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Abstract

In the Prediction Error Identification framework, it is essential that the experiment yields informative data with respect to the chosen model structure to get a consistent estimate. In this work, we focus on the data informativity property for the identification of Multi-Inputs Multi-Outputs system in closed-loop and we derive conditions to verify if a given external excitation combined with the feedback introduced by the controller yields informative data with respect to the model structure. This study covers the case of the classical model structures used in prediction-error identification and the classical types of external excitation vectors, i.e., vectors whose elements are either multisine or filtered white noises.

Key words: System Identification; Closed-loop Identification; MIMO Systems; Prediction Error Methods; Consistency

1 Introduction

For an identification within the Prediction Error framework, it is crucial to choose an excitation that yields informative data. Indeed, the prediction error estimate is then guaranteed to be consistent (provided the chosen model structure is globally identifiable at the true parameter vector) [10,14]. In this work, we study the data informativity with respect to (w.r.t.) Multi-Input Multi-Output (MIMO) model structures for the identification of MIMO systems in the closed-loop configuration (direct closed-loop identification).

Data informativity is obtained when the input excitation is sufficiently rich to guarantee that the prediction error is distinctive for different models in the considered model structure. For Single-Input Single-Output (SISO) systems, this property has been largely studied. In this case, the data are informative if the input signal is sufficiently rich of an order that depends on the type and the complexity of the considered model structure. More precisely, an input signal is sufficiently rich of an order η if and only if its input power spectrum has a non-zero amplitude in η different frequencies in the interval $]-\pi,\pi]$ (see e.g. [10]). Due to renewed interest in optimal experiment design (see e.g. [3,9]), where the covariance matrix of the identified model is involved, there has been a lot of works to connect the positive definiteness of the covariance matrix to the data informativity [1,8]. In addition, necessary and sufficient conditions have been derived for the data informativity in both the open-loop and closed-loop case in [7,8]. In particular, these papers derive the minimal order of signal richness that the excitation signal must have to ensure data informativity and this is done for all classical model structures (BJ, OE, ARX, ARMAX, FIR). In the closed-loop case, this minimal order is related to the complexity of the controller present in the loop during the identification. In [7,8], it is also shown that, if the controller is sufficiently complex, the data can be informative even if the excitation signal is equal to zero (costless identification).

While the data informativity seems to be a grown-up research area in the SISO case, this cannot be said for the MIMO case. In [12], and more recently in [2], attention has been given to determine the minimal order that the controller must have to ensure that informative data are obtained when the external excitation is zero. Two conditions have been derived, one being sufficient and the other one necessary. In the majority of the cases, though, a (nonzero) external excitation will be required to yield informative data. Up to our knowledge, in the MIMO case, there is (almost) no result about the minimal richness this external excitation must have to yield

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informative data and about how this minimal signal richness relates to the complexity of the controller. Perhaps, the only result in that matter is given in [2]. In [2], it is indeed said that an external excitation signal $r(t)$ with a strictly positive definite power spectrum matrix $\Phi_r(\omega)$ at all ω always yields informative data for direct closed-loop identification. This condition is of course only sufficient and is moreover very restrictive. As an example, a multisine excitation will never respect this condition.

In this paper, we will therefore derive a condition allowing to verify whether, for a given MIMO controller, an arbitrary external excitation $r(t)$ yields informative data for the direct closed-loop identification of a given MIMO system in a full-order model structure. We will do that for the classical model structures used in the Prediction Error framework (FIR, ARX, ARMAX, OE, BJ) and for both multisine external excitations and filtered white noise external excitations.

As we will see in this paper, data informativity will be guaranteed in the MIMO closed-loop case if, for each channel/output, a certain matrix is full row rank. This matrix depends on the model structure complexity, on the controller coefficients and on the external excitation parametrization (i.e., amplitude, phase-shift and frequencies for multisine and filter coefficients for filtered white noise) . We will also observe that this matrix clearly separates the contributions of the controller and of the external excitation to the informativity of the collected input-output data.

This paper builds upon our previous contributions where we consider the data informativity problem for the open-loop identification of MIMO systems [4–6].

2 Notations

For a complex-valued matrix A, A^T denotes its transpose and A^* its conjugate transpose. A positive semi-definite (resp. definite) matrix A is denoted $A \succeq 0$ (resp. $A \succ 0$). We will denote A_{ik} the (i,k)-entry of the matrix A, $A_{i:}$ the i-th row of A and $A_{:k}$ the k-th column of A. The identity matrix of size $n \times n$ is denoted $\mathbf{I}_n$ and $\mathbf{0}_{n\times p}$ is the $n \times p$ matrix full of zeros. For the sake of simplicity, we will often drop the index $n \times p$ and just write $\mathbf{0}$. The notation $\text{diag}(a_1,\cdots,a_n)$ refers to the $n\times n$ diagonal matrix whose elements in its diagonal are the scalars $a_1,\cdots,a_n$. For a vector $x \in \mathbb{R}^n$, the notation $||x||$ refers to the Euclidean norm, i.e. $||x|| = \sqrt{x_1^2+\cdots+x_n^2}$. For two integers $n \leq p$, the set $[\![n,p]\!]$ is the set of consecutive integers between n and p.

For quasi-stationary signals $x(t)$ [10], we define the operator $\bar{E}[x(t)] = \lim_{N\to+\infty} \frac{1}{N}\sum_{t=1}^{N} E[x(t)]$ where E is the expectation operator.

For discrete-time model, z is the forward-shift operator. The degree of a polynomial $P(X)$ is denoted $\deg(P(X))$. When $X = z^{-1}$, we say that ρ is the delay of $P(z^{-1})$ when the first non-zero coefficient is linked to $z^{-\rho}$, i.e. $P(z^{-1}) = p_\rho z^{-\rho} + p_{\rho+1} z^{-(\rho+1)} + \cdots + p_n z^{-n}$ with $p_\rho \neq 0$.

3 Prediction Error Framework in closed-loop

Consider a MIMO system $\mathcal{S}$ with an input vector $u \in \mathbb{R}^{n_u}$ and an output vector $y \in \mathbb{R}^{n_y}$, described by

$$\mathcal{S}: \; y(t) = G_0(z)u(t) + H_0(z)e(t) \tag{1}$$

where $G_0(z)$ is a stable matrix of transfer functions of dimension $n_y \times n_u$, $H_0(z)$ a stable, inversely stable and monic[1] matrix of transfer functions of dimension $n_y \times n_y$ and $e \in \mathbb{R}^{n_y}$ is a vector made up of zero-mean white noise signals such that $\bar{E}\left[e(t)e^T(t)\right] = \Sigma_0 \succ 0$. We will make the following assumption for the sake of simplicity.

Assumption 1 *Assume that $H_0(z)$ is a diagonal transfer function matrix, i.e., $H_0(z) = diag(H_{0,1}(z),\cdots,H_{0,n_y}(z))$ where each scalar transfer function $H_{0,i}(z)$ $(i = 1,\cdots,n_y)$ is stable, inversely stable and monic.*

As shown in Fig. 1, the system $\mathcal{S}$ is under feedback control with a stabilizing controller described by a matrix of (rational) transfer functions $K(z)$ of dimension $n_u \times n_y$. The reference signal is set to $\mathbf{0}$. For identification purpose, a quasi-stationary external excitation $r \in \mathbb{R}^{n_u}$ can be added to the control effort such that the input u is given by

$$u(t) = -K(z)y(t) + r(t) \tag{2}$$

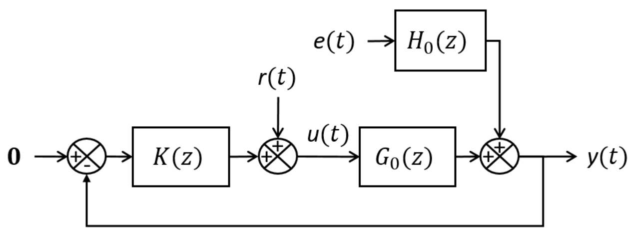

Fig. 1. Closed-loop configuration.

We will assume that the excitation vector r is uncorrelated with the vector of white noises e and that there is no algebraic loop in the closed loop made up of $K(z)$ and $G_0(z)$, i.e., there is at least a delay in all entries of the matrix $K(z)G_0(z)$. In this work, we focus on developing conditions to get a consistent estimate of $(G_0(z), H_0(z))$ when considering the direct closed-loop identification approach, i.e., by using the data $x(t) = (y^T(t), u^T(t))^T$ for the identification.

We will consider two types of quasi-stationary external excitation r. The first type is a multisine where each entry

[1] This means that H_0 and H_0^{-1} are stable and $H_0(z=\infty) = \mathbf{I}_{n_y}$.

r_k of r ($k=1,\cdots,n_u$) is a multisine made up of sinusoids at s different frequencies ω_l ($l=1,\cdots,s$), i.e.,

$$r_k(t)=\sum_{l=1}^{s}\Lambda_{kl}\cos(\omega_l t+\Psi_{kl}) \qquad k=1,\cdots,n_u \tag{3}$$

where Λ_{kl} and Ψ_{kl} are respectively the amplitude and phase shift of the sinusoid at the frequency ω_l. Note that Λ_{kl} and Ψ_{kl} can be zero for some k ($k=1,\cdots,n_u$) but, for each $l=1,\cdots,s$, there exists (at least) a value of k for which $\Lambda_{kl}\neq 0$. In the second type, r is generated as $r=Mv$ via a stable transfer matrix $M(z)=(M_{kq}(z))_{(k,q)\in[\![1,n_u]\!]\times[\![1,f]\!]}$ and a vector $v=(v_1,\cdots,v_f)^T$ containing f independent white noises signals v_q ($q=1,\cdots,f$) of covariance $\Sigma_v\succ 0$. In other words, each entry r_k of r is given by:

$$r_k(t)=\sum_{q=1}^{f}M_{kq}(z)v_q(t) \qquad k=1,\cdots,n_u \tag{4}$$

Note that some $M_{kq}(z)$ can be identically zero and that f is not necessarily equal to n_u but, for each $q=1,\cdots,f$, there exists (at least) a value of k for which $M_{kq}(z)\neq 0$.

The system $\mathcal{S}$ is identified within a full-order model structure $\mathcal{M}=\{(G(z,\theta),H(z,\theta))\mid\theta\in\mathcal{D}_\theta\}$ where $\theta\in\mathbb{R}^n$ is a parameter vector and the set $\mathcal{D}_\theta$ restricts the parameter vector θ to those values for which $G(z,\theta)$ is stable and $H(z,\theta)$ is stable and inversely stable. The model structure is said to be full-order if $\exists\theta_0\in\mathcal{D}_\theta$ such that $(G(z,\theta_0),H(z,\theta_0))=(G_0(z),H_0(z))$. We will suppose that $\mathcal{M}$ is globally identifiable at the true parameter vector θ_0, i.e. $(G(z,\theta),H(z,\theta))=(G_0(z),H_0(z))\Rightarrow\theta=\theta_0$ [1,2].

Assume that we have collected a set of N input-output data:

$$Z^N=\left\{x(t)=\left(y^T(t),\ u^T(t)\right)^T \mid t=1,\cdots,N\right\}$$

For each $(G(z,\theta),H(z,\theta))\in\mathcal{M}$, we can define the one-step ahead predictor $\hat{y}(t,\theta)$ for the output $y(t)$ using Z^N:

$$\hat{y}(t,\theta)=W_y(z,\theta)y(t)+W_u(z,\theta)u(t)=W(z,\theta)x(t) \tag{5}$$
$$W_u(z,\theta)=H^{-1}(z,\theta)G(z,\theta) \tag{6}$$
$$W_y(z,\theta)=\mathbf{I}_{n_y}-H^{-1}(z,\theta) \tag{7}$$

where $W(z,\theta)=(W_y(z,\theta),W_u(z,\theta))$.

As will be illustrated in the sequel, the data $x(t)=(y^T(t),u^T(t))^T$ must be informative with respect to $\mathcal{M}$ to get the consistency of the estimate.

Definition 2 (Data Informativity [2]) *Consider the framework defined above with the data $x(t)=(y^T(t),u^T(t))^T$ collected on the true system $\mathcal{S}$ operated in closed-loop with a controller $K(z)$ and a quasi-stationary external excitation r (see* (2)*). Consider also a model structure $\mathcal{M}$ yielding the predictor $\hat{y}(t,\theta)=W(z,\theta)x(t)$. Define the set $\Delta_{\mathbf{W}}=\{\Delta W(z)=W(z,\theta')-W(z,\theta'')\mid\theta'$ and θ'' in $\mathcal{D}_\theta\}$. The data $x(t)$ are said to be informative w.r.t. the model structure $\mathcal{M}$ if, for all $\Delta W(z)\in\Delta_{\mathbf{W}}$, we have*

$$\bar{E}\left[||\Delta W(z)x(t)||^2\right]=0\Longrightarrow\Delta W(z)\equiv\mathbf{0} \tag{8}$$

where the notation $\Delta W(z)\equiv\mathbf{0}$ means that $\Delta W(e^{j\omega})=\mathbf{0}$ at all or almost all $\omega\in]-\pi,\pi]$.

Data informativity combined with global identifiability at θ_0 are important properties since they imply that the prediction error criterion [10,14] defined below yields a consistent estimate $\hat{\theta}_N$ for θ_0:

$$\hat{\theta}_N=\arg\min_{\theta\in\mathcal{D}_\theta}V_N\left(\theta,Z^N\right) \tag{9}$$

$$V_N\left(\theta,Z^N\right)=\frac{1}{N}\sum_{t=1}^{N}\epsilon^T(t,\theta)\Sigma_0^{-1}\epsilon(t,\theta) \tag{10}$$

where $\epsilon(t,\theta)=y(t)-\hat{y}(t,\theta)$ and where Σ_0 is assumed known for simplicity[2]. The estimate $\hat{\theta}_N$ is consistent if it converges to θ_0 with probability equal to 1 when $N\to+\infty$.

In [4–6], we derived conditions to ensure the data informativity in open-loop ($K(z)\equiv\mathbf{0}$) for input vectors $u=r$ of the types (3)-(4) and for the classical types of model structures $\mathcal{M}$ in open-loop i.e. FIR, ARX, OE and BJ model structures. In the sequel, we will study the closed-loop case. In the next section, we present the model structures considered in this study.

4 Considered model structures

For the sake of simplicity, we will here also restrict attention to the classical types of model structures i.e. FIR, ARX, ARMAX, OE and BJ. For this purpose, let us decompose θ as follows

$$\theta=\begin{pmatrix}\tilde{\theta}\\ \eta\end{pmatrix} \tag{11}$$

where $\tilde{\theta}$ is made up by the parameters uniquely found in $G(z,\theta)$ and η contains the rest of the parameters.

The ARMAX model structure can be described as follows

$$\begin{cases}G(z,\theta) &= A(z,\eta)^{-1}B(z,\tilde{\theta})\\ H(z,\theta) &= A(z,\eta)^{-1}C(z,\eta)\end{cases} \tag{12}$$

where $A(z,\eta)$ and $C(z,\eta)$ are diagonal and monic polynomial matrices of dimension $n_y\times n_y$ and $B(z,\theta)$ is a polynomial matrix of dimension $n_y\times n_u$. Note that the ARX and the FIR model structures are special cases of

[2] It can however be estimated together with $\hat{\theta}_N$ (see e.g. [10, Chapter 15])

the ARMAX one with $C(z,\eta) = \mathbf{I}_{n_y}$ in the ARX case and $C(z,\eta) = A(z,\eta) = \mathbf{I}_{n_y}$ in the FIR case.

For BJ model structures, $G(z,\theta)$ and $H(z,\theta)$ do not share common parameters:

$$\begin{cases} G(z,\theta) = G(z,\tilde{\theta}) \\ H(z,\theta) = H(z,\eta) \end{cases} \tag{13}$$

with $H(z,\eta) = \text{diag}(H_1(z,\eta),\cdots,H_{n_y}(z,\eta))$.

Let us introduce some further notations for these MIMO model structures.

ARMAX/ARX/FIR: For the MIMO ARMAX/ARX/FIR model structure in (12), $B(z,\tilde{\theta})$ is a matrix made up of $n_y n_u$ independently parametrized polynomials $B_{ik}(z,\tilde{\theta}_{ik}) = z^{-\rho_{ik}}\mathring{B}_{ik}(z,\tilde{\theta}_{ik})$ $(i = 1,\cdots,n_y,\ k = 1,\cdots,n_u)$ where the delays ρ_{ik} can be all different and $\mathring{B}_{ik}(z,\tilde{\theta}_{ik})$ is a polynomial. All $H_i(z,\eta) = C_i(z,\eta)/A_i(z,\eta)$ are parametrized independently with a parameter vector η_i, i.e, $H_i(z,\eta) = H_i(z,\eta_i) = C_i(z,\eta_i)/A_i(z,\eta_i)$ where $C_i(z,\eta_i)$ and $A_i(z,\eta_i)$ are monic polynomials.

The coefficients of the polynomial $\mathring{B}_{ik}$ are the parameters in $\tilde{\theta}_{ik}$ while the ones in the polynomials $A_i(z,\eta_i)$ and $C_i(z,\eta_i)$ are the ones in η_i. Consequently, we have that:

$$\mathring{B}_{ik}(z,\tilde{\theta}_{ik}) = \tilde{\theta}_{ik,1} + \sum_{m=1}^{\deg(\mathring{B}_{ik})} \tilde{\theta}_{ik,(m+1)} z^{-m} \tag{14}$$

$$C_i(z,\eta_i) = 1 + \sum_{m=1}^{\deg(C_i)} \eta_{i,m} z^{-m} \tag{15}$$

$$A_i(z,\eta_i) = 1 + \sum_{m=1}^{\deg(A_i)} \eta_{i,(m+\deg(C_i))} z^{-m} \tag{16}$$

where $\tilde{\theta}_{ik,m}$ denotes the m-th entry of $\tilde{\theta}_{ik}$ and $\eta_{i,m}$ the m-th entry of η_i. The number of parameters to identify in $\mathring{B}_{ik}$, C_i and A_i is thus equal to $\deg(\mathring{B}_{ik})+1$, $\deg(C_i)$ and $\deg(A_i)$ respectively.

BJ/OE: For the MIMO BJ/OE model structure in (13), $G(z,\tilde{\theta})$ is a matrix made up of $n_y n_u$ independently parametrized transfer functions $G_{ik}(z,\tilde{\theta}_{ik}) = z^{-\rho_{ik}}\mathring{B}_{ik}(z,\tilde{\theta}_{ik})/F_{ik}(z,\tilde{\theta}_{ik})$ where $\mathring{B}_{ik}(z,\tilde{\theta}_{ik})$ and $F_{ik}(z,\tilde{\theta}_{ik})$ are polynomials. The diagonal matrix $H(z,\theta)$ is composed by n_y independently parametrized transfer functions $H_i(z,\eta_i)$ with $H_i(z,\eta_i) = C_i(z,\eta_i)/D_i(z,\eta_i)$ where $C_i(z,\eta_i)$ and $D_i(z,\eta_i)$ are monic polynomials.

The coefficients of the polynomials $\mathring{B}_{ik}$ and F_{ik} are the parameters in $\tilde{\theta}_{ik}$ while the coefficients of the polynomials $C_i(z,\eta_i)$ and $D_i(z,\eta_i)$ are the ones in η_i. Consequently, we have that:

$$\mathring{B}_{ik}(z,\tilde{\theta}_{ik}) = \tilde{\theta}_{ik,1} + \sum_{m=1}^{\deg(\mathring{B}_{ik})} \tilde{\theta}_{ik,(m+1)} z^{-m} \tag{17}$$

$$F_{ik}(z,\tilde{\theta}_{ik}) = 1 + \sum_{m=1}^{\deg(\mathring{F}_{ik})} \tilde{\theta}_{ik,(m+\deg(\mathring{B}_{ik})+1)} z^{-m} \tag{18}$$

$$C_i(z,\eta_i) = 1 + \sum_{m=1}^{\deg(C_i)} \eta_{i,m} z^{-m} \tag{19}$$

$$D_i(z,\eta_i) = 1 + \sum_{m=1}^{\deg(D_i)} \eta_{i,(m+\deg(C_i))} z^{-m} \tag{20}$$

where $\tilde{\theta}_{ik,m}$ denotes the m-th entry of $\tilde{\theta}_{ik}$ and $\eta_{i,m}$ the m-th entry of η_i. The number of parameters to identify in $\mathring{B}_{ik}$, F_{ik}, D_i and C_i is thus equal to $\deg(\mathring{B}_{ik})+1$, $\deg(F_{ik})$, $\deg(D_i)$ and $\deg(C_i)$ respectively.

For all classical model structures, the parameter vector $\tilde{\theta} \in \mathbb{R}^{\tilde{n}}$ is the concatenation of $\tilde{\theta}_{ik}$ $(i = 1,\cdots,n_y,\ k = 1,\cdots,n_u)$ i.e. $\tilde{\theta} = (\tilde{\theta}_{11}^T,\tilde{\theta}_{12}^T,\cdots,\tilde{\theta}_{1n_u}^T,\cdots,\cdots,\tilde{\theta}_{n_y1}^T,\tilde{\theta}_{n_y2}^T,\cdots,\tilde{\theta}_{n_yn_u}^T)^T$. The parameter vector $\eta \in \mathbb{R}^{n_\eta}$ is also the concatenation of η_i $(i = 1,\cdots,n_y)$ i.e. $\eta = (\eta_1^T,\eta_2^T,\cdots,\eta_{n_y}^T)^T$.

These classical model structures are globally identifiable at the true parameter vector θ_0 if θ_0 does not lead to a pole/zero cancellation.

Index notations: In the sequel, we will use the index $i \in [\![1,n_y]\!]$ to specify the entries y_i of y while $k \in [\![1,n_u]\!]$ will be used to specify the entries u_k of u.

5 Data informativity for MIMO systems in closed-loop

5.1 Simplification of the study

We will first see in the next theorem how the controller $K(z)$ and the external excitation r contribute to the data informativity in the closed-loop MIMO case.

Theorem 3 *Consider Definition 2 and one of the model structures $\mathcal{M}$ defined in Section 4. Recall that r and e are independent. For each $\Delta W(z) \in \Delta_{\mathbf{W}}$, we define similarly $\Delta W_y(z)$ and $\Delta W_u(z)$. Then, the data $x(t)$ are informative w.r.t. the model structure $\mathcal{M}$ if and only if, for all $\Delta W(z) = (\Delta W_y(z),\Delta W_u(z)) \in \Delta_{\mathbf{W}}$,*

$$\begin{cases} \Delta W_y(z) - \Delta W_u(z)K(z) \equiv \mathbf{0} \\ \bar{E}\left[||\Delta W_u(z)r(t)||^2\right] = 0 \end{cases} \Longrightarrow (\Delta W_y,\Delta W_u) \equiv (\mathbf{0},\mathbf{0}) \tag{21}$$

PROOF. See Appendix A. ∎

Remark 4 *As mentioned in the introduction, a sufficient condition for the informativity is to choose r such that* *its power spectrum matrix is strictly positive definite at all frequencies, i.e.* $\Phi_r(\omega) \succ 0\ \forall\omega$. *Indeed, by using Parseval theorem* $\bar{E}[||\Delta W_u(z)r(t)||^2] = 0$ *is equivalent to* $tr\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\Delta W_u(e^{j\omega})\Phi_r(\omega)\Delta W_u^*(e^{j\omega})d\omega\right) = 0$ *where tr is the trace operator. Consequently, if* $\Phi_r(\omega) \succ 0$ *for all* ω, *then* $\bar{E}[||\Delta W_u(z)r(t)||^2] = 0$ *implies* $\Delta W_u \equiv \mathbf{0}$. *When* $\Delta W_u \equiv 0$, *the first equation of the left hand side of* (21) *is equivalent to* $\Delta W_y \equiv \mathbf{0}$. *Consequently, we have informativity when* $\Phi_r(\omega) \succ 0\ \forall\omega$ *(or at almost all* ω*).*

However, this sufficient condition for data informativity is in fact very restrictive. It will indeed never be verified for multisine excitation such as (3) *or for filtered white noise excitation of the type* (4) *when* $f < n_u$, *while such excitation signals can of course yield informative data (see, e.g., Section 9).*

In the next theorem, we will show that we can simplify the result of Theorem 3.

Theorem 5 *Consider Definition 2, Theorem 3 and one of the model structures* $\mathcal{M}$ *defined in Section 4. Define the sets* $\Delta_{\mathbf{W,i}} = \{\Delta W_{i:}(z)\ |\ \Delta W_{i:}(z)$ *is the ith row of* $\Delta W(z) \in \Delta_{\mathbf{W}}\}$ $(i = 1,\cdots,n_y)$. *For each* $\Delta W_{i:}(z) \in \Delta_{\mathbf{W,i}}$, *we define similarly* $\Delta W_{u,i:}(z)$ *and* $\Delta W_{y,i:}(z)$ *which are the* i*-th row of* $\Delta W_u(z)$ *and* $\Delta W_y(z)$ *respectively. Then, the data* $x(t)$ *are informative w.r.t. the model structure* $\mathcal{M}$ *if and only if, for all* $i = 1,\cdots,n_y$, *the following property holds for all* $\Delta W_{i:}(z) = (\Delta W_{y,i:}(z), \Delta W_{u,i:}(z)) \in \Delta_{\mathbf{W,i}}$*:*

$$\begin{cases} \Delta W_{y,i:}(z) - \Delta W_{u,i:}(z)K(z) \equiv \mathbf{0} \\ \bar{E}\left[||\Delta W_{u,i:}(z)r(t)||^2\right] = 0 \end{cases} \Longrightarrow (\Delta W_{y,i:}, \Delta W_{u,i:}) \equiv (\mathbf{0},\mathbf{0}) \tag{22}$$

PROOF. See Appendix B. ■

Theorem 5 allows to simplify the data informativity verification: it can be done channel-by-channel (or output-by-output). Hence, in the sequel we will consider an arbitrary i and restrict attention to (22) for that particular i.

In its actual form, (22) is function of the rational transfer functions matrix $(\Delta W_{y,i:}(z), \Delta W_{u,i:}(z)) \in \Delta_{\mathbf{W,i}}$. We need to transform (22) into a polynomial matrix form for the development of the conditions on data informativity. Let us for this purpose first observe that the first equation of the left hand side of (22) can be rewritten as follows

$$\left(\Delta W_{y,i:}(z), -\Delta W_{u,i:}(z)\right)\begin{pmatrix}\mathbf{I}_{n_y}\\ K(z)\end{pmatrix} \equiv \mathbf{0} \tag{23}$$

By considering a right-factorization of $K(z)$ and a left-factorization of $(\Delta W_{y,i:}(z), -\Delta W_{u,i:}(z))$, we will prove that we can transform (23) into a polynomial matrix form. Moreover, we will see that this left-factorization of $(\Delta W_{y,i:}(z), -\Delta W_{u,i:}(z))$ will allow us to transform the second equation of the left hand side of (22) into a polynomial matrix form.

To simplify the presentation in the sequel, we will often use shorthand notations for each transfer of the form $M(z)$, $M(z,\theta')$ and $M(z,\theta'')$ by dropping the argument and we will denote them by M, M' and M'' respectively.

5.2 Factorization of (22) *into a polynomial matrix form*

We are going to consider a right-factorization for $K(z)$ and a left-factorization for the rational block matrix $(\Delta W_{y,i:}(z), -\Delta W_{u,i:}(z))$ which should be valid for all $\Delta W_{i:}(z) = (\Delta W_{y,i:}(z), \Delta W_{u,i:}(z)) \in \Delta_{\mathbf{W,i}}$. The one that we will choose is obtained by putting all entries of $(\Delta W_{y,i:}(z), -\Delta W_{u,i:}(z))$ on the same denominator. It is given in the next lemma.

Lemma 6 *Consider the model structures defined in Section 4. For these model structures, the rational block-matrix* $(\Delta W_{y,i:}(z), -\Delta W_{u,i:}(z))$ *can be left-factorized into* $(\Delta W_{y,i:}(z), -\Delta W_{u,i:}(z)) = Q_i^{-1}(z)(\Upsilon_{y,i}(z), \Upsilon_{u,i}(z))$ *where the row polynomial vector* $\Upsilon_{y,i}(z)$ *of dimension* n_y *has all its entries equal to 0 except possibly the* i*-th one denoted* $\Upsilon_{y,ii}(z)$. *The row polynomial vector* $\Upsilon_{u,i}(z)$ *is of dimension* n_u. *The scalar polynomials* $Q_i(z)$ *and* $\Upsilon_{y,ii}(z)$ *and the row polynomial vector* $\Upsilon_{u,i}(z)$ *are given by*

- *for FIR:* $Q_i = 1$, $\Upsilon_{y,ii} = 0$, $\Upsilon_{u,i} = B''_{i:} - B'_{i:}$.
- *for ARX:* $Q_i = 1$, $\Upsilon_{y,ii} = A''_i - A'_i$, $\Upsilon_{u,i} = B''_{i:} - B'_{i:}$.
- *for ARMAX:* $Q_i = C'_iC''_i$, $\Upsilon_{y,ii} = C'_iA''_i - C''_iA'_i$, $\Upsilon_{u,i} = C'_iB''_{i:} - C''_iB'_{i:}$.
- *for OE:* $Q_i = \prod_{k=1}^{n_u} F'_{ik}F''_{ik}$, $\Upsilon_{y,ii} = 0$ *and* $\Upsilon_{u,i} = (\Upsilon_{u,ik})_{k\in\times[\![1,n_u]\!]}$ *with*

$$\Upsilon_{u,ik} = (B''_{ik}F'_{ik} - B'_{ik}F''_{ik})\prod_{l=1,l\neq k}^{n_u} F'_{il}F''_{il}$$

- *for BJ:* $Q_i = C'_iC''_i\prod_{k=1}^{n_u} F'_{ik}F''_{ik}$, $\Upsilon_{y,ii} = D''_iC'_i\prod_{k=1}^{n_u} F'_{ik}F''_{ik} - D'_iC''_i\prod_{k=1}^{n_u} F'_{ik}F''_{ik}$ *and* $\Upsilon_{u,i} = (\Upsilon_{u,ik})_{k\in[\![1,n_u]\!]}$ *with*

$$\Upsilon_{u,ik} = D''_iC'_iB''_{ik}F'_{ik}\prod_{l=1,l\neq k}^{n_u} F'_{il}F''_{il} - D'_iC''_iB'_{ik}F''_{ik}\prod_{l=1,l\neq k}^{n_u} F'_{il}F''_{il}$$

PROOF. See Appendix C. ■

Based on the set $\Delta_{\mathbf{W,i}}$ $(i = 1,\cdots,n_y)$ (see Theorem 5), let us define the set $\mathbf{Y_i}$ made up of all polynomial matrices $(\Upsilon_{y,ii}(z), \Upsilon_{u,i}(z))$ obtained by considering the left-factorization of $\Delta W_{i:}(z) \in \Delta_{\mathbf{W,i}}$ (see Lemma 6).

For the controller $K(z)$, we will consider the right-factorization consisting in putting all entries of $K(z)$ on the least common multiple $J(z)$ of the denominator of the entries of $K(z)$ ($J(z)$ is a polynomial). Therefore, $K(z)$

can be rewritten as

$$K(z) = N(z)V^{-1}(z) \tag{24}$$

where $N(z)$ is a FIR matrix of dimension $n_u \times n_y$ and $V(z) = \text{diag}(\underbrace{J(z), \cdots, J(z)}_{n_y \text{ times}})$.

Based on the factorization of $K(z)$ in (24) and of $\Delta W_{i:}(z) \in \Delta_{\mathbf{W,i}}$ in Lemma 6, we can transform (22) into a polynomial matrix form.

Theorem 7 *Consider Definition 2, Theorem 5 and one of the model structures $\mathcal{M}$ defined in Section 4. Consider the right-factorization of $K(z)$ in (24). For all $i = 1, \cdots, n_y$, denote*

$$\Xi_i(z) = \begin{pmatrix} V_{i:}(z) \\ N(z) \end{pmatrix} \tag{25}$$

where $V_{i:}(z)$ is the i-th row of $V(z)$.

For each $\Delta W_{i:}(z) = (\Delta W_{y,i:}(z), \Delta W_{u,i:}(z)) \in \Delta_{\mathbf{W,i}}$, consider the left-factorization of $(\Delta W_{y,i:}(z), -\Delta W_{u,i:}(z)) = Q_i^{-1}(z)(\Upsilon_{y,i}(z), \Upsilon_{u,i}(z))$ given in Lemma 6 and denote $\Upsilon_{y,ii}$ the i-th element of the row polynomial vector $\Upsilon_{y,i}$. Then, the data $x(t)$ are informative w.r.t. the model structure $\mathcal{M}$ if and only if, for all $i = 1, \cdots, n_y$,

$$\begin{cases} (\Upsilon_{y,ii}(z), \Upsilon_{u,i}(z))\Xi_i(z) \equiv \mathbf{0} \\ \bar{E}\left[||\Upsilon_{u,i}(z)r(t)||^2\right] = 0 \end{cases} \Longrightarrow (\Upsilon_{y,ii}, \Upsilon_{u,i}) \equiv (0, \mathbf{0}) \tag{26}$$

for all $(\Upsilon_{y,ii}, \Upsilon_{u,i}) \in \mathbf{Y_i}$.

PROOF. See Appendix D. □

5.3 Main result for data informativity

In this section, we derive the main result of this paper. This result will allow us to check data informativity by verifying for each channel $i = 1, \cdots, n_y$ whether a given matrix is full row rank. As we will see in the sequel, this matrix will depend on the complexity of the model structure, on the controller coefficients and on the external excitation parametrization (amplitudes, phase-shifts, frequencies for multisine excitation and filter coefficients for filtered white noise excitation).

For this purpose, a first step is to give a formal expression of the polynomial $\Upsilon_{y,ii}$ and the row vector of polynomials $\Upsilon_{u,i} = (\Upsilon_{u,i1}, \cdots, \Upsilon_{u,in_u})$ in $\mathbf{Y_i}$. Using Lemma 6 and the notations introduced in Section 4, we can determine the scalars $\eta_{y,i}$, $\eta_{u,ik}$ $(i = 1, \cdots, n_y,\ k = 1, \cdots, n_u)$ such that all $\Upsilon_{y,11}$ and the entries of $\Upsilon_{u,i}$ in $\mathbf{Y_i}$ can be expressed as follows:

$$\Upsilon_{y,ii}(z) = \tilde{\delta}_{y,i}^T Z_{y,i}(z) \tag{27}$$

$$\Upsilon_{u,ik}(z) = \tilde{\delta}_{u,ik}^T Z_{u,ik}(z) \tag{28}$$

where $Z_{y,i}(z) = (z^{-1}, \cdots, z^{-\eta_{y,i}})^T$, $Z_{u,ik}(z) = (z^{-\rho_{ik}}, \cdots, z^{-\eta_{u,ik}})^T$ and $\tilde{\delta}_{y,i}$, $\tilde{\delta}_{u,ik}$ are vectors of coefficients. The values of $\eta_{y,i}$, $\eta_{u,ik}$ $(i = 1, \cdots, n_y,\ k = 1, \cdots, n_u)$ as a function of the model structure type and of the model structure complexity are given in Appendix E.

Using (27) and (28) and defining $\tilde{\delta}_{u,i} = (\tilde{\delta}_{u,i1}^T, \cdots, \tilde{\delta}_{u,in_u}^T)^T$, the vectors of polynomials $(\Upsilon_{y,ii}, \Upsilon_{u,i})$ in the set $\mathbf{Y_i}$ can be rewritten as follows:

$$(\Upsilon_{y,ii}(z), \Upsilon_{u,i}(z)) = \tilde{\delta}_i^T \text{bdiag}(Z_{y,i}(z), Z_{u,i}(z)) \tag{29}$$

where $\tilde{\delta}_i = (\tilde{\delta}_{y,i}^T, \tilde{\delta}_{u,i}^T)^T$ and $Z_{u,i} = \text{bdiag}(Z_{u,i1}, \cdots, Z_{u,in_u})$ (with $\text{bdiag}(X_1, \cdots, X_n)$ a block diagonal matrix whose blocks are given by X_i $(i = 1, \cdots, n)$). We will denote by ζ_i (resp. $\zeta_{u,i}$) the dimension of $\tilde{\delta}_i$ (resp. $\tilde{\delta}_{u,i}$). These dimensions can be easily deduced based on $\eta_{y,i}$, $\eta_{u,ik}$ $(i = 1, \cdots, n_y,\ k = 1, \cdots, n_u)$. Let us also introduce the set $\mathbf{D_i} = \{\tilde{\delta}_i \mid (29) \in \mathbf{Y_i}\}$. Note that, except in the FIR case where $\mathbf{D_i} = \mathbb{R}^{\zeta_i}$, $\mathbf{D_i}$ is a subset of $\mathbb{R}^{\zeta_i}$.

In Sections 6 and 7, we will show that, using (27)-(29), we can determine matrices $\mathscr{A}_i$, $\mathscr{B}_i$ and $\mathscr{C}_i$ such that the left hand side of (26) for a given $(\Upsilon_{y,ii}, \Upsilon_{u,i}) \in \mathbf{Y_i}$ can be expressed as a function of the parameter vector $\tilde{\delta}_i \in \mathbf{D_i}$ defining $(\Upsilon_{y,ii}, \Upsilon_{u,i})$ (see (29)). In particular, the first equation of the left hand side of (26) is equivalent to:

$$\begin{cases} \underbrace{\begin{pmatrix} \tilde{\delta}_{y,i}^T & \tilde{\delta}_{u,i}^T \end{pmatrix}}_{\tilde{\delta}_i^T} \begin{pmatrix} \mathscr{A}_i \\ \mathscr{B}_i \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \end{pmatrix} & \text{(BJ/ARX/ARMAX case)} \\ \tilde{\delta}_{u,i}^T \mathscr{B}_i = \mathbf{0} & \text{(OE/FIR case)} \end{cases} \tag{30}$$

The difference in the above equation follows from the fact that $\Upsilon_{y,ii} = 0$ for OE/FIR model structures (see Lemma 6). The second equation of the left hand side of (26) is equivalent to:

$$\tilde{\delta}_{u,i}^T \mathscr{C}_i = \mathbf{0} \tag{31}$$

As will be shown in Sections 6 and 7, the dimension and the elements of the matrices $\mathscr{A}_i$, $\mathscr{B}_i$ and $\mathscr{C}_i$ will be function of the complexity of the model structure, the controller coefficients and the external excitation parametrization.

Theorem 8 *Consider the data $x(t)$ generated as in Section 3 with an external excitation $r(t)$. Consider also Definition 2, Theorem 7 and one of the model structures $\mathcal{M}$ defined in Section 4. Consider finally the notations introduced in Section 5.3 and, in particular, the equations* (30) *and* (31) *that are respectively equivalent to the first and second equations of the left hand side of* (26). *Then, the data $x(t)$ are informative with respect to $\mathcal{M}$ if, for each $i = 1, \cdots, n_y$, at least one of the matrices $\mathscr{P}_i^{(a)}$, $\mathscr{P}_i^{(b)}$ and $\mathscr{P}_i^{(c)}$ defined below are full row rank.*

- (a) *The matrix* $\mathscr{P}_i^{(a)}$ *is equal to* $\begin{pmatrix} \mathscr{A}_i \\ \mathscr{B}_i \end{pmatrix}$ *for the BJ/ARX/ARMAX case and to* $\mathscr{B}_i$ *for the OE/FIR case.*
- (b) *The matrix* $\mathscr{P}_i^{(b)}$ *is equal to* $\mathscr{C}_i$.
- (c) *The matrix* $\mathscr{P}_i^{(c)}$ *is equal to* $\begin{pmatrix} \mathscr{A}_i & \mathbf{0} \\ \mathscr{B}_i & \mathscr{C}_i \end{pmatrix}$ *for the BJ/ARX/ARMAX case and to* $\begin{pmatrix} \mathscr{B}_i & \mathscr{C}_i \end{pmatrix}$ *for the OE/FIR case.*

Moreover, when $r = \mathbf{0}$*, the data* $x(t)$ *are informative with respect to* $\mathscr{M}$ *if, for each* $i = 1, \cdots, n_y$*, the matrix* $\mathscr{P}_i^{(a)}$ *defined above is full row rank.*

PROOF. We will prove the theorem in the BJ/ARX/ARMAX case. The proof for the OE/FIR case can be derived using the same argumentation. Note first that $\tilde{\delta}_i = (\tilde{\delta}_{y,i}, \tilde{\delta}_{u,i})$ in (30)-(31) is constrained to lie in $\mathbf{D_i}$. When $\tilde{\delta}_i \in \mathbf{D_i}$, having a full row rank $\mathscr{P}_i^{(a)}$ is a sufficient condition[3] for $\tilde{\delta}_i = \mathbf{0}$ to be the unique solution of (30). Note also that $\tilde{\delta}_i = \mathbf{0}$ is equivalent to $(\Upsilon_{y,ii} \Upsilon_{u,i}) = (0, \mathbf{0})$ (see (29)). Consequently, using the equivalence recalled in the statement of the theorem, when $\mathscr{P}_i^{(a)}$ is full row rank, we have also that $(\Upsilon_{y,ii}(z), \Upsilon_{u,i}(z))\Xi_i(z) \equiv 0 \Longrightarrow (\Upsilon_{y,ii}, \Upsilon_{u,i}) \equiv (0, \mathbf{0})$ for all $(\Upsilon_{y,ii}, \Upsilon_{u,i}) \in \mathbf{Y_i}$. It is clear by Theorem 7 that the latter implies that $x(t)$ is informative (in particular when $r = \mathbf{0}$).

Note now that (30) and (31) can be combined into $\tilde{\delta}_i^T \mathscr{P}_i^{(c)} = \mathbf{0}$. If $\mathscr{P}_i^{(c)}$ is full row rank, then $\tilde{\delta}_i^T \mathscr{P}_i^{(c)} = \mathbf{0}$ implies $\tilde{\delta}_i = \mathbf{0}$. Consequently, using the equivalence recalled in the statement of the theorem, when $\mathscr{P}_i^{(c)}$ is full row rank, we have also that (26) holds for all $(\Upsilon_{y,ii}, \Upsilon_{u,i}) \in \mathbf{Y_i}$. It is clear by Theorem 7 that the latter implies that $x(t)$ is informative.

Finally, if $\mathscr{P}_i^{(b)}$ is full row rank, the equation (31) implies $\tilde{\delta}_{u,i} = \mathbf{0}$. Consequently, using the equivalence recalled in the statement of the theorem, when $\mathscr{P}_i^{(b)}$ is full row rank, we have also that: $\bar{E}\left[||\Upsilon_{u,i}(z) r(t)||^2\right] = 0 \Longrightarrow \Upsilon_{u,i} \equiv \mathbf{0}$ for all $\Upsilon_{u,i}(z) \in \mathbf{Y_i}$. Since $\Upsilon_{u,i} \equiv \mathbf{0}$, the first equation of the left hand side of (26) reduces to $\Upsilon_{y,ii}(z) J(z) \equiv \mathbf{0}$ where $J(z)$ is a given polynomial defined before in (24). The latter always implies that $\Upsilon_{y,ii} \equiv 0$. Consequently, by virtue of Theorem 7, we have also data informativity when $\mathscr{P}_i^{(b)}$ is full row rank. □

Let us now show how we can rewrite the left hand side of (26) into (30)-(31).

[3] If $\mathbf{D_i}$ would be equal to $\mathbb{R}^{\zeta_i}$, this would be a necessary and sufficient condition.

6 Rewriting of $(\Upsilon_{y,ii}, \Upsilon_{u,i})\Xi_i(z) \equiv \mathbf{0}$ into (30)

We will consider the BJ/ARX/ARMAX case since it is the more general. Using (29) and (25), we have that:

$$(\Upsilon_{y,ii}(z), \Upsilon_{u,i}(z))\Xi_i(z) = (\tilde{\delta}_{y,i}^T, \tilde{\delta}_{u,i}^T) L(z) \tag{32}$$

with $L(z)$ a polynomial matrix of dimension $\zeta_i \times n_y$:

$$L(z) = \begin{pmatrix} Z_{y,i}(z) V_{i:}(z) \\ Z_{u,i}(z) N(z) \end{pmatrix}$$

Each entry $L_{mj}(z)$ of $L(z)$ $(m = 1, \cdots, \zeta_i,\ j = 1, \cdots, n_y)$ is a polynomial in z^{-1} and can therefore be rewritten as $\lambda_{mj}^T Z_L(z)$ with $Z_L(z) = (z^{-\beta_{min}}, z^{-\beta_{min+1}}, \cdots, z^{-\beta_{max}})^T$ where λ_{mj} is a vector containing the coefficients[4] of the polynomial $L_{mj}(z)$ and $z^{-\beta_{min}}$, $z^{-\beta_{max}}$ are the smallest and largest value of the monomials $z^{-\beta}$ among all entries L_{mj} of L. This yields to the following expression for $L(z)$:

$$L(z) = \begin{pmatrix} Z_{y,i}(z) V_{i:}(z) \\ Z_{u,i}(z) N(z) \end{pmatrix} = \begin{pmatrix} \mathscr{A}_i \\ \mathscr{B}_i \end{pmatrix} (\mathbf{I}_{n_y} \otimes Z_L(z)) \tag{33}$$

where $\otimes$ represents the Kronecker product and the matrix $(\mathscr{A}_i^T, \mathscr{B}_i^T)^T$ is such that its m-th row $(m = 1, \cdots, \zeta_i)$ is given by $(\lambda_{m1}^T, \cdots, \lambda_{mn_y}^T)$. Using now (32) and (33), we see that $(\Upsilon_{y,ii}(z), \Upsilon_{u,i}(z))\Xi_i(z) \equiv \mathbf{0}$ is equivalent to (30). The matrix $(\mathscr{A}_i^T, \mathscr{B}_i^T)^T$ has ζ_i rows and a number of columns equal to $n_y(\beta_{max} - \beta_{min} + 1)$. This number of columns therefore depends both on the complexity of the controller and on the complexity of the model structures (via β_{min}, β_{max}, $\eta_{u,ik}$, $\eta_{y,i}$). The more complex the model structure and the controller, the larger this number of columns is.

Example 9 *Consider the following ARX model structure* $\mathscr{M}$ *with* $n_u = 2$ *inputs and* $n_y = 1$ *output:*

$$G(z,\theta) = \left(\frac{\tilde{\theta}_{11,1} z^{-1}}{1 + \eta_{1,1} z^{-1}}, \frac{\tilde{\theta}_{12,1} z^{-1} + \tilde{\theta}_{12,2} z^{-2}}{1 + \eta_{1,1} z^{-1}} \right)$$
$$H(z,\theta) = \frac{1}{1 + \eta_{1,1} z^{-1}}$$

where $\theta = (\tilde{\theta}_{11,1}, \tilde{\theta}_{12,1}, \tilde{\theta}_{12,2}, \eta_{1,1})^T \in \mathscr{D}_\theta$*. In this case, we have the left-factorization of* $(\Delta W_{y,1:}(z), -\Delta W_{u,1:}(z))$ *in Lemma 6 of the form* $(\Delta W_{y,1:}(z), -\Delta W_{u,1:}(z)) = Q_1^{-1}(z)(\Upsilon_{y,1}(z), \Upsilon_{u,1}(z))$ *with* $Q_1(z) = 1$*. From Appendix E, the polynomial* $\Upsilon_{y,1}(z) = \Upsilon_{y,11}(z)$ *has an order of* $\eta_{y,1} = 1$ *and is given by*

[4] These coefficients correspond to the ones in $N(z)$ and $J(z)$ of the right-factorization of the controller $K(z)$ in (24).

$$\Upsilon_{y,11}(z) = {\delta_{y,1}}^T Z_{y,1}(z)$$

with $\delta_{y,1} = \eta''_{1,1} - \eta'_{1,1}$ *and* $Z_{y,1}(z) = z^{-1}$.

From Appendix E, the polynomial entries $\Upsilon_{u,11}(z)$ *and* $\Upsilon_{u,12}(z)$ *of* $\Upsilon_{u,1}(z)$ *have a degree of* $\eta_{u,11} = 1$ *and* $\eta_{u,12} = 2$ *respectively and are successively given by*

$$\Upsilon_{u,11}(z) = \delta_{u,11}^T Z_{u,11}(z)$$
$$\Upsilon_{u,12}(z) = \delta_{u,12}^T Z_{u,12}(z)$$

where

- $\delta_{u,11} = \tilde{\theta}''_{11,1} - \tilde{\theta}'_{11,1}$ *and* $Z_{u,11} = z^{-1}$.
- $\delta_{u,12} = (\tilde{\theta}''_{12,1} - \tilde{\theta}'_{12,1}, \tilde{\theta}''_{12,2} - \tilde{\theta}'_{12,2})^T$ *and* $Z_{u,12} = (z^{-1}, z^{-2})^T$.

Then, we have that

$$(\Upsilon_{y,11}(z), \Upsilon_{u,1}(z)) = \underbrace{(\tilde{\delta}_{y,1}^T, \tilde{\delta}_{u,1}^T)}_{\tilde{\delta}_1^T}\, bdiag(Z_{y,1}, Z_{u,1}) \tag{34}$$

where $Z_{u,1} = bdiag(Z_{u,11}, Z_{u,12})$ *and* $\delta_{u,1} = (\delta_{u,11}^T, \delta_{u,12}^T)^T$.

The true system $\mathscr{S}$ *is put under feedback control with the following stabilizing controller* $K(z)$*:*

$$K(z) = \underbrace{\begin{pmatrix} 0.4 + 0.29z^{-1} - 0.07z^{-2} \\ 0.04 + 0.01z^{-1} - 0.3z^{-2} \end{pmatrix}}_{N(z)} \underbrace{\frac{1}{1 - 0.62z^{-1} + 0.07z^{-2}}}_{V^{-1}(z)}$$

In this case, the least common multiple of the denominators of $K(z)$ *is directly* $J(z) = 1 - 0.62z^{-1} + 0.07z^{-2}$. *Consequently, the matrix* $\Xi_1(z)$ *in Theorem 7 is equal to* $\Xi_1(z) = (J^T(z), N^T(z))^T$. *Therefore,* $(\Upsilon_{y,11}(z), \Upsilon_{u,1}(z))\Xi_1(z)$ *can be rewritten as*

$$(\tilde{\delta}_{y,1}^T, \tilde{\delta}_{u,1}^T) \underbrace{\begin{pmatrix} Z_{y,1}(z)J(z) \\ \hline Z_{u,1}(z)N(z) \end{pmatrix}}_{L(z)} = (\tilde{\delta}_{y,1}^T, \tilde{\delta}_{u,1}^T) \underbrace{\begin{pmatrix} z^{-1} - 0.62z^{-2} + 0.07z^{-3} \\ \hline 0.4z^{-1} + 0.29z^{-2} - 0.07z^{-3} \\ 0.04z^{-1} + 0.01z^{-2} - 0.3z^{-3} \\ 0.04z^{-2} + 0.01z^{-3} - 0.3z^{-4} \end{pmatrix}}_{L(z)}$$

The smallest and largest value of the monomials $z^{-\beta}$ *among the entries of* $L(z)$ *are* $\beta_{min} = 1$ *and* $\beta_{max} = 4$ *respectively. Therefore, the latter can be rewritten as follows*

$$(\tilde{\delta}_{y,1}^T, \tilde{\delta}_{u,1}^T) \begin{pmatrix} \mathscr{A}_1 \\ \hline \mathscr{B}_1 \end{pmatrix} \begin{pmatrix} z^{-\beta_{min}} \\ \vdots \\ z^{-\beta_{max}} \end{pmatrix} = (\tilde{\delta}_{y,1}^T, \tilde{\delta}_{u,1}^T) \begin{pmatrix} 1 & -0.62 & 0.07 & 0 \\ \hline 0.4 & 0.29 & -0.07 & 0 \\ 0.04 & 0.01 & -0.3 & 0 \\ 0 & 0.04 & 0.01 & -0.3 \end{pmatrix} \begin{pmatrix} z^{-1} \\ z^{-2} \\ z^{-3} \\ z^{-4} \end{pmatrix}$$

The entries in the matrices $\mathscr{A}_1$ *and* $\mathscr{B}_1$ *are the polynomial coefficients of the numerators in* $N(z)$ *and the least common multiple denominator* $J(z)$ *of the controller* $K(z)$. *The number of rows of* $\mathscr{A}_1$ *and* $\mathscr{B}_1$ *are respectively equal to the dimension of* $\tilde{\delta}_{y,1}$ *and* $\tilde{\delta}_{u,1}$, *which are 1 and 3 respectively.*

The matrix $\mathscr{P}_1^{(a)} = (\mathscr{A}_1^T, \mathscr{B}_1^T)^T$ *has a rank of 4: it is full-row rank. Therefore, from Theorem 8, the data* $x(t)$ *are informative with respect to* $\mathscr{M}$ *in the costless framework* ($r = \mathbf{0}$)*. The paper [2] proposes another (sufficient) condition to verify whether* $r = \mathbf{0}$ *can yield informative data. Here, this condition is not satisfied. Consequently, the sufficient condition of Theorem 8 is in this example less restrictive than the one in [2].*

7 Rewriting of $\bar{E}\left[||\Upsilon_{u,i}(z)r(t)||^2\right] = 0$ into (31)

7.1 Main idea of the rewriting of $\bar{E}\left[||\Upsilon_{u,i}(z)r(t)||^2\right] = 0$

This equation has already been studied in the open-loop case [5]. The idea for the rewriting of $\bar{E}\left[||\Upsilon_{u,i}(z)r(t)||^2\right] = 0$ is based on the introduction of the regressor concept [5,7]. For this purpose, let us observe that the n_u entries $\Upsilon_{u,ik}$ of the polynomial vector $\Upsilon_{u,i}$ are the FIR filters given in (28) with the orders $\eta_{u,ik}$ given in appendix E. Hence, we have that

$$\begin{aligned} \Upsilon_{u,i}(z)r(t) &= \sum_{k=1}^{n_u} \Upsilon_{u,ik}(z) r_k(t) \\ &= \sum_{k=1}^{n_u} \tilde{\delta}_{u,ik}^T \phi_{r_k,i}(t) \\ &= \underbrace{\begin{pmatrix} \tilde{\delta}_{u,i1}^T & \tilde{\delta}_{u,i2}^T & \cdots & \tilde{\delta}_{u,in_u}^T \end{pmatrix}}_{\tilde{\delta}_{u,i}^T} \phi_{r,i}(t) \end{aligned} \tag{35}$$

where

$$\phi_{r,i}(t) = \begin{pmatrix} \phi_{r_1,i}(t) \\ \phi_{r_2,i}(t) \\ \vdots \\ \phi_{r_{n_u},i}(t) \end{pmatrix} \quad \text{with} \quad \phi_{r_k,i}(t) = \begin{pmatrix} r_k(t - \rho_{ik}) \\ r_k(t - \rho_{ik} - 1) \\ \vdots \\ r_k(t - \eta_{u,ik}) \end{pmatrix} \tag{36}$$

The signal vector $\phi_{r,i}(t)$ is is generally called regressor or regressor vector in the literature [7]. We have that $\bar{E}\left[||\Upsilon_{u,i}(z)r(t)||^2\right] = 0 \Leftrightarrow \tilde{\delta}_{u,i}^T \bar{E}\left[\phi_{r,i}(t)\phi_{r,i}^*(t)\right] \tilde{\delta}_{u,i} = 0$. In [5], we have shown that, when $r(t)$ is given either by (3) or (4), the regressor vector $\phi_{r,i}$ can be rewritten as $\phi_{r,i}(t) = \mathscr{C}_i \varphi_i(t)$ where $\varphi_i(t)$ is a complex or real-valued signal vector of dimension d such that

$\bar{E}\left[\varphi_i(t)\varphi_i^*(t)\right] \succ 0$ and $\mathscr{C}_i$ a complex or real-valued time independent deterministic matrix of dimension $\zeta_{u,i} \times d$. Consequently, we have that $\bar{E}\left[||\Upsilon_{u,i}(z)r(t)||^2\right] = 0 \Leftrightarrow \tilde{\delta}_{u,i}^T \mathscr{C}_i \bar{E}\left[\varphi_i(t)\varphi_i^*(t)\right] \mathscr{C}_i^* \tilde{\delta}_{u,i} = 0 \Leftrightarrow \tilde{\delta}_{u,i}^T \mathscr{C}_i = \mathbf{0}_{1\times\zeta_{u,i}}$. Therefore, $\bar{E}\left[||\Upsilon_{u,i}(z)r(t)||^2\right] = 0$ is rewritten as

$$\tilde{\delta}_{u,i}^T \mathscr{C}_i = \mathbf{0}_{1\times\zeta_{u,i}} \tag{37}$$

As shown in [5], when the external signal is given by (3), we have that $\phi_{r,i} = \mathscr{C}_i\varphi_i$ with $\varphi_i(t) = \frac{1}{2}(e^{j\omega_1 t}, e^{-j\omega_1 t}, \cdots, e^{j\omega_s t}, e^{-j\omega_s t})$ and $\mathscr{C}_i = (\mathscr{C}_{i,1}^T, \cdots, \mathscr{C}_{i,n_u}^T)^T$ a column block matrix of dimension $\zeta_{u,i} \times 2s$ such that:

$$\mathscr{C}_{i,k}^T = \begin{pmatrix} \overline{\Lambda}_{k1} e^{-j\rho_{ik}\omega_1} & \dots & \overline{\Lambda}_{k1} e^{-j\eta_{u,ik}\omega_1} \\ \overline{\Lambda}_{k1}^* e^{j\rho_{ik}\omega_1} & \dots & \overline{\Lambda}_{k1} e^{j\eta_{u,ik}\omega_1} \\ \vdots & \dots & \vdots \\ \overline{\Lambda}_{ks} e^{-j\rho_{ik}\omega_s} & \dots & \overline{\Lambda}_{ks} e^{-j\eta_{u,ik}\omega_s} \\ \overline{\Lambda}_{ks}^* e^{j\rho_{ik}\omega_s} & \dots & \overline{\Lambda}_{ks}^* e^{j\eta_{u,ik}\omega_s} \end{pmatrix}$$

where the phasors $\bar{\Lambda}_{kl}$ are given by $\bar{\Lambda}_{kl} = \Lambda_{kl}e^{j\Psi_{kl}}$ ($k = 1, \cdots, n_u$, $l = 1, \cdots, s$). Note that the larger the number s of sinusoids in r, the larger the number of columns in $\mathscr{C}_i$.

As shown in [5], when r is given by (4), we can also derive an expression for the matrix $\mathscr{C}_i$ [5]. In this case, the larger the complexity of the $M(z)$ and the number f of white noises generating r are, the larger the number of columns of $\mathscr{C}_i$ is.

8 Interpretation of Theorem 8 for data informativity

Theorem 8 shows that we can verify whether a given external excitation $r(t)$ yields informative data by checking, for each channel, if one of the three matrices $\mathscr{P}_i^{(a)}$, $\mathscr{P}_i^{(b)}$ or $\mathscr{P}_i^{(c)}$ is full row rank.

In particular, if $\mathscr{P}_i^{(a)}$ is full row rank for each channel i, then the data $x(t)$ generated as in Section 3 will yield informative data even if the external excitation $r(t)$ is equal to zero.

To have that $\mathscr{P}_i^{(a)}$ is full row rank, the number of its columns should be larger than ζ_i. If the complexity of the controller and of the model structure (which determines the number of columns of $\mathscr{P}_i^{(a)}$) is not sufficient, $\mathscr{P}_i^{(a)}$ will not be full row rank.

Even if $\mathscr{P}_i^{(a)}$ is not full row rank, we can of course obtain informative data by adding a nonzero excitation $r(t)$ of the type (3) or (4). In this case, the data informativity can be checked by verifying whether $\mathscr{P}_i^{(c)}$ is full row rank.

[5] In [5], this matrix is denoted $\mathscr{L}$.

We observe that $\mathscr{P}_i^{(a)}$ and $\mathscr{P}_i^{(c)}$ have the same number of rows, but the number of columns of $\mathscr{P}_i^{(c)}$ is larger than the one of $\mathscr{P}_i^{(a)}$ (due to the matrix $\mathscr{C}_i$ linked to the external excitation). Consequently, even if $\mathscr{P}_i^{(a)}$ has too few columns, the addition of the external excitation can allow $\mathscr{P}_i^{(c)}$ to have more columns than rows (and thus to imply (in the vast majority of the cases) that the data $x(t)$ are informative.

To do that, we can choose an excitation signal yielding a full row rank matrix $\mathscr{C}_i$, but it is as such not necessary since it is in theory sufficient to have that the sum of the number of columns in $\mathscr{C}_i$ and the number of columns in $\mathscr{P}_i^{(a)}$ is larger than ζ_i. A similar phenomenon was observed in the SISO case [7,8] with the external excitation.

As already mentioned, in the vast majority of the cases, the matrices $\mathscr{P}_i^{(a)}$, $\mathscr{P}_i^{(b)}$, $\mathscr{P}_i^{(c)}$ will be full row rank when the number of rows is smaller than the number of columns. However, for some badly chosen external excitations, controllers, ..., a rank deficiency can occur (as was also observed in in the open loop case [5]) and it is thus important to formally verify the rank of these matrices.

9 Numerical example

9.1 True system to be identified

Consider the following BJ system $\mathscr{S}$ with $n_u = 2$ inputs and $n_y = 1$ output given by

$$y_1(t) = \underbrace{\begin{pmatrix} \frac{-z^{-1}}{1-0.4z^{-1}} & 2z^{-1} \end{pmatrix}}_{=G_0(z)} u(t) + \frac{1}{1+0.5z^{-1}} e_1(t)$$

where $u(t) = (u_1(t), u_2(t))^T$. The system $\mathscr{S}$ is put under feedback control with a stabilizing controller $K(z)$ given by

$$K(z) = \begin{pmatrix} N_{11}(z) \\ N_{21}(z) \end{pmatrix} \underbrace{\frac{1}{J(z)}}_{V^{-1}(z)}$$

$$\begin{aligned} J(z) &= 1 - 0.3z^{-1} - 0.49z^{-2} + 0.155z^{-3} + 0.06z^{-4} - 0.02z^{-5} \\ N_{11}(z) &= 1 + 0.1z^{-1} - 0.24z^{-2} - 0.004z^{-3} + 0.008z^{-4} \\ N_{21}(z) &= 0.7 + 0.13z^{-1} - 0.033z^{-2} + 0.0116z^{-3} - 0.004z^{-4} \end{aligned}$$

We will identify $\mathscr{S}$ within a full-order model structure $\mathscr{M}$ as in (13) with $\rho_{11} = \rho_{12} = 1$,

$$\begin{aligned} \mathring{B}_{11} &= \tilde{\theta}_{11,1} & \mathring{B}_{12} &= \tilde{\theta}_{12,1} \\ F_{11} &= 1+\tilde{\theta}_{11,2}z^{-1} & F_{12} &= 1 \\ C_1 &= 1 & D_1 &= 1+\eta_{1,1}z^{-1} \end{aligned}$$

where $\theta = (\tilde{\theta}_{11,1}, \tilde{\theta}_{11,2}, \tilde{\theta}_{12,1}, \eta_{1,1})^T$. With this model structure, from Lemma 6, the vector $\tilde{\delta}_1 = (\tilde{\delta}_{y,1}^T, \tilde{\delta}_{u,1}^T)^T \in \mathbf{D_1}$ contains 10 polynomial coefficients with 3 in $\tilde{\delta}_{y,1}$ and 7 in $\tilde{\delta}_{u,1}$ since $\eta_{y,1} = 3$, $\eta_{u,11} = 3$ and $\eta_{u,12} = 4$ (see Appendix E).

9.2 Costless identification

Let us study if we can get informative data with a costless experiment, i.e., without external excitation ($r = \mathbf{0}$). For that, we should calculate the rank of the matrix $\mathscr{P}_1^{(a)}$ in Theorem 8 since $r = \mathbf{0}$. But, first, let us observe that the controller has been chosen such that

$$\underbrace{H^{-1}(z,\theta'') - H_0^{-1}(z)}_{\Delta W_y(z)} = \underbrace{(H_0^{-1}(z)G_0(z) - H^{-1}(z,\theta'')G(z,\theta''))}_{\Delta W_u(z)} K(z)$$

with $\theta'' = (0.2, 0.7, 0, 0.7)^T$. From Theorem 3, a zero excitation (i.e, $r = \mathbf{0}$) will therefore not yield informative data with respect to $\mathscr{M}$. Let us verify that the condition in Theorem 8 allows one to check this fact. By following the steps given in Section 5.3, we obtain the following matrix $\mathscr{P}_1^{(a)} = \begin{pmatrix} \mathscr{A}_1 \\ \hline \mathscr{B}_1 \end{pmatrix}$:

$$\mathscr{P}_1^{(a)} = \left(\begin{array}{cccccccc} 1 & -0.3 & 0.49 & -0.155 & 0.06 & -0.02 & 0 & 0 \\ 0 & 1 & -0.3 & 0.49 & -0.155 & 0.06 & -0.02 & 0 \\ 0 & 0 & 1 & -0.3 & 0.49 & -0.155 & 0.06 & -0.02 \\ \hline 1 & 0.1 & -0.24 & -0.04 & 0.008 & 0 & 0 & 0 \\ 0 & 1 & 0.1 & -0.24 & -0.04 & 0.008 & 0 & 0 \\ 0 & 0 & 1 & 0.1 & -0.24 & -0.04 & 0.008 & 0 \\ 0.7 & 0.13 & -0.033 & 0.0116 & -0.004 & 0 & 0 & 0 \\ 0 & 0.7 & 0.13 & -0.033 & 0.0116 & -0.004 & 0 & 0 \\ 0 & 0 & 0.7 & 0.13 & -0.033 & 0.0116 & -0.004 & 0 \\ 0 & 0 & 0 & 0.7 & 0.13 & -0.033 & 0.0116 & -0.004 \end{array}\right)$$

The matrix $\mathscr{P}_1^{(a)}$ has 8 columns for 10 rows. Consequently, it cannot be full row rank. In this example, the (sufficient) condition for data informativity given in Theorem 8 allows to verify without any problem that $r = \mathbf{0}$ cannot yield informative data.

9.3 Identification with external excitation ($r \neq \mathbf{0}$)

As the costless identification cannot yield informative data, we need to add an external excitation r. In this paragraph, we will propose 2 cases.

Case 1: Let us consider a multisine excitation. To make $\mathscr{P}_1^{(b)}$ full row rank, we need at least four sinusoids in $r(t)$ since $\zeta_{u,1} = 7$. However, to make $\mathscr{P}_1^{(c)}$ full row rank, one sinusoid should be sufficient (two columns in $\mathscr{C}_1$) since $\mathscr{P}_1^{(a)}$ has already 8 columns. Let us verify this property. Let us therefore choose a signal $(r_1(t), r_2(t))^T$ with one sinusoid at $\omega_1 = 0.1$rad/s and with the following phasors $\overline{\Lambda}_{kl}$ $(k = 1,2,\ l = 1)$

$$\overline{\Lambda}_{11} = 2e^{j0.2} \qquad \overline{\Lambda}_{21} = 2e^{j0.2}$$

i.e., $r_1(t) = r_2(t) = 2\cos(\omega_1 t + 0.2)\ \forall t$. Here, the matrix $\mathscr{C}_1$ is given by

$$\mathscr{C}_1 = \begin{pmatrix} \overline{\Lambda}_{11}e^{-j\omega_1} & \overline{\Lambda}_{11}^* e^{j\omega_1} \\ \overline{\Lambda}_{11}e^{-2j\omega_1} & \overline{\Lambda}_{11}^* e^{2j\omega_1} \\ \overline{\Lambda}_{11}e^{-3j\omega_1} & \overline{\Lambda}_{11}^* e^{3j\omega_1} \\ \overline{\Lambda}_{21}e^{-j\omega_1} & \overline{\Lambda}_{21}^* e^{j\omega_1} \\ \overline{\Lambda}_{21}e^{-2j\omega_1} & \overline{\Lambda}_{21}^* e^{2j\omega_1} \\ \overline{\Lambda}_{21}e^{-3j\omega_1} & \overline{\Lambda}_{21}^* e^{3j\omega_1} \\ \overline{\Lambda}_{21}e^{-4j\omega_1} & \overline{\Lambda}_{21}^* e^{4j\omega_1} \end{pmatrix} = \begin{pmatrix} 2e^{j0.1} & 2e^{-j0.1} \\ 2 & 2 \\ 2e^{-j0.1} & 2e^{j0.1} \\ 2e^{j0.1} & 2e^{-j0.1} \\ 2 & 2 \\ 2e^{-j0.1} & 2e^{j0.1} \\ 2e^{-j0.2} & 2e^{j0.2} \end{pmatrix}$$

Let us calculate the rank of the corresponding matrix $\mathscr{P}_1^{(c)}$. With this excitation, the matrix $\mathscr{P}_1^{(c)}$ has a rank of 10: it is full-row rank. From Theorem 8, the data $x(t) = (y^T(t), u^T(t))^T$ generated with this excitation r are thus informative with respect to $\mathscr{M}$.

Case 2: Consider now the same frequency as in Case 1 with these following phasors $\overline{\Lambda}_{kl}$ $(k = 1,2,\ l = 1)$

$$\overline{\Lambda}_{11} = 2e^{j0.2} \qquad \overline{\Lambda}_{21} = 0$$

i.e., only one entry of r is excited and the other is set to 0. The matrix $\mathscr{C}_1$ is given by

$$\mathscr{C}_1 = \begin{pmatrix} \overline{\Lambda}_{11}e^{-j\omega_1} & \overline{\Lambda}_{11}^* e^{j\omega_1} \\ \overline{\Lambda}_{11}e^{-2j\omega_1} & \overline{\Lambda}_{11}^* e^{2j\omega_1} \\ \overline{\Lambda}_{11}e^{-3j\omega_1} & \overline{\Lambda}_{11}^* e^{3j\omega_1} \\ \overline{\Lambda}_{21}e^{-j\omega_1} & \overline{\Lambda}_{21}^* e^{j\omega_1} \\ \overline{\Lambda}_{21}e^{-2j\omega_1} & \overline{\Lambda}_{21}^* e^{2j\omega_1} \\ \overline{\Lambda}_{21}e^{-3j\omega_1} & \overline{\Lambda}_{21}^* e^{3j\omega_1} \\ \overline{\Lambda}_{21}e^{-4j\omega_1} & \overline{\Lambda}_{21}^* e^{4j\omega_1} \end{pmatrix} = \begin{pmatrix} 2e^{j0.1} & 2e^{-j0.1} \\ 2 & 2 \\ 2e^{-j0.1} & 2e^{j0.1} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

The rank of the corresponding matrix $\mathscr{P}_1^{(c)}$ is equal to 10: the data $x(t)$ generated with this excitation r are thus also informative with respect to $\mathscr{M}$. It is important to see that we can generate informative data by only exciting one signal in r (as was observed in [11]) or, as in the previous case, by putting $r_1 = r_2$.

9.4 Monte-Carlo simulations

In order to confirm that the input choices in Cases 1 and 2 yield informative data, we have applied each of these external excitation vectors to the true system in 1000 identification experiments (with different realizations of the white noise e, assumed to be Gaussian with of variance of $\sigma_0^2 = 0.01$) and we have identified $\hat{\theta}_N$ (see (9)) for each experiment. For each input vector, we have computed the mean of these 1000 estimates and we have in each case observed that this mean is almost equal to θ_0, given in Table 1, illustrating the consistency.

Table 1
Mean of the identified parameter vector with $N = 10000$ for different cases over 1000 experiments.

$\hat{\theta}_N$	$\tilde{\theta}_{11,1}$	$\tilde{\theta}_{11,1}$	$\tilde{\theta}_{12,1}$	$\eta_{1,1}$
θ_0	−1	−0.4	2	0.5
Case 1	−1.0010	−0.3999	1.9996	0.4994
Case 2	−1.0045	−0.3987	2.0034	0.5033

10 Conclusion

In this work, we have developed a condition that allows one to check if a closed-loop identification experiment could yield informative data with respect to MIMO model structures. We have addressed this study for classical model structures and classical external excitations (multisine or filtered white noise). This condition can be easily verified and depends on the controller complexity, the external excitation parametrization and the complexity of the model structure. Based on this condition, we give also hints to design the experiment to yield informative data.

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A Proof of Theorem 3

The proof is adapted from the one in [2]. The idea is to prove that the left hand sides of (8) and (21) are equivalent and that the same holds for the right hand sides. Note that the latter is straightforward since $\Delta W(z) \equiv \mathbf{0} \Leftrightarrow (\Delta W_y(z), \Delta W_u(z)) \equiv (\mathbf{0}, \mathbf{0})$.

First, combining (1) and (2) leads to

$$x(t) = \begin{pmatrix} y(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} S_y(z)G_0(z)r(t) + S_y(z)H_0(z)e(t) \\ S_u(z)r(t) - S_u(z)K(z)H_0(z)e(t) \end{pmatrix} \quad \text{(A.1)}$$

where S_u and S_y are respectively the input and output sensitivity transfer functions of the closed-loop depicted in Fig. 1, defined respectively by $S_u(z) = (\mathbf{I}_{n_u} + K(z)G_0(z))^{-1}$ and $S_y(z) = (\mathbf{I}_{n_y} + G_0(z)K(z))^{-1}$. Using (A.1) and the independence assumption between r and e, $\bar{E}\left[||\Delta W(z)x(t)||^2\right] = 0$ leads to

$$\begin{cases} \bar{E}\left[||\left(\Delta W_y(z)S_y(z) - \Delta W_u(z)S_u(z)K(z)\right)H_0(z)e(t)||^2\right] = 0 \\ \bar{E}\left[||\left(\Delta W_u(z)S_u(z) + \Delta W_y(z)S_y(z)G_0(z)\right)r(t)||^2\right] = 0 \end{cases} \quad \text{(A.2)}$$

We are going to prove that (A.2) is equivalent to the left hand side of (21).

Let us first prove that the first equation of (A.2) is equivalent to $\Delta W_y(z) - \Delta W_u(z)K(z) \equiv \mathbf{0}$. Since $\bar{E}\left[e(t)e^T(t)\right] \succ 0$ and $H_0(z)$ is stable and inversely stable, we have that the power spectrum matrix $\Phi_{\bar{v}}$ of $\bar{v}(t) = H_0(z)e(t)$ is strictly positive definite at all frequencies. Consequently, the first equation of (A.2) is equivalent to

$$\Delta W_y(z)S_y(z) - \Delta W_u(z)S_u(z)K(z) \equiv \mathbf{0} \tag{A.3}$$

From the push-through rule [13, Chapter 3], $S_u(z)K(z) = K(z)S_y(z)$. Consequently, (A.3) is equivalent to

$$\left(\Delta W_y(z) - \Delta W_u(z)K(z)\right) S_y(z) \equiv \mathbf{0} \tag{A.4}$$

Finally, by post-multiplying by $S_y^{-1}(z)$, we obtain

$$\Delta W_y(z) - \Delta W_u(z)K(z) \equiv \mathbf{0} \tag{A.5}$$

which is the first equation of the left hand side of (21).

Let us now prove that the second equation of (A.2) is equivalent to $\bar{E}\left[||\Delta W_u(z)r(t)||^2\right] = 0$. Combining the second equation of (A.2) with (A.5) leads to

$$\bar{E}\left[||\Delta W_u(z)\left(S_u(z) + K(z)S_y(z)G_0(z)\right) r(t)||^2\right] = 0 \tag{A.6}$$

Again, from the push-through rule [13, Chapter 3], $S_y(z)G_0(z) = G_0(z)S_u(z)$. Consequently, (A.6) is equivalent to

$$\bar{E}\left[||\Delta W_u(z)\left(\mathbf{I}_{n_u} + K(z)G_0(z)\right) S_u(z)r(t)||^2\right] = 0 \tag{A.7}$$

Finally, by observing[6] that $\mathbf{I}_{n_u} + K(z)G_0(z) = S_u^{-1}(z)$, we obtain that the second equation of (A.2) is equivalent to $\bar{E}\left[||\Delta W_u(z)r(t)||^2\right] = 0$. This concludes the proof. ■

B Proof of Theorem 5

We are going to prove that the property (21) for all $\Delta W(z) = (\Delta W_y(z), \Delta W_u(z)) \in \Delta_{\mathbf{W}}$ is equivalent to the fact that the property (22) holds for all $\Delta W_{i:}(z) = (\Delta W_{y,i:}(z), \Delta W_{u,i:}(z)) \in \Delta_{\mathbf{W,i}}$ and for all $i = 1, \cdots, n_y$.

Let us first observe that $\Delta W(z) \equiv \mathbf{0}$ is equivalent to $\Delta W_{i:}(z) \equiv \mathbf{0}\ \forall i \in [\![1, n_y]\!]$. Secondly, it is also straightforward to see that $\Delta W_y(z) - \Delta W_u(z)K(z) \equiv \mathbf{0}$ is equivalent to $\Delta W_{y,i:}(z) - \Delta W_{u,i:}(z)K(z) \equiv 0\ \forall i \in [\![1, n_y]\!]$. Finally, let us prove that $\bar{E}\left[||\Delta W_u(z)r(t)||^2\right] = 0$ is equivalent to $\bar{E}\left[||\Delta W_{u,i:}(z)r(t)||^2\right] = 0\ \forall i \in [\![1, n_y]\!]$. For this purpose, observe that the term $\bar{E}\left[||\Delta W_u(z)r(t)||^2\right] = 0$ can be recast as follows

$$\bar{E}\left[||\Delta W_u(z)r(t)||^2\right] = \sum_{i=1}^{n_y} \bar{E}\left[||\Delta W_{u,i:}(z)r(t)||^2\right]$$

Since the term $\bar{E}\left[||\Delta W_{u,i:}(z)r(t)||^2\right]$ is non-negative ($i = 1, \cdots, n_y$), we have indeed that $\bar{E}\left[||\Delta W_u(z)r(t)||^2\right] = 0$ is equivalent to $\bar{E}\left[||\Delta W_{u,i:}(z)r(t)||^2\right] = 0$ ($i = 1, \cdots, n_y$).

We have thus proven that the property (21) for all $\Delta W(z) \in \Delta_{\mathbf{W}}$ is equivalent to

$$\begin{cases} \Delta W_{y,i:}(z) - \Delta W_{u,i:}(z)K(z) \equiv \mathbf{0} \\ \bar{E}\left[||\Delta W_{u,i:}(z)u(t)||^2\right] = 0 \end{cases} (i = 1, \cdots, n_y) \tag{B.1}$$
$$\implies \Delta W_{i:}(z) = (\Delta W_{y,i:}(z), \Delta W_{u,i:}(z)) = \mathbf{0}\ (i = 1, \cdots, n_y)$$

for all $\Delta W_{i:}(z) = (\Delta W_{y,i:}(z), \Delta W_{u,i:}(z)) \in \Delta_{\mathbf{W,i}}$ ($i = 1, \cdots, n_y$).

Using the parametrization introduced in Section 4, we observe that there are no common parameters in $\Delta W_{i:}(z) = (\Delta W_{y,i:}(z), \Delta W_{u,i:}(z))$ and $\Delta W_{j:}(z) = (\Delta W_{u,j:}(z), \Delta W_{y,j:}(z))$ ($j \neq i$). Therefore, (B.1) is equivalent to the fact that property (22) holds for all $\Delta W_{i:}(z) = (\Delta W_{y,i:}(z), \Delta W_{u,i:}(z)) \in \Delta_{\mathbf{W,i}}$ and for all $i = 1, \cdots, n_y$, completing the proof. ■

C Proof of Lemma 6

For the right factorization of $(\Delta W_{y,i:}(z), -\Delta W_{u,i:}(z)) \in \Delta_{\mathbf{W,i}}$, we will consider the one that consists on putting all entries of $(\Delta W_{y,i:}(z), -\Delta W_{u,i:}(z))$ on the same denominators.

First, the matrix $H(z,\theta)$ is diagonal for all $\theta \in \mathscr{D}_\theta$ and so only the i-th entry of the $1 \times n_y$ row-matrix $\Delta W_{y,i:}$ is possibly non-zero. Let us denote $\Delta W_{y,ii}$ the i-th entry of $\Delta W_{y,i:}$. Observe first that $\Delta W_{y,ii}(z) = W_{y,ii}(z,\theta') - W_{y,ii}(z,\theta'') = 1 - H_i^{-1}(z,\theta') - \left(1 - H_i^{-1}(z,\theta'')\right) = H_i^{-1}(z,\theta'') - H_i^{-1}(z,\theta')$.

Let us first study the case of the ARMAX model structure (see (12)). For all $\Delta W_{i:}(z) = (\Delta W_{y,i:}(z),\ \Delta W_{u,i:}(z)) \in \Delta_{\mathbf{W,i}}$, we have that

$$\begin{cases} \Delta W_{y,ii}(z) = C_i^{-1}(z,\theta'')A_i(z,\theta'') - C_i^{-1}(z,\theta')A_i(z,\theta') \\ -\Delta W_{u,i:}(z) = C_i^{-1}(z,\theta'')B_{i:}(z,\theta'') - C_i^{-1}(z,\theta')B_{i:}(z,\theta') \end{cases} \tag{C.1}$$

Therefore, by posing $Q_i = C_i' C_i''$, we obtain the factorization proposed in the lemma. Since ARX and FIR model structures are special cases of ARMAX model structures with $C_i(z,\eta) = 1$ and $C_i(z,\eta) = A_i(z,\eta) = 1$ ($i = 1, \cdots, n_y$) respectively, we also obtain the factorization proposed in the lemma for these two model structures.

Let us now consider the BJ model structures in (13). For

[6] And by assuming that $S_u(z)$ does not filter out any part of r.

all $\Delta W_{i:}(z) = (\Delta W_{y,i:}(z),\ \Delta W_{u,i:}(z)) \in \Delta_{\mathbf{W,i}}$, we have that

$$\begin{cases} \Delta W_{y,ii}(z) = C_i^{-1''} D_i'' - C_i^{-1'} D_i' \\ \Delta W_{u,i:}(z) = C_i^{-1'} D_i' G_{i:}' - C_i^{-1''} D_i'' G_{i:}'' \end{cases} \tag{C.2}$$

where $G_{i:}'$ and $G_{i:}''$ are respectively the i-th row of G' and G''. Since C_i', C_i'', D_i' and D_i'' are scalars, we have that

$$\begin{aligned} &(\Delta W_{y,ii}(z), -\Delta W_{u,i:}(z)) \\ &= C_i^{-1'} C_i^{-1''} (C_i' D_i'' - C_i'' D_i' \ ,\ C_i' D_i'' G_{i:}'' - C_i'' D_i' G_{i:}') \end{aligned} \tag{C.3}$$

Let us put each entry of $G_{i:}(z,\theta')$ and $G_{i:}(z,\theta'')$ on the same denominator as follows

$$G_{i:}(z,\theta') = \mathscr{F}_i^{-1}(z,\theta')\mathscr{G}_i(z,\theta') \tag{C.4}$$

$$G_{i:}(z,\theta'') = \mathscr{F}_i^{-1}(z,\theta'')\mathscr{G}_i(z,\theta'') \tag{C.5}$$

where $\mathscr{F}_i(z,\theta') = \prod_{k=1}^{n_u} F_{ik}'$ and $\mathscr{G}_i(z,\theta') = (\mathscr{G}_{i,k}')_{k\in[\![1,n_u]\!]}$ is a row vector of polynomials with $\mathscr{G}_{i,k}(z,\theta') = B_{ik}(z,\theta')\prod_{l=1,l\neq k}^{n_u} F_{il}(z,\theta')$. The matrices $\mathscr{F}_i(z,\theta'')$ and $\mathscr{G}_i(z,\theta'')$ are defined similarly.

Since $\mathscr{F}_i(z,\theta')$ and $\mathscr{F}_i(z,\theta'')$ are scalars, (C.3) can be recast as follows

$$\begin{aligned} \Delta W_{y,ii} &= Q_i^{-1}\Upsilon_{y,ii} = Q_i^{-1}(\mathscr{F}_i'\mathscr{F}_i'' C_i' D_i'' - \mathscr{F}_i'\mathscr{F}_i'' C_i'' D_i') \\ -\Delta W_{u,i:} &= Q_i^{-1}\Upsilon_{u,i} = Q_i^{-1}(C_i' D_i'' \mathscr{F}_i' \mathscr{G}_i'' - C_i'' D_i' \mathscr{F}_i'' \mathscr{G}_i') \end{aligned}$$

where $Q_i = C_i' C_i'' \mathscr{F}_i' \mathscr{F}_i''$. For OE, $C_i' = C_i'' = D_i' = D_i'' = 1$. Hence, for OE model structures, $\Upsilon_{y,ii} \equiv 0$ and $\Upsilon_{u,i}$ is the one defined in the statement. This concludes the proof. ■

D Proof of Theorem 7

We will prove that the property (22) is equivalent to (26). First, $(\Upsilon_{y,ii}, \Upsilon_{u,i}) \equiv (0, \mathbf{0})$ is equivalent to $(\Delta W_{y,i:}, \Delta W_{u,i:}) = (\mathbf{0}, \mathbf{0})$ since Q_i is invertible. Hence the right hand sides of (22) and (26) are equivalent.

Secondly, the equation $\Delta W_{y,i:}(z) - \Delta W_{u,i:}(z)K(z) = \mathbf{0}$ can be rewritten as follows

$$\left(\Delta W_{y,i:}(z),\ -\Delta W_{u,i:}(z)\right)\begin{pmatrix} \mathbf{I}_{n_y} \\ N(z)V^{-1}(z) \end{pmatrix} \equiv (\mathbf{0}, \mathbf{0})$$

By post-multiplying by $V(z)$ and by pre-multiplying by $Q_i(z)$, the latter is equivalent to

$$\left(\Upsilon_{y,i}(z),\ \Upsilon_{u,i}(z)\right)\begin{pmatrix} V(z) \\ N(z) \end{pmatrix} \equiv (\mathbf{0}, \mathbf{0})$$

By observing that all entries $\Upsilon_{y,ij}$ $(j \neq i)$ of the row vector $\Upsilon_{y,i}$ are equal to 0, the latter is equivalent to

$$\left(\Upsilon_{y,ii}(z),\ \Upsilon_{u,i}(z)\right)\underbrace{\begin{pmatrix} V_{i:}(z) \\ N(z) \end{pmatrix}}_{\Xi_i(z)} \equiv (0, \mathbf{0}) \tag{D.1}$$

which is the first equation of the left hand side of (26).

Finally, $\bar{E}[||\Delta W_{u,i:}(z)r(t)||^2] = 0$ is equivalent to $\bar{E}[||Q_i^{-1}(z)\Upsilon_{u,i}(z)r(t)||^2] = 0$. The latter is equivalent to $\bar{E}[||Q_i(z)Q_i^{-1}(z)\Upsilon_{u,i}(z)r(t)||^2] = 0$ which is in turn equivalent to $\bar{E}[||\Upsilon_{u,i}(z)r(t)||^2] = 0$, which is the desired result. ■

E Degrees of the polynomials $\Upsilon_{y,ii}$ and $\Upsilon_{u,ik}$

When non-zero (i.e., for the BJ/ARX/ARMAX case), $\Upsilon_{y,ii}(z)$ ca be written as in (27) with $\eta_{y,i}$ (see Lemma 6):

- $\eta_{y,i} = \deg(A_i)$ for the ARX case.
- $\eta_{y,i} = \deg(A_i) + \deg(C_i)$ for the ARMAX case.
- $\eta_{y,i} = \deg(C_i) + \deg(D_i) + 2\sum_{k=1}^{n_u} \deg(F_{ik})$ for the BJ case.

The polynomials $\Upsilon_{u,ik}(z)$ can be written as in (28) with ρ_{ik} as defined in Section 4 and with $\eta_{u,ik}$ (see Lemma 6):

- $\eta_{u,ik} = \deg(B_{ik})$ for the FIR/ARX case.
- $\eta_{u,ik} = \deg(B_{ik}) + \deg(C_i)$ for the ARMAX case.
- $\eta_{u,ik} = \deg(B_{ik}) + \deg(F_{ik}) + 2\sum_{l=1,l\neq k}^{n_u} \deg(F_{il})$ for the OE case.
- $\eta_{u,ik} = \deg(C_i) + \deg(D_i) + \deg(B_{ik}) + \deg(F_{ik}) + 2\sum_{l=1,l\neq k}^{n_u} \deg(F_{il})$ for the BJ case.