness of the Hall-MHD system for small data in $\dot B^{\frac3p-1}_{p,1}$, and conservation of higher order Sobolev regularity. It is valid for all positive coefficients μ, ν and ε.

Theorem 2.1. *Let $1\leq p<\infty$ and $(u_0,B_0)\in \dot B^{\frac3p-1}_{p,1}$ with $\operatorname{div} u_0=\operatorname{div} B_0=0$ and $J_0:=\nabla\times B_0\in \dot B^{\frac3p-1}_{p,1}$. There exists a constant $c>0$ depending only on p, μ, ν and ε such that if*

$$\|u_0\|_{\dot B^{\frac3p-1}_{p,1}}+\|B_0\|_{\dot B^{\frac3p-1}_{p,1}}+\|J_0\|_{\dot B^{\frac3p-1}_{p,1}}<c, \tag{2.1}$$

then the Cauchy problem (1.1)-(1.4) has a unique global solution $(u,B)\in E_p$, with $J:=\nabla\times B\in E_p$. Furthermore,

$$\|u\|_{E_p}+\|B\|_{E_p}+\|J\|_{E_p}<2c. \tag{2.2}$$

If, in addition, $u_0\in H^s$ and $B_0\in H^r$ with

$$\frac3p-1<s\leq r \quad \text{and} \quad \frac3p<r\leq 1+s, \tag{2.3}$$

then $(u,B)\in \mathcal{C}_b(\mathbb{R}_+;H^s\times H^r)$, $\nabla u\in L^2(\mathbb{R}_+;H^s)$ and $\nabla B\in L^2(\mathbb{R}_+;H^r)$ and the following energy balance is fulfilled for all $t\geq 0$:

$$\|u(t)\|_{L^2}^2+\|B(t)\|_{L^2}^2+2\int_0^t\big(\mu\|\nabla u\|_{L^2}^2+\nu\|\nabla B\|_{L^2}^2\big)\,d\tau=\|u_0\|_{L^2}^2+\|B_0\|_{L^2}^2. \tag{2.4}$$

Finally, in the case where only J_0 fulfills (2.1), there exists some time $T>0$ such that (1.1)-(1.4) has a unique local-in-time solution on $[0,T]$ with (u,B,J) in $E_p(T)$, and additional Sobolev regularity is preserved.

Whether the smallness condition on J_0 may be omitted in the context of general critical regularity spaces $\dot B^{\frac3p-1}_{p,1}$ is an open question. We shall see that the difficulty not only comes from Hall term but also from the coupling between u and B through the term $\nabla\times(u\times B)$. For essentially the same reason, we do not know how to solve the system in $\dot B^{\frac3p-1}_{p,r}$ if $r>1$, unless $p=2$ and $\mu=\nu$.

The key to the proof is to consider the extended Hall-MHD system, suitably rewritten in the form of a generalized Navier-Stokes system that may be solved by implementing the classical fixed point theorem in the (complete) space E_p, as in Chemin's work [9]. In order to derive an appropriate formulation of the system, one has to recall some algebraic identities. The first one is that for any couple of C^1 divergence free vector-fields v and w on $\mathbb{R}^3$, we have

$$w\cdot\nabla v=\operatorname{div}(v\otimes w), \quad \text{where} \quad \big(\operatorname{div}(v\otimes w)\big)^j:=\sum_{k=1}^3\partial_k(v^jw^k). \tag{2.5}$$

Observe also that

$$(\nabla\times w)\times w=(w\cdot\nabla)w-\nabla\Big(\frac{|w|^2}{2}\Big)\cdot \tag{2.6}$$

Hence, setting $Q:=P+|B|^2/2$, equation (1.1) recasts in

$$\partial_t u+\operatorname{div}(u\otimes u)+\nabla Q=\operatorname{div}(B\otimes B)+\mu\Delta u. \tag{2.7}$$

After projecting (2.7) onto the set of divergence free vector fields by means of the *Leray projector* $\mathcal{P} := \mathrm{Id} - \nabla(-\Delta)^{-1}\mathrm{div}$, we get

$$\partial_t u - \mu\Delta u = Q_a(B,B) - Q_a(u,u), \tag{2.8}$$

where the bilinear form Q_a is defined by

$$Q_a(v,w) := \frac{1}{2}\mathcal{P}(\mathrm{div}(v\otimes w) + \mathrm{div}(w\otimes v)).$$

Next, by using the identity

$$\nabla\times(w\times v) = v\cdot\nabla w - w\cdot\nabla v, \tag{2.9}$$

one can rewrite Hall term as

$$\nabla\times(J\times B) = B\cdot\nabla J - J\cdot\nabla B.$$

Hence, combining with (2.5), equation (1.3) recasts in

$$\partial_t B - \nu\Delta B = Q_b(B, \varepsilon J - u),$$

where

$$Q_b(v,w) := \mathrm{div}(v\otimes w) - \mathrm{div}(w\otimes v) = w\cdot\nabla v - v\cdot\nabla w,$$

and the equation for J may thus be written

$$\partial_t J - \varepsilon\Delta J = \nabla\times Q_b(\mathrm{curl}^{-1}J, \varepsilon J - u).$$

Altogether, we conclude that the extended Hall-MHD system recasts in

$$\begin{cases} \partial_t u - \mu\Delta u = Q_a(B,B) - Q_a(u,u),\\ \partial_t B - \nu\Delta B = Q_b(B, \varepsilon J - u),\\ \partial_t J - \nu\Delta J = \nabla\times Q_b(\mathrm{curl}^{-1}J, \varepsilon J - u),\\ (u(0,x), B(0,x), J(0,x)) = (u_0, B_0, J_0). \end{cases}$$

Set $U := (U_1, U_2, U_3)$ with $U_1 := u$, $U_2 := B$ and $U_3 := J$. Then, the above system may be shortened into:

$$\begin{cases} \partial_t U - \Delta U = Q(U,U),\\ U|_{t=0} = U_0, \end{cases} \tag{2.10}$$

where $Q : \mathbb{R}^3\times\mathbb{R}^3 \to \mathbb{R}^9$ is defined by

$$Q(V,W) := \begin{pmatrix} Q_a(V_2,W_2) - Q_a(V_1,W_1)\\ Q_b(V_2, \varepsilon W_3 - W_1)\\ \nabla\times Q_b(\mathrm{curl}^{-1}V_3, \varepsilon W_3 - W_1) \end{pmatrix}\cdot \tag{2.11}$$

The gain of considering the above extended system rather than the initial one is that it is semi-linear, while the Hall-MHD system for (u,B) is quasi-linear. Furthermore, the quadratic terms in the first two lines of (2.10) are of the same type of that of the incompressible Navier-Stokes equation. Only the last line forces us to go beyond the theory of the generalized Navier-Stokes equations as presented in e.g. [4, Chap. 5] since the differentiation is *outside* instead of being inside the first variable of Q_b (this actually prevents us from considering large J_0's and to handle regularity in Besov spaces $\dot B^{\frac{3}{p}-1}_{p,r}$ with $r > 1$).

The Hall term makes the Hall-MHD system more nonlinear than the usual MHD system which explains while, somehow, it is difficult to recover exactly the same results. In the case $\mu=\nu$ however, it is possible to take advantage of some cancellation property that eliminates the Hall term when performing an energy method. This will enable us to prove the local well-posedness for large data in $\dot B^{\frac12}_{2,1}$ and the global well-posedness for small data in all spaces $\dot B^{\frac12}_{2,r}$ with $r\in[1,\infty]$.

At this stage, it is convenient to introduce the function $v:=u-\varepsilon J$ (that, physically, may be interpreted as the velocity of an electron, see [2] page 5) and to use another extended formulation for the Hall-MHD system that is valid only if $\mu=\nu$. To achieve it, we need the vector identities:

$$\nabla(w\cdot z)=(\nabla w)^Tz+(\nabla v)^Tw \quad\text{and}\quad (\nabla w-(\nabla w)^T)z=(\nabla\times w)\times z,$$

where $(\nabla w)_{ij}:=\partial_j w^i$ for $1\leq i,j\leq 3$.

Combining with (2.9) yields

$$\begin{aligned}\nabla\times(w\times z)&=z\cdot\nabla w-w\cdot\nabla z\\&=(\nabla w-(\nabla w)^T)z+(\nabla z-(\nabla z)^T)w-2w\cdot\nabla z+\nabla(w\cdot z)\\&=(\nabla\times w)\times z+(\nabla\times z)\times w-2w\cdot\nabla z+\nabla(w\cdot z).\end{aligned}\tag{2.12}$$

Then, applying Identity (2.12) to the term $\nabla\times(v\times B)$, equation (1.3) turns into

$$\partial_t B-\nu\Delta B=(\nabla\times v)\times B-v\times u-2v\cdot\nabla B+\nabla(v\cdot B).$$

Taking $\varepsilon\cdot$curl of the above equation, and subtracting it from (1.1), we get

$$\begin{aligned}\partial_t v-\mu\Delta v=B\cdot\nabla B-u\cdot\nabla u-\nabla\times((\nabla\times v)\times B)\\+\nabla\times(v\times u)+2\nabla\times(v\cdot\nabla B)-\nabla Q.\end{aligned}$$

Therefore, in terms of unknowns (u,B,v), the extended Hall-MHD system reads

$$\begin{cases}\partial_t u-\mu\Delta u=B\cdot\nabla B-u\cdot\nabla u-\nabla Q,\\ \operatorname{div}u=0,\\ \partial_t B-\mu\Delta B=\nabla\times(v\times B),\\ \partial_t v-\mu\Delta v=B\cdot\nabla B-u\cdot\nabla u-\nabla\times((\nabla\times v)\times B)\\ \qquad\qquad+\nabla\times(v\times u)+2\nabla\times(v\cdot\nabla B)-\nabla Q.\end{cases}\tag{2.13}$$

That system is still quasilinear. However, the most nonlinear term cancels out when performing an energy method, since

$$(\nabla\times((\nabla\times v)\times B),v)_{L^2}=0.\tag{2.14}$$

After localization of the system by means of the Littlewood-Paley spectral cut-off operators $\dot\Delta_j$ defined in the Appendix, the above identity still holds, up to some lower order commutator term. This will enable us to prove the following local well-posedness result *for large data* in the critical Besov space $\dot B^{\frac12}_{2,1}$, together with a blow-up criterion.

Theorem 2.2. *Assume that $\mu=\nu$. For any initial data (u_0,B_0) in $\dot B^{\frac12}_{2,1}$ with $\operatorname{div}u_0=\operatorname{div}B_0=0$ and $J_0:=\nabla\times B_0\in\dot B^{\frac12}_{2,1}$, there exists a positive time T such that the Cauchy problem* (1.1)-(1.4) *has a unique solution $(u,B)\in E_2(T)$ with*

$J := \nabla \times B \in E_2(T)$. *Moreover, if the maximal time of existence* T^* *of that solution is finite, then*

$$\int_0^{T^*} \|(u, B, \nabla B)(t)\|^2_{L^\infty}\, dt = \infty \tag{2.15}$$

$$\int_0^{T^*} \|(u, B, \nabla B)(t)\|_{\dot B^{\frac52}_{2,1}}\, dt = \infty \tag{2.16}$$

and, for any $\rho \in (2, \infty)$,

$$\int_0^{T^*} \|(u, B, \nabla B)(t)\|^\rho_{\dot B^{\frac{2}{\rho}-1}_{\infty,\infty}}\, dt = \infty. \tag{2.17}$$

Still for $\mu = \nu$, one can prove well-posedness in *any* critical space $\dot B^{\frac12}_{2,r}$ with $r \in [1, \infty]$. Then, the components of the solution will belong to the following space[1]:

$$E_{2,r}(T) := \Big\{ v \in \widetilde{\mathcal C}_T(\dot B^{\frac12}_{2,r}),\ \nabla_x^2 v \in \widetilde L^1_T(\dot B^{\frac12}_{2,r}) \quad \text{and} \quad \operatorname{div}_x v = 0 \Big\},$$

where the letter T is omitted if the time interval is $\mathbb{R}_+$.

Theorem 2.3. *Assume that* $\mu = \nu$. *Consider initial data* (u_0, B_0) *in* $\dot B^{\frac12}_{2,r}$ *with* $\operatorname{div} u_0 = \operatorname{div} B_0 = 0$ *and* $J_0 := \nabla \times B_0 \in \dot B^{\frac12}_{2,r}$ *for some* $r \in [1, \infty]$. *Then, the following results hold true:*

(1) there exists $c > 0$ *such that if*

$$\|u_0\|_{\dot B^{\frac12}_{2,r}} + \|B_0\|_{\dot B^{\frac12}_{2,r}} + \|u_0 - \varepsilon J_0\|_{\dot B^{\frac12}_{2,r}} < c\mu, \tag{2.18}$$

then the Hall-MHD equations have a unique global solution (u, B) *with* (u, B, J) *in* $E_{2,r}$.

(2) If only $\|u_0 - \varepsilon J_0\|_{\dot B^{\frac12}_{2,r}} < c\mu$, *then there exists* $T > 0$ *such that the Hall-MHD system has a unique solution* (u, B) *on* $[0, T]$, *with* (u, B, J) *in* $E_{2,r}(T)$.

The rest of the paper unfolds as follows. The next section is devoted to the proof of Theorem 2.1. In Section 4, we focus on the case $\mu = \nu$ and prove Theorem 2.2 by taking advantage of the cancellation property pointed out above. The proof of Theorem 2.3 is carried out in Section 5. For the reader's convenience, a few results concerning Besov spaces, Littlewood-Paley decomposition and commutator estimates are recalled in Appendix.

3. Well-posedness in general critical Besov spaces with third index 1

Here we prove Th. 2.1. For expository purpose, we assume that[2] $\mu = \nu = \varepsilon = 1$. Throughout this section and the following ones, we shall repeatedly use the fact that, as a consequence of Proposition A.2 (vi), one has the following equivalence of norms for all $s \in \mathbb{R}$ and $(p, r) \in [1, +\infty]^2$:

$$\|\nabla B\|_{\dot B^s_{p,r}} \sim \|J\|_{\dot B^s_{p,r}} \quad \text{and} \quad \|\nabla B\|_{\dot H^s} = \|J\|_{\dot H^s}. \tag{3.1}$$

In order to establish the global existence of a solution of the Hall-MHD system in the case of small data, we shall first prove the corresponding result for the

[1] The reader may refer to Definition A.3 for the definition of 'tilde spaces'

[2] Note that $\nu = \varepsilon = 1$ can be achieved after suitable rescaling, and that having $\mu \neq 1$ would not affect the final result: it is just a matter of changing the definition of $\mathcal{B}$ in (3.2) accordingly.

extended system (2.10). It relies on the following well known corollary of the fixed point theorem in complete metric spaces.

Lemma 3.1. *Let $(X, \|\cdot\|_X)$ be a Banach space and $\mathcal{B} : X \times X \to X$, a bilinear continuous operator with norm K. Then, for all $y \in X$ such that $4K\|y\|_X < 1$, equation*

$$x = y + \mathcal{B}(x, x)$$

has a unique solution x in the ball $B(0, \frac{1}{2K})$. Besides, x satisfies $\|x\|_X \leq 2\|y\|_X$.

We shall take for X the set of triplets of (time dependent) divergence free vector-fields with components in E_p endowed with the norm

$$\|V\|_X := \|V\|_{L^1(\dot{B}^{\frac{3}{p}+1}_{p,1})} + \|V\|_{L^\infty(\dot{B}^{\frac{3}{p}-1}_{p,1})}.$$

Let $(e^{t\Delta})_{t\geq 0}$ denote the heat semi-group defined in (A.3). We set $y : \ t \mapsto e^{t\Delta}U_0$ and define the bilinear functional $\mathcal{B}$ by the formula

$$\mathcal{B}(V, W)(t) = \int_0^t e^{(t-\tau)\Delta} Q(V, W)\, d\tau. \tag{3.2}$$

By virtue of (A.2), System (2.10) recasts in

$$U(t) = y(t) + \mathcal{B}(U, U)(t). \tag{3.3}$$

In order to apply Lemma 3.1, it suffices to show that y is small in X, and that $\mathcal{B}$ maps $X \times X$ to X. The former property holds true if Condition (2.1) is fulfilled for a small enough $c > 0$, as Proposition A.4 ensures that y belongs to X and that

$$\|y\|_X \leq C\|U_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}.$$

In order to prove the latter property, one can use the fact that, by virtue of Identity (2.5), Proposition A.2 (i), (iii), (vi), and of Inequality (A.6), we have

$$\begin{aligned} \|\mathrm{div}(v \otimes w)\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} &\lesssim \|v \otimes w\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \\ &\lesssim \|v\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \|w\|_{\dot{B}^{\frac{3}{p}}_{p,1}}, \end{aligned} \tag{3.4}$$

$$\begin{aligned} \|\mathrm{div}((\mathrm{curl}^{-1}v) \otimes w)\|_{\dot{B}^{\frac{3}{p}}_{p,1}} &= \|w \cdot \nabla(\mathrm{curl}^{-1}v)\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \\ &\lesssim \|\nabla \mathrm{curl}^{-1}v\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \|w\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \\ &\lesssim \|v\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \|w\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \\ &\lesssim \|v\|^{\frac{1}{2}}_{\dot{B}^{\frac{3}{p}-1}_{p,1}} \|w\|^{\frac{1}{2}}_{\dot{B}^{\frac{3}{p}-1}_{p,1}} \|v\|^{\frac{1}{2}}_{\dot{B}^{\frac{3}{p}+1}_{p,1}} \|w\|^{\frac{1}{2}}_{\dot{B}^{\frac{3}{p}+1}_{p,1}}, \end{aligned} \tag{3.5}$$

and, since $\mathrm{div}(\mathrm{curl}^{-1}v) = 0$, owing to Proposition A.2 *(vii)*,

$$\begin{aligned} \|\mathrm{div}(w \otimes (\mathrm{curl}^{-1}v))\|_{\dot{B}^{\frac{3}{p}}_{p,1}} &= \|(\mathrm{curl}^{-1}v) \cdot \nabla w\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \\ &\lesssim \|\mathrm{curl}^{-1}v\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \|\nabla w\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \\ &\lesssim \|v\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} \|w\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}}. \end{aligned} \tag{3.6}$$

Hence, integrating on $\mathbb{R}_+$ and observing that the Leray projector $\mathcal{P}$ maps $\dot{B}^{\frac{3}{p}}_{p,1}$ to itself according to Proposition A.2 (vi), we get

$$\begin{aligned}\|Q_a(v,w)\|_{L^1(\dot{B}^{\frac{3}{p}-1}_{p,1})} &\lesssim \|\mathrm{div}(v\otimes w)+\mathrm{div}(w\otimes v)\|_{L^1(\dot{B}^{\frac{3}{p}-1}_{p,1})}\\ &\lesssim \|v\|_X\|w\|_X, \end{aligned}\tag{3.7}$$

$$\begin{aligned}\|Q_b(v,w)\|_{L^1(\dot{B}^{\frac{3}{p}-1}_{p,1})} &= \|\mathrm{div}(v\otimes w)-\mathrm{div}(w\otimes v)\|_{L^1(\dot{B}^{\frac{3}{p}-1}_{p,1})}\\ &\lesssim \|v\|_X\|w\|_X, \end{aligned}\tag{3.8}$$

$$\begin{aligned}\|\nabla\times Q_b(\mathrm{Curl}^{-1}v,w)\|_{L^1(\dot{B}^{\frac{3}{p}-1}_{p,1})} &\lesssim \|Q_b(\mathrm{Curl}^{-1}v,w)\|_{L^1(\dot{B}^{\frac{3}{p}}_{p,1})}\\ &\lesssim \|\mathrm{div}((\mathrm{Curl}^{-1}v)\otimes w)\|_{L^1(\dot{B}^{\frac{3}{p}}_{p,1})} + \|\mathrm{div}(w\otimes(\mathrm{Curl}^{-1}v))\|_{L^1(\dot{B}^{\frac{3}{p}}_{p,1})}\\ &\lesssim \|v\|_X\|w\|_X. \end{aligned}\tag{3.9}$$

Now, by definition of $\mathcal{B}(V,W)$, we have

$$\begin{cases}\partial_t\mathcal{B}(V,W)-\Delta\mathcal{B}(V,W)=Q(V,W),\\ \mathcal{B}(V,W)|_{t=0}=0.\end{cases}$$

Hence, by Proposition A.4 and the definition of Q in (2.11), we get

$$\begin{aligned}\|\mathcal{B}(V,W)\|_X \lesssim\ & \|Q_a(V_2,W_2)-Q_a(V_1,W_1)\|_{L^1(\dot{B}^{\frac{3}{p}-1}_{p,1})}\\ &+\|Q_b(V_2,W_3-W_1)\|_{L^1(\dot{B}^{\frac{3}{p}-1}_{p,1})} + \|\nabla\times Q_b(\mathrm{curl}^{-1}V_3,W_3-W_1)\|_{L^1(\dot{B}^{\frac{3}{p}-1}_{p,1})}.\end{aligned}$$

Remembering (3.7)-(3.9), one can conclude that $\mathcal{B}$ maps $X\times X$ to X. Hence, System (2.10) has a global solution (u,B,J) in X.

For completing the proof of the global existence for the original Hall-MHD system, we have to check that if $J_0=\nabla\times B_0$, then $J=\nabla\times B$ so that (u,B) is indeed a distributional solution of (1.1)–(1.4). Actually, we have

$$(\partial_t-\Delta)(\nabla\times B-J)=\nabla\times Q_b(\mathrm{curl}^{-1}(\nabla\times B-J),J-u).$$

Hence, using (3.5) (before interpolation), (3.6) and Proposition A.4, one gets for all $t\geq 0$,

$$\begin{aligned}&\|(\nabla\times B-J)(t)\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}+\int_0^t\|\nabla\times B-J\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}}\,d\tau\\ &\qquad\leq C\int_0^t\Big(\|J-u\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\|\nabla\times B-J\|_{\dot{B}^{\frac{3}{p}}_{p,1}}+\|J-u\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}}\|\nabla\times B-J\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}\Big)\,d\tau.\end{aligned}$$

Then, combining interpolation and Gronwall lemma ensures that $\nabla\times B-J\equiv 0$ on $\mathbb{R}_+\times\mathbb{R}^3$. This yields the existence part of Theorem 2.1 in the small data case.

Let us explain how the above arguments have to be modified so as to prove local existence in the case where only J_0 is small. The idea is to control the existence time according to the solution U^L of the heat equation:

$$\begin{cases}\partial_tU^L-\Delta U^L=0,\\ U^L|_{t=0}=U_0.\end{cases}$$

By Proposition A.4, we have

$$\|J^L\|_{L^\infty_T(\dot B^{\frac3p-1}_{p,1})} \leq C\|J_0\|_{\dot B^{\frac3p-1}_{p,1}}, \tag{3.10}$$

and, using also the dominated convergence theorem yields

$$\lim_{T\to0}\|U^L\|_{L^\rho_T(\dot B^{\frac3p+\frac2\rho-1}_{p,1})}=0, \text{ whenever } 1\leq\rho<\infty.$$

Clearly, U is a solution of (2.10) on $[0,T]\times\mathbb{R}^3$ with data U_0 if and only if

$$U := U^L+\widetilde U \tag{3.11}$$

with, for all $t\in[0,T]$,

$$\widetilde U(t) := \int_0^t e^{(t-\tau)\Delta}(Q(\widetilde U,U^L)+Q(U^L,\widetilde U)+Q(\widetilde U,\widetilde U)+Q(U^L,U^L))\,d\tau.$$

Then, proving local existence relies on the following generalization of Lemma 3.1.

Lemma 3.2. *Let $(X,\|\cdot\|_X)$ be a Banach space, $\mathcal B : X\times X\to X$, a bilinear continuous operator with norm K and $\mathcal L : X\to X$, a continuous linear operator with norm $M<1$. Let $y\in X$ satisfy $4K\|y\|_X<(1-M)^2$. Then, equation*

$$x = y+\mathcal L(x)+\mathcal B(x,x)$$

has a unique solution x in the ball $B(0,\frac{1-M}{2K})$·

Take $\mathcal B$ as in (3.2), set $y:=\mathcal B(U^L,U^L)$ and define the linear map $\mathcal L$ by

$$\mathcal L(V) := \mathcal B(V,U^L)+\mathcal B(U^L,V). \tag{3.12}$$

Our problem recasts in

$$\widetilde U = y+\mathcal L(\widetilde U)+\mathcal B(\widetilde U,\widetilde U). \tag{3.13}$$

For X, we now take the space (denoted by X_T) of triplets of divergence free vector-fields with components in $E_p(T)$. Then, arguing as for getting (3.4), (3.5), integrating on $[0,T]$ and using Cauchy-Schwarz inequality, we get

$$\|\mathrm{div}\,(v\otimes w)\|_{L^1_T(\dot B^{\frac3p-1}_{p,1})}+\|\mathrm{div}\,(\mathrm{curl}^{-1}v\otimes w)\|_{L^1_T(\dot B^{\frac3p}_{p,1})}\lesssim\|v\|_{L^2_T(\dot B^{\frac3p}_{p,1})}\|w\|_{L^2_T(\dot B^{\frac3p}_{p,1})}.$$

Hence, using also (3.6) and the definition of $\mathcal B(V,W)$, we end up with

$$\begin{aligned}\|\mathcal B(V,W)\|_{X_T}\lesssim{}&\|V\|_{L^2_T(\dot B^{\frac3p}_{p,1})}\|W\|_{L^2_T(\dot B^{\frac3p}_{p,1})}\\&+(\|W_1\|_{L^1_T(\dot B^{\frac3p+1}_{p,1})}+\|W_3\|_{L^1_T(\dot B^{\frac3p+1}_{p,1})})\|V_3\|_{L^\infty_T(\dot B^{\frac3p-1}_{p,1})}.\end{aligned} \tag{3.14}$$

For justifying that $\mathcal L$ defined in (3.12) is indeed a continuous linear operator on X_T *with small norm if* $T\to0$, the troublemakers in the right-hand side of (3.14) are

$$\|\widetilde u\|_{L^1_T(\dot B^{\frac3p+1}_{p,1})}\|J^L\|_{L^\infty_T(\dot B^{\frac3p-1}_{p,1})}\quad\text{and}\quad\|\widetilde J\|_{L^1_T(\dot B^{\frac3p+1}_{p,1})}\|J^L\|_{L^\infty_T(\dot B^{\frac3p-1}_{p,1})}$$

since, for large J_0, the term $\|J^L\|_{L^\infty_T(\dot B^{\frac3p-1}_{p,1})}$ need not to be small. One thus have to assume that $\|J_0\|_{\dot B^{\frac3p-1}_{p,1}}$ is small so as to guarantee that the norm of $\mathcal L$ is smaller than 1 for T small enough. Then, one can conclude thanks to Lemma 3.2, to the local-in-time existence statement of Theorem 2.1.

Let us next prove the uniqueness part of Theorem 2.1. Consider two solutions (u^1,B^1) and (u^2,B^2) of (1.1)–(1.3) emanating from the same data, and denote by U^1 and U^2 the corresponding solutions of the extended system (2.10). Since one can take (with no loss of generality) for U^2 the solution built previously, and as $\|J_0\|_{\dot B^{\frac3p-1}_{p,1}}\leq c$ is assumed, we have

$$\|J^2\|_{X_T}\leq 2c. \tag{3.15}$$

Denoting $\delta U:=U^2-U^1$, we find that δU satisfies

$$\partial_t\delta U-\Delta\delta U=Q(U^2,\delta U)+Q(\delta U,U^1)$$

with $\delta U|_{t=0}$=0, and thus

$$\delta U=\mathcal{B}(U^2,\delta U)+\mathcal{B}(\delta U,U^1).$$

Arguing as in the proof of (3.14) yields

$$\begin{aligned}\|\mathcal{B}(U^2,\delta U)\|_{X_T}&\lesssim\int_0^T\|U^2\|_{\dot B^{\frac3p}_{p,1}}\|\delta U\|_{\dot B^{\frac3p}_{p,1}}\,dt+\int_0^T\|J^2\|_{\dot B^{\frac3p-1}_{p,1}}\|\delta U\|_{\dot B^{\frac3p+1}_{p,1}}\,dt\\&\lesssim\int_0^T\|U^2\|_{\dot B^{\frac3p}_{p,1}}\|\delta U\|^{\frac12}_{\dot B^{\frac3p-1}_{p,1}}\|\delta U\|^{\frac12}_{\dot B^{\frac3p+1}_{p,1}}\,dt+\int_0^T\|J^2\|_{\dot B^{\frac3p-1}_{p,1}}\|\delta U\|_{\dot B^{\frac3p+1}_{p,1}}\,dt\end{aligned}$$

whence there exists $C>0$ such that for all $\eta>0$,

$$\begin{aligned}\|\mathcal{B}(U^2,\delta U)\|_{X_T}\leq\big(\eta+C\|J^2\|_{L^\infty_T(\dot B^{\frac3p-1}_{p,1})}\big)\|\delta U\|_{L^1_T(\dot B^{\frac3p+1}_{p,1})}&\\+C\eta^{-1}\int_0^T\|U^2\|^2_{\dot B^{\frac3p}_{p,1}}\|\delta U\|_{\dot B^{\frac3p-1}_{p,1}}\,dt.&\end{aligned}$$

Similarly, we have

$$\begin{aligned}\|\mathcal{B}(\delta U,U^1)\|_{X_T}&\leq C\bigg(\int_0^T\|U^1\|_{\dot B^{\frac3p}_{p,1}}\|\delta U\|_{\dot B^{\frac3p}_{p,1}}\,dt+\int_0^T\|U^1\|_{\dot B^{\frac3p+1}_{p,1}}\|\delta J\|_{\dot B^{\frac3p-1}_{p,1}}\,dt\bigg)\\&\leq\eta\|\delta U\|_{L^1_T(\dot B^{\frac3p+1}_{p,1})}+C\int_0^T\big(\|U^1\|_{\dot B^{\frac3p+1}_{p,1}}+\eta^{-1}\|U^1\|^2_{\dot B^{\frac3p}_{p,1}}\big)\|\delta U\|_{\dot B^{\frac3p-1}_{p,1}}\,dt.\end{aligned}$$

Hence, taking η small enough, and remembering (3.15), one gets

$$\|\delta U\|_{X_T}\leq C\int_0^T\big(\|U^1\|_{\dot B^{\frac3p+1}_{p,1}}+\|U^1\|^2_{\dot B^{\frac3p}_{p,1}}+\|U^2\|^2_{\dot B^{\frac3p}_{p,1}}\big)\|\delta U\|_{\dot B^{\frac3p-1}_{p,1}}\,dt.$$

Gronwall lemma thus implies that $\delta U\equiv0$ in X_T, whence uniqueness on $[0,T]\times\mathbb{R}^3$. Of course, in the case where the data are small, then J^2 remains small for all $T>0$, and one gets uniqueness on $\mathbb{R}_+\times\mathbb{R}^3$.

Let us finally justify the propagation of Sobolev regularity in the case where, additionally, (u_0,B_0) is in $H^s\times H^r$ with (r,s) satisfying (2.3). For expository purpose, assume that the data fulfill (2.1) (the case where only J_0 is small being left to the reader). Our aim is to prove that the solution (u,B) we constructed above satisfies

$$(u,B)\in\mathcal{C}_b(\mathbb{R}_+;H^s\times H^r)\quad\text{and}\quad(\nabla u,\nabla B)\in L^2(\mathbb{R}_+;H^s\times H^r).$$

For the time being, let us assume that (u, B) is smooth. Then, taking the L^2 scalar product of (1.1) and (1.3) by u and B, respectively, adding up the resulting identities, and using the fact that

$$(\nabla \times (J \times B),\ B) = (J \times B,\ J) = 0,$$

one gets the following energy balance:

$$\frac{1}{2}\frac{d}{dt}(\|u\|_{L^2}^2 + \|B\|_{L^2}^2) + \|\nabla u\|_{L^2}^2 + \|\nabla B\|_{L^2}^2 = 0. \tag{3.16}$$

Since $\|z\|_{\dot H^a} = \|\Lambda^a z\|_{L^2}$ and $\|z\|_{H^a} \sim \|z\|_{L^2} + \|z\|_{\dot H^a}$, in order to prove estimates in $H^s \times H^r$, it suffices to get a suitable control on $\|\Lambda^s u\|_{L^2}$ and on $\|\Lambda^r B\|_{L^2}$. To this end, apply Λ^s to (1.1), then take the L^2 scalar product with $\Lambda^s u$:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s \nabla u\|_{L^2}^2 &= (\Lambda^s(B\cdot\nabla B), \Lambda^s u) - (\Lambda^s(u\cdot\nabla u), \Lambda^s u)\\ &=: A_1 + A_2.\end{aligned}$$

Similarly, apply Λ^r to (1.3) and taking the L^2 scalar product with $\Lambda^r B$:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Lambda^r B\|_{L^2}^2 + \|\Lambda^r \nabla B\|_{L^2}^2 &= (\Lambda^r(u\times B), \Lambda^r J) - (\Lambda^r(J\times B), \Lambda^r J)\\ &=: A_3 + A_4.\end{aligned}$$

To bound A_1, A_2, A_3 and A_4, we shall use repeatedly the following classical tame estimate in homogeneous Sobolev spaces:

$$\|\Lambda^\sigma(fg)\|_{L^2} \lesssim \|f\|_{L^\infty}\|\Lambda^\sigma g\|_{L^2} + \|g\|_{L^\infty}\|\Lambda^\sigma f\|_{L^2}, \qquad \sigma \geq 0. \tag{3.17}$$

Using first the Cauchy-Schwarz inequality, then (3.17), the fact that $s \leq r \leq 1+s$ and Young inequality, we readily get

$$\begin{aligned}|A_1| &\leq C(\|\Lambda^s B\|_{L^2}\|\nabla B\|_{L^\infty} + \|B\|_{L^\infty}\|\Lambda^s\nabla B\|_{L^2})\|u\|_{H^s}\\ &\leq C(\|B\|_{H^s}^2 + \|u\|_{H^s}^2)\|\nabla B\|_{L^\infty} + \frac{1}{8}\|\nabla B\|_{H^r}^2 + C\|B\|_{L^\infty}^2\|u\|_{H^s}^2,\\ |A_2| &\leq C\|\nabla u\|_{L^\infty}\|u\|_{H^s}^2,\\ |A_3| &\leq C\|\Lambda^r(u\times B)\|_{L^2}\|\Lambda^r J\|_{L^2}\\ &\leq C(\|\Lambda^r u\|_{L^2}^2\|B\|_{L^\infty}^2 + \|\Lambda^r B\|_{L^2}^2\|u\|_{L^\infty}^2) + \frac{1}{8}\|\nabla B\|_{H^r}^2\\ &\leq C\big(\|u\|_{L^2}^2 + \|\nabla u\|_{H^s}^2\big)\|B\|_{L^\infty}^2 + C\|B\|_{H^r}^2\|u\|_{L^\infty}^2 + \frac{1}{8}\|\nabla B\|_{H^r}^2,\\ |A_4| &\leq C\|J\times B\|_{H^r}\|J\|_{H^r}\\ &\leq C(\|J\|_{H^r}^2\|B\|_{L^\infty} + \|J\|_{L^\infty}\|B\|_{H^r}\|J\|_{H^r})\\ &\leq C\|B\|_{L^\infty}\|\nabla B\|_{H^r}^2 + C\|J\|_{L^\infty}^2\|B\|_{H^r}^2 + \frac{1}{8}\|J\|_{H^r}^2.\end{aligned}$$

Putting the above estimates and (3.16) together, and using the fact that $\|B\|_{L^\infty}$ is small since, according to Proposition A.2 and the first part of the proof, we have

$$\|B\|_{L^\infty} \lesssim \|B\|_{\dot B^{\frac{3}{p}}_{p,1}} \lesssim \|J\|_{\dot B^{\frac{3}{p}-1}_{p,1}} \lesssim \|(u_0, B_0, J_0)\|_{\dot B^{\frac{3}{p}-1}_{p,1}},$$

one gets

$$\frac{1}{2}\frac{d}{dt}(\|u\|_{H^s}^2 + \|B\|_{H^r}^2) + \|\nabla u\|_{H^s}^2 + \|\nabla B\|_{H^r}^2 \leq C(\|u\|_{H^s}^2 + \|B\|_{H^r}^2)S(t),$$

with

$$S(t) := \|\nabla u(t)\|_{L^\infty} + \|\nabla B(t)\|_{L^\infty} + \|u(t)\|_{L^\infty}^2 + \|B(t)\|_{L^\infty}^2 + \|J(t)\|_{L^\infty}^2.$$

By Gronwall's inequality, we conclude that for all $t \geq 0$,

$$\|u(t)\|_{H^s}^2 + \|B(t)\|_{H^r}^2 + \int_0^t (\|\nabla u(\tau)\|_{H^s}^2 + \|\nabla B(\tau)\|_{H^r}^2)\, d\tau \leq \big(\|u_0\|_{H^s}^2 + \|B_0\|_{H^r}^2\big) \exp\bigg(C\int_0^t S(\tau)\, d\tau\bigg)\cdot$$

As $\int_0^t S(\tau)\, d\tau$ is bounded thanks to the first part of the theorem and embedding (use Proposition A.2 (ii)), we get a control of the Sobolev norms for all time.

Let us briefly explain how those latter computations may be made rigorous. Let us consider data (u_0, B_0) fulfilling (2.1) and such that, additionally, we have u_0 in H^s and B_0 in H^r with (r, s) satisfying (2.3). Then, there exists a sequence (u_0^n, B_0^n) in the Schwartz space $\mathcal{S}$ such that

$$(u_0^n, B_0^n) \to (u_0, B_0) \text{ in } \big(\dot{B}_{p,1}^{\frac{3}{p}-1} \cap H^s\big) \times \big(\dot{B}_{p,1}^{\frac{3}{p}-1} \cap H^r\big)\cdot$$

The classical well-posedness theory in Sobolev spaces (see e.g. [7]) ensures that the Hall-MHD system with data (u_0^n, B_0^n) has a unique maximal solution (u^n, B^n) on some interval $[0, T^n)$ belonging to all Sobolev spaces. For that solution, the previous computations hold, and one ends up for all $t < T^n$ with

$$\|u^n(t)\|_{H^s}^2 + \|B^n(t)\|_{H^r}^2 + \int_0^t (\|\nabla u^n(\tau)\|_{H^s}^2 + \|\nabla B^n(\tau)\|_{H^r}^2)\, d\tau \leq \big(\|u_0^n\|_{H^s}^2 + \|B_0^n\|_{H^r}^2\big) \exp\bigg(C\int_0^t S^n(\tau)\, d\tau\bigg),$$

where

$$S^n(t) := \|\nabla u^n(t)\|_{L^\infty} + \|\nabla B^n(t)\|_{L^\infty} + \|u^n(t)\|_{L^\infty}^2 + \|B^n(t)\|_{L^\infty}^2 + \|J^n(t)\|_{L^\infty}^2.$$

Since the regularized data (u_0^n, B_0^n) fulfill (2.3) for large enough n, they generate a global solution $(\widetilde{u}^n, \widetilde{B}^n)$ in E_p which, actually, coincides with (u^n, B^n) on $[0, T^n)$ by virtue of the uniqueness result that has been proved before. Therefore, S^n belongs to $L^1(0, T^n)$ and thus (u^n, B^n) is in $L^\infty(0, T^n; H^s \times H^r)$. Combining with the continuation argument of e.g. [7], one can conclude that $T^n = +\infty$.

At this stage, one can assert that:

i) $(u^n, B^n, J^n)_{n\in\mathbb{N}}$ is bounded in E_p;
ii) $(u^n, B^n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{C}(\mathbb{R}_+; H^s \times H^r)$ and $(\nabla u^n, \nabla B^n)_{n\in\mathbb{N}}$ is bounded in $L^2(\mathbb{R}_+; H^s \times H^r)$.

Hence, up to subsequence,

i) (u^n, B^n, J^n) converges weakly $*$ in E_p;
ii) (u^n, B^n) converges weakly $*$ in $L^\infty(\mathbb{R}_+; H^s \times H^r)$;
iii) $(\nabla u^n, \nabla B^n)$ converges weakly in $L^2(\mathbb{R}_+; H^s \times H^r)$.

Clearly, a small variation of the proof of uniqueness in E_p allows to prove the continuity of the flow map. Hence, given that (u_0^n, B_0^n, J_0^n) converges to (u_0, B_0, J_0) in $\dot{B}_{p,1}^{\frac{3}{p}-1}$, one gets $(u^n, B^n, J^n) \to (u, B, J)$ strongly in E_p, where (u, B, J) stands for the solution of (2.10) with data (u_0, B_0, J_0).

Since the weak convergence results listed above imply the convergence in the sense of distributions, one can conclude that the weak limit coincides with the strong one in E_p. Hence (u,B) (resp. $(\nabla u,\nabla B)$) is indeed in $L^\infty(\mathbb{R}_+;H^s\times H^r)$ (resp. $L^2(\mathbb{R}_+;H^s\times H^r)$). Then, looking at (u,B) as the solution of a heat equation yields the time continuity with values in Sobolev spaces (use for instance Proposition A.4). This completes the proof of Theorem 2.1. □

4. Local existence for large data in $\dot B^{\frac12}_{2,1}$, and blow-up criteria

Proving Theorem 2.2 is based on a priori estimates in the space $E_2(T)$ for smooth solutions (u,B,v) of (2.13). Those estimates will be obtained by implementing an energy method on (2.13) after localization in the Fourier space. A slight modification of the method will yield uniqueness and blow-up criteria.

Throughout that section, we shall assume with no loss of generality that $\mu=\nu=\varepsilon=1$ (remember that we have $\mu=\nu$ in Theorem 2.2).

First step: A priori estimates.

Our main aim here is to prove the following result.

Proposition 4.1. *Consider a smooth solution (u,B,P) to the Hall-MHD System on $[0,T]\times\mathbb{R}^3$ for some $T>0$, and denote $v:=u-\nabla\times B$. Let $u^L:=e^{t\Delta}u_0$, $B^L:=e^{t\Delta}B_0$, $v^L:=e^{t\Delta}v_0$ and $(\widetilde u,\widetilde B,\widetilde v):=(u-u^L,B-B^L,v-v^L)$. Let*

$$c_1(t):=\|v^L(t)\|_{\dot B^{\frac52}_{2,1}},$$
$$c_2(t):=\|u^L(t)\|^2_{\dot B^{\frac32}_{2,1}}+\|B^L(t)\|^2_{\dot B^{\frac32}_{2,1}}+\big(\|u_0\|_{\dot B^{\frac12}_{2,1}}+\|v_0\|_{\dot B^{\frac12}_{2,1}}\big)\|v^L(t)\|_{\dot B^{\frac52}_{2,1}}.$$

There exist three positive constants κ, C and C_1 such that if

$$\int_0^T c_2(\tau)e^{C\int_\tau^T c_1(\tau')\,d\tau'}\,d\tau<\kappa, \tag{4.1}$$

then we have

$$\|(\widetilde u,\widetilde B,\widetilde v)\|_{L^\infty_T(\dot B^{\frac12}_{2,1})}+C_1\|(\widetilde u,\widetilde B,\widetilde v)\|_{L^1_T(\dot B^{\frac52}_{2,1})}\le C\kappa\quad\text{and} \tag{4.2}$$
$$\|(u,B,v)\|_{L^\infty_T(\dot B^{\frac12}_{2,1})}+C_1\|(u,B,v)\|_{L^1_T(\dot B^{\frac52}_{2,1})}\le \|(u_0,B_0,v_0)\|_{\dot B^{\frac12}_{2,1}}+C\kappa. \tag{4.3}$$

Proof. From (A.3), Plancherel identity and the definition of $\|\cdot\|_{\dot B^s_{2,1}}$, we have

$$\|z\|_{L^\infty_T(\dot B^{\frac12}_{2,1})}+C_1\|z\|_{L^1_T(\dot B^{\frac52}_{2,1})}\le\|z_0\|_{\dot B^{\frac12}_{2,1}}\quad\text{for}\quad z=u^L,B^L,v^L. \tag{4.4}$$

Hence Inequality (4.3) follows from Inequality (4.2).

In order to prove (4.2), we use the fact that $(\widetilde u,\widetilde B,\widetilde v,Q)$ satisfies

$$\begin{cases}\partial_t\widetilde u-\Delta\widetilde u=B\cdot\nabla B-u\cdot\nabla u-\nabla Q,\\ \partial_t\widetilde B-\Delta\widetilde B=\nabla\times(v\times B),\\ \partial_t\widetilde v-\Delta\widetilde v=B\cdot\nabla B-u\cdot\nabla u-\nabla\times((\nabla\times\widetilde v)\times B)\\ \qquad-\nabla\times((\nabla\times v^L)\times B)+\nabla\times(v\times u)+2\nabla\times(v\cdot\nabla B)-\nabla Q,\end{cases} \tag{4.5}$$

with initial condition

$$(\widetilde u,\widetilde B,\widetilde v)=(0,0,0).$$

Apply operator $\dot\Delta_j$ to both sides of (4.5), then take the L^2 scalar product with $\dot\Delta_j\widetilde u$, $\dot\Delta_j\widetilde B$, $\dot\Delta_j\widetilde v$, respectively. To handle the third equation of (4.5), let us use that

$$\nabla\times\dot\Delta_j((\nabla\times\widetilde v)\times B)=\nabla\times([\dot\Delta_j,B\times](\nabla\times\widetilde v))+\nabla\times(B\times\dot\Delta_j(\nabla\times\widetilde v)),$$

and that the L^2 scalar product of the last term with $\dot\Delta_j\widetilde v$ is 0. Then, we get

$$\begin{aligned}
&\frac12\frac{d}{dt}\|\dot\Delta_j\widetilde u\|_{L^2}^2+\|\nabla\dot\Delta_j\widetilde u\|_{L^2}^2\le(\|\dot\Delta_j(B\cdot\nabla B)\|_{L^2}+\|\dot\Delta_j(u\cdot\nabla u)\|_{L^2})\|\dot\Delta_j\widetilde u\|_{L^2},\\
&\frac12\frac{d}{dt}\|\dot\Delta_j\widetilde B\|_{L^2}^2+\|\nabla\dot\Delta_j\widetilde B\|_{L^2}^2\le\|\nabla\times\dot\Delta_j(v\times B)\|_{L^2}\|\dot\Delta_j\widetilde B\|_{L^2},\\
&\frac12\frac{d}{dt}\|\dot\Delta_j\widetilde v\|_{L^2}^2+\|\nabla\dot\Delta_j\widetilde v\|_{L^2}^2\le\big(\|\dot\Delta_j(B\cdot\nabla B)\|_{L^2}+\|\dot\Delta_j(u\cdot\nabla u)\|_{L^2}\big)\|\dot\Delta_j\widetilde v\|_{L^2}\\
&\qquad+\big(\|[\dot\Delta_j,B\times](\nabla\times\widetilde v)\|_{L^2}+\|\dot\Delta_j((\nabla\times v^L)\times B)\|_{L^2}+\|\dot\Delta_j(v\times u)\|_{L^2}\\
&\qquad+2\|\dot\Delta_j(v\cdot\nabla B)\|_{L^2}\big)\|\nabla\times\dot\Delta_j\widetilde v\|_{L^2}.
\end{aligned}$$

Hence, using Bernstein inequalities, one can deduce after time integration that for some universal constants C_1 and C_2,

$$\begin{aligned}
&\|(\dot\Delta_j\widetilde u,\dot\Delta_j\widetilde B,\dot\Delta_j\widetilde v)(t)\|_{L^2}+C_12^{2j}\int_0^t\|(\dot\Delta_j\widetilde u,\dot\Delta_j\widetilde B,\dot\Delta_j\widetilde v)\|_{L^2}\,d\tau\\
&\quad\le\int_0^t\Big(\|\dot\Delta_j(B\cdot\nabla B)\|_{L^2}+\|\dot\Delta_j(u\cdot\nabla u)\|_{L^2}+C_22^j\Big(\|[\dot\Delta_j,B\times](\nabla\times\widetilde v)\|_{L^2}\\
&\qquad+\|\dot\Delta_j((\nabla\times v^L)\times B)\|_{L^2}+\|\dot\Delta_j(v\times u)\|_{L^2}\\
&\qquad+\|\dot\Delta_j(v\cdot\nabla B)\|_{L^2}+\|\dot\Delta_j(v\times B)\|_{L^2})\,d\tau\Big)\Big)\,d\tau.
\end{aligned}\tag{4.6}$$

Multiplying both sides of (4.6) by $2^{\frac j2}$ and summing up over $j\in\mathbb Z$, we obtain that

$$\begin{aligned}
&\|(\widetilde u,\widetilde B,\widetilde v)(t)\|_{\dot B^{\frac12}_{2,1}}+C_1\int_0^t\|(\widetilde u,\widetilde B,\widetilde v)\|_{\dot B^{\frac52}_{2,1}}\,d\tau\\
&\quad\le C_2\int_0^t\Big(\|B\cdot\nabla B\|_{\dot B^{\frac12}_{2,1}}+\|u\cdot\nabla u\|_{\dot B^{\frac12}_{2,1}}+\|v\times B\|_{\dot B^{\frac32}_{2,1}}+\|v\times u\|_{\dot B^{\frac32}_{2,1}}\\
&+\|v\cdot\nabla B\|_{\dot B^{\frac32}_{2,1}}+\|(\nabla\times v^L)\times B\|_{\dot B^{\frac32}_{2,1}}+\sum_j2^{\frac{3j}{2}}\|[\dot\Delta_j,B\times](\nabla\times\widetilde v)\|_{L^2}\Big)\,d\tau.
\end{aligned}\tag{4.7}$$

Using (A.6), Proposition A.2 (i), (ii), (iii) and Young's inequality yields

$$\begin{aligned}
\|B\cdot\nabla B\|_{\dot B^{\frac12}_{2,1}}&\lesssim\|B^L\|^2_{\dot B^{\frac32}_{2,1}}+\|\tilde B\|^2_{\dot B^{\frac32}_{2,1}}\\
&\lesssim\|B^L\|^2_{\dot B^{\frac32}_{2,1}}+\|\widetilde B\|_{\dot B^{\frac12}_{2,1}}\|\widetilde B\|_{\dot B^{\frac52}_{2,1}},
\end{aligned}$$

$$\begin{aligned}
\|u\cdot\nabla u\|_{\dot B^{\frac12}_{2,1}}+\|v\times B\|_{\dot B^{\frac32}_{2,1}}+\|v\times u\|_{\dot B^{\frac32}_{2,1}}&\lesssim\|u\|^2_{\dot B^{\frac32}_{2,1}}+\big(\|B\|_{\dot B^{\frac32}_{2,1}}+\|u\|_{\dot B^{\frac32}_{2,1}}\big)\|v\|_{\dot B^{\frac32}_{2,1}}\\
&\lesssim\|u^L\|^2_{\dot B^{\frac32}_{2,1}}+\|B^L\|^2_{\dot B^{\frac32}_{2,1}}+\|v^L\|_{\dot B^{\frac12}_{2,1}}\|v^L\|_{\dot B^{\frac52}_{2,1}}\\
&\quad+\|\widetilde u\|_{\dot B^{\frac12}_{2,1}}\|\widetilde u\|_{\dot B^{\frac52}_{2,1}}+\|\widetilde B\|_{\dot B^{\frac12}_{2,1}}\|\widetilde B\|_{\dot B^{\frac52}_{2,1}}+\|\widetilde v\|_{\dot B^{\frac12}_{2,1}}\|\widetilde v\|_{\dot B^{\frac52}_{2,1}}.
\end{aligned}$$

Using that $B=\mathrm{curl}^{-1}(u-v)$ and that $\nabla\mathrm{curl}^{-1}$ is a self-map on $\dot B^{\frac32}_{2,1}$ (see Proposition A.2 (vi)) yields

$$\begin{aligned}
\|v\cdot\nabla B\|_{\dot B^{\frac32}_{2,1}}&\lesssim\|v\|_{\dot B^{\frac32}_{2,1}}\|\nabla\mathrm{curl}^{-1}(u-v)\|_{\dot B^{\frac32}_{2,1}}\\
&\lesssim\|v\|^2_{\dot B^{\frac32}_{2,1}}+\|u\|^2_{\dot B^{\frac32}_{2,1}}\\
&\lesssim\|u^L\|^2_{\dot B^{\frac32}_{2,1}}+\|v^L\|_{\dot B^{\frac12}_{2,1}}\|v^L\|_{\dot B^{\frac52}_{2,1}}+\|\widetilde u\|_{\dot B^{\frac12}_{2,1}}\|\widetilde u\|_{\dot B^{\frac52}_{2,1}}+\|\widetilde v\|_{\dot B^{\frac12}_{2,1}}\|\widetilde v\|_{\dot B^{\frac52}_{2,1}}
\end{aligned}$$

and, using also (A.6),

$$\begin{aligned}\|(\nabla\times v^L)\times B\|_{\dot B^{\frac32}_{2,1}} &\lesssim \|\nabla\times v^L\|_{\dot B^{\frac32}_{2,1}}\|B\|_{\dot B^{\frac32}_{2,1}}\\ &\lesssim \|v^L\|_{\dot B^{\frac52}_{2,1}}\|\mathrm{curl}^{-1}(u-v)\|_{\dot B^{\frac32}_{2,1}}\\ &\lesssim \|v^L\|_{\dot B^{\frac52}_{2,1}}\big(\|u^L\|_{\dot B^{\frac12}_{2,1}}+\|v^L\|_{\dot B^{\frac12}_{2,1}}+\|\widetilde u\|_{\dot B^{\frac12}_{2,1}}+\|\widetilde v\|_{\dot B^{\frac12}_{2,1}}\big)\cdot\end{aligned}$$

From the estimate (A.8) with $s=3/2$ and the embedding $\dot B^{\frac32}_{2,1}\hookrightarrow L^\infty$, we get

$$\sum_j 2^{\frac{3j}{2}}\|[\dot\Delta_j,b]a\|_{L^2}\lesssim\|\nabla b\|_{\dot B^{\frac32}_{2,1}}\|a\|_{\dot B^{\frac12}_{2,1}},\tag{4.8}$$

whence

$$\begin{aligned}\sum_j 2^{\frac{3j}{2}}\|[\dot\Delta_j,B\times](\nabla\times\widetilde v)\|_{L^2} &\lesssim \|v-u\|^2_{\dot B^{\frac32}_{2,1}}+\|\widetilde v\|^2_{\dot B^{\frac32}_{2,1}}\\ &\lesssim \|u^L\|^2_{\dot B^{\frac32}_{2,1}}+\|v^L\|_{\dot B^{\frac12}_{2,1}}\|v^L\|_{\dot B^{\frac52}_{2,1}}\\ &\qquad+\|\widetilde u\|_{\dot B^{\frac12}_{2,1}}\|\widetilde u\|_{\dot B^{\frac52}_{2,1}}+\|\widetilde v\|_{\dot B^{\frac12}_{2,1}}\|\widetilde v\|_{\dot B^{\frac52}_{2,1}}.\end{aligned}$$

Plugging the above estimates into the right-hand side of (4.7) and using (4.4), we end up with

$$X(t)+C_1\int_0^t D(\tau)\,d\tau\le C\int_0^t X(\tau)D(\tau)\,d\tau+C\int_0^t(c_1(\tau)X(\tau)+c_2(\tau))\,d\tau,\tag{4.9}$$

where c_1 and c_2 have been defined in the proposition,

$$\begin{aligned}X(t)&:=\|\widetilde u(t)\|_{\dot B^{\frac12}_{2,1}}+\|\widetilde B(t)\|_{\dot B^{\frac12}_{2,1}}+\|\widetilde v(t)\|_{\dot B^{\frac12}_{2,1}}\\ \text{and}\quad D(t)&:=\|\widetilde u(t)\|_{\dot B^{\frac52}_{2,1}}+\|\widetilde B(t)\|_{\dot B^{\frac52}_{2,1}}+\|\widetilde v(t)\|_{\dot B^{\frac52}_{2,1}}.\end{aligned}$$

Note that whenever

$$2C\sup_{\tau\in[0,t]}X(\tau)\le C_1,\tag{4.10}$$

Inequality (4.9) combined with Gronwall lemma implies that

$$X(t)+\frac{C_1}{2}\int_0^t D(\tau)\,d\tau\le C\int_0^t c_2(\tau)e^{C\int_\tau^t c_1(\tau')\,d\tau'}\,d\tau.\tag{4.11}$$

Now, if Condition (4.1) is satisfied with $\kappa:=C_1/2C^2$, then the fact that the left-hand side of (4.9) is a continuous function on $[0,T]$ that vanishes at 0 combined with a standard bootstrap argument allows to prove that (4.10) and thus (4.1) is satisfied. Renaming the constants completes the proof of the proposition. □

Second step: Constructing approximate solutions. It is based on Friedrichs' method : consider the spectral cut-off operator $\mathbb{E}_n$ defined by

$$\mathcal{F}(\mathbb{E}_n f)(\xi)=1_{\{n^{-1}\le|\xi|\le n\}}(\xi)\mathcal{F}(f)(\xi).$$

We want to solve the following truncated system:

$$\begin{cases}\partial_t u-\Delta u=\mathbb{E}_n\mathcal{P}(\mathbb{E}_nB\cdot\mathbb{E}_n\nabla B-\mathbb{E}_nu\cdot\nabla\mathbb{E}_nu),\\ \partial_tB-\Delta B=\nabla\times\mathbb{E}_n(\mathbb{E}_n(u-\nabla\times B)\times\mathbb{E}_nB),\end{cases}\tag{4.12}$$

supplemented with initial data $(\mathbb{E}_nu_0,\mathbb{E}_nB_0)$.

We need the following obvious lemma:

Lemma 4.2. *Let $s\in\mathbb{R}$ and $k\geq 0$. Let $f\in\dot B^s_{2,1}$. Then, for all $n\geq 1$, we have*

$$\|\mathbb{E}_n f\|_{\dot B^{s+k}_{2,1}}\lesssim n^k\|f\|_{\dot B^s_{2,1}},\tag{4.13}$$

$$\lim_{n\to\infty}\|\mathbb{E}_n f-f\|_{\dot B^s_{2,1}}=0,\tag{4.14}$$

$$\|\mathbb{E}_n f-f\|_{\dot B^s_{2,1}}\lesssim\frac{1}{n^k}\|f\|_{\dot B^{s+k}_{2,1}}.\tag{4.15}$$

We claim that (4.12) is an ODE in the Banach space $L^2(\mathbb{R}^3;\mathbb{R}^3\times\mathbb{R}^3)$ for which the standard Cauchy-Lipschitz theorem applies. Indeed, the above lemma ensures that $\mathbb{E}_n$ maps L^2 to all Besov spaces, and that the right-hand side of (4.12) is a continuous bilinear map from $L^2(\mathbb{R}^3;\mathbb{R}^3\times\mathbb{R}^3)$ to itself. We thus deduce that (4.12) admits a unique maximal solution $(u^n,B^n)\in\mathcal{C}^1([0,T^n);L^2(\mathbb{R}^3;\mathbb{R}^3\times\mathbb{R}^3))$. Furthermore, as $\mathbb{E}_n^2=\mathbb{E}_n$, uniqueness implies $\mathbb{E}_n u^n=u^n$ and $\mathbb{E}_n B^n=B^n$, and we clearly have $\operatorname{div}u^n=\operatorname{div}B^n=0$. Being spectrally supported in the annulus $\{n^{-1}\leq|\xi|\leq n\}$, one can also deduce that the solution belongs to $\mathcal{C}^1([0,T^n);\dot B^s_{2,1})$ for all $s\in\mathbb{R}$. Hence, setting $J^n:=\nabla\times B^n$ and $v^n:=u^n-J^n$, we see that u^n, B^n and v^n belong to the space $E_2(T)$ for all $T<T^n$ and fulfill:

$$\begin{cases}\partial_t u^n-\Delta u^n=\mathbb{E}_n\mathcal{P}(B^n\cdot\nabla B^n-u^n\cdot\nabla u^n),\\ \partial_t B^n-\Delta B^n=\nabla\times\mathbb{E}_n(v^n\times B^n),\\ \partial_t v^n-\Delta v^n=\mathbb{E}_n\mathcal{P}\Big(B^n\cdot\nabla B^n-u^n\cdot\nabla u^n-\nabla\times((\nabla\times v^n)\times B^n)\\ \qquad\qquad\qquad\qquad+\nabla\times(v^n\times u^n)+2\nabla\times(v^n\cdot\nabla B^n)\Big).\end{cases}\tag{4.16}$$

Third step: uniform estimates

We want to apply Proposition 4.1 to our approximate solution (u^n,B^n,v^n). The key point is that since $\mathbb{E}_n$ is an L^2 orthogonal projector, it has no effect on the energy estimates. We claim that T^n may be bounded from below by the supremum T of all the times satisfying (4.1), and that $(u^n,B^n,v^n)_{n\geq1}$ is bounded in $E_2(T)$. To prove our claim, , we split (u^n,B^n,v^n) into

$$(u^n,B^n,v^n)=(u^{n,L},B^{n,L},v^{n,L})+(\widetilde u^n,\widetilde B^n,\widetilde v^n),$$

where

$$u^{n,L}:=\mathbb{E}_n e^{t\Delta}u_0,\quad B^{n,L}:=\mathbb{E}_n e^{t\Delta}B_0\quad\text{and}\quad v^{n,L}:=\mathbb{E}_n e^{t\Delta}v_0.$$

Since $\mathbb{E}_n$ maps any Besov space $\dot B^s_{2,1}$ to itself with norm 1, Condition (4.1) may be made independent of n and thus, so does the corresponding time T. Now, as $(\widetilde u^n,\widetilde B^n,\widetilde v^n)$ is spectrally supported in $\{\xi\in\mathbb{R}^3\,|\,n^{-1}\leq|\xi|\leq n\}$, the estimate (4.2) ensures that it belongs to $L^\infty([0,T];L^2(\mathbb{R}^3))$. So, finally, the standard continuation criterion for ordinary differential equations implies that T^n is greater than any time T satisfying (4.1) and that we have, for all $n\geq1$,

$$\|(\widetilde u^n,\widetilde B^n,\widetilde v^n)\|_{L^\infty_T(\dot B^{\frac12}_{2,1})}+C_1\|(\widetilde u^n,\widetilde B^n,\widetilde v^n)\|_{L^1_T(\dot B^{\frac52}_{2,1})}\leq C\kappa\quad\text{and}\tag{4.17}$$

$$\|(u^n,B^n,v^n)\|_{L^\infty_T(\dot B^{\frac12}_{2,1})}+C_1\|(u^n,B^n,v^n)\|_{L^1_T(\dot B^{\frac52}_{2,1})}\leq\|(u_0,B_0,v_0)\|_{\dot B^{\frac12}_{2,1}}+C\kappa.\tag{4.18}$$

Fourth step: existence of a solution

We claim that, up to an extraction, the sequence $(u^n, B^n, v^n)_{n\in\mathbb{N}}$ converges in $\mathcal{D}'(\mathbb{R}^+\times\mathbb{R}^3)$ to a solution (u,B,v) of (2.13) supplemented with data (u_0,B_0,v_0) having the desired regularity properties. The definition of $\mathbb{E}_n$ entails that

$$(\mathbb{E}_n u_0, \mathbb{E}_n B_0, \mathbb{E}_n v_0) \to (u_0, B_0, v_0) \quad \text{in} \quad \dot B^{\frac12}_{2,1},$$

and Proposition A.4 thus ensures that $(u^{n,L}, B^{n,L}, v^{n,L}) \to (u^L, B^L, v^L)$ in $E_2(T)$.

Proving the convergence of $(\widetilde u^n, \widetilde B^n, \widetilde v^n)$ will be achieved from compactness arguments : we shall exhibit uniform bounds in suitable spaces for $(\partial_t u^n, \partial_t B^n, \partial_t v^n)_{n\in\mathbb{N}}$ so as to glean some Hölder regularity with respect to the time variable. Then, combining with compact embedding will enable us to apply Ascoli's theorem and to get the existence of a limit (u,B,v) for a subsequence. Furthermore, the uniform bounds of previous steps provide us with additional regularity and convergence properties so that we may pass to the limit in (4.16). Let us start with a lemma.

Lemma 4.3. *Sequence $(\widetilde u^n, \widetilde B^n, \widetilde v^n)_{n\geq1}$ is bounded in $\mathcal{C}^{\frac12}([0,T];\dot B^{-\frac12}_{2,1})$.*

Proof. Observe that $(\widetilde u^n, \widetilde B^n, \widetilde v^n)$ satisfies

$$\begin{cases}\partial_t \widetilde u^n = \Delta \widetilde u^n + \mathbb{E}_n\mathcal{P}(B^n\cdot\nabla B^n - u^n\cdot\nabla u^n),\\ \partial_t \widetilde B^n = \Delta \widetilde B^n + \nabla\times\mathbb{E}_n(v^n\times B^n),\\ \partial_t \widetilde v^n = \Delta \widetilde v^n + \mathbb{E}_n\mathcal{P}\Big(B^n\cdot\nabla B^n - u^n\cdot\nabla u^n - \nabla\times((\nabla\times v^n)\times B^n)\\ \qquad\qquad + \nabla\times(v^n\times u^n) + 2\nabla\times(v^n\cdot\nabla B^n)\Big).\end{cases} \tag{4.19}$$

According to the uniform bounds (4.17), (4.18) and to the product laws:

$$\|ab\|_{\dot B^{-\frac12}_{2,1}} \lesssim \|a\|_{\dot B^{\frac12}_{2,1}}\|b\|_{\dot B^{\frac12}_{2,1}} \quad \text{and} \quad \|ab\|_{\dot B^{\frac12}_{2,1}} \lesssim \|a\|_{\dot B^{\frac12}_{2,1}}\|b\|_{\dot B^{\frac32}_{2,1}},$$

the right-hand side of (4.19) is uniformly bounded in $L^2_T(\dot B^{-\frac12}_{2,1})$. Hence, since $\widetilde u^n(0) = \widetilde B^n(0) = \widetilde b^n(0) = 0$, applying Hölder inequality completes the proof of the lemma. □

We can now come to the proof of the existence of a solution. Let $(\phi_j)_{j\in\mathbb{N}}$ be a sequence of $\mathcal{C}^\infty_0(\mathbb{R}^3)$ cut-off functions supported in the ball $B(0,j+1)$ of $\mathbb{R}^3$ and equal to 1 in a neighborhood of $B(0,j)$. Lemma 4.3 tells us that $(\widetilde u^n, \widetilde B^n, \widetilde v^n)_{n\geq1}$ is uniformly equicontinuous in the space $\mathcal{C}([0,T];\dot B^{-\frac12}_{2,1})$ and (4.17) ensures that it is bounded in $L^\infty([0,T];\dot B^{\frac12}_{2,1})$. Using the fact that the application $u\mapsto\phi_j u$ is compact from $\dot B^{\frac12}_{2,1}$ into $\dot B^{-\frac12}_{2,1}$, combining Ascoli's theorem and Cantor's diagonal process ensures that there exists some triplet $(\widetilde u, \widetilde B, \widetilde v)$ such that for all $j\in\mathbb{N}$,

$$(\phi_j\widetilde u^n, \phi_j\widetilde B^n, \phi_j\widetilde v^n) \to (\phi_j\widetilde u, \phi_j\widetilde B, \phi_j\widetilde v) \quad \text{in} \quad \mathcal{C}([0,T];\dot B^{-\frac12}_{2,1}). \tag{4.20}$$

This obviously entails that $(\widetilde u^n, \widetilde B^n, \widetilde v^n)$ tends to $(\widetilde u, \widetilde B, \widetilde v)$ in $\mathcal{D}'(\mathbb{R}^+\times\mathbb{R}^3)$.

Coming back to the uniform estimates of third step and using the argument of [4, p. 443] to justify that there is no time concentration, we get that $(\widetilde u, \widetilde B, \widetilde v)$ belongs to $L^\infty(0,T;\dot B^{\frac12}_{2,1}) \cap L^1(0,T;\dot B^{\frac52}_{2,1})$ and to $\mathcal{C}^{\frac12}([0,T];\dot B^{-\frac12}_{2,1})$.

Let us now prove that $(u,B,v) := (u^L+\widetilde u, B^L+\widetilde B, v^L+\widetilde v)$ solves (2.13). The only problem is to pass to the limit in the non-linear terms. By way of example, let us explain how to handle the term $\mathbb{E}_n\mathcal{P}\nabla\times((\nabla\times v^n)\times B^n)$ in (4.16) (actually, $\mathcal{P}$ may be omitted as a curl is divergence free). Let $\theta\in\mathcal{C}_0^\infty(\mathbb{R}^+\times\mathbb{R}^3;\mathbb{R}^3)$ and $j\in\mathbb{N}$ be such that $\operatorname{Supp}\theta\subset[0,\,j]\times B(0,j)$. We use the decomposition

$$\begin{aligned}\langle\nabla\times\mathbb{E}_n((\nabla\times v^n)\times B^n),\,\theta\rangle&-\langle\nabla\times((\nabla\times v)\times B),\theta\rangle\\ &=\langle(\nabla\times v^n)\times\phi_j(B^n-B),\,\nabla\times\mathbb{E}_n\theta\rangle+\langle(\nabla\times\phi_j(v^n-v))\times B,\,\nabla\times\mathbb{E}_n\theta\rangle\\ &\qquad+\langle\mathbb{E}_n((\nabla\times v)\times B)-(\nabla\times v)\times B,\,\nabla\times\theta\rangle.\end{aligned}$$

As $\nabla\times v^n$ is uniformly bounded in $L^1_T(\dot B^{\frac32}_{2,1})$ and ϕ_jB^n tends to ϕ_jB in $L^\infty_T(\dot B^{\frac12}_{2,1})$, the first term tends to 0. According to the uniform estimates (4.18) and (4.20), $\nabla\times\phi_j(v^n-v)$ tends to 0 in $L^1_T(\dot B^{\frac12}_{2,1})$ so that the second term tends to 0 as well. Finally, thanks to (4.14), the third term tends to 0.

The other non-linear terms can be treated similarly, and the continuity of (u,B,v) stems from Proposition A.4 since the right-hand side of (2.13) belongs to $L^1_T(\dot B^{\frac12}_{2,1})$.

Fifth step: uniqueness

Let (u_1,B_1) and (u_2,B_2) be two solutions of the Hall-MHD system on $[0,T]\times\mathbb{R}^3$, with the same initial data, and such that $(u_i,B_i,v_i)\in E_2(T)$ for $i=1,2$. Then, the difference $(\delta u,\delta B,\delta v):=(u_1-u_2,B_1-B_2,v_1-v_2)$ is in $E_2(T)$ and satisfies

$$\begin{cases}\partial_t\delta u-\Delta\delta u:=R_1,\\ \partial_t\delta B-\Delta\delta B:=R_2,\\ \partial_t\delta v-\Delta\delta v:=R_1+R_3+R_4+R_5,\end{cases}\tag{4.21}$$

where

$$\begin{aligned}R_1&:=\mathcal{P}(B_1\cdot\nabla\delta B+\delta B\cdot\nabla B_2-u_1\cdot\nabla\delta u-\delta u\cdot\nabla u_2),\\ R_2&:=\nabla\times(v_1\times\delta B+\delta v\times B_2),\\ R_3&:=-\nabla\times((\nabla\times v_1)\times\delta B+(\nabla\times\delta v)\times B_2),\\ R_4&:=\nabla\times(v_1\times\delta u+\delta v\times u_2),\\ R_5&:=2\nabla\times(v_1\cdot\nabla\delta B+\delta v\cdot\nabla B_2).\end{aligned}$$

Hence, arguing as in the first step of the proof gives for all $t\in[0,T]$,

$$\begin{aligned}\|(\delta u,\delta B,\delta v)(t)\|_{\dot B^{\frac12}_{2,1}}+\int_0^t\|(\delta u,\delta B,\delta v)\|_{\dot B^{\frac12}_{2,1}}\,d\tau\lesssim\int_0^t\Big(\|(R_1,R_2,R_4,R_5)\|_{\dot B^{\frac12}_{2,1}}\\ +\|\nabla\times((\nabla\times v_1)\times\delta B)\|_{\dot B^{\frac12}_{2,1}}+\sum_{j\in\mathbb{Z}}2^{\frac{3j}{2}}\|[\dot\Delta_j,B_2\times](\nabla\times\delta v)\|_{L^2}\Big)d\tau.\end{aligned}\tag{4.22}$$

Putting together the product laws (A.6) and the commutator estimate (4.8) yields

$$\begin{aligned}\|R_1\|_{\dot B^{\frac12}_{2,1}}&\lesssim\|(u_1,B_1,u_2,B_2)\|_{\dot B^{\frac32}_{2,1}}\|(\delta u,\delta B)\|_{\dot B^{\frac32}_{2,1}},\\ \|R_2\|_{\dot B^{\frac12}_{2,1}}&\lesssim\|(B_2,v_1)\|_{\dot B^{\frac32}_{2,1}}\|(\delta B,\delta v)\|_{\dot B^{\frac32}_{2,1}},\\ \|R_4\|_{\dot B^{\frac12}_{2,1}}&\lesssim\|(u_2,v_1)\|_{\dot B^{\frac32}_{2,1}}\|(\delta u,\delta v)\|_{\dot B^{\frac32}_{2,1}},\\ \|R_5\|_{\dot B^{\frac12}_{2,1}}&\lesssim\|(\nabla B_2,v_1)\|_{\dot B^{\frac32}_{2,1}}\|(\nabla\delta B,\delta v)\|_{\dot B^{\frac32}_{2,1}}\\ &\lesssim\|(u_2,v_1,v_2)\|_{\dot B^{\frac32}_{2,1}}\|(\delta u,\delta v)\|_{\dot B^{\frac32}_{2,1}},\end{aligned}$$

$$\begin{aligned}\|\nabla\times((\nabla\times v_1)\times\delta B)\|_{\dot B^{\frac12}_{2,1}} &\lesssim \|\nabla\times v_1\|_{\dot B^{\frac32}_{2,1}}\|\delta B\|_{\dot B^{\frac32}_{2,1}}\\ &\lesssim \|v_1\|_{\dot B^{\frac52}_{2,1}}\|(\delta u,\delta v)\|_{\dot B^{\frac12}_{2,1}}\end{aligned}$$

and

$$\begin{aligned}\sum_{j\in\mathbb{Z}}2^{\frac{3j}{2}}\|[\dot\Delta_j,B_2\times](\nabla\times\delta v)\|_{L^2} &\lesssim \|\nabla B_2\|_{\dot B^{\frac32}_{2,1}}\|\nabla\times\delta v\|_{\dot B^{\frac12}_{2,1}}\\ &\lesssim \|(u_2,v_2)\|_{\dot B^{\frac32}_{2,1}}\|\delta v\|_{\dot B^{\frac32}_{2,1}}\cdot\end{aligned}$$

Hence, by interpolation and Young's inequality, Inequality (4.22) becomes

$$\|(\delta u,\delta B,\delta v)(t)\|_{\dot B^{\frac12}_{2,1}}+\int_0^t\|(\delta u,\delta B,\delta v)(\tau)\|_{\dot B^{\frac52}_{2,1}}\,d\tau\ \le\int_0^t Z(\tau)\|(\delta u,\delta B,\delta v)(\tau)\|_{\dot B^{\frac12}_{2,1}}\,d\tau$$

with $Z(t):=C\big(\|(u_1,u_2,B_1,B_2,v_1,v_2)\|^2_{\dot B^{\frac32}_{2,1}}+\|v_1\|_{\dot B^{\frac52}_{2,1}}\big)\cdot$

Thus, Gronwall lemma and our assumptions on the solutions ensure that

$$(\delta u,\delta B,\delta v)\equiv 0\quad\text{on}\quad[0,T].$$

Sixth step: Blow-up criterion

Let us assume that we are given a solution (u,B) on some *finite* time interval $[0,T^*)$ fulfilling the regularity properties listed in Theorem 2.2 for all $t<T^*$. Then, applying the method of the first step to (2.13) yields for all $t<T^*$,

$$\begin{aligned}\|(u,B,v)(t)\|_{\dot B^{\frac12}_{2,1}}+C_1\int_0^t\|(u,B,v)\|_{\dot B^{\frac52}_{2,1}}\,d\tau &\le \|(u,B,v)(0)\|_{\dot B^{\frac12}_{2,1}}\\ +\int_0^t\Big(\|B\cdot\nabla B\|_{\dot B^{\frac12}_{2,1}}+\|u\cdot\nabla u\|_{\dot B^{\frac12}_{2,1}}&+\big(\|v\times B\|_{\dot B^{\frac32}_{2,1}}+\|v\times u\|_{\dot B^{\frac32}_{2,1}}\\ +\|v\cdot\nabla B\|_{\dot B^{\frac32}_{2,1}}&+\sum_j 2^{\frac{3j}{2}}\|[\dot\Delta_j,B\times](\nabla\times v)\|_{L^2}\big)\Big)d\tau.\end{aligned}\tag{4.23}$$

Using the tame estimates (A.5), the fact that $\dot B^{\frac32}_{2,1}$ is an algebra embedded in L^∞, interpolation inequalities and Young's inequality, we get for all $\eta>0$,

$$\begin{aligned}\|B\cdot\nabla B\|_{\dot B^{\frac12}_{2,1}} &\le C\|B\otimes B\|_{\dot B^{\frac32}_{2,1}}\\ &\le C\|B\|_{L^\infty}\|B\|_{\dot B^{\frac32}_{2,1}}\\ &\le \frac{C}{\eta}\|B\|^2_{L^\infty}\|B\|_{\dot B^{\frac12}_{2,1}}+\eta\|B\|_{\dot B^{\frac52}_{2,1}},\end{aligned}$$

and, similarly,

$$\|u\cdot\nabla u\|_{\dot B^{\frac12}_{2,1}}\le\frac{C}{\eta}\|u\|^2_{L^\infty}\|u\|_{\dot B^{\frac12}_{2,1}}+\eta\|u\|_{\dot B^{\frac52}_{2,1}}.$$

We also have

$$\begin{aligned}\|v\times B\|_{\dot B^{\frac32}_{2,1}} &\le C(\|v\|_{L^\infty}\|B\|_{\dot B^{\frac32}_{2,1}}+\|B\|_{L^\infty}\|v\|_{\dot B^{\frac32}_{2,1}})\\ &\le\frac{C}{\eta}\|(B,v)\|^2_{L^\infty}\|(B,v)\|_{\dot B^{\frac12}_{2,1}}+\eta\|(B,v)\|_{\dot B^{\frac52}_{2,1}},\\ \|v\times u\|_{\dot B^{\frac32}_{2,1}} &\le\frac{C}{\eta}\|(u,v)\|^2_{L^\infty}\|(u,v)\|_{\dot B^{\frac12}_{2,1}}+\eta\|(u,v)\|_{\dot B^{\frac52}_{2,1}},\end{aligned}$$

$$\|v\cdot\nabla B\|_{\dot B^{\frac32}_{2,1}}\leq\frac{C}{\eta}\|(\nabla B,v)\|^2_{L^\infty}\|(\nabla B,v)\|_{\dot B^{\frac12}_{2,1}}+\eta\|(\nabla B,v)\|_{\dot B^{\frac52}_{2,1}}\cdot$$

As, according to (A.8) with $s=3/2$ and to the fact that $\nabla:L^\infty\to\dot B^{-1}_{\infty,\infty}$, we have

$$\sum_j 2^{\frac{3j}{2}}\|[\dot\Delta_j,B\times](\nabla\times v)\|_{L^2}\leq C\big(\|\nabla B\|_{L^\infty}\|v\|_{\dot B^{\frac32}_{2,1}}+\|v\|_{L^\infty}\|\nabla B\|_{\dot B^{\frac32}_{2,1}}\big),\tag{4.24}$$

that term may be bounded as $v\cdot\nabla B$.

Therefore, if we choose η small enough, then (4.23) becomes:

$$\|(u,B,v)(t)\|_{\dot B^{\frac12}_{2,1}}+\frac{C_1}{2}\int_0^t\|(u,B,v)\|_{\dot B^{\frac52}_{2,1}}\,d\tau\leq\|(u,B,v)(0)\|_{\dot B^{\frac12}_{2,1}}$$
$$+C\int_0^t\|(u,B,\nabla B)\|^2_{L^\infty}\|(u,B,v)\|_{\dot B^{\frac12}_{2,1}}\,d\tau$$

and Gronwall's inequality implies that for all $t\in[0,T^*)$,

$$\|(u,B,v)(t)\|_{\dot B^{\frac12}_{2,1}}+\frac{C_1}{2}\int_0^t\|(u,B,v)\|_{\dot B^{\frac52}_{2,1}}\,d\tau$$
$$\leq\|(u,B,v)(0)\|_{\dot B^{\frac12}_{2,1}}\exp\bigg(C\int_0^t\|(u,B,\nabla B)\|^2_{L^\infty}\,dt\bigg)\cdot$$

Now, if one assumes that

$$\int_0^{T^*}\|(u,B,\nabla B)(t)\|^2_{L^\infty}\,dt<\infty,$$

then the above inequality ensures that (u,B,v) belongs to $L^\infty(0,T^*;\dot B^{\frac12}_{2,1})$ and one may conclude by classical arguments that the solution may be continued beyond T^*.

In order to prove the second blow-up criterion, one uses the following inequalities, based on (A.6) and interpolation inequalities:

$$\begin{aligned}
\|B\cdot\nabla B\|_{\dot B^{\frac12}_{2,1}}&\lesssim\|B\|_{\dot B^{\frac12}_{2,1}}\|B\|_{\dot B^{\frac52}_{2,1}},\\
\|u\cdot\nabla u\|_{\dot B^{\frac12}_{2,1}}&\lesssim\|u\|_{\dot B^{\frac12}_{2,1}}\|u\|_{\dot B^{\frac52}_{2,1}},\\
\|v\times B\|_{\dot B^{\frac32}_{2,1}}&\lesssim\|v\|_{\dot B^{\frac32}_{2,1}}\|B\|_{\dot B^{\frac32}_{2,1}}\\
&\lesssim\|v\|_{\dot B^{\frac12}_{2,1}}\|v\|_{\dot B^{\frac52}_{2,1}}+\|B\|_{\dot B^{\frac12}_{2,1}}\|B\|_{\dot B^{\frac52}_{2,1}},\\
\|v\times u\|_{\dot B^{\frac32}_{2,1}}&\lesssim\|v\|_{\dot B^{\frac32}_{2,1}}\|u\|_{\dot B^{\frac32}_{2,1}}\\
&\lesssim\|v\|_{\dot B^{\frac12}_{2,1}}\|v\|_{\dot B^{\frac52}_{2,1}}+\|u\|_{\dot B^{\frac12}_{2,1}}\|u\|_{\dot B^{\frac52}_{2,1}},\\
\|v\cdot\nabla B\|_{\dot B^{\frac32}_{2,1}}&\lesssim\|v\|_{\dot B^{\frac32}_{2,1}}\|\nabla B\|_{\dot B^{\frac32}_{2,1}}\\
&\lesssim\|v\|^2_{\dot B^{\frac32}_{2,1}}+\|J\|^2_{\dot B^{\frac32}_{2,1}}\\
&\lesssim\|v\|^2_{\dot B^{\frac32}_{2,1}}+\|u\|^2_{\dot B^{\frac32}_{2,1}}\\
&\lesssim\|v\|_{\dot B^{\frac12}_{2,1}}\|v\|_{\dot B^{\frac52}_{2,1}}+\|u\|_{\dot B^{\frac12}_{2,1}}\|u\|_{\dot B^{\frac52}_{2,1}}
\end{aligned}$$

and by (4.8) and Proposition A.2 *(iii) (vi)*,

$$\sum_j 2^{\frac{3j}{2}}\|[\dot\Delta_j, B\times](\nabla\times v)\|_{L^2} \lesssim \|\nabla B\|_{\dot B^{\frac32}_{2,1}}\|v\|_{\dot B^{\frac32}_{2,1}}$$
$$\lesssim \|v\|_{\dot B^{\frac12}_{2,1}}\|v\|_{\dot B^{\frac52}_{2,1}} + \|u\|_{\dot B^{\frac12}_{2,1}}\|u\|_{\dot B^{\frac52}_{2,1}}.$$

Plugging those estimates in (4.23), we find that

$$\|(u,B,v)(t)\|_{\dot B^{\frac12}_{2,1}} + C_1\int_0^t \|(u,B,v)\|_{\dot B^{\frac52}_{2,1}}\,d\tau \le \|(u,B,v)(0)\|_{\dot B^{\frac12}_{2,1}} + \int_0^t \|(u,B,v)\|_{\dot B^{\frac12}_{2,1}}\|(u,B,v)\|_{\dot B^{\frac52}_{2,1}}\,d\tau.$$

Hence, if

$$\int_0^{T^*}\|(u,B,J)\|_{\dot B^{\frac52}_{2,1}}\,dt<\infty,$$

then the solution may be continued beyond T^*.

For proving the last blow-up criterion, one can use that for $\rho\in(2,\infty]$, most of the terms of (4.23) may be bounded by means of Inequality (A.7). The last commutator term may be bounded from (A.9) (without time integration) with $r=1$ and $s=3/2$ as follows:

$$\sum_j 2^{\frac{3j}{2}}\|[\dot\Delta_j, B\times](\nabla\times v)\|_{L^2} \lesssim \|\nabla B\|_{\dot B^{\frac2\rho-1}_{\infty,\infty}}\|v\|_{\dot B^{\frac52-\frac2\rho}_{2,1}} + \|v\|_{\dot B^{\frac2\rho-1}_{\infty,\infty}}\|\nabla B\|_{\dot B^{\frac52-\frac2\rho}_{2,1}}.$$

Since, by interpolation, we have

$$\|Z\|_{\dot B^{\frac52-\frac2\rho}_{2,1}} \lesssim \|Z\|^{\frac1\rho}_{\dot B^{\frac12}_{2,1}}\|Z\|^{\frac1{\rho'}}_{\dot B^{\frac52}_{2,1}}\quad\text{with}\quad \frac1{\rho'}=1-\frac1\rho,$$

using Young inequality and reverting to (4.23) yields

$$\|(u,B,v)(t)\|_{\dot B^{\frac12}_{2,1}} + \int_0^t \|(u,B,v)\|_{\dot B^{\frac52}_{2,1}}\,d\tau \le \|(u,B,v)(0)\|_{\dot B^{\frac12}_{2,1}} + C\int_0^t \|(u,B,v)\|^{\rho}_{\dot B^{\frac2\rho-1}_{\infty,\infty}}\|(u,B,v)\|_{\dot B^{\frac12}_{2,1}}\,d\tau.$$

As before, one can conclude that if $T^*<\infty$ and (2.17) is fulfilled, then the solution may be continued beyond T^*. This completes the proof of the theorem. □

5. The well-posedness theory in spaces $\dot B^{\frac12}_{2,r}$ for general r

Let us first prove the a priori estimates leading to global existence.

Proposition 5.1. *Assume that (u,B) is a smooth solution of Hall-MHD system on $[0,T]\times\mathbb{R}^3$ with $\varepsilon=\mu=\nu=1$. Let $v:=u-\nabla\times B$. There exists a universal constant C such that for any $r\in[1,\infty]$, we have*

$$\|(u,B,v)\|_{E_{2,r}(T)} \le C\big(\|(u_0,B_0,v_0)\|_{\dot B^{\frac12}_{2,r}} + \|(u,B,v)\|^2_{E_{2,r}(T)}\big)\cdot \tag{5.1}$$

Proof. We argue as in the proof of Inequality (4.23), but take the $\ell^r(\mathbb{Z})$ norm instead of the $\ell^1(\mathbb{Z})$ norm. We get for all $t\in[0,T]$,

$$\|(u,B,v)\|_{\widetilde L^\infty_t(\dot B^{\frac12}_{2,r})} + \|(u,B,v)\|_{\widetilde L^1_t(\dot B^{\frac52}_{2,r})} \lesssim \|(u_0,B_0,v_0)\|_{\dot B^{\frac12}_{2,r}} + \|B\cdot\nabla B\|_{\widetilde L^1_t(\dot B^{\frac12}_{2,r})}$$

$$+\|u\cdot\nabla u\|_{\widetilde L^1_t(\dot B^{\frac12}_{2,r})}+\|v\cdot\nabla B\|_{\widetilde L^1_t(\dot B^{\frac12}_{2,r})}+\|B\cdot\nabla v\|_{\widetilde L^1_t(\dot B^{\frac12}_{2,r})}+\|v\cdot\nabla u\|_{\widetilde L^1_t(\dot B^{\frac12}_{2,r})}$$
$$+\|u\cdot\nabla v\|_{\widetilde L^1_t(\dot B^{\frac12}_{2,r})}+\|v\cdot\nabla B\|_{\widetilde L^1_t(\dot B^{\frac32}_{2,r})}+\Big\|2^{\frac{3j}{2}}\|[\dot\Delta_j,B\times](\nabla\times v)\|_{L^1_t(L^2)}\Big\|_{\ell^r(\mathbb Z)}.$$

The first six nonlinear terms in the right-hand side may be bounded according to the following product law that is proved in Appendix:

$$\|a\,b\|_{\widetilde L^1_t(\dot B^{\frac12}_{2,r})}\lesssim\|a\|_{\widetilde L^{\frac43}_t(\dot B^1_{2,r})}\|b\|_{\widetilde L^4_t(\dot B^1_{2,r})}. \tag{5.2}$$

The last but one term may be bounded as follows:

$$\|v\cdot\nabla B\|_{\widetilde L^1_t(\dot B^{\frac32}_{2,r})}\lesssim\|v\|_{\widetilde L^4_t(\dot B^1_{2,r})}\|\nabla B\|_{\widetilde L^{\frac43}_t(\dot B^2_{2,r})}+\|\nabla B\|_{\widetilde L^4_t(\dot B^1_{2,r})}\|v\|_{\widetilde L^{\frac43}_t(\dot B^2_{2,r})}. \tag{5.3}$$

Finally, in light of (A.9) with $b=B$, $a=\nabla\times v$, $s=3/2$ and $\rho=4$, and embedding, one discovers that the commutator term may be bounded exactly as $v\cdot\nabla B$.

Putting together all the above inequalities eventually yields for all $t\geq0$,

$$\|(u,B,v)\|_{E_{2,r}(t)}\lesssim\|(u_0,B_0,v_0)\|_{\dot B^{\frac12}_{2,r}}+\|(u,B,v)\|_{\widetilde L^{\frac43}_t(\dot B^2_{2,r})}\|(u,B,v)\|_{\widetilde L^4_t(\dot B^1_{2,r})}. \tag{5.4}$$

Since one can prove by making use of Hölder inequality and interpolation that

$$\|z\|_{\widetilde L^\rho_t(\dot B^{\frac12+\frac2\rho}_{2,r})}\leq\|z\|_{E_{2,r}(t)}\quad\text{for all}\quad\rho\in[1,+\infty],$$

Inequality (5.4) implies (5.1). □

In order to prove Theorem 2.3, we proceed as follows:

(1) smooth out the data and get a sequence $(u^n,B^n)_{n\in\mathbb N}$ of global smooth solutions to Hall-MHD system;
(2) apply Proposition 5.1 to $(u^n,B^n)_{n\in\mathbb N}$ and obtain uniform estimates for $(u^n,B^n,v^n)_{n\in\mathbb N}$ in the space $E_{2,r}$;
(3) use compactness to prove that $(u^n,B^n)_{n\in\mathbb N}$ converges, up to extraction, to a solution of Hall-MHD system supplemented with initial data (u_0,B_0);
(4) prove stability estimates *in a larger space* to get the uniqueness of the solution.

To proceed, let us smooth out the initial data as follows[3]:

$$u_0^n:=(\dot S_n-\dot S_{-n})u_0\quad\text{and}\quad B_0^n:=(\dot S_n-\dot S_{-n})B_0.$$

Clearly, u_0^n and B_0^n belong to all Sobolev spaces, and we have for $z=u,B,v$ and all $n\in\mathbb N$,

$$\forall j\in\mathbb Z,\quad\|\dot\Delta_jz_0^n\|_{L^2}\leq\|\dot\Delta_jz_0\|_{L^2}\quad\text{and}\quad\|z_0^n\|_{\dot B^{\frac12}_{2,r}}\leq\|z_0\|_{\dot B^{\frac12}_{2,r}}. \tag{5.5}$$

Since in particular (u_0^n,B_0^n,v_0^n) is in $\dot B^{\frac12}_{2,1}$, Theorem 2.2 guarantees that the Hall-MHD system with data (u_0^n,B_0^n) has a unique maximal solution on $[0,T^n)$ for some $T^n>0$, that belongs to $E_{2,1}(T)$ for all $T<T^n$. Now, take some positive real number M to be chosen later on and define

$$T_n:=\sup\big\{t\in[0,T^n)\,,\,\|(u^n,B^n,v^n)\|_{E_{2,r}(t)}\leq Mc\big\}\cdot$$

We are going to show first that $T_n=T^n$, then that $T^n=+\infty$.

[3]The reader may refer to the appendix for the definition of $\dot S_j$

According to Proposition 5.1 and to (5.5), we have

$$\|(u^n,B^n,v^n)\|_{E_{2,r}(T_n)} \leq C\big(\|(u_0,B_0,v_0)\|_{\dot B^{\frac12}_{2,r}} + \|(u^n,B^n,v^n)\|^2_{E_{2,r}(T_n)}\big)\cdot$$

Hence, using the smallness condition on (u_0,B_0,v_0) and the definition of T_n,

$$\|(u^n,B^n,v^n)\|_{E_{2,r}(T_n)} \leq Cc(1+M^2c).$$

If we take $M=2C$, then c so that $4C^2c<1$, then we have

$$\|(u^n,B^n,v^n)\|_{E_{2,r}(T_n)} < Mc,$$

and thus, by a classical continuity argument, $T_n=T^n$.

Now, using functional embedding and interpolation arguments, we discover that

$$\left(\int_0^{T^n}\|(u^n,B^n,v^n)\|^4_{\dot B^{-\frac12}_{\infty,\infty}}\,dt\right)^{\frac14} \lesssim \|(u^n,B^n,v^n)\|_{\widetilde L^4_{T^n}(\dot B^1_{2,r})} \lesssim \|(u^n,B^n,v^n)\|_{E_{2,r}(T^n)}.$$

Hence, the continuation criterion (2.17) guarantees that, indeed, $T^n=+\infty$. This means that the solution is global and that, furthermore,

$$\|(u^n,B^n,v^n)\|_{E_{2,r}} \leq Mc \quad \text{for all} \quad n\in\mathbb{N}. \tag{5.6}$$

At this stage, proving that $(u^n,B^n)_{n\in\mathbb{N}}$ converges (up to subsequence) to a global solution (u,B) of the Hall-MHD system with data (u_0,B_0) and (u,B,v) in $E_{2,r}$ follows from the same arguments as in the previous section.

Let us finally prove the uniqueness part of the theorem. Suppose that (u_1,B_1) and (u_2,B_2) are two solutions of Hall-MHD system on $[0,T]\times\mathbb{R}^3$ supplemented with the same initial data (u_0,B_0) and such that

$$(u_i,B_i,v_i)\in\widetilde{\mathcal C}([0,T];\dot B^{\frac12}_{2,r})\cap\widetilde L^1(0,T;\dot B^{\frac52}_{2,r}),\quad i=1,2.$$

In order to prove the uniqueness, we look at the difference $(\delta u,\delta B,\delta v)=(u_1-u_2,B_1-B_2,v_1-v_2)$ as a solution of System (4.21). In contrast with the previous section however, we do not know how to estimate the difference in the space $E_{2,r}(T)$ since the term $\nabla\times((\nabla\times v_1)\times\delta B)$ cannot be bounded in the space $\widetilde L^1_T(\dot B^{\frac12}_{2,r})$ from the norm of v_1 and δB in $E_{2,r}(T)$ (this is due to the fact that the norm of $E_{2,r}(T)$ fails to control $\|\cdot\|_{L^\infty(0,T\times\mathbb{R}^3)}$ by a little if $r>1$).

For that reason, we shall accept to lose some regularity in the stability estimates and prove uniqueness in the space

$$F_{2,r}(T) := \widetilde L^\infty_T(\dot B^{-\frac12}_{2,r}).$$

We need first to justify that $(\delta u,\delta B,\delta v)$ belongs to that space, though. According to Proposition A.4, it is enough to check that the terms R_1 to R_5 defined just below (4.21) belong to $\widetilde L^1_T(\dot B^{-\frac12}_{2,r})$. Now, from (5.2) and Holder inequality, we have

$$\|R_1\|_{\widetilde L^1_T(\dot B^{-\frac12}_{2,r})} \lesssim T^{\frac12}\|(u_1,B_1,u_2,B_2)\|_{\widetilde L^4_T(\dot B^1_{2,r})}\|(\delta u,\delta B)\|_{\widetilde L^4_T(\dot B^1_{2,r})},$$

$$\|R_2\|_{\widetilde L^1_T(\dot B^{-\frac12}_{2,r})} \lesssim T^{\frac12}\|(B_2,v_1)\|_{\widetilde L^4_T(\dot B^1_{2,r})}\|(\delta B,\delta v)\|_{\widetilde L^4_T(\dot B^1_{2,r})},$$

$$\|R_3\|_{\widetilde L^1_T(\dot B^{-\frac12}_{2,r})} \lesssim \|(\nabla v_1,\nabla\delta v)\|_{\widetilde L^{\frac43}_T(\dot B^1_{2,r})}\|(\delta B,B_2)\|_{\widetilde L^4_T(\dot B^1_{2,r})},$$

$$\|R_4\|_{\widetilde L^1_T(\dot B^{-\frac12}_{2,r})} \lesssim T^{\frac12}\|(u_2,v_1)\|_{\widetilde L^4_T(\dot B^1_{2,r})}\|(\delta u,\delta v)\|_{\widetilde L^4_T(\dot B^1_{2,r})},$$

$$\|R_5\|_{\widetilde{L}^1_T(\dot{B}^{-\frac12}_{2,r})} \lesssim \|(\nabla B_2, \nabla \delta B)\|_{\widetilde{L}^{\frac43}_T(\dot{B}^1_{2,r})} \|(\delta v, v_1)\|_{\widetilde{L}^4_T(\dot{B}^1_{2,r})}.$$

Since the norm in $E_{2,r}(T)$ bounds the norm in $\widetilde{L}^4_T(\dot{B}^1_{2,r}) \cap \widetilde{L}^{\frac43}_T(\dot{B}^2_{2,r})$, one can indeed conclude that the terms R_1 to R_5 are in $\widetilde{L}^1_T(\dot{B}^{-\frac12}_{2,r})$.

Next, estimating $(\delta u, \delta B, \delta v)$ in $F_{2,r}(T)$ may be achieved by a slight modification of the beginning of the proof of Proposition 5.1. We get for all $t \in [0, T]$,

$$\begin{aligned}\|(\delta u, \delta B, \delta v)\|_{F_{2,r}(t)} &\lesssim \|B_1 \cdot \nabla \delta B\|_{\widetilde{L}^1_t(\dot{B}^{-\frac12}_{2,r})} + \|\delta B \cdot \nabla B_2\|_{\widetilde{L}^1_t(\dot{B}^{-\frac12}_{2,r})} \\ &+ \|u_1 \cdot \nabla \delta u\|_{\widetilde{L}^1_t(\dot{B}^{-\frac12}_{2,r})} + \|\delta u \cdot \nabla u_2\|_{\widetilde{L}^1_t(\dot{B}^{-\frac12}_{2,r})} + \|v_1 \cdot \nabla \delta B\|_{\widetilde{L}^1_t(\dot{B}^{-\frac12}_{2,r})} \\ &+ \|\delta B \cdot \nabla v_1\|_{\widetilde{L}^1_t(\dot{B}^{-\frac12}_{2,r})} + \|B_2 \cdot \nabla \delta v\|_{\widetilde{L}^1_t(\dot{B}^{-\frac12}_{2,r})} + \|\delta v \cdot \nabla B_2\|_{\widetilde{L}^1_t(\dot{B}^{-\frac12}_{2,r})} \\ &+ \|v_1 \cdot \nabla \delta u\|_{\widetilde{L}^1_t(\dot{B}^{-\frac12}_{2,r})} + \|\delta u \cdot \nabla v_1\|_{\widetilde{L}^1_t(\dot{B}^{-\frac12}_{2,r})} + \|u_2 \cdot \nabla \delta v\|_{\widetilde{L}^1_t(\dot{B}^{-\frac12}_{2,r})} \\ &+ \|\delta v \cdot \nabla u_2\|_{\widetilde{L}^1_t(\dot{B}^{-\frac12}_{2,r})} + \|(\nabla \times v_1) \times \delta B\|_{\widetilde{L}^1_t(\dot{B}^{\frac12}_{2,r})} + \|v_1 \cdot \nabla \delta B\|_{\widetilde{L}^1_t(\dot{B}^{\frac12}_{2,r})} \\ &+ \|\delta v \cdot \nabla B_2\|_{\widetilde{L}^1_t(\dot{B}^{\frac12}_{2,r})} + \Big\|2^{\frac j2} \|[\dot{\Delta}_j, B_2 \times](\nabla \times \delta v)\|_{L^1_t(L^2)}\Big\|_{\ell^r(\mathbb{Z})}.\end{aligned}$$

Most of the terms on the right-hand side can be bounded by means of the following inequalities that are proved in appendix:

$$\|ab\|_{\widetilde{L}^1_t(\dot{B}^{-\frac12}_{2,r})} \lesssim \|a\|_{\widetilde{L}^4_t(\dot{B}^1_{2,r})} \|b\|_{\widetilde{L}^{\frac43}_t(\dot{B}^0_{2,r})}, \tag{5.7}$$

$$\|ab\|_{\widetilde{L}^1_t(\dot{B}^{-\frac12}_{2,r})} \lesssim \|a\|_{\widetilde{L}^{\frac43}_t(\dot{B}^1_{2,r})} \|b\|_{\widetilde{L}^4_t(\dot{B}^0_{2,r})}. \tag{5.8}$$

Next, owing to Inequality (5.2) and interpolation, we have

$$\begin{aligned}\|(\nabla \times v_1) \times \delta B\|_{\widetilde{L}^1_t(\dot{B}^{\frac12}_{2,r})} &+ \|v_1 \cdot \nabla \delta B\|_{\widetilde{L}^1_t(\dot{B}^{\frac12}_{2,r})} + \|\delta v \cdot \nabla B_2\|_{\widetilde{L}^1_t(\dot{B}^{\frac12}_{2,r})} \\ &\lesssim \|\delta B\|_{\widetilde{L}^4_t(\dot{B}^1_{2,r})} \|\nabla v_1\|_{\widetilde{L}^{\frac43}_t(\dot{B}^1_{2,r})} + \|\nabla \delta B\|_{\widetilde{L}^{\frac43}_t(\dot{B}^1_{2,r})} \|v_1\|_{\widetilde{L}^4_t(\dot{B}^1_{2,r})} \\ &\qquad + \|\delta v\|_{\widetilde{L}^{\frac43}_t(\dot{B}^1_{2,r})} \|\nabla B_2\|_{\widetilde{L}^4_t(\dot{B}^1_{2,r})} \\ &\lesssim \Big(\|(u_2, v_2, v_1)\|_{\widetilde{L}^4_t(\dot{B}^1_{2,r})} + \|v_1\|_{\widetilde{L}^{\frac43}_t(\dot{B}^2_{2,r})}\Big) \|(\delta u, \delta v)\|_{F_{2,r}(t)}.\end{aligned}$$

Finally, applying (A.9) with $\rho = 4$, $s = 1/2$ and using the embedding $\dot{B}^0_{2,r} \hookrightarrow \dot{B}^{-\frac32}_{\infty,\infty}$ and $\dot{B}^1_{2,r} \hookrightarrow \dot{B}^{-\frac12}_{\infty,\infty}$ yields

$$\Big\|2^{\frac j2} \|[\dot{\Delta}_j, B_2 \times](\nabla \times \delta v)\|_{L^1_t(L^2)}\Big\|_{\ell^r(\mathbb{Z})} \lesssim \|\nabla B_2\|_{L^4_t(\dot{B}^1_{2,r})} \|\nabla \times \delta v\|_{\widetilde{L}^{\frac43}_t(\dot{B}^0_{2,r})}.$$

Thus, one can conclude that

$$\|(\delta u, \delta B, \delta v)\|_{F_{2,r}(t)} \leq Y(t) \|(\delta u, \delta B, \delta v)\|_{F_{2,r}(t)}$$

with $Y(t) := \sum_{i=1,2} \|(u_i, B_i, v_i)\|_{\widetilde{L}^4_t(\dot{B}^1_{2,r})} + \|v_1\|_{\widetilde{L}^{\frac43}_t(\dot{B}^2_{2,r})}$.

Now, Lebesgue dominated convergence theorem ensures that Y is a continuous nondecreasing function which vanishes at zero. Hence $(\delta u, \delta B, \delta v) \equiv 0$ in $\widetilde{L}^\infty_t(\dot{B}^{-\frac12}_{2,r}) \cap \widetilde{L}^1_t(\dot{B}^{\frac32}_{2,r})$ for small enough t. Combining with a standard connectivity argument allows to conclude that $(\delta u, \delta B, \delta v) \equiv 0$ on $\mathbb{R}^+$. This completes the proof of the theorem in the small data case. □

Let us briefly explain how the above arguments have to be modified so as to handle the case where only v_0 is small. Note that no smallness condition is needed whatsoever in the proof of uniqueness. As regards the existence part, we split u and B (not v) into $u=u^L+\widetilde{u}$ and $B=B^L+\widetilde{B}$ and repeat the proof of Proposition 5.1 on the system fulfilled by $(\widetilde{u},\widetilde{B},v)$ rather than (2.13). Instead of (5.4), we get

$$\|(\widetilde{u},\widetilde{B},v)\|_{E_{2,r}(t)}\lesssim\|v_0\|_{\dot B^{\frac12}_{2,r}}+\|(u,B,v)\|_{\widetilde L^{\frac43}_t(\dot B^2_{2,r})}\|(u,B,v)\|_{\widetilde L^4_t(\dot B^1_{2,r})}$$

from which we deduce that

$$\begin{aligned}\|(\widetilde{u},\widetilde{B},v)\|_{E_{2,r}(t)}\lesssim\|v_0\|_{\dot B^{\frac12}_{2,r}}&+\|(u^L,B^L)\|_{\widetilde L^{\frac43}_t(\dot B^2_{2,r})}\|(u^L,B^L)\|_{\widetilde L^4_t(\dot B^1_{2,r})}\\&+\|(u^L,B^L)\|_{\widetilde L^{\frac43}_t(\dot B^2_{2,r})\cap\widetilde L^4_t(\dot B^1_{2,r})}\|(\widetilde{u},\widetilde{B},v)\|_{E_{2,r}(t)}+\|(\widetilde{u},\widetilde{B},v)\|^2_{E_{2,r}(t)}.\end{aligned}$$

Since, by dominated convergence theorem, we have

$$\lim_{t\to0}\Big(\|(u^L,B^L)\|_{\widetilde L^{\frac43}_t(\dot B^2_{2,r})}+\|(u^L,B^L)\|_{\widetilde L^4_t(\dot B^1_{2,r})}\Big)=0,$$

it is easy to see that if $\|v_0\|_{\dot B^{\frac12}_{2,r}}$ is small enough, then one can get a control on $\|(\widetilde{u},\widetilde{B},v)\|_{E_{2,r}(t)}$ for small enough t. From this, repeating essentially the same arguments as in the small data case, one gets a local-in-time existence statement.

Appendix A. Besov Spaces and commutator estimates

Here, we briefly recall the definition of the Littlewood-Paley decomposition, define Besov spaces and list some properties that have been used repeatedly in the paper. For the reader's convenience, we also prove some nonlinear and commutator estimates. More details and proofs may be found in e.g. [4].

The Littlewood-Paley decomposition is a dyadic localization procedure in the frequency space for tempered distributions over $\mathbb{R}^d$. On can define it from any nonincreasing smooth radial function χ on $\mathbb{R}^d$, supported in, say, $B(0,4/3)$ and with value 1 on $B(0,3/4)$. Let $\varphi:=\chi(\cdot/2)-\chi$. Then, we have

$$\forall\xi\in\mathbb{R}^d,\ \chi(\xi)+\sum_{j\geq0}\varphi(2^{-j}\xi)=1\quad\text{and}\quad\forall\xi\in\mathbb{R}^d\setminus\{0\},\ \sum_{j\in\mathbb{Z}}\varphi(2^{-j}\xi)=1.$$

The homogeneous dyadic blocks $\dot\Delta_j$ and low-frequency cut-off operator $\dot S_j$ are defined for all $j\in\mathbb{Z}$ by

$$\begin{aligned}\dot\Delta_ju\ &:=\varphi(2^{-j}D)u=2^{jd}\int_{\mathbb{R}^d}h(2^jy)u(x-y)\,dy\quad\text{with}\quad h{:=}\mathcal{F}^{-1}\varphi,\\ \dot S_ju\ &:=\chi(2^{-j}D)u=2^{jd}\int_{\mathbb{R}^d}\tilde h(2^jy)u(x-y)\,dy\quad\text{with}\quad \tilde h{:=}\mathcal{F}^{-1}\chi.\end{aligned}$$

The following *Littlewood-Paley decomposition* of u:

$$u=\sum_{j\in\mathbb{Z}}\dot\Delta_ju$$

holds true modulo polynomials for any tempered distribution u. In order to have an equality in the sense of tempered distributions, we consider only elements of the set $\mathcal{S}'_h(\mathbb{R}^d)$ of tempered distributions u such that

$$\lim_{j\to-\infty}\|\dot S_ju\|_{L^\infty}=0.$$

Definition A.1. *Let s be a real number and (p,r) be in $[1,\infty]^2$. The homogeneous Besov space $\dot B^s_{p,r}$ is the set of distributions u in $\mathcal{S}'_h$ such that*

$$\|u\|_{\dot B^s_{p,r}} := \|2^{js}\|\dot\Delta_j u\|_{L^p(\mathbb{R}^d)}\|_{\ell^r(\mathbb{Z})} < \infty.$$

Proposition A.2. *The following properties hold true:*

(i) *Derivatives: for all $s \in \mathbb{R}$ and $1 \le p, r \le \infty$, we have*

$$\sup_{|\alpha|=k} \|\partial^\alpha u\|_{\dot B^s_{p,r}} \sim \|u\|_{\dot B^{s+k}_{p,r}}.$$

(ii) *Embedding: we have the following continuous embedding*

$$\dot B^s_{p,r} \hookrightarrow \dot B^{s-d(\frac{1}{p}-\frac{1}{\tilde p})}_{\tilde p,\tilde r} \quad \text{whenever } \tilde p \ge p \quad \textit{and} \quad \tilde r \ge r,$$

and the space $\dot B^{\frac{d}{p}}_{p,1}$ is embedded in the set of bounded continuous functions.

(iii) *Real interpolation: for any $\theta \in (0,1)$ and $s < \tilde s$, we have*

$$\|u\|_{\dot B^{\theta s+(1-\theta)\tilde s}_{p,1}} \lesssim \|u\|^\theta_{\dot B^s_{p,\infty}} \|u\|^{1-\theta}_{\dot B^{\tilde s}_{p,\infty}}.$$

(iv) *Completeness: the space $\dot B^s_{p,r}$ is complete if (and only if) (s,p,r) satisfies*

$$s < \frac{d}{p}, \quad \text{or } s = \frac{d}{p} \quad \textit{and} \quad r = 1. \tag{A.1}$$

(v) *Density: the space $\mathcal{S}_0(\mathbb{R}^d)$ of Schwartz functions on $\mathbb{R}^d$ with Fourier transform supported away from the origin is dense in $\dot B^s_{p,r}$ whenever both p and r are finite.*

(vi) *Let f be a smooth function on $\mathbb{R}^d \setminus \{0\}$ which is homogeneous of degree 0. Define $f(D)$ on $\mathcal{S}(\mathbb{R}^d)$ by*

$$\mathcal{F}(f(D)u)(\xi) := f(\xi)\mathcal{F}u(\xi),$$

Then, for all exponents (s,p,r), we have the estimate

$$\|f(D)u\|_{\dot B^s_{p,r}} \lesssim \|u\|_{\dot B^s_{p,r}}.$$

If in addition $f(D)$ extends to a map from $\mathcal{S}'_h(\mathbb{R}^d)$ to itself and (A.1) *is fulfilled, then $f(D)$ is continuous from $\dot B^s_{p,r}$ to $\dot B^s_{p,r}$.*

(vii) *Operator* curl^{-1} *maps $\dot B^{s-1}_{p,1}$ to $\dot B^s_{p,1}$ if $1 \le p < \infty$ and $s \le d/p$.*

Proof. We only prove the last item as it is fundamental in our analysis. Owing to the definition in (1.10), it is obvious that curl^{-1} maps $\mathcal{S}_0(\mathbb{R}^d)$ to itself, and homogeneity of degree -1 implies that we have for all u in $\mathcal{S}_0(\mathbb{R}^d)$:

$$\|\mathrm{curl}^{-1} u\|_{\dot B^s_{p,1}} \lesssim \|u\|_{\dot B^{s-1}_{p,1}}.$$

As $\mathcal{S}_0(\mathbb{R}^d)$ is dense in $\dot B^{s-1}_{p,1}$ and since the space $\dot B^s_{p,1}$ is complete (owing to $s \le d/p$), we get the result. □

A great deal of our analysis relies on regularity estimates for the heat equation:

$$(H) \qquad \begin{cases} \partial_t u - \Delta u = f, \\ u|_{t=0} = u_0. \end{cases}$$

It is classical that for all $u_0 \in \mathcal{S}'(\mathbb{R}^d)$ and $f \in L^1_{loc}(\mathbb{R}^+; \mathcal{S}'(\mathbb{R}^d))$, equation (H) has a unique tempered distribution solution, given by the following Duhamel formula:

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-\tau)\Delta} f(\tau)\, d\tau, \qquad t \geq 0. \tag{A.2}$$

Above, $(e^{t\Delta})_{t\geq 0}$ stands for the heat semi-group. It is defined on $\mathcal{S}(\mathbb{R}^d)$ by

$$\mathcal{F}(e^{t\Delta}z)(\xi) := e^{-t|\xi|^2}\widehat{z}(\xi), \tag{A.3}$$

and is extended to the set of tempered distributions by duality.

As observed by Chemin in [9], the following spaces are suitable for describing the maximal regularity properties of the heat equation.

Definition A.3. *For $T > 0$, $s \in \mathbb{R}$, $1 \leq \rho \leq \infty$, we set*

$$\|u\|_{\widetilde{L}^\rho_T(\dot{B}^s_{p,r})} := \big\|2^{js}\|\dot{\Delta}_j u\|_{L^\rho_T(L^p)}\big\|_{\ell^r(\mathbb{Z})}.$$

We define the space $\widetilde{L}^\rho_T(\dot{B}^s_{p,r})$ to be the set of tempered distribution u on $(0,T)\times\mathbb{R}^d$ such that $\lim_{j\to-\infty}\|\dot{S}_j u(t)\|_{L^\infty} = 0$ a.e. in $(0,T)$, and $\|u\|_{\widetilde{L}^\rho_T(\dot{B}^s_{p,r})} < \infty$. The space $\widetilde{L}^\rho_T(\dot{B}^s_{p,r}) \cap \mathcal{C}([0,T]; \dot{B}^s_{p,r})$ is denoted by $\widetilde{\mathcal{C}}_T(\dot{B}^s_{p,r})$. In the case $T = +\infty$, one denotes the corresponding space and norm by $\widetilde{L}^\rho(\dot{B}^s_{p,r})$ and $\|\cdot\|_{\widetilde{L}^\rho(\dot{B}^s_{p,r})}$, respectively.

The above spaces or norms may be compared to more classical ones according to Minkowski's inequality:

$$\|u\|_{\widetilde{L}^\rho_T(\dot{B}^s_{p,r})} \leq \|u\|_{L^\rho_T(\dot{B}^s_{p,r})} \text{ if } r \geq \rho \quad \text{and} \quad \|u\|_{\widetilde{L}^\rho_T(\dot{B}^s_{p,r})} \geq \|u\|_{L^\rho_T(\dot{B}^s_{p,r})} \text{ if } r \leq \rho.$$

The following fundamental result has been proved in [9].

Proposition A.4. *Let $T > 0$, $s \in \mathbb{R}$ and $1 \leq \rho, p, r \leq \infty$. Assume that $u_0 \in \dot{B}^s_{p,r}$ and $f \in \widetilde{L}^\rho_T(\dot{B}^{s-2+\frac{2}{\rho}}_{p,r})$. Then, (H) has a unique solution u in $\widetilde{L}^\rho_T(\dot{B}^{s+\frac{2}{\rho}}_{p,r}) \cap \widetilde{L}^\infty_T(\dot{B}^s_{p,r})$ and there exists a constant C depending only on d and such that for all $\rho_1 \in [\rho, \infty]$, we have*

$$\|u\|_{\widetilde{L}^{\rho_1}_T(\dot{B}^{s+\frac{2}{\rho_1}}_{p,r})} \leq C(\|u_0\|_{\dot{B}^s_{p,r}} + \|f\|_{\widetilde{L}^\rho_T(\dot{B}^{s-2+\frac{2}{\rho}}_{p,r})}). \tag{A.4}$$

Furthermore, if r is finite, then u belongs to $\mathcal{C}([0,T]; \dot{B}^s_{p,r})$.

Let us now recall a few nonlinear estimates in Besov spaces, that we used in the paper. They all may be easily proved by using the following so-called Bony decomposition (from [6]) for the (formal) product of two distributions u and v:

$$u\,v = T_u v + T_v u + R(u,v).$$

Above, T designates the paraproduct bilinear operator defined by

$$T_u v := \sum_j \dot{S}_{j-1}u\dot{\Delta}_j v, \quad T_v u := \sum_j \dot{S}_{j-1}v\,\dot{\Delta}_j u$$

and R stands for the remainder operator given by

$$R(u,v) := \sum_j \sum_{|j'-j|\leq 1} \dot{\Delta}_j u\,\dot{\Delta}_{j'} v.$$

The following properties of the paraproduct and remainder operators are classical:

Proposition A.5. *For any $(s,p,r)\in\mathbb{R}\times[1,\infty]^2$ and $t<0$, there exists a constant C such that*

$$\|T_uv\|_{\dot B^s_{p,r}}\leq C\,\|u\|_{L^\infty}\|v\|_{\dot B^s_{p,r}}\quad\text{and}\quad \|T_uv\|_{\dot B^{s+t}_{p,r}}\leq C\,\|u\|_{\dot B^t_{\infty,\infty}}\|v\|_{\dot B^s_{p,r}}.$$

For any (s_1,p_1,r_1) and (s_2,p_2,r_2) in $\mathbb{R}\times[1,\infty]^2$ satisfying

$$s_1+s_2>0,\quad \frac1p:=\frac1{p_1}+\frac1{p_2}\leq1\quad\textit{and}\quad\frac1r:=\frac1{r_1}+\frac1{r_2}\leq1,$$

there exists a constant C such that

$$\|R(u,v)\|_{\dot B^{s_1+s_2}_{p,r}}\leq C\,\|u\|_{\dot B^{s_1}_{p_1,r_1}}\|v\|_{\dot B^{s_2}_{p_2,r_2}}.$$

Combining the above Proposition with the Bony decomposition allows to get a number of inequalities like, for instance:

- tame estimates: for any $s>0$ and $1\leq p,r\leq\infty$,
$$\|uv\|_{\dot B^s_{p,r}}\lesssim\|u\|_{L^\infty}\|v\|_{\dot B^s_{p,r}}+\|v\|_{L^\infty}\|u\|_{\dot B^s_{p,r}};\tag{A.5}$$
- the following product estimate:
$$\|uv\|_{\dot B^{s_1+s_2-\frac dp}_{p,1}}\lesssim\|u\|_{\dot B^{s_1}_{p,1}}\|v\|_{\dot B^{s_2}_{p,1}}\tag{A.6}$$
that holds true whenever[4] $s_1,s_2\leq\frac dp$ satisfy $s_1+s_2>d\max(0,\frac2p-1)$;
- the following inequality (in the case $d=3$ and $\rho>2$) that has been used in the proof of (2.17):
$$\|ab\|_{\dot B^{\frac32}_{2,1}}\lesssim\|a\|_{\dot B^{\frac2\rho-1}_{\infty,\infty}}\|b\|_{\dot B^{\frac52-\frac2\rho}_{2,1}}+\|b\|_{\dot B^{\frac2\rho-1}_{\infty,\infty}}\|a\|_{\dot B^{\frac52-\frac2\rho}_{2,1}}.\tag{A.7}$$

Remark 1. Proposition A.5 and estimates like (A.6) or (A.7) may be adapted to the spaces $\widetilde L^\rho_T(\dot B^s_{p,r})$. The general principle is that the time exponent behaves according to Hölder inequality. For example, we have

$$\|T_ab\|_{\widetilde L^1_t(\dot B^{\frac12}_{2,r})}+\|T_ba\|_{\widetilde L^1_t(\dot B^{\frac12}_{2,r})}+\|R(a,b)\|_{\widetilde L^1_t(\dot B^{\frac12}_{2,r})}\lesssim\|a\|_{\widetilde L^{\frac43}_t(\dot B^{-\frac12}_{\infty,\infty})}\|b\|_{\widetilde L^4_t(\dot B^1_{2,r})}.$$

Then, combining with embedding (in the case $d=3$) gives Inequality (5.2).

Similarly, Inequality (5.3) stems from

$$\|T_ab\|_{\widetilde L^1_t(\dot B^{\frac32}_{2,r})}+\|T_ba\|_{\widetilde L^1_t(\dot B^{\frac32}_{2,r})}+\|R(a,b)\|_{\widetilde L^1_t(\dot B^{\frac32}_{2,r})}$$
$$\lesssim\|a\|_{\widetilde L^4_t(\dot B^{-\frac12}_{\infty,\infty})}\|b\|_{\widetilde L^{\frac43}_t(\dot B^2_{2,r})}+\|b\|_{\widetilde L^4_t(\dot B^{-\frac12}_{\infty,\infty})}\|a\|_{\widetilde L^{\frac43}_t(\dot B^2_{2,r})}.$$

In order to prove Inequality (5.7), it suffices to use the fact that

$$\|T_ab\|_{\widetilde L^1_t(\dot B^{-\frac12}_{2,r})}\lesssim\|a\|_{\widetilde L^4_t(\dot B^{-\frac12}_{\infty,\infty})}\|b\|_{\widetilde L^{\frac43}_t(\dot B^0_{2,r})},$$
$$\|T_ba\|_{\widetilde L^1_t(\dot B^{-\frac12}_{2,r})}\lesssim\|b\|_{\widetilde L^{\frac43}_t(\dot B^{-\frac32}_{\infty,\infty})}\|a\|_{\widetilde L^4_t(\dot B^1_{2,r})},$$
$$\|R(a,b)\|_{\widetilde L^1_t(\dot B^{-\frac12}_{2,r})}\lesssim\|a\|_{\widetilde L^4_t(\dot B^1_{2,r})}\|b\|_{\widetilde L^{\frac43}_t(\dot B^0_{2,r})}.$$

Proving Inequality (5.8) is similar.

We end this appendix with the proof of commutator estimates that were crucial in our analysis.

[4]In particular, $\dot B^{\frac dp}_{p,1}$ is an algebra for any $1\leq p<\infty$.

Proposition A.6. *Let s be in $(0,d/2]$. Then we have:*

$$\sum_{j\in\mathbb{Z}} 2^{js}\|[\dot\Delta_j,b]a\|_{L^2}\lesssim \|\nabla b\|_{L^\infty}\|a\|_{\dot B^{s-1}_{2,1}}+\|a\|_{\dot B^{-1}_{\infty,\infty}}\|\nabla b\|_{\dot B^{s}_{2,1}}. \tag{A.8}$$

Furthermore, for all $r\in[1,\infty]$ and $\rho\in(2,\infty]$, we have if we set $1/\rho':=1-1/\rho$,

$$\big\|2^{js}\|[\dot\Delta_j,b]a\|_{L^1_t(L^2)}\big\|_{\ell^r(\mathbb{Z})}\lesssim \|\nabla b\|_{\widetilde L^\rho_t(\dot B^{\frac2\rho-1}_{\infty,\infty})}\|a\|_{\widetilde L^{\rho'}_t(\dot B^{s-\frac2\rho}_{2,r})} + \|\nabla b\|_{\widetilde L^{\rho'}_t(\dot B^{s+1-\frac2\rho}_{2,r})}\|a\|_{\widetilde L^{\rho}_t(\dot B^{\frac2\rho-2}_{\infty,\infty})}. \tag{A.9}$$

Proof. Proving the two inequalities relies on the decomposition

$$[\dot\Delta_j,b]a=[\dot\Delta_j,T_b]a+\dot\Delta_j(T_ab+R(a,b))-(T_{\dot\Delta_ja}b+R(\dot\Delta_ja,b)). \tag{A.10}$$

For getting (A.8), we bound the first term of (A.10) as follows (use [4, Ineq. (2.58)]):

$$\sum_{j\in\mathbb{Z}}2^{js}\|[\dot\Delta_j,T_b]a\|_{L^2}\lesssim\|\nabla b\|_{L^\infty}\|a\|_{\dot B^{s-1}_{2,1}}.$$

The next two terms of (A.10) may be bounded by using the fact that the remainder and paraproduct operator map $\dot B^{-1}_{\infty,\infty}\times\dot B^{s+1}_{2,1}$ to $\dot B^{s}_{2,1}$. Finally, owing to the properties of localization of the Littlewood-Paley decomposition, we have

$$T_{\dot\Delta_ja}b+R(\dot\Delta_ja,b)=\sum_{j'\geq j-2}\dot S_{j'+2}\dot\Delta_ja\,\dot\Delta_{j'}b. \tag{A.11}$$

From Bernstein inequality and $\|\dot S_{j'+2}a\|_{L^\infty}\lesssim 2^{j'}\|a\|_{\dot B^{-1}_{\infty,\infty}}$, we gather

$$\begin{aligned}\sum_j 2^{js}\|T_{\dot\Delta_ja}b+R(\dot\Delta_ja,b)\|_{L^2}&\lesssim\sum_j\sum_{j'\geq j-2}2^{js}\|\dot S_{j'+2}a\|_{L^\infty}\|\dot\Delta_{j'}b\|_{L^2}\\&\lesssim\|a\|_{\dot B^{-1}_{\infty,\infty}}\sum_j\sum_{j'\geq j-2}2^{s(j-j')}\,2^{j's}\|\nabla\dot\Delta_{j'}b\|_{L^2}\\&\lesssim\|a\|_{\dot B^{-1}_{\infty,\infty}}\|\nabla b\|_{\dot B^{s}_{2,1}}.\end{aligned}$$

To prove (A.9), we observe that owing to the localization properties of the Littlewood-Paley decomposition, the first term of (A.10) may be decomposed into

$$[\dot\Delta_j,T_b]a=\sum_{|j'-j|\leq4}[\dot\Delta_j,\dot S_{j'-1}b]\dot\Delta_{j'}a.$$

Now, according to [4, Lem. 2.97], we have

$$\|[\dot\Delta_j,\dot S_{j'-1}b]\dot\Delta_{j'}a\|_{L^2}\lesssim 2^{-j}\|\nabla\dot S_{j'-1}b\|_{L^\infty}\|\dot\Delta_{j'}a\|_{L^2},$$

and, since $\frac2\rho-1<0$,

$$\|\nabla S_{j'-1}b\|_{L^\rho_t(L^\infty)}\lesssim 2^{j'(1-\frac2\rho)}\|\nabla b\|_{\widetilde L^\rho_t(\dot B^{\frac2\rho-1}_{\infty,\infty})}.$$

Hence, for all $(j,j')\in\mathbb{Z}^2$ such that $|j-j'|\leq4$,

$$2^{js}\|[\dot\Delta_j,\dot S_{j'-1}b]\dot\Delta_{j'}a\|_{L^1_t(L^2)}\lesssim 2^{js}2^{-\frac2\rho j'}\|\dot\Delta_{j'}a\|_{L^{\rho'}_t(L^2)}\|\nabla b\|_{\widetilde L^\rho_t(\dot B^{\frac2\rho-1}_{\infty,\infty})}.$$

Therefore, summing up on $j'\in\{j-4,j+4\}$, then taking the $\ell^r(\mathbb{Z})$ norm,

$$\big\|2^{js}\|[\dot\Delta_j,T_b]a\|_{L^1_t(L^2)}\big\|_{\ell^r}\lesssim\|\nabla b\|_{\widetilde L^\rho_t(\dot B^{\frac2\rho-1}_{\infty,\infty})}\|a\|_{\widetilde L^{\rho'}_t(\dot B^{s-\frac2\rho}_{2,r})}.$$

The next two terms may be bounded according to Proposition A.5 and Remark 1:
$$\left\|2^{js}\|\dot\Delta_j T_a b\|_{L^1_t(L^2)}\right\|_{\ell^r}+\left\|2^{js}\|\dot\Delta_j R(a,b)\|_{L^1_t(L^2)}\right\|_{\ell^r}\lesssim \|a\|_{\widetilde L^\rho_t(\dot B^{\frac{2}{\rho}-2}_{\infty,\infty})}\|\nabla b\|_{\widetilde L^{\rho'}_t(\dot B^{s+1-\frac{2}{\rho}}_{2,r})}.$$
Finally, use (A.11) and the fact that
$$\|\dot S_{j'+2}a\|_{L^\rho_t(L^\infty)}\lesssim 2^{(2-\frac{2}{\rho})j'}\|a\|_{\widetilde L^\rho_t(\dot B^{\frac{2}{\rho}-2}_{\infty,\infty})}$$
to get
$$\begin{aligned}2^{js}\|T_{\dot\Delta_j a}b+R(\dot\Delta_j a,b)\|_{L^1_t(L^2)}&\lesssim\sum_{j'\geq j-2}2^{js}\|\dot S_{j'+2}a\|_{L^\rho_t(L^\infty)}\|\dot\Delta_{j'}b\|_{L^{\rho'}_t(L^2)}\\&\lesssim\|a\|_{\widetilde L^\rho_t(\dot B^{\frac{2}{\rho}-2}_{\infty,\infty})}\sum_{j'\geq j-2}2^{s(j-j')}\,2^{(s+2-\frac{2}{\rho})j'}\|\dot\Delta_{j'}b\|_{L^{\rho'}_t(L^2)}.\end{aligned}$$
Taking the $\ell^r(\mathbb{Z})$ norm of both sides and using a convolution inequality for series (remember that $s>0$), we end up with
$$\left\|2^{js}\|T_{\dot\Delta_j a}b+R(\dot\Delta_j a,b)\|_{L^1_t(L^2)}\right\|_{\ell^r(\mathbb{Z})}\lesssim\|a\|_{\widetilde L^\rho_t(\dot B^{\frac{2}{\rho}-2}_{\infty,\infty})}\|\nabla b\|_{\widetilde L^{\rho'}_t(\dot B^{s+1-\frac{2}{\rho}}_{2,r})}.$$
This completes the proof of Inequality (A.9). □

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Université Paris-Est Créteil, LAMA UMR 8050, 61 avenue du Général de Gaulle, 94010 Créteil et Sorbonne Université, LJLL UMR 7598, 4 Place Jussieu, 75005 Paris

E-mail address: raphael.danchin@u-pec.fr

Université Paris-Est Créteil, LAMA UMR 8050, 61 avenue du Général de Gaulle, 94010 Créteil

E-mail address: jin.tan@u-pec.fr