Study of a stable hybridization method to couple FVTD,GD scheme with FDTD/FEM scheme

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Abstract

The purpose of this paper is the study of a stable method for coupling two types of schemes using or not numerical fluxes for the resolution of Maxwell's equations in time domain. The methods considered in the hybridization process are Finite Volume (FV), Finite Element (FEM), Finite Difference (FD) and Discontinuous Galerkin (DG). We give the principle of the hybrid method and by the study of an energetic quantity, one specifies the conditions of stability of it. Finally, one example is given to validate the proposed hybrid method.

Keywords: Hybrid method, finite element scheme, Galerkin discontinuous scheme, Maxwell's equations in time domain

Introduction

Current electromagnetic problems require the processing of complex structures in which geometrical details, computational time and size of the discrete problem are important factors to consider in the simulation. To increase the performances in terms of computation time and storage memory, the use of cartesian meshes is prefered, whereas near the geometry, to guarantee the precision, it is rather necessary to use a non-structured mesh. For each type of mesh or class of problems, there are different kinds of schemes more or less appropriate to solve it. This is due to the approximations made in the scheme which induce different kinds of errors on the solution. Then, to improve the simulation, a good idea consists in dividing the computational domain into different sub-domains where we apply the most appropriate numerical method to solve the Maxwell equations. In this strategy, different numerical schemes are coupled and the difficulty lies in having a stable and consistent hybrid scheme. In this paper, we propose a stable hybrid method allowing to take into account FV, FD, FEM and DG schemes. After the presentation of the hybridization principle, by considering a leapfrog time discretization for the hybrid method, one shows that it exists a condition according to the schemes taken in the hybridization which ensures the conservation of an energetic quantity. This condition gives a stability condition of the hybrid method. Next, we give some results on the study of a propagative mode inside a wave guide to validate the proposed hybrid method.

Principle of the hybrid method

In this paper we present a hybrid method based on the coupling of some numerical schemes (FEM, DG, FD etc.) to solve the transient Maxwell equations. One of the main difficulties is to ensure the stability of the method at the space and time level. In the literature, it exists, for exemple, several approaches constructed from interpolation techniques which suffer from numerical instabilities. To overcome this problem, we propose to use a DG formalism in order to couple together in a stable manner the subdomains in which different numerical methods are used. Now, we briefly explain our approach. Let $(\Omega_i)_{i=1,\cdots,N}$ be a partition of the computational domain Ω, $\Sigma_{i,j} := \Omega_i \cap \Omega_j$, $n_{i,j}$ the unit normal to $\Sigma_{i,j}$ directed from Ω_i et Ω_j and $V(i)$ is the set of neighbour subdomains of Ω_i. In a first step, we write the Maxwell problem in Ω as a transmission one on the subdomains : $\forall i = 1, \cdots, N$,

$$\varepsilon \frac{\partial E_i}{\partial t} - \nabla \times H_i = J \text{ in } \Omega_i \tag{1a}$$

$$\mu \frac{\partial H_i}{\partial t} + \nabla \times E_i = 0 \text{ in } \Omega_i \tag{1b}$$

$$n_{i,j} \times E_i = n_{i,j} \times E_j \text{ on } \Sigma_{i,j},\ \forall j \in V(i) \tag{1c}$$

$$n_{i,j} \times H_i = n_{i,j} \times H_j \text{ on } \Sigma_{i,j},\ \forall j \in V(i) \tag{1d}$$

Next, we rewrite (1) in a weak form by using a DG approach : $\forall \varphi,\ \psi \in H(curl, \Omega_i)$,

$$\int_{\Omega_i} \left(\varepsilon \frac{\partial E_i}{\partial t} - \nabla \times H_i \right) \cdot \varphi \, dx = \int_{\Omega_i} J \cdot \varphi dx + \sum_{j \in V(i)} \int_{\Sigma_{i,j}} \alpha_{i,j} (n_{i,j} \times H_i - n_{i,j} \times H_j) \cdot \varphi d\gamma \tag{2a}$$

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$$\int_{\Omega_i}\left(\mu\frac{\partial H_i}{\partial t}+\nabla\times E_i\right)\cdot\psi\,dx$$
$$=\sum_{j\in V(i)}\int_{\Sigma_{i,j}}\gamma_{i,j}(n_{i,j}\times E_i-n_{i,j}\times E_j)\cdot\psi d\gamma \tag{2b}$$

To ensure the equivalence of this reformulation with the original problem, the coefficients $\alpha_{i,j}$ and $\gamma_{i,j}$ must be as follows :

$$1-\alpha_{i,j}+\alpha_{j,i}=0 \text{ and } 1-\gamma_{i,j}+\gamma_{j,i}=0$$
$$\alpha_{i,j}+\gamma_{j,i}=0 \text{ and } \alpha_{i,j}+\gamma_{j,i}=0$$

Stability

We now use in each subdomain Ω_i an adapted spatial numerical scheme (FEM, DG, FDTD, FV) and a second order leapfrog scheme for the time discretization. To ensure the stability of the hybrid scheme, we study the evolution in time of a quantity given by $\int_\Omega \varepsilon E(x)\cdot E(x)dx+\int_\Omega \nu H(x)\cdot H(x)dx$.

More precisely, we have to prove that this quantity is maintained or decreased along the time. For example, by considering an hybridization between GD [1] and FEM [2] methods, we have [3] :

$$\Delta t\le\frac{2}{c_0\Big[\max(A_1,A_2,A_3)\Big]} \qquad \text{with}$$

$$A_1=\sqrt{\rho\Big(\hat{\mathscr{M}}_1^{-\frac{1}{2}}\hat{\mathscr{R}}_1\hat{\mathscr{M}}_1^{-\frac{1}{2}}\Big)}+\frac{\sqrt{\mu}}{\sqrt{\varepsilon}}\max\Big(\rho\Big(\hat{\mathscr{M}}_1^{-\frac{1}{2}}\hat{\mathscr{B}}_1^{E}\hat{\mathscr{M}}_1^{-\frac{1}{2}}\Big)\Big)$$

$$A_2=\rho\Big(\hat{\mathscr{M}}_1^{-\frac{1}{2}}\hat{\mathscr{B}}_1^{H}\hat{\mathscr{M}}_1^{-\frac{1}{2}}\Big)$$

$$A_3=\frac{1}{\Lambda_{\mathcal{K}}}\Big(\sqrt{\rho\Big(\hat{\mathscr{M}}_2^{-\frac{1}{2}}\hat{\mathscr{R}}_2\hat{\mathscr{M}}_2^{-\frac{1}{2}}\Big)}+\frac{\sqrt{\mu}}{\sqrt{\varepsilon}}\rho\Big(\hat{\mathscr{M}}_2^{-\frac{1}{2}}\hat{\mathscr{B}}_2\hat{\mathscr{M}}_2^{-\frac{1}{2}}\Big)$$

with $\rho(A)$ the spectrum radius, c_0 the waves speed, the matrices $\hat{\mathscr{M}}_2,\hat{\mathscr{R}}_2,\hat{\mathscr{B}}_2$ the mass, stiffness and jump matrices for the DG scheme and $\hat{\mathscr{M}}_1,\hat{\mathscr{R}}_1,\hat{\mathscr{B}}_1^E,\hat{\mathscr{B}}_1^H$ the same for FEM. $\Lambda_{\mathcal{K}}$ is a coefficient which depends on the Jacobian matrix of the transformation between $\mathcal{K}$ and the unit element for the DG scheme.

Numerical validation

To validate our hydrid method, we propose to study a propagative mode inside a curved wave guide. The guide is divided into two parts on which a FEM and a DG scheme are respectively applied (see Figure 1). In Figure 2, we compare the results obtained with our hybrid FEM/DG method and with a DG method applied on all the problem. We can show on this figure the good agreement between the solutions obtained by our FEM/DG and the DG methods at a test-point located inside the curved guide.

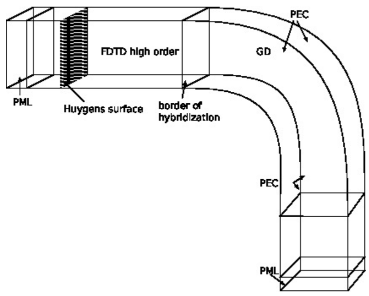

Figure 1: Curved guide geometry with the two domains for FEM/GD Hybrid method.

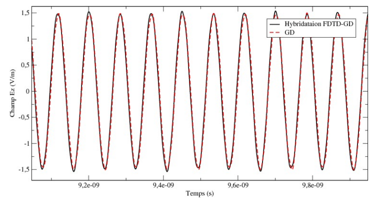

Figure 2: Comparison between FEM/DG and DG approaches.

Conclusion

In this paper, we proposed a stable hybrid method between continuous and discontinuous Gakerkin schemes, in order to solve the Maxwell equations in time domain. We have specified how to establish a stability condition for two combined methods and we give, as example, a stability condition for a hybrid FEM/DG method. Finally, we showed a validation of this method by comparison to a reference DG solution, based on the computation of a propagative mode inside a curved wave guide.

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