On space-time methods based on Isogeometric analysis for structures analysis

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Abstract

Despite some early pioneering attempts in the 1970s and 1980s, space-time methods did not experience the same success as finite-elements combined with discrete time-integration scheme (implicit/explicit finite-differences, Newmark, HHT, etc). Since these years, the software and hardware context has changed and we propose to rediscuss the interest of these methods in the context of the Isogeometric Analysis method (IGA). We show the construction of different formulations adapted to different situations in solid mechanics: elastodynamics, viscoelasticity, thermo-mechanical coupling, etc. From representative numerical examples, we show that we can obtain promising numerical performances (optimum convergence rate, …) that make space-time methods a credible alternative to standard paradigms for structures analysis.

Introduction

Space-time methods based on finite elements were first proposed by Argyris & al [1969], Oden [1969], Hughes & al [1988], Hulbert & al [1990] for elastodynamics problems. These authors have proposed original discontinuous Galerkin formulations (in time) with application to 1D (in space) cases. Continuous Galerkin formulation have also been proposed by French [1993], Idesman [2006] and other authors. From these works, it can be seen that the development of an appropriate weak formulation plays a central role for the stability of the numerical scheme and for capturing appropriately sharp solutions in time. Since the pioneer works in the 70s, the progress made on computer hardware has allowed to break the limitations in terms of memory usage and space-time methods have been applied for 2D and 3D applications (e.g. Dumont & al [2018]). Nevertheless, all these works have had a limited impact on the mechanical community, and exception made of the work of Tezduyar & al [1992] in fluid mechanics there are very few developments in solid mechanics. In parallel, to FE approaches, space-time Isogeometric Analysis has been recently studied from a mathematical point of view (e.g. Langer & al [2016]). Compare to FE, IGA allows more possibilities and flexibility in the control and construction of approximations spaces (continuity and order of the approximation). The numerical properties of NURBS based approximation offers the possibility to obtain higher numerical performances. Nevertheless, to the authors knowledge, space-time methods have been only developed in the context of linear elasticity or linear viscoelasticity. In this communication, we first propose to illustrate and study the construction of stable and convergent IGA space-time formulations for these linear problems: linear elastodynamics and linear viscoelasticity. We show that we can achieve optimum rate of convergence in space and time and that we can appropriately capture sharp solution in time. By studying viscoelasticity formulations, we will show that space-time methods allow to change the classical paradigm of a local numerical integration of evolution laws obtained from continuum thermodynamics approaches of dissipative phenomena. We also discuss the interest of applying space-time methods to other linear problems with different characteristic times such as thermo-mechanical ones. As perspective, we will give some elements of discussion for the extension to non-linear problems.

Methods

The central idea of space-time methods is based on the simultaneous discretization (approximation) of both space and time with an appropriate basis of functions. Time dependence is therefore treated in the same manner as spatial dependence. A linear evolution problem therefore leads to the resolution of a linear system. With this paradigm, choices made about approximation spaces and the construction of a weak form play a central role both for the numerical properties obtained and for taking boundary conditions into account in a simple way.

For instance, if we consider a linear elastodynamics problem that can be express from the following equations:

$$\begin{aligned}
&\rho\frac{\partial^2 u}{\partial t^2}-\mathrm{div}_x\,\sigma=f \quad \text{in}\,\Omega\times[0,T]\\
&\sigma=C:\epsilon\\
&\epsilon=(\nabla_x u)_{sym}\\
&u(x,t=0)=u_0(x),\ \frac{\partial u}{\partial t}(x,t=0)=v_0(x)\\
&u(x,t)=u_d \ \ on\,\partial\Omega_u,\ \ \sigma.n=t \ \ on\,\partial\Omega_f,\ \frac{\partial u}{\partial t}=v_d \ \ on\,\partial\Omega_v
\end{aligned} \tag{1}$$

a good approach is to used the following weak form with a two fields formulation:

$$\begin{gathered}
\int_Q \rho\frac{\partial v}{\partial t}.\,\delta\dot{u}\,dQ+\int_Q \epsilon:C:(\nabla\,\delta\dot{u})_{sym}\,dQ-\int_S(\sigma.n).\,\delta\dot{u}\,dS-\int_Q f\,\delta\dot{u}\,dQ=0\\
\int_Q \rho\left(\frac{\partial u}{\partial t}\delta\dot{v}-v\,\delta\dot{v}\right)dQ=0
\end{gathered} \tag{2}$$

where $Q=\Omega\times[0,T]$ is the space time domain and $S=\partial\Omega_f\times[0,T]$. The previous weak form has the advantage, first to account of Neumann boundary conditions on S and Dirichlet boundary conditions for $u(x,t), v(x,t)$, and second to lead to a symmetric system. Therefore, using approximation functions for $u(x,t), v(x,t)$ such as:

$$u_h(x,t)=\sum_i N_i(x,t)u_i,\ \ v_h(x,t)=\sum_i N_i(x,t)v_i \tag{3}$$

leads to the resolution of the following linear system:

$$\begin{bmatrix} K_{uu} & K_{uv}\\ K_{vu} & K_{vv}\end{bmatrix}\begin{Bmatrix} u_i\\ v_i\end{Bmatrix}=\begin{Bmatrix} f_u\\ 0\end{Bmatrix} \tag{4}$$

Equations (1) to (4) give the basic scheme of building (linear) space-time formulations within a continuous Galerkin formulation. The same approach can be used with viscoelasticity or thermo-elasticity problems. For each problems we need to develop a specific formulation. Discontinuous Galerkin formulations can also be used without any limitations.

In this work, we use NURBS-based functions to build the approximation base for unknown fields. This makes it possible to obtain higher continuity properties for a given degree of polynomials and thus to achieve better numerical performance, especially in the time domain.

Results

As a typical example of application we can consider the case of an elastic bar with an initial homogeneous velocity that impacts a rigid wall. This test involve a sharp solution in time: propagation of a discontinuous compression wave just after the impact. For this test, the weak formulation of eq. (2) needs to be slightly modified by adding a supplementary Galerkin Least Square term on the first equation such as to limit the numerical oscillations that appears in such a problem (purely elastic). Figure 1 shows the results of the space-time IGA solution compared to the exact solution. It can be seen that increasing the polynomial degree while keeping the same mesh size leads to a better agreement of the numerical solution and the exact one. This result illustrates the ability of NURBS functions to capture such sharp solution.

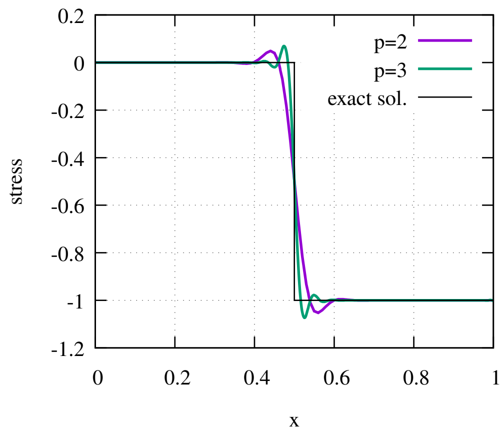

Figure 1: Bar impact: normalized compressive stress upon the position along the bar at t=0.5s

The same example can also be considered for a 2D plane strain solid. Boundary conditions are defined such as the solid can not separate from the wall after impact, but transversal displacements are free. Figures 1 and 2 show the results of space time computations with NURBS functions of order 2 with and without stabilization. It can be seen that, as expected, the stabilization limits the apparition of numerical oscillations.

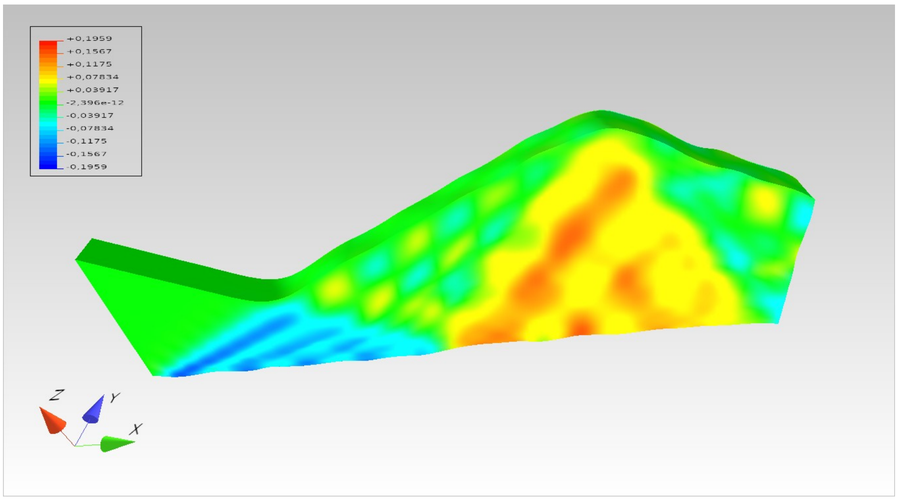

Figure 2: Impact of a plane strain elastic beam-like structures: x axis corresponds to time, y and z to space. Isocolor of the first component of the kinematic field. Non stabilized solution.

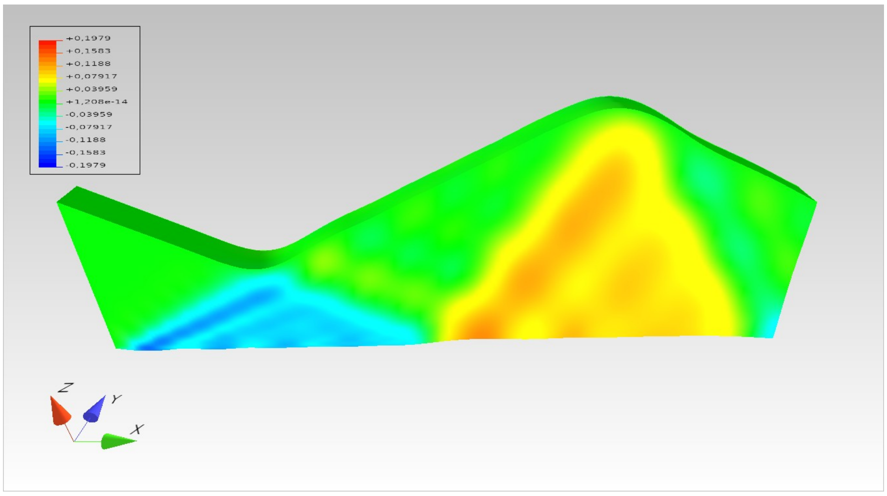

Figure 3: Impact of a plane strain elastic beam-like structures: x axis corresponds to time, y and z to space. Isocolor of the first component of the kinematic field. Stabilized solution with Galerkin Least Square terms.

Conclusions & Contributions

To conclude, space-time methods combined with isogeometric analysis offer promising open fields of developments. These methods and the latest software and hardware developments may in a near future represent a credible alternative to standard simulation methods of time evolving problems. Nevertheless, their applications to real industrial problems require further developments in different areas : automatic space and time mesh refinement to capture localization in time and space, parallelisation in time with domain decomposition methods, non-linear and dissipative problems, etc.

Argyris J.H., Scharpf D.W., Finite elements in time and space. Nuclear Engineering and Design, 10, 456-464, 1969

Oden J.T., A general theory of finite elements II. Applications. International Journal of Numerical Methods in Engineering. 1, 247-259, 1969

Hughes T.J.R, Hulbert G.M., Space-time finite elements methods for elastodynamics: formulations and error estimates. Computer Methods in Applied Mechanics and Engineering. 66, 339-363, 1988

Hulbert G.M., Hughes T.J.R, Space-time finite elements methods for second-order hyperbolic equations. Computer Methods in Applied Mechanics and Engineering. 84, 327-348, 1990

French D.A., A space-time finite element method for the wave equation. Computer Methods in Applied Mechanics and Engineering. 107, 145-157, 1993

Idesman, A.V., Solution of linear elastodynamics problems with space-time finite elements on structured and unstructured meshes. Computer Methods in Applied Mechanics and Engineering. 196, 1787-1815, 2007

Langer, U., Moore S.E., Neumüller M., Space-time isogeometric analysis of parabolic evolution problems. Computer Methods in Applied Mechanics and Engineering. 306, 342-363, 2016

Dumont S., Jourdan F., Madani T., A space-time finite element method for elastodynamics problems: elementary examples of 4D remeshing using simplex elements. Mathematical and Computational Application, MDPI, 23, pp. 29, 2018

Tezduyar T.E., Behr M., Liou J. , A new strategy for finite element computations involving moving boundaries and interfaces- The deforming-spatial-domain/space-time procedure: I. The concept and the preliminary numerical tests, Comput. Methods Appl. Mech. Engrg., 94 (1992) pp. 339-351