Improved bounds for reaction-diffusion propagation with a line of nonlocal diffusion

Anne-Charline Chalmin, Jean-Michel Roquejoffre

Improved bounds for reaction-diffusion propagation with a line of nonlocal diffusion

Anne-Charline Chalmin
Institut de Mathématiques de Toulouse (UMR CNRS 5219)
Université Toulouse III, 118 route de Narbonne
31062 Toulouse cedex, France
abchalmin@gmail.com

Jean-Michel Roquejoffre
Institut de Mathématiques de Toulouse (UMR CNRS 5219)
Université Toulouse III, 118 route de Narbonne
31062 Toulouse cedex, France
jean-michel.roquejoffre@math.univ-toulouse.fr

November 7, 2019

Abstract

We consider here a model of accelerating fronts, introduced in [2], consisting of one equation with nonlocal diffusion on a line, coupled via the boundary condition with a reaction-diffusion equation of the Fisher-KPP type in the upper half-plane. It was proved in [2] that the propagation is accelerated in the direction of the line exponentially fast in time. We make this estimate more precise by computing an explicit correction that is algebraic in time. Unexpectedly, the solution mimicks the behaviour of the solution of the equation linearised around the rest state 0 in a closer way than in the classical fractional Fisher-KPP model.

This paper is dedicated to S. Salsa, as the expression of our friendship and respect.

1 Introduction

1.1 Model and question

Consider the following system, with unknowns $u(t,x)$ and $v(t,x,y)$, where $(t,x,y)\in\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+$

$$
\left\{\begin{array}{rcll}
\partial_t v-\Delta v &=& f(v), & t>0, x\in\mathbb{R},\ y>0,\\
\partial_t u+(-\partial_{xx})^\alpha u &=& -\mu u+\nu v & t>0, x\in\mathbb{R},\ y=0,\ t>0\\
-\partial_y v &=& \mu u-\nu v, & t>0, x\in\mathbb{R},\ y=0.
\end{array}\right. \tag{1.1}
$$

The real number μ is a positive given parameter, and the nonlinear term f is chosen as

$$f(v) = av - g(v),$$

with $a > 0$ and g of class C^2, with $g \geq 0$, convex, $g(0) = g'(0) = 0$, $g'(+\infty) > a$. The equation for v in the upper half-plane is therefore a variant of the Fisher-KPP equation, in reference to the pioneering works of Fisher [11] and of Kolmogorov, Petrovskii and Piskunov [14]. The operator $(-\partial_{xx})^\alpha$ is the fractional Laplacian of order $\alpha \in (0, 1)$:

$$(-\partial_{xx})^\alpha u(x) = c_\alpha \text{ P.V.}\Big(\int_{\mathbb{R}} \frac{u(y) - u(x)}{|x - y|^{1+2\alpha}} dy\Big),$$

the constant $c_\alpha > 0$ being chosen so that the symbol of $(-\partial_{xx})^\alpha$ is $|\xi|^{1+2\alpha}$. The initial datum is chosen as

$$(u(0, x), v(0, x, y)) = (\delta_0 \mathbf{1}_{(-x_0,x_0)}(x), 0) \tag{1.2}$$

where x_0 and δ_0 are given positive constants. Their value is not relevant for the discussion, one may think them as small. Under the listed assumptions, system (1.1) has a unique global smooth solution, that is also globally bounded as well as its derivatives, see [2]. The question under study is the behaviour of $(u(t, x), v(t, x, y))$ for large t.

1.2 Motivation, context, known results

System (1.1) is relevant in the study of the influence of a line having a fast diffusion of its own, that exchanges with an adjacent domain of the plane (here, the upper half plane), in which reactive and diffusive phenomena occur. The application is the modelling of how biological invasions can be enhanced by transportation networks, see [4] for an overview. In this context, $u(t, x)$ represents the density of individuals on the line, and $v(t, x, y)$ represents the density of individuals in the upper half-plane. Exchanges occur through the Robin condition $-\partial_y v(t, x, 0) = \mu u(t, x) - v(t, x, 0)$.

System (1.1) was first introduced by H. Berestycki, L. Rossi and the second author in [7]. There, the diffusion on the line (that we called "the road", while the upper half plane was called "the field") took the form $-D\partial_{xx}$, with $D > 0$, possibly large. The effect of the line may be accounted for as follows: when not present, the model amounts to the single Fisher-KPP equation with unknown $v(t, X)$, $X \in \mathbb{R}^2$:

$$\begin{cases} v_t - \Delta v = & f(v), \quad t > 0, \ X \in \mathbb{R}^2 \\ v(0, X) = & \delta_0 \mathbf{1}_{(-x_0,x_0)^2}(X), \quad X \in \mathbb{R}^2. \end{cases} \tag{1.3}$$

Note that here, we need to shift the mass from the line to the plane in order to avoid the trivial solution $v \equiv 0$. We have (Aronson, Weinberger [1])

$$\begin{aligned} &\text{for all } \varepsilon > 0, \quad \lim_{t\to+\infty} \inf_{|X|\leq(c_*-\varepsilon)t} v(t, X) = v_0 \\ &\text{for all } \varepsilon > 0, \quad \lim_{t\to+\infty} \sup_{|X|\geq(c_*+\varepsilon)t} v(t, X) = 0, \end{aligned} \tag{1.4}$$

where $c_* = 2\sqrt{a}$, and v_0 is the unique positive zero of f, whose existence is granted by the assumptions. In other words, the stable state v_0 invades the whole space at speed c_*. Reverting to (1.1), and concentrating on what happens on the line (or its vicinity), first when the diffusion is $-D\partial_{xx}$, then when it is $(-\partial_{xx})^\alpha$. In the first case, the main result of

[7] is the existence of $c_*(D) > 0$, with $\liminf \dfrac{c_*(D)}{\sqrt{D}} > 0$, such that invasion occurs at speed $c_*(D)$ on the line and in the upper half plane, at finite distance from the line. This ashows the importance of the line on the overall propagation. The limiting states for u and v are $u_\infty \equiv \dfrac{v_0}{\mu}$, $v_\infty \equiv v_0$, a property that is not entirely trivial, and also proved in [7].

The effect of the nonlocal diffusion $(-\partial_{xx})^\alpha$ was studied for the first time in [2] by H. Berestycki, L. Rossi and the two authors of the present paper. The main result of [2] is the following.

Theorem 1.1 *Define* $\lambda_* = \dfrac{a}{1+2\alpha}$. *Then we have*

$$\text{(1.5)}\qquad \begin{array}{ll} \textit{for all } \varepsilon > 0, & \displaystyle\lim_{t\to+\infty} \inf_{|x|\le e^{(\lambda_*-\varepsilon)t}} (u(t,x), v(t,x,y)) = \left(\frac{v_0}{\mu}, v_0\right) \\ \textit{for all } \varepsilon > 0, & \displaystyle\lim_{t\to+\infty} \sup_{|x|\ge e^{(\lambda_*+\varepsilon)t}} (u(t,x), v(t,x,y)) = 0. \end{array}$$

In (1.5)*, the limits of v should be understood pointwise in y.*

Let us note that this result may be parallelled by the following one: let us bluntly replace the exchange term $\mu u - v$ in the equation for u by the reaction term $f(u)$ (so that we shift the whole weight of the reaction from the upper half plane to the line), so as to obtain

$$\text{(1.6)}\qquad \left\{ \begin{array}{rl} u_t + (-\partial_{xx})^\alpha u = & f(u) \quad (t>0,\ x\in\mathbb{R}) \\ u(0,x) = & \delta_0 \mathbf{1}_{(-x_0,x_0)}(x). \end{array} \right.$$

Then, Cabré and the second author [8] proved that invasion at the same rate as in Theorem 1.1 occurs. Thus, $u(t,x)$ actually behaves just as in equation (1.6) at the leading order.

While Theorem 1.1 captures the essence of the main features of the invasion phenomenon, it is interesting to ask whether the asymptotics can be made a little more precise. Indeed there is, in Theorem 1.1, a lot of room between the upper and lower bound. For instance the level sets of u may advance like $t^p e^{\lambda_* t}$, where p could be any real number. This question can also be asked for the simpler model (1.6), all the more as one may give the following heuristics : the dynamics of (1.6) being driven by the small values of u (given the concavity of u they are, loosely speaking, the most unstable ones in the range of f), so that the dynamics of the level sets is really given by the linear equation

$$u_t + (-\partial_{xx})^\alpha u = au.$$

Call $G_\alpha(t,x)$ the fractional heat kernel, we have $G_\alpha(t,x) \lesssim \dfrac{t}{|x|^{1+2\alpha}}$ for large t and x, see [15] for instance. Then we have

$$u(t,x) \lesssim \frac{te^{at}}{|x|^{1+2\alpha}},$$

still for large t and x. So, a level set of u will move like $t^{\frac{1}{1+2\alpha}} e^{\frac{at}{1+2\alpha}}$. This heuristics does not give the correct sharper behaviour, as was proved by Cabré and the two authors of the paper [9]: a level set $\{x(t)\}$ of u will in fact be such that $|x(t)|e^{-\frac{at}{1+2\alpha}}$ is bounded, that is, there is no polynomial correction in the expansion of $x(t)$.

Consider now the linearised version of (1.1):

$$\text{(1.7)}\qquad \left\{ \begin{array}{rcll} \partial_t v - \Delta v & = & av, & t>0, x\in\mathbb{R},\ y>0, \\ \partial_t u + (-\partial_{xx})^\alpha u & = & -\mu u + \nu v & t>0, x\in\mathbb{R},\ y=0,\ t>0 \\ -\partial_y v & = & \mu u - \nu v, & t>0, x\in\mathbb{R},\ y=0. \end{array} \right.$$

Let us call this time $G_\alpha(t,x)$ the solution $u(t,x)$ with the initial datum $\delta_{x=0}$, that is, the u-component of the fundamental solution. Then the first author proved [10] (a more precise estimate will be stated later).

$$G_\alpha(t,x) \lesssim \frac{e^{at}}{t^{\frac{3}{2}}|x|^{1+2\alpha}}, \quad t \to +\infty, \ |x| \to +\infty.$$

And so, a level set $\{x(t)\}$ of the solution $u(t,x)$ of (1.7) will move like $t^{-\frac{3}{2(1+2\alpha)}} e^{\frac{at}{1+2\alpha}}$. The question that we want to address in this paper is whether a discrepancy of the same kind holds between the linear and nonlinear equation.

1.3 Result and organisation of the paper

Surprisingly, and in contrast to what happens with (1.6), the linear equation (1.7) mimicks the behaviour of the nonlinear one (1.1) in a better fashion than for the fractional Fisher-KPP equation. The result that we are going to prove is the following.

Theorem 1.2 *Consider any $\lambda \in \left(0, \dfrac{v_0}{\mu}\right)$. Let $x_\lambda(t)$ be the largest x such that $u(t,x) = \lambda$ or $u(t,-x) = \lambda$. Then, for all $\delta > 0$, there is $T_{\lambda,\delta} > 0$ such that, for all $t \geq T_{\lambda,\delta}$ we have*

$$\frac{e^{\frac{a}{1+2\alpha}t}}{t^{\frac{3}{2(1+2\alpha)}+\delta}} \leq x(t) \leq \frac{e^{\frac{a}{1+2\alpha}t}}{t^{\frac{3}{2(1+2\alpha)}-\delta}} \tag{1.8}$$

In fact, the upper bound is more precise, as we may choose $\delta = 0$ there. To improve the lower bound seems to us more challenging, and will be addressed in a future work.

The paper is organised as follows. In Section 2, we explain the strategy of the proof of Theorem 1.2 and discuss some perspectives that our work has opened. In Section 3 we address the underlying mechanism of Theorem 1.2, namely, the transients of the one-dimensional Fisher-KPP equation with Dirichlet boundary conditions, this is a result of independent interest for the Fisher-KPP equation. We then devote a short section to quantify how the exchanges between the road and the field are organised. The proof of Theorem 1.2 is then displayed in Section 5. In the whole paper, the computations will be greatly simplified when we take a function $g \geq 0$, smooth, convex, supported in $(\theta, 1]$ for some $\theta \in (0,1)$, with $g(1) = 1$. Threfore, the computations will sometimes be carried out with this type of nonlinearity in order to highlight the main ideas, before being extended to the general Fisher-KPP nonlinearity. Also, from now on we will assume, without loss of generality, that $a = 1$.

2 The underlying mechanism of Theorem 1.2, discussion

The starting point of this paper was the following numerical simulations, carried out in the PhD thesis of the first author [10]. The figure represents the evolution of $u(t, t^{-m}e^{\frac{t}{1+2\alpha}})$, with, from left to right: $m = 0$, $m = \dfrac{3}{2(1+2\alpha)}$ and $m = \dfrac{3}{1+2\alpha}$. The gradation of colours from blue to red represents the advance in time, blue standing for the earlier stages of the development. One clearly sees the stabilisation mechanism for the middle value of m, and this came to us as a surprise. However, this suggests the following idea: the $t^{-3/2}$ term being

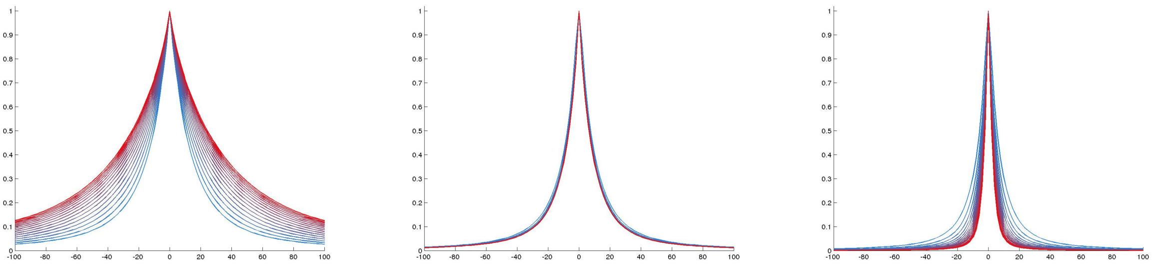

Figure 1: The different renormalisations

typical of the one-dimensional Dirichlet heat equation, we thought that it was interesting to understand this feature in a little more depth.

Assuming - which will turn out to be a good approximation - that $\partial_{xx} v$ is small, this suggests in fact that, for a fixed x, the function $v(t, x, y)$ behaves like a solution of the one dimensional Fisher-KPP equation

$$\begin{aligned} w_t - w_{yy} = & \quad f(v), \quad t > 0, \ y > 0 \\ w(t, 0) = & \quad 0, \end{aligned}$$

This is even more evident when one takes $f(v) = av - g(v)$, g vanishing on a small interval to the right of 0. The initial value (or, at least, the value of v at any small positive time) is small, dictated by the size of $u(1, x)$. The Dirichlet boundary condition is the most convenient one that allows to put below the solution $v(t, x, y)$ of (1.1) a barrier devised on the model of $w(t, y)$, with an initial datum suitably dictated by the behaviour of $u(t, x)$ (the solution of (1.1) on the line) at infinity. Of course, with this particular condition, the role of the line seems to be forgotten, such is not exactly the case, as long as we prove - as will be done in the course of this work - some easy lemmas that describe how the communication between the road and the field is organised.

Let us briefly discuss the optimality of our estimates. Of course, the corrections of the exponents in (1.8) by a small $\delta > 0$ shows that there is still a room for improvement. In particular, one could ask whether replacing the Dirichlet boundary condition by the exchange condition $-\partial_y v + \nu v$ in the 1D Fisher-KPP equation would lead to the optimal bounds. In fact, the best strategy would probably be to investigate the full one-dimensional problem with unknowns $(u(t), v(t, y))$

$$\left\{ \begin{aligned} v_t - v_{yy} = & \quad f(v), \quad t > 0, \ y > 0 \\ -v_y(t, 0) = & \quad \mu u(t) - \nu v(t, 0) \quad t > 0 \\ \dot{u}(t) = & \quad \nu v(t, 0) - \mu u(t). \end{aligned} \right.$$

We choose no to do it here, as it would involve, in our opinion, heavier computations with possibly no real further understanding of the mechanisms at work. So, we leave this task for a future contribution.

3 The transients for 1D Fisher-KPP propagation with small initial data

In this section we consider a function $g \geq 0$, smooth, convex, supported in $(\theta, 1]$, with $g(1) = 1$ (this last assumption is in fact unnecessary). Pick a small $\varepsilon > 0$ with $\varepsilon < \theta$. The goal of this section is to understand how much time it will take to the solution of the model Fisher-KPP equation

$$
(3.1)\qquad \left\{\begin{array}{rl} v_t - v_{yy} = & v - g(v) \quad (t > 0,\ y \geq 0) \\ v(0,y) = & \varepsilon \mathbf{1}_{[1/2,1]}(y) \\ v(t,0) = & 0 \end{array}\right.
$$

to reach the value θ at finite distance from $y = 0$. First, let us note that the value θ will eventually be reached at a finite distance (both with respect to t and ε) from $y = 0$. Indeed, a classical sub-solution argument (see for instance Berestycki-Hamel-Roques [6]) implies that v will converge to the unique nontrivial solution v_∞ of

$$
(3.2)\qquad \begin{array}{rl} -v''_\infty = & v'_\infty - g(v'_\infty) \quad (y > 0) \\ v'_\infty(0) = & 0, \end{array}
$$

which satisfies $v'_\infty(0) > 0$, hence uniformly bounded from below on every set of the form $[y_0, +\infty)$, $y_0 > 0$. On the other hand, as $\varepsilon \to 0$, the time that it will take to v to come close to v_∞ will grow infinitely, and our aim is to devise an upper bound that will be precise up to algebraic powers of ε.

Theorem 3.1 *Let v_ε be the solution to* (3.1)*, and $\varepsilon < \lambda < v_\infty(1)$. Define T_ε as the first time t such that*

$$
(3.3)\qquad v_\varepsilon(t,1) = \lambda.
$$

Then, for all $\delta > 0$, there is $Q_\delta > 0$, possibly blowing up as $\delta \to 0$, such that

$$
(3.4)\qquad \frac{e^{T_\varepsilon}}{T_\varepsilon^{\frac{3}{2}+\delta}} \leq \frac{Q_\delta}{\varepsilon}.
$$

It is worth saying a word on the scenario leading to (3.4), and the special structure of the nonlinearity $f(v) = v - g(v)$ will make it especially obvious: the region where the solution will first reach a nontrivial value is not close to 0, but at a large distance from 0. At this stage, one could think of invoking classical results on Fisher-KPP propagation for studying how much more time v_ε will take to be nontrivial near $y = 0$. This is not the correct intuition, because it would lead to a T_ε that would be largely over-estimated. The mechanism is in fact closer to that of nonlocal Fisher-KPP propagation [8], [9]. It is also not so far from what happens with the classical Fisher-KPP with slowly decreasing initial data, [12], [13].

Proof of Theorem 3.1. For small, or, even, finite t, (for instance $t \in [1,2]$) we have $v(t,y) < \theta$ as soon as $\varepsilon > 0$ is small enough. Let us make this assumption; as soon as $v_\varepsilon \leq \theta$ everywhere we have $g(v_\varepsilon) \equiv 0$ and, thus:

$$
(3.5)\qquad v_\varepsilon(t,y) = \varepsilon e^t \int_{\frac{1}{2}}^1 \frac{e^{-\frac{(y-y')^2}{4t}} - e^{-\frac{(y+y')^2}{4t}}}{\sqrt{4\pi t}} dy'.
$$

For $t \geq 1$ and $y' \in [\frac{1}{2}, 1]$ we have for all $y > 1$

$$
e^{-\frac{(y-y')^2}{4t}} - e^{-\frac{(y+y')^2}{4t}} \leq C\frac{yy'}{t} e^{-\frac{y^2}{5t}},
$$

$C > 0$ universal. So we have, for a possibly different $C > 0$:

$$
(3.6)\qquad v_\varepsilon(t,y) \leq C\varepsilon \frac{e^t}{t} . \frac{y}{\sqrt{t}} e^{-\frac{y^2}{5t}}.
$$

For all fixed $t \geq 2$, the maximum in y of the right handside of (3.6) is taken at

$$y = z_0\sqrt{t},$$

where z_0 is the maximum of $m(z) := ze^{-z^2/5}$. Call m_0 the (easily computable) maximum of m, a sufficient condition to have $v(t,y) \leq \theta$ everywhere is to have, from (3.6):

$$Cm_0\varepsilon\frac{e^{at}}{t} \leq \theta.$$

Define $T^1_\varepsilon \geq 2$ as

$$\frac{e^{T^1_\varepsilon}}{T^1_\varepsilon} = \frac{\theta}{m_0 C\varepsilon}. \tag{3.7}$$

So, we have easily proved that v_ε reaches a nontrivial value in a time of (roughly) the order of $\ln(\varepsilon^{-1})$, but this value is reached at $y \sim \left(\ln\varepsilon^{-1}\right)^{1/2}$. To study what happens at finite distance to $y = 0$, consider $L > 0$ large. There is $c_L > 0$, with, in the worst case scenario

$$\lim_{L\to+\infty} c_L = 0,$$

such that, in the limit $\varepsilon \to 0$, we have, from (3.5):

$$v_\varepsilon(T^1_\varepsilon, y) \geq \frac{\varepsilon c_L}{\sqrt{T^1_\varepsilon}}, \text{ for } 1 \leq y \leq L. \tag{3.8}$$

Let $e_L(y)$ be the first Dirichlet eigenfunction of $-\partial_{xx}$ on $(1, L)$, we normalise so that its maximum is 1. Thus we have

$$e_L(y) = \sin\left(\frac{\pi}{L-1}(y-1)\right),$$

with first eigenvalue

$$\lambda_1(L) = \frac{\pi^2}{(L-1)^2}.$$

Let $\underline{v}_{\varepsilon,L}(t,y)$ solve

$$\begin{aligned} (\partial_t - \partial_{xx} - 1)\underline{v}_{\varepsilon,L} &= 0 \quad (t > T^1_\varepsilon,\ y \in (1,L)) \\ \underline{v}_{\varepsilon,L}(t,1) = \underline{v}_{\varepsilon,L}(t,L) &= 0 \quad (t \geq T^1_\varepsilon) \\ \underline{v}_{\varepsilon,L}(T^1_\varepsilon, y) &= \frac{\varepsilon c_L}{\sqrt{T^1_\varepsilon}} e_L(y). \end{aligned} \tag{3.9}$$

On the one hand, we have

$$\underline{v}_{\varepsilon,L}(t,y) = \frac{\varepsilon c_L e^{(1-\lambda_1(L))(t-T^1_\varepsilon)}}{\sqrt{T^\varepsilon_1}} e_L(y). \tag{3.10}$$

On the other hand we have $\underline{v}_{\varepsilon,L}(t,y) \leq v_\varepsilon(t,y)$ for $t \geq T^1_\varepsilon$, as long as $\underline{v}_{\varepsilon,L}(t,y)$ is globally less than θ. This last condition is fulfilled as long as

$$t - T^1_\varepsilon \leq \frac{1}{2(1-\lambda_1(L))}\ln\left(\frac{\theta^2 T^\varepsilon_1}{c_L}\right), \tag{3.11}$$

and the maximum is exactly θ at equality. Note that it is attained far away from the origin, that is, at $y_L = \dfrac{L-1}{2}+1$. However, the situation is not as bad as before, because we now have

$$e_L(y) \sim \frac{\pi}{L-1}(y-1), \text{ for } y-1 << L.$$

This is certainly a small quantity, but it is independent of ε. Let us set

$$T_\varepsilon^2 = T_\varepsilon^1 + \frac{1}{2(1-\lambda_1(L))}\ln(\theta^2 T_\varepsilon^1). \tag{3.12}$$

We have now, from (3.11):

$$v_\varepsilon(T_\varepsilon^2, y) \geq \frac{\theta\pi}{L-1}(y-1), \quad \text{for } y << L.$$

From now on, once again by a classical sub-solution argument, there is $\tilde{T}_L > 0$ (independent of ε), blowing up as $L \to +\infty$, such that

$$v_\varepsilon(T_\varepsilon^1 + T_\varepsilon^2 + \tilde{T}_L, 2) = \lambda.$$

This is not exactly (3.3), but we are now quite close to it: from the Harnack inequality we have

$$v_\varepsilon(T_\varepsilon^1 + T_\varepsilon^2 + \tilde{T}_L + 1, 1) \geq q\lambda,$$

for some universal $q > 0$, and the same sub-solution argument yields the (3.3), at a time of the form $T_\varepsilon^1 + T_\varepsilon^2 + \tilde{T}_L + \tilde{T}'_L$, the new constant $\tilde{T}'_L$ being ε-indeendent. Set $T_\varepsilon = T_\varepsilon^1 + T_\varepsilon^2 + \tilde{T}_L + \tilde{T}'_L$; it now suffices to notice that (3.11) implies that

$$T_\varepsilon = T_\varepsilon^1 + \frac{1}{2}\left(1 + O\left(\frac{1}{L^2}\right)\right)\ln\left(\frac{\theta^2 T_1^\varepsilon}{c_L}\right) + \tilde{T}_L + \tilde{T}'_L,$$

which, combined to (3.7), implies

$$\frac{1}{\varepsilon} = \frac{\sqrt{c_L}e^{T_\varepsilon - \tilde{T}_L}}{\theta^{1+O(1/L^2)}T_\varepsilon^{3/2+O(1/L^2)}(1 + T_\varepsilon^{-1}\ln T_\varepsilon - T_\varepsilon^{-1}(\tilde{T}_L + \tilde{T}'_L))}.$$

Let us denote by μ_L a common bound for the two $O(\dfrac{1}{L^2})$ appearing in the above expression. We now pick a small δ and choose $L > 0$, denoted by L_δ, such that

$$\mu_{L_\delta} = \delta, \quad Q_\delta = \frac{e^{\tilde{T}_{L_\delta}}}{\theta^{1+\delta\mu_{L_\delta}}},$$

which is exactly (3.4). □

Remark 3.2 *Using $f(v) \leq v$, and the solution $\overline{v}(t,x)$ of (3.1) with $f(v) = v - g(v)$ replaced by v, we obtain the (sharper) converse inequality*

$$\frac{e^{T_\varepsilon}}{T_\varepsilon^{3/2}} \geq \frac{C}{\varepsilon},$$

thus an asymptotic expansion of T_ε:

$$T_\varepsilon = \ln\frac{1}{\varepsilon} + \frac{3}{2}\ln\ln\left(\frac{1}{\varepsilon}\right) + o_{\varepsilon\to 0}\left(\ln\ln\left(\frac{1}{\varepsilon}\right)\right).$$

4 Communications between the road and the field

The goal of this section is to prove that, if the solution on the road is of a certain order at some time and on a certain interval, then the solution in the field will be of the same order, possibly in a square with a smaller size and a little later in time. We also want to prove that the converse holds: if the solution is of some order at some time and some point in the field, this is transmitted to the road. Such results can be seen as weak versions of the Harnack inequality (a bound at a certain time and point would entail the same bound in a whole neighbourhood, possibly at later times) but this will be sufficient for our purpose. See [5] for estimates that are more in the spirit of the Harnack inequality.

Lemma 4.1 *Consider $t_0 \geq 1$, $x_0 \in \mathbb{R}$, $L \geq 1$ and $\varepsilon > 0$ (not necessarily small) such that*

$$u(t_0, x) \geq \varepsilon \text{ on } [x_0 - L, x_0 + L].$$

There is $c_L > 0$ (universal otherwise) such that

$$v(t, x, y) \geq c_L\varepsilon \text{ on } [t_0 + 1, t_0 + 2] \times [x_0 - L, x_0 + L] \times [0, 1].$$

Proof. Without loss of generality, we may translate time and space so as to have $t_0 = 1$, $x_0 = 0$. Notice then that, because $v(t, x, 0) \geq 0$ we have

$$u_t + (-\partial_{xx})^\alpha u + \mu u \geq 0, \quad t \geq 1, \ x \in \mathbb{R}.$$

Recall that the fundamental solution of the fractional heat equation of order α, that we call $G_\alpha(t, x)$, is uniformly bounded away from 0 on $[1, 2] \times [-L - 1, L + 1]$. Thus

$$u(t, x) \geq e^{-\mu(t-1)} \int_{|y|\leq 1} G(t, x - x')u(1, x')dx', \text{ for } t > 1 \text{ and } |x| \leq L.$$

This implies

$$u(t, x) \geq c_L\varepsilon, \text{ for } t \in [1, 2] \text{ and } |x| \leq L.$$

Then, recall that $\dfrac{f(v)}{v}$ is bounded from below - say, by $-\Lambda > 0$, and that

$$\tilde{v}(t, x, y) = e^{\Lambda t}v(t, x, y)$$

is a super-solution to the heat equation, while the boundary condition reads

$$\partial_y\tilde{v} + \nu\tilde{v} \geq c_L\varepsilon,$$

for a possibly different c_L. Thus we have $\tilde{v} \geq \underline{v}$, where

$$\begin{aligned} (\partial_t - \Delta)\underline{v} = &\ 0 \quad (t \in (1, 2], x \in \mathbb{R}, y > 0) \\ \partial_y\underline{v} + \nu\underline{v} = &\ c_L\varepsilon\mathbf{1}_{[-2L,2L]}(x), \quad y = 0 \\ \underline{v}(1, x, y) = &\ 0. \end{aligned}$$

By parabolic regularity we have, for some universal $C > 0$:

$$|\nabla\underline{v}(t, x, y)| \leq C\varepsilon, \quad t \in [1, 2], |x| \leq 3L/2, 0 \leq y \leq 1.$$

Thus, there is $y_0 \in (0, 1)$, independent of x_0 - and thus of ε such that

$$\underline{v}(t, x, y_0) \geq \frac{c_L}{2}\varepsilon, \quad 1/2 \leq t \leq 1, -x_0 - 3L/2 \leq x \leq x_0 + 3L/2.$$

And the classical parabolic Harnack inequality implies the lemma. Note that, due to [3], Section 3, one may push it to the boundary thanks to the Robin condition, at the expense of considering $\underline{\tilde{v}}(t, x, y) := e^{\nu y}\underline{v}(t, x, y)$. $\square$

Lemma 4.2 *Consider $t_0 \geq 1$, $x_0 \in \mathbb{R}$, and $\varepsilon > 0$ such that*

$$v(t_0, x_0, 1) \geq \varepsilon.$$

For all $L > 0$, there is $c_L > 0$ (universal otherwise) such that

$$u(t,x), v(t,x,y) \geq c_L\varepsilon \text{ on } [t_0+1, t_0+2] \times [x_0 - 2L, x_0 + 2L] \times [0,1].$$

Proof. Once again there is no loss in generality by assuming $t_0 = 1$, $x_0 = 0$. The classical Harnack inequality applied to v entails a lower bound of the order ε at least for $v(t,x,1)$ for $t \in [1,2]$ and $-2L \leq x \leq 2L$. Fix now $L > 0$, for all $\delta \in (0,1)$ there is $c_\delta > 0$ (we omit the dependence in L) such that

$$v(t,x,y) \geq c_\delta \varepsilon, \quad (t,x,y) \in [1,2] \times [-L, L] \times [\delta, 1].$$

Assume the existence of $x_1 \in [-L, L]$ and $t_1 \in [1,2]$ such that $v(t_1, x_1, 0)$ is much smaller than its order of magnitude in the field. This is equivalent to assuming the existence of a sequence of solutions (u_n, v_n) of (1.1), such that the following situation holds:

- for $t \in [1,2]$, $x \in [x_1 - L/2, x_1 + L/2]$ and $y = -1$, then $v_n(t,x,y) \geq c\varepsilon$ (dependence on L omitted),
- there is $t_1 \in [1,2]$ such that $v_n(t_1, x_1, 0) \leq 1/n$.

Remember that v_n is uniformly bounded from above. So, by parabolic regularity, (a subsequence of) the sequence $(u_n, v_n)_n$ converges, on $[1,2] \times [x_1 - L/2, x_1 + L/2] \times [-1, 0]$ to a limiting function (u_∞, v_∞) which is not identically equal to 0 due to the first assumption on v_n. The Hopf Lemma implies $\partial_y v_\infty(t,x,0) < 0$, thus the exchange condition yields

$$\mu u_\infty(t_1, x_1) - \nu v_\infty(t_1, x_1, 0) < 0.$$

This contradicts the fact that $v_\infty(t_1, x_1, 0) = 0$. Now, we have $u(t,x) \geq \underline{u}(t,x)$, with

$$\begin{cases} \underline{u}_t + (-\partial_{xx})^\alpha \underline{u} + \mu \underline{u} = & \nu c\varepsilon \mathbf{1}_{[-L,L]}(x), \quad t > 0,\ x \in \mathbb{R} \\ \underline{u}(0,x) = & 0. \end{cases}$$

Thus, for $t \in [1,2]$ we have

$$u(t,x) \geq c\varepsilon e^{-\mu t} \int_0^1 \int_{-L}^{L} G_\alpha(t-s, x-x') dx' ds \geq c'\varepsilon$$

for a constant c' that only depends on L. □

5 Bounds to the full model

The starting point of the analysis is the (computationally non trivial) result, whose main line of the proof are given in [2], and proved in full length in [10].

Theorem 5.1 ([10], Chapter 4) *Let $(\overline{u}(t,x), \overline{v}(t,x,y))$ solve*

$$(5.1) \qquad \begin{cases} \partial_t \overline{v} - \Delta \overline{v} &= \overline{v}, & t > 0, x \in \mathbb{R}, y > 0 \\ \partial_t \overline{u} + (-\partial_{xx})^\alpha \overline{u} &= -\mu \overline{u} + \overline{v} - k\overline{u}, & t > 0, x \in \mathbb{R} \\ \partial_y \overline{v} &= \mu \overline{u} - v, & x \in t > 0, \mathbb{R}, y = 0, \end{cases}$$

with $(\overline{u}(0,x), \overline{v}(0,x,y)) = (u_0(x), 0)$ and $u_0 \not\equiv 0$ nonnegative and compactly supported. There exists a function $R(t,x)$ and constants $\delta > 0$, $C > 0$ such that

1. *we have, for large x:*

$$\left|\overline{u}(t,x) - \frac{8\alpha\mu\sin(\alpha\pi)\Gamma(2\alpha)\Gamma(3/2)}{\pi}\frac{e^t}{t^{3/2}|x|^{1+2\alpha}}\right| \leq R(t,x), \tag{5.2}$$

2. *and the function $R(t,x)$ is estimated as*

$$0 \leq R(t,x) \leq C\left(e^{-\delta t} + \frac{e^t}{|x|^{\min(1+4\alpha,3)}} + \frac{e^t}{|x|^{1+2\alpha}t^{\frac{5}{2}}}\right).$$

Note that this result readily entails the upper bound in Theorem 1.2, so that it suffices to prove the lower bound. We first prove it when g is compactly supported in $(0,1]$, then indicate the necessary changes for a general g.

5.1 Proof of Theorem 1.2 when g is compactly supported in $(0,1]$

Let us pick $\lambda \in (\varepsilon, 1/\mu)$ and $x_0 > 0$ (the same argument would apply for $x_0 < 0$) very large, we set

$$u(1,x_0) := \varepsilon.$$

From Theorem 5.1, applied at time $t=1$, the function $u(1,x)$ is of the order ε (and also of the order $1/x_0^{1+2\alpha}$) on any interval around x_0 whose length will not exceed, say, $\sqrt{x_0}$. Thus, from Lemma 4.1, we have

$$v(1,x,y) \geq c\varepsilon \text{ for } (x,y) \in [x_0 - \sqrt{x}_0, x_0 + \sqrt{x}_0]\times[0,1]. \tag{5.3}$$

We ask how much time it will take for u to reach the value λ at x_0. From (5.2) we have, as soon as $\varepsilon< \theta$ is small enough - that is, if x_0 is large enough - and for all $L>0$:

$$u(1/2,x) \leq c_L\varepsilon \quad \text{for all } x \in [-x_0 - 2L, x_0 + 2L].$$

Then, translate the point $(x_0,0)$ to the origin, and let this time $\underline{v}(t,x,y)$ solve the two-dimensional Fisher-KPP equation with Dirichlet conditions on the road:

$$\begin{cases} (\partial_t - \Delta - 1)v + g(v) = & 0, \quad t \geq 1, x \in \mathbb{R}, y>0 \\ \underline{v}(t,x,0) = & 0, \quad t \geq 1,\ x \in \mathbb{R} \\ \underline{v}(1,x,y) = & c\varepsilon\mathbf{1}_{[-\sqrt{x}_0,\sqrt{x}_0]}(x)\mathbf{1}_{[0,1]}(y). \end{cases} \tag{5.4}$$

As long as $\underline{v} \leq \theta$ everywhere, it solves the linear equation, that is, with $g(v)=0$. In such a case it consists of the product of two solutions of the heat equation times the exponential:

$$\underline{v}(t,x,y) = \frac{\underline{v}^{1D}(t,y)}{\sqrt{\pi}}\left(\int_{[-\sqrt{x}_0,\sqrt{x}_0]} e^{-(x-x')^2/4t}dx'\right), \tag{5.5}$$

where $\underline{v}^{1D}(t,y)$ is the solution of the Dirichlet heat equation in y, and is exactly given by (3.5). The function v^{1D} reaches θ in a time T_ε^1 given by equation (3.7), in other words

$$T_\varepsilon^1 = O\left(\ln\left(\frac{1}{\varepsilon}\right)\right).$$

This time is too short for the solution of the heat equation in x to decay significantly on, say, the interval $[-L, L]$ with L large but finite (any size L which is an $o(\sqrt{x_0})$ will do). We have indeed, for $|x| \leq L$ and $t \leq T_\varepsilon^1$:

$$\begin{aligned} \frac{1}{\sqrt{t}} \int_{-\sqrt{x_0}}^{\sqrt{x_0}} e^{-\frac{(x-x')^2}{4t}} dx' \geq & \; C \int_{-\frac{(\sqrt{x_0}+L)^2}{2\sqrt{t}}}^{\frac{(\sqrt{x_0}-L)^2}{2\sqrt{t}}} e^{-\xi^2} d\xi \\ \sim & \; C, \end{aligned}$$

simply because $x_0 \sim \varepsilon^{-1/(1+2\alpha)}$ and T_ε^1 is of the order $\ln\left(\frac{1}{\varepsilon}\right)$. Thus, the function $y \mapsto \underline{v}(T_\varepsilon^1, 0, y)$ reaches maximum of the order θ, at a point of the order $\sqrt{T_\varepsilon^1}$, while it is of the order $\frac{1}{\sqrt{T_\varepsilon^1}}$ for $y \sim 1$. Then, we run (5.4) again, from T_ε^1, with

$$\underline{v}(T_\varepsilon^1, x, y) = \mathbf{1}_{[-L',L']}(x)\mathbf{1}_{[1,L']}\frac{e_{L'}(x,y)}{\sqrt{T_\varepsilon^1}},$$

the function $e_{L'}(x, y)$ being the first eigenfunction of the Dirichlet Laplacian in the rectangle $(-L', L') \times (1, L')$, and L' a large number (in fact we could take $L' = L$). We have

$$e_{L'}(x,y) = \sin\left(\frac{\pi}{2L'}x\right)\sin\left(\frac{\pi}{L'-1}(y-1)\right),$$

the first eigenvalue being still an $O\left(\frac{1}{L'^2}\right)$. And so, for a time T_ε given by (3.12), there is $q > 0$ independent of ε such that $\underline{v}(T_\varepsilon, 0, 1) \geq q$.

To conclude the proof, it remains to prove the existence of $q' > 0$ universal such that $u(T_\varepsilon, 0) \geq q'$. This is, however, easy: because x_0 is arbitrary, we have

$$v(T_\varepsilon, x, 1) \geq q \quad \text{for } x_0 - L \leq x \leq x_0 + L,$$

and Lemma 4.2 implies the desired bound for u.

5.2 The lower bound for a general concave nonlinearity

We write again

$$f(v) = v - g(v), \quad g(0) = g'(0) = 0,\; g(1) = 1,\; g'' > 0 \text{ on [0,1]}.$$

Thus we have, for all $v \in [0, 1]$: $g(v) = O(v^2)$, and this is what we will really use. In view of what we have already done when g vanishes in a vicinity of 0, what we really have to do is study the function $v(t, x, y)$ solving

$$(5.6) \qquad \begin{aligned} v_t - \Delta v - v = &\; g(v) \quad (t > 0, x > 0, y>0) \\ v(t,x,0) = &\; 0 \\ v(0,x,y) = &\; c\varepsilon\mathbf{1}_{[-\sqrt{x_0},\sqrt{x_0}]}(x)\mathbf{1}[-1,0](y), \end{aligned}$$

with $\varepsilon = \dfrac{1}{1+x_0^{1+2\alpha}}$. In view of the proof of Theorem 3.1, and Section 4, the main property that we have to prove is the following.

Lemma 5.2 *Let T^1_ε be given by* (3.7)*. There is $q>0$ universal such that*

$$v(T^1_\varepsilon,0,1)\geq\frac{q}{\sqrt{T^\varepsilon_1}}. \tag{5.7}$$

It then suffices, as in the preceding section, to put v above the solution $\underline{v}_{\delta,L,L'}$ of

$$\begin{cases}\Big(\partial_t-\Delta-(1-\delta)\Big)v_{\delta,L,L'}= & 0\quad (t>T^1_\varepsilon,-L<x<L,1<y<L')\\ v_{\delta,L,L'}(0,x,y)= & \dfrac{q}{\sqrt{T^1_\varepsilon}}e_{L,L'}(x,y),\end{cases}$$

with δ small, and $e_{L,L'}(y)$ the first eigenfunction of the Dirichlet Laplacian in $(-L,L)\times(1,L')$. At time

$$T^2_\varepsilon=T^1_\varepsilon+\left(\frac12+O(\delta)+O\left(\frac{1}{L^2}\right)+O\left(\frac{1}{L'^2}\right)\right)\ln T^1_\varepsilon,$$

we have $\underline{v}_{\delta,L,L'}(t,x,y)\geq C\delta$ on $(-L,L)\times(1,L')$, and one finishes the proof of Theorem 1.2 by Lemmas 4.1 and 4.2.

Let us therefore present the

Proof of Lemma 5.2. Call $X=(x,y)\in\mathbb{R}\times\mathbb{R}_+$ a generic point of the upper half-plane $\mathbb{R}\times\mathbb{R}_-$. Let $G(t,X)$ be the fundamental solution of the Dirichlet heat equation in the upper half-plane, we have

$$G(t,X,X')=G_0(t,y,y')G_1(t,x),$$

the function G_0 being the Dirichlet fundamental solution (see (3.5)) whereas G_1 is the standard Gaussian $G_1(t,x)=\dfrac{e^{-x^2/4t}}{\sqrt{4\pi t}}$. The Duhamel formula yields

$$v(t,X)=\varepsilon e^t\int_{\mathbb{R}^2_+}G(t,X,X')v(0,X')dX'-\int_0^t\int_{\mathbb{R}^2_+}e^{t-s}G(t-s,X,X')g(v(s,X'))dsdX'.$$

We call $e^{t-s}D(t-s,s,X,X')$ the integrand of the second integral in the right handside of the above inequality; we have

$$v(s,X)\leq e^s\varepsilon\int_{\mathbb{R}^2_+}G(s,X,X')v(0,X')dX'\leq C\varepsilon^2e^s\int_{-1}^0G_0(s,y,y')dy'.$$

Because $g(v)=O(v^2)$ we get, taking (5.5) and (3.6) into account:

$$D(t-s,s,X,X')\leq C\varepsilon^4G(t-s,X,X')\frac{y'^2e^{-2(y-y')^2/5s}}{s(1+s^2)}e^{2s},$$

$C>0$ universal. Note that we have only estimated the integral for $s\geq1$, the integral for $s\leq1$ being negligible. Integrating in x' and specialising at $x=0$ we get

$$\int_0^t\int_{\mathbb{R}^2_+}e^{t-s}G(t-s,X,X')g(v(s,X'))dsdX'\leq C\varepsilon^4\int_0^t\int_{\mathbb{R}_+}e^{t+s}E(t-s,s,y')dsdy',$$

where

$$E(t-s,s,y')\lesssim\frac{|1+y'|^3e^{-(1+y')^2/4(t-s)}e^{-y'^2/3s}}{(1+|t-s|)(t-s)^{1/2}s(1+s^2)}. \tag{5.8}$$

We are going to prove the inequality

$$\int_0^{T_1^\varepsilon}\int_{\mathbb{R}\times\mathbb{R}_+} e^{T_1^\varepsilon-s}D(T_1^\varepsilon-s,s,(0,1),X')dsdX' \leq C(\sqrt{\varepsilon}+\frac{1}{\ln\frac{1}{\varepsilon}})\frac{1}{\sqrt{T_1^\varepsilon}}, \tag{5.9}$$

$C>0$ universal. There is nothing special about the point $X=(0,1)$ the inequality would be valid for all neighbouring points, at the expense of increasing C. Recall the inequality (see Sections 2 and 3) for v:

$$v^{1D}(T_1^\varepsilon,1) \geq \frac{q}{\sqrt{T_1^\varepsilon}}.$$

This, combined to (5.9), will imply the lemma. As we will set, eventually, $t=T_1^\varepsilon$, we will always assume

$$t = O(\ln\frac{1}{\varepsilon}).$$

we cut the time interval $(0,t)$ into two.

1. $s\in(0,\kappa t)$**,** $\kappa>0$ **small.** We will use the factor $e^{-y'^2/3s}$ to make the integral convergent, and make the change of variables $y'\mapsto z'=y'/\sqrt{s}$.Thus we have

$$E(t-s,s,y') \lesssim \frac{1+|z'|^3e^{-z'^2/3}}{(1+(t-s))\sqrt{t-s}s},$$

using $e^{t+s}\leq e^{(1+\kappa)t}$ and the definition of T_1^ε we end up with

$$\varepsilon^2\int_0^{\kappa t}\int_{\mathbb{R}\times\mathbb{R}_+} e^{t-s}D(t-s,s,(0,1),X')dsdX' \lesssim \frac{\varepsilon^4te^{(1+\kappa)t}}{t^{3/2}}.$$

And so,

$$\varepsilon^2\int_0^{(1+\kappa)T_1^\varepsilon}\int_{\mathbb{R}\times\mathbb{R}_+} e^{T_1^\varepsilon}D(T_1^\varepsilon,s,(0,-1),X')dsdX' \lesssim \frac{\varepsilon^{1-\kappa}}{\sqrt{T_\varepsilon^1}}. \tag{5.10}$$

2. The range $s\geq\kappa t$**.** This time we rely on the part $e^{-\frac{(1+y')^2}{4(t-s)}}$ to make the spatial integral convergent, and we will have to be a little careful about the e^{t+s} factor. As for the powers y'^2, we dominate them by $1+|1+y'|^2$. We make the change of variables $y'\mapsto\dfrac{z'}{\sqrt{t-s}}$ and we have

$$\begin{aligned} e^{t-s}D(t,s,(0,-1),X') &\lesssim \varepsilon^2G_1(x')\tfrac{e^{t+s}e^{-\frac{z'^2}{4}}\sqrt{t-s}}{(1+|z'|^3)s^3} \\ &\lesssim \varepsilon^2G_1(x')\tfrac{e^{t+s}e^{-\frac{z'^2}{4}}}{t^{5/2}}. \end{aligned}$$

Integrating on $(0,t)\times\mathbb{R}\times\mathbb{R}_+$ yields

$$\varepsilon^2\int_0^t\int_{\mathbb{R}\times\mathbb{R}_+} e^{t-s}D(t,s,(0,-1),X')dsdX' \leq \frac{Ce^{2t}}{(T_1^\varepsilon)^{3/2}} \leq \frac{C}{\ln\frac{1}{\varepsilon}}.\frac{1}{\sqrt{T_1^\varepsilon}}.$$

Putting everything together yields (5.9), hence the lemma. □

Acknowledgement. The research leading to these results has received funding from the ERC under the European Union's Seventh Frame work Programme (FP/2007-2013) / ERC Grant Agreement 321186 - ReaDi.

References

[1] D.G. Aronson, H.F. Weinberger, *Multidimensional nonlinear diffusions arising in population genetics*, Adv. Math. **30** (1978), 33–76.

[2] H. Berestycki, A.-C. Coulon, J.-M. Roquejoffre, L. Rossi, *The effect of a line with non-local diffusion on Fisher-KPP propagation*, Math. Models Meth. Appl. Sci. **25** (2015), 2519–2562.

[3] H. Berestycki, L.A. Caffarelli, L. Nirenberg, *Uniform estimates for the regularization of free boundary problems,* Analysis and Partial Differential Equations, 567–619, Lecture Notes in Pure and Applied Mathematics, **122**, Dekker, New York, 1990.

[4] H. Berestycki, A.-C. Coulon, J.-M. Roquejoffre, L. Rossi, *Speed-up of reaction fronts by a line of fast diffusion,* Séminaire Laurent Schwartz, Equations aux Dérivés Partielles et Applications, 2013-2014, exposé XIX, Ed. Ecole Polytechnique, Palaiseau, 2014.

[5] H. Berestycki, R. Ducasse, L. Rossi, *Generalized principal eigenvalues for heterogeneous road-field systems*, Comm. Contemp. Math., to appear.

[6] H. Berestycki, F. Hamel, L. Roques, *Analysis of the periodically fragmented environment model : I - Species persistence,* J. Math. Biol. **51** (2005), 75–113.

[7] H. Berestycki, J.-M. Roquejoffre, L. Rossi, *The influence of a line with fast diffusion on Fisher-KPP propagation.* J. Math. Biol. **66** No. 4-5 (2013), 743–766.

[8] X. Cabre, J.-M. Roquejoffre, *The influence of fractional diffusion on front propagation in Fisher-KPP equations*, Comm. Math. Phys., **320** (2013), 679–722.

[9] X. Cabre, A.-C. Coulon, J.-M. Roquejoffre, *Propagation in Fisher-KPP type equations with fractional laplacian in periodic media*, C.R. Acad. Sci. Paris, **350** (2012), 885–890.

[10] A.-C. Coulon Chalmin, *Fast propagation in reaction-diffusion equations with fractional diffusion*, PhD thesis, 2014. Online manuscript: thesesups.ups-tlse.fr/2427/

[11] R.A. Fisher, *The advance of advantageous genes*, Ann. Eugenics **7** (1937), 335–369.

[12] C. Henderson, *Propagation of solutions to the Fisher-KPP equation with slowly decaying initial data*, Nonlinearity, **29** (2016), 3215–3240.

[13] F. Hamel, L. Roques, *Fast propagation for KPP equations with slowly decaying initial conditions,* J. Differential Equations **249** (2010), 1726–1745.

[14] A.N. Kolmogorov, I.G. Petrovskii, and N.S. Piskunov, *Etude de l'équation de diffusion avec accroissement de la quantité de matière, et son application à un problème biologique.* Bjul. Moskowskogo Gos. Univ., **17** (1937), pp. 1–26.

[15] V. Kolokoltsov, *Symmetric stable laws and stable-like jump-diffusions.* Proc. London Math. Soc. **80** (2000), 725–768.

[16] J.-M. Roquejoffre, A. Tarfulea, *Gradient estimates and symmetrization for Fisher-KPP front propagation with fractional diffusion,* J. Maths Pures Appl., **108** (2017) 399–424.