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Alessio Moreschini [*,**], **Salvatore Monaco** [*], and
Dorothée Normand-Cyrot [**]

[*] *Dipartimento di Ingegneria Informatica, Automatica e Gestionale A. Ruberti (Sapienza University of Rome), Via Ariosto 25, 00185 Rome, Italy (e-mail:{ alessio.moreschini,salvatore.monaco} @uniroma1.it).*
[**] *Laboratoire des Signaux et Systèmes (L2S, CNRS), 3, rue Joliot Curie, 91192, Gif-sur-Yvette, France (e-mail:{ alessio.moreschini,cyrot} @l2s.centralesupelec.fr)*

Abstract: In this paper gradient and Hamiltonian dynamics are investigated in both discrete-time and sampled-data contexts. At first, the discrete gradient function is profitably employed to define discrete gradient and Hamiltonian dynamics. On these bases, it is shown that representations of these forms can be recovered when computing the sampled-data equivalent models to gradient and Hamiltonian continuous-time dynamics.

Keywords: Sampled-data systems; Discrete gradient methods; Discrete Hamiltonian dynamics.

1. INTRODUCTION

Gradient and Hamiltonian dynamics have straight relations with fundamental properties of physical systems such as conservation and/or variational principles; they are widely investigated and are at the basis of ad hoc design approaches (e.g. Wiggins (2003), van der Schaft et al. (2014)). Digital analysis and design methods, which make use of discrete-time models representing a given plant, possibly under sampling, are faced with the preservation of such properties; properties which are lost under usual sampling techniques (e.g. Stramigioli et al. (2005), Tiefensee et al. (2010), Monaco et al. (2011), Mattioni et al. (2019)).

Discrete gradient methods, introduced in Gonzalez (1996) and McLachlan et al. (1999) to solve numerical integration problems, have been employed in the last decade to characterize discrete Hamiltonian structures. Several approaches have been proposed (e.g. Laila and Astolfi (2006), Sümer and Yalçın (2011) Yalçın et al. (2015), and Aoues et al. (2017)). Such solutions, although preserving dissipative and conservative properties, are based on approximated, essentially Euler type, sampled-data models which do not reproduce the continuous-time behaviours at the sampling instants. A different approach is then proposed by Talasila et al. (2006) which directly models the systems in a discrete setting. The aim of this paper is to go further answering the following question: does the exact sampled-data equivalent dynamics exhibit a discrete gradient or discrete Hamiltonian form?

To properly address the problem, a precise characterization of gradient and Hamiltonian dynamics in discrete-time must be given; it turns out that the discrete gradient function can be profitably used to this purpose. The problem is addressed in this paper making reference to dynamics associated with quadratic real valued functions in order to explicitly compute the discrete gradient. It is shown that to preserve both the energetic properties and to match the state trajectories of the continuous-time system it is necessary to modify the interconnection and dissipation terms in the state space representation of the discrete dynamics. More precisely, one defines those parts through suitable matrices depending on the sampling period δ. The proof is constructive in the sense that these matrices are described by their asymptotic expansions in powers of δ around the continuous-time solutions. It comes out that the approaches proposed in the literature (Aoues et al. (2017), Sümer and Yalçın (2011), and McLachlan et al. (1999)) correspond to first-order approximations of the solution here proposed. This preliminary study is performed for gradient dynamics and then extended to conservative and dissipative Hamiltonian dynamics.

The paper is organized as follows. In Section 2 preliminary concepts are introduced. In Section 3 the problem is addressed for gradient dynamics. Section 4 addresses the problem for conservative and dissipative Hamiltonian dynamics. An elementary example is used to stress the different behaviors. The paper ends with some concluding remarks.

2. NOTATIONS AND PRELIMINARIES

Throughout the paper all the functions and vector fields defining the dynamics are assumed smooth and complete over the respective definition spaces. The sets $\mathbb{R}$ and $\mathbb{N}$ denote, respectively, the set of real and natural numbers including 0. For any vector $v \in \mathbb{R}^n$, $|v|$ and $v^\top$ define the norm and transpose of v respectively. For $v, w \in \mathbb{R}^n$, $\langle v, w \rangle$ denotes the inner product, i.e. $\langle v, w \rangle = v^\top w$. I_d denotes the identity function or identity matrix while I

denotes the identity operator. Given a real-valued function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}$ assumed differentiable, ∇V is used to represent the gradient vector, ∇ denoting the differential operator vector. "$(\cdot) > 0$" and "$(\cdot) < 0$" denote functions or matrices positive or negative definite (let recall that for a function such definition may be local or global and holds with respect to a point where the function takes the zero value). Given a smooth vector field over $\mathbb{R}^n$, e^f (indifferently e^{L_f}) denotes the exponential Lie operator $e^f := I + \sum_{i \geq 1} \frac{\mathrm{L}_f^i}{i!}$ in the Lie operator $\mathrm{L}_f = \sum_{i=1}^n f_i(x) \frac{\partial}{\partial x_i}$; for a linear vector field the exponential Lie operator recovers the exponential of the matrix representing the operator. For any smooth function $h(\cdot) : \mathbb{R}^n \to \mathbb{R}$ then $e^f h(x) = h(e^f(x)) = e^f h|_x$ where $|_x$ denotes the evaluation of the function at x. For sampled-data systems, $x_k := x(k\delta)$ and $x_{k+1} := x((k+1)\delta)$, $\forall k \in \mathbb{N}$ and $\delta \in]0, T[$, a finite time interval; $x_k := x(k)$ and $x_{k+1} := x(k+1)$ in a pure discrete-time context. The arguments of the functions are dropped when clear from the context.

Let us recall from the concerned literature (see Gonzalez (1996), McLachlan et al. (1999)) the following definition.

Definition 2.1. (Discrete gradient). Given $V(\cdot) : \mathbb{R}^n \to \mathbb{R}$ a differentiable real-valued function, its discrete gradient is a vector-valued function $\bar\nabla V(v, w) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ which satisfies the equality

$$\langle (w - v), \bar\nabla V(v, w) \rangle = V(w) - V(v),$$

with $\bar\nabla V(v, v) = \nabla V(v)$ for continuity argument.

According to Definition 2.1 and setting

$$\bar\nabla V(v, w) = \begin{bmatrix} \bar\nabla V(v_1, w_1) & \cdots & \bar\nabla V(v_n, w_n) \end{bmatrix}^\top$$

for $v = (v_1, \ldots, v_n)^\top$ and $w = (w_1, \ldots, w_n)^\top$, then the discrete gradient can be computed according to

$$\bar\nabla V(v_i, w_i) = \frac{1}{w_i - v_i} \int_{v_i}^{w_i} \frac{\partial V(v_1, ..., v_{i-1}, \xi, w_{i+1}, ..., w_n)}{\partial \xi} \, \mathrm{d}\xi.$$

Lemma 2.1. Given $V(v) = \frac{1}{2} v^\top P v$, with symmetric P matrix, then the associated discrete gradient verifies

$$\bar\nabla V(v, w) = \frac{1}{2} P(v + w) = \frac{1}{2}(\nabla V(v) + \nabla V(w)). \quad (1)$$

Proof. This is directly deduced from the equality

$$V(w) - V(v) = \frac{1}{2} w^\top P w - \frac{1}{2} v^\top P v$$
$$= \frac{1}{2}(w - v)^\top P(v + w).$$

3. GRADIENT DYNAMICS

Gradient dynamics are preliminarily described in both continuous and discrete time. It is shown that sampled-data equivalent models recover discrete gradient forms thanks to the introduction of suitable interconnection matrices which depend on the sampling period. The problem, set in the nonlinear context, is solved when considering gradient dynamics associated with quadratic forms.

3.1 Continuous-time and discrete-time gradient dynamics

Given a C^r ($r \geq 2$) real-valued function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}$, a *continuous-time gradient dynamics* is defined as

$$\dot{x}(t) = f(x(t)) = -\nabla V(x(t)) \quad (2)$$

so directly concluding that by construction

- any equilibrium x_e of (2) ($f(x_e) = 0$) coincides with a local extremum of $V(x)$ ($\nabla V(x_e) = 0$);
- $\dot{V}(x) = \mathrm{L}_f V(x) = -|\nabla V(x)|^2$;
- x_e is an asymptotically stable equilibrium of (2) provided $V(x)$ is positive-definite.

Similarly, a *discrete-time gradient dynamics* can be defined in terms of the discrete gradient of $V(x)$ as

$$x_{k+1} - x_k = -\bar\nabla V(x_k, x_{k+1}) \quad (3)$$

so verifying by construction that:

- any equilibrium x_e of (3) coincides with a local extremum of $V(\cdot)$ ($\bar\nabla V(x_e, x_e) = \nabla V(x_e) = 0$);
- $V(x_{k+1}) - V(x_k) = -|\bar\nabla V(x_k, x_{k+1})|^2$;
- x_e is an asymptotically stable equilibrium provided $V(\cdot)$ is positive-definite.

When $V(\cdot)$ is a quadratic form then the discrete gradient dynamics (3) can be explicitly computed from (1).

Proposition 1. Assume $V(x) = \frac{1}{2} x^\top P x$, with symmetric positive-definite matrix $P = P^\top > 0$, then (3) rewrites

$$x_{k+1} = x_k + F x_k = \left(I + \frac{1}{2} P\right)^{-1} \left(I - \frac{1}{2} P\right) x_k$$

where for any square matrix X over $\mathbb{R}^n$ so that $I + X$ is invertible, the inverse matrix is formally defined as $(I + X)^{-1} = I + \sum_{p \geq 1} (-1)^p X^p$.

3.2 Problem statement and motivating example

Let us first recall from (Monaco and Normand-Cyrot, 1990) that the equivalent sampled-data model to the nonlinear dynamics (2) admits for any $\delta \in]0, T[$ (T small enough) the discrete-time representation

$$x_{k+1} = F^\delta(x_k) \quad (4)$$

where $F^\delta(x_k)$ is given by its asymptotic expansion in powers of δ

$$F^\delta(x_k) = e^{\delta f} x_k = x_k + \delta \mathrm{L}_f x|_k + \frac{\delta^2}{2} \mathrm{L}_f^2 x|_k + \ldots.$$

Setting $f(x) = -\nabla V(x)$, we address the question: does the equivalent sampled-data dynamics (4) admit a discrete gradient form?

When $V(x)$ is a quadratic function, it is shown in the sequel that an "equivalent" discrete gradient form can be computed; it turns out to be a quadratic δ-dependent function which is specified as a series expansion in powers of δ. The approximation at the first order recovers the discrete gradient dynamics usually adopted in the current literature.

The following elementary example is used to better point out the posed question. Given

$$\dot{x}(t) = -x(t) = -\nabla V(x(t)) \quad (5)$$

with $V(x) = \frac{1}{2} x^2$, does the sampled data equivalent model

$$x_{k+1} = e^{-\delta} x_k \quad (6)$$

admit a discrete gradient form?

As said before, it is quite usual (see Aoues et al. (2017), Sümer and Yalçın (2011), McLachlan et al. (1999), etc.)

to associate to (5) the discrete gradient dynamics below making reference to the same function $V(\cdot)$, i.e.

$$x_{k+1} - x_k = -\delta \bar{\nabla} V(x_k, x_{k+1}). \qquad (7)$$

Such a form preserves the stability of the evolutions at the sampling instants $t = k\delta$, since one gets along (7)

$$V(x_{k+1}) - V(x_k) = -\delta(\bar{\nabla} V(x_k, x_{k+1}))^\top \bar{\nabla} V(x_k, x_{k+1}) < 0,$$

but its equivalent explicit representation does not match the state evolutions of (2) at the sampling instants. As a matter of fact, as $\bar{\nabla} V(x_k, x_{k+1}) = \frac{1}{2}(x_k + x_{k+1})$, (7) rewrites as

$$x_{k+1} = (1 + \frac{\delta}{2})^{-1}(1 - \frac{\delta}{2})x_k \qquad (8)$$

which is not equivalent to (6).

We will show in the sequel that the exact sampled-data dynamics (6) does satisfy a new discrete gradient form

$$x_{k+1} - x_k = -\delta \mathcal{I}(\delta, -I)\bar{\nabla} V(x_k, x_{k+1}) \qquad (9)$$

with a suitably defined δ-dependent matrix $\mathcal{I}(\delta, -I)$. Moreover, (7) recovers (8) in first approximation in δ.

3.3 Gradient dynamics under sampling

Given the continuous-time dynamics (2) with quadratic form $V(x) = \frac{1}{2}x^\top P x$ and $P = P^\top > 0$, then its sampled equivalent dynamics

$$x_{k+1} := e^{-\delta P} x_k \qquad (10)$$

satisfies the forward difference inequality

$$V(x_{k+1}) - V(x_k) = -\int_{k\delta}^{(k+1)\delta} |\nabla V(x(\tau))|^2 \mathrm{d}\tau < 0. \quad (11)$$

The question now relies on the possibility to rewrite (10) into a discrete gradient form (3) with respect to a suitably defined $V(\cdot)$ function.

The following matrix will be instrumental throughout the rest of the paper. Given $X \in \mathbb{R}^{n \times n}$ and $\delta \in]0, T[$, we denote by $\mathcal{I}(\delta, X) \in \mathbb{R}^{n \times n}$, the matrix which satisfies the algebraic equality below

$$\delta \mathcal{I}(\delta, X)X = 2(e^{\delta X} - I)(I + e^{\delta X})^{-1} \qquad (12)$$

where the inverse is again formally defined by the series

$$(I + e^{\delta X})^{-1} = I + \sum_{p \geq 1} (-1)^p e^{p\delta X}.$$

Accordingly, one gets by construction the description of $\mathcal{I}(\delta, X)$ as the series expansion in δ below

$$\mathcal{I}(\delta, X) = \sum_{p \geq 0} \sum_{j_0 \geq 0, j_1, \ldots, j_p \geq 1} \frac{(-1)^p (\delta X)^{\sum_{i=0}^p j_i}}{2^p (j_0 + 1)! j_1! \ldots j_p!}. \qquad (13)$$

For the first terms one computes

$$\mathcal{I}(\delta, X) = I - \frac{\delta^2}{3!2}X^2 + \frac{\delta^4}{5!}X^4 + O(\delta^6)$$

so verifying that the coefficients of the odd powers in X $((\delta X)^{2i+1}, i \geq 0)$ are equal to zero in the expansion (13).

On these bases the following result can be proved.

Theorem 3.1. Given the gradient dynamics (2) with function $V(x) = \frac{1}{2}x^\top P x$, then for any fixed $\delta \in]0, T[$, its sampled-data equivalent dynamics (10) admits,

(a) the discrete gradient form

$$x_{k+1} - x_k = -\delta \mathcal{I}(\delta, -P)\bar{\nabla} V(x_k, x_{k+1}) \qquad (14)$$

with matrix $\mathcal{I}(\delta, -P)$ defined as in (13);

(b) equivalently, the discrete gradient form

$$x_{k+1} - x_k = -\delta \bar{\nabla} V(\delta)(x_k, x_{k+1}) \qquad (15)$$

with respect to the new quadratic function $V(\delta)(x) = \frac{1}{2}x^\top (-P)(\delta)x$ with symmetric positive-definite square matrix $(-P)(\delta) = \mathcal{I}(\delta, -P)P$;

(c) according to (14), one gets

$$V(x_{k+1}) - V(x_k) = -\delta(\bar{\nabla} V)^\top \mathcal{I}(\delta, -P)\bar{\nabla} V \qquad (16)$$

$$= -\int_{k\delta}^{(k+1)\delta} |\nabla V(x(\tau))|^2 \mathrm{d}\tau < 0.$$

Proof. As far as (a) is concerned, according to the definition of discrete gradient and considering the sampled-data equivalent dynamics (10), the equality (14) rewrites as

$$e^{-\delta P} x = x - \frac{\delta}{2} \mathcal{I}(\delta, -P)P(I + e^{-\delta P})x$$

which holds true by construction of $\mathcal{I}(\delta, -P)$ given in (13) when replacing the X matrix with $-P$, so that the following equality is verified

$$\delta \mathcal{I}(\delta, -P)P = 2(I - e^{-\delta P})(I + e^{-\delta P})^{-1}.$$

Regarding (b), (15) follows from (14) when setting $V(\delta)(x) = \frac{1}{2}x^\top (-P)(\delta)x$ with $(-P)(\delta) = \mathcal{I}(\delta, -P)P$. Moreover $(-P)(\delta)$, which is given by the expansion

$$(-P)(\delta) = \sum_{p \geq 0} \sum_{j_0 \geq 0, j_1, \ldots, j_p \geq 1} \frac{(-1)^p (-\delta P)^{\sum_{i=0}^p j_i}}{2^p (j_0 + 1)! j_1! \ldots j_p!} P$$

$$= P - \frac{\delta^2}{3!2}P^3 + \frac{\delta^4}{5!}P^5 + O(\delta^6),$$

is by construction a symmetric matrix. Its positivity for all $\delta \in]0, T[$ follows from the equality (16) in (c). Such equality is a direct consequence of (11), of the discrete gradient definition, and of the gradient form of the dynamics (14). $\square$

The results in Theorem 3.1 show that:

- the dynamics (10) exhibits a gradient form (14) with respect to the same real-valued function $V(\cdot)$ as in the continuous-time case through a new connection matrix $\mathcal{I}(\delta, -P)$ which depends on the function $V(\cdot)$ itself and is described by its series expansion in powers of δ. The negativity of the forward difference $V(x_{k+1}) - V(x_k)$ follows as it exactly matches the continuous-time $V(x)$ evolution at the sampling instants according to (16);

- the dynamics (15) (equivalent to (14)) is defined with respect to a different real-valued function $V(\delta)(\cdot)$ parameterized by δ which preserves symmetry and positivity of $V(\cdot)$. The negativity of the forward difference $V(\delta)(x_{k+1}) - V(\delta)(x_k)$ follows by definition of the gradient form itself, i.e.

$$V(\delta)(x_{k+1}) - V(\delta)(x_k) = -\delta |\bar{\nabla} V(\delta)|^2 < 0$$

but differs from (16).

For completeness, it is worth mentioning that usual sampled-data gradient structures proposed in Yalçin et al. (2015), Aoues et al. (2017), Sümer and Yalçin (2011),

and McLachlan et al. (1999) correspond to Euler type approximations of the proposed results. More precisely, one can solve in $\mathcal{I}(\delta, -P)$ the equation (14) when setting $x_{k+1} - x_k = -\delta P x_k$ so restricting the dynamics to the Euler approximation (first-order in δ) of the exact sampled dynamics (10). Equivalently, this corresponds to set in (15) $V(\delta)(x) = V(x)$ so getting approximated results at the first-order in δ. To conclude, we note that the discrete gradient form associated to the exact sampled data model of the elementary dynamics (5) is given by

$$x_{k+1} = x_k - \frac{\delta}{2} \mathcal{I}(\delta, -1)(x_k + x_{k+1})$$
$$= x_k - 2\left(\frac{1 - e^{-\delta}}{1 + e^{-\delta}}\right) \bar{\nabla} V(x_k, x_{k+1})$$

which clearly generalizes the usually proposed approximated sampled gradient dynamics (7).

4. HAMILTONIAN DYNAMICS

The previous results are now extended to Hamiltonian dynamics described over $\mathbb{R}^{2n}$ when the real-valued function $H(\cdot)$ is an "energy-like" function associated with a dynamics expressed in the canonical position and momenta coordinates (q, p). In Section 4.2, the problem of preserving Hamiltonian forms under sampling is discussed for both conservative and dissipative Hamiltonian dynamics with quadratic energy function and constant interconnection and damping matrices.

4.1 Continuous and discrete Hamiltonian dynamics

A continuous-time Hamiltonian dynamics is given by
$$\dot{x} = f(x) = (J(x) - R(x))\nabla H(x) \tag{17}$$
where $H(\cdot) : \mathbb{R}^{2n} \to \mathbb{R}$ is assumed a C^r ($r \geq 2$) function, $J(x) \in \mathbb{R}^{2n \times 2n}$ a skew-symmetric non-degenerate matrix, and $R(x) \geq 0 \in \mathbb{R}^{2n \times 2n}$ a symmetric matrix whose entries are functions of x and characterizing the stored and dissipated energy respectively. The following comments are in order:

- any equilibrium x_e of (17) coincides with a local extremum of $H(x)$ ($\nabla H(x_e) = 0$);
- assuming $H(x)$ positive-definite, one gets
$$\dot{H}(x) = \mathrm{L}_f H(x) = -(\nabla H(x))^\top R(x) \nabla H(x) < 0$$
which implies asymptotic stability of x_e;
- when $R(x) = 0$,
$$\dot{H}(x) = \mathrm{L}_f H(x) = 0$$
corresponding to the conservative property of the Hamiltonian along (17).

Along the same lines, discrete Hamiltonian dynamics can be defined in terms of the discrete gradient of $H(\cdot)$.

Definition 4.1. Given a C^r ($r \geq 2$) real-valued function $H(\cdot) : \mathbb{R}^{2n} \to \mathbb{R}$, a discrete Hamiltonian dynamics is given by
$$x_{k+1} = x_k + (J(x_k) - R(x_k))\bar{\nabla} H(x_k, x_{k+1}) \tag{18}$$
where $J(x), R(x)$ are square matrices satisfying $J(x) = -J^\top(x)$, $R(x) = R^\top(x) \geq 0$.

Analogously to the continuous-time case, the following holds true:

- any equilibrium of (18) coincides with a local extremum of $H(\cdot)$ ($\bar{\nabla} H(x_e, x_e) = \nabla H(x_e) = 0$);
- assuming $H(x)$ positive-definite, one gets from (18)
$$H(x_{k+1}) - H(x_k) = -(\bar{\nabla} H)^\top R(x_k) \bar{\nabla} H < 0$$
which implies asymptotic stability of x_e;
- when $R(x) = 0$, the discrete dynamics is energy conservative
$$H(x_{k+1}) - H(x_k) = -(\bar{\nabla} H)^\top J(x_k) \bar{\nabla} H = 0,$$

with $\bar{\nabla} H = \bar{\nabla} H(x_k, x_{k+1})$.

An explicit representation of (18) can be computed when $H(x)$ is a quadratic form and the damping and interconnection matrices J and R are constant matrices.

Proposition 2. Set $V(x) = \frac{1}{2} x^\top P x$, $J = -J^\top$, and $R = R^\top \geq 0$, then the discrete Hamiltonian dynamics (18) equivalently satisfies the difference equation

$$x_{k+1} = x_k + F x_k = \left(I - \frac{1}{2}(J - R)P\right)^{-1} \left(I + \frac{1}{2}(J - R)P\right) x_k.$$

4.2 Hamiltonian dynamics under sampling

In what follows it is shown that the exact sampled equivalent dynamics to (17) recovers a discrete Hamiltonian form when assuming a quadratic energy function $H(x) = \frac{1}{2} x^\top P x$ with symmetric positive-definite matrix P, and constant interconnection and dissipation matrices $J = -J^\top$, $R = R^\top \geq 0$. Consider the continuous-time dynamics
$$\dot{x}(t) = (J - R)Px(t) \tag{19}$$
with exact sampled equivalent dynamics described by
$$x_{k+1} := e^{\delta(J-R)P} x_k. \tag{20}$$
Since by construction the evolutions of (19) and (20) from the initial state $x(0) = x_0$ coincide at the sampling instants $t = k\delta$, the sampled dynamics (20) satisfies the forward difference

$$H(x_{k+1}) - H(x_k) = -\int_{k\delta}^{(k+1)\delta} (\nabla H(x(\tau)))^\top R \nabla H(x(\tau)) \mathrm{d}\tau < 0$$

so matching, at the sampling instants, the energy behavior of the continuous-time dynamics as well.

Remark 4.1. If the matrix JP characterizing the Hamiltonian dynamics (19) with $R = 0$ is infinitesimally symplectic (Marsden and Ratiu (2013)) (i.e. $(JP)^\top \Omega = -\Omega JP$ for a skew-symmetric non-degenerate matrix Ω) then its equivalent sampled-data dynamics
$$x_{k+1} = e^{\delta JP} x_k$$
is defined by a symplectic matrix (i.e. $(e^{\delta JP})^\top \Omega e^{\delta JP} = \Omega$).

The result below extends the result of Section 3 to conservative dynamics (19) ($R = 0$).

Theorem 4.1. Given a conservative Hamiltonian dynamics (19) with $R = 0$, then for any $\delta \in]0, T[$ its sampled equivalent model (20) admits a conservative discrete Hamiltonian form

$$x_{k+1} = x_k + \delta \mathcal{J}(\delta) \bar{\nabla} H(x_k, x_{k+1}) \tag{21}$$
with skew-symmetric $\mathcal{J}(\delta) = \mathcal{I}(\delta, JP)J \in \mathbb{R}^{2n \times 2n}$ satisfying the equality
$$\delta \mathcal{J}(\delta)P = 2(e^{\delta JP} - I)(I + e^{\delta JP})^{-1}. \tag{22}$$

Proof. By definition of $\mathcal{I}(\delta, JP)$, the matrix $\mathcal{J}(\delta) = \mathcal{I}(\delta, JP)J$ can be computed to satisfy the equality

$$e^{\delta JP}x = x + \frac{\delta}{2}\mathcal{J}(\delta)P(I + e^{\delta JP})x$$

so recovering (22). Moreover it is a matter of computation to verify that the matrix $\mathcal{J}(\delta)$ admits the series expansion

$$\mathcal{J}(\delta) = \sum_{p \geq 0} \sum_{j_0 \geq 0, j_1, \ldots, j_p \geq 1} \frac{(-1)^p (\delta JP)^{\sum_{i=0}^{p} j_i}}{2^p (j_0 + 1)! j_1! \ldots j_p!} J$$

$$= J - \frac{\delta^2}{3!2}(JP)^2 J + \frac{\delta^4}{5!}(JP)^4 J + O(\delta^6)$$

which is characterized by terms in J at odd power indices only so proving that the skew-symmetry of J is preserved for $\mathcal{J}(\delta)$. $\square$

By construction, the discrete Hamiltonian dynamics (21) is energy preserving, i.e.

$$H(x_{k+1}) - H(x_k) = \delta(\bar{\nabla}H)^\top \mathcal{J}(\delta)\bar{\nabla}H = 0.$$

Remark 4.2. Defining the matrix $P(\delta) \in \mathbb{R}^{2n \times 2n}$ as
$$JP(\delta) = \mathcal{I}(\delta, JP)JP$$
then (21) can be rewritten as
$$x_{k+1} = x_k + \delta J\bar{\nabla}H(\delta)(x_k, x_{k+1})$$
so defining a new energy function $H(\delta) = \frac{1}{2}x^\top P(\delta)x$ depending on δ. It can be proven that $P(\delta) \in \mathbb{R}^{2n \times 2n}$ is again a symmetric positive-definite matrix for any $\delta \in]0, T[$ that is described by an infinite sum of matrices of the form $P(JP)^i$, $i = 2, 4, \ldots$, which result to be alternatively negative and positive definite, starting from $i = 2$. For the first terms, one gets

$$P(\delta) = P - \frac{\delta^2}{3!2}P(JP)^2 + \frac{\delta^4}{5!}P(JP)^4 + O(\delta^6)$$

with $P(JP)^2 < 0$ and $P(JP)^4 > 0$.

Let us now address the case of dissipative Hamiltonian dynamics. According to Theorem 4.1, it is easy to show that the sampled equivalent dynamics to (19) can be rewritten in the Hamiltonian form below

$$x_{k+1} = x_k + \delta Q(\delta)\bar{\nabla}H(x_k, x_{k+1})$$

with by definition $Q(\delta) := \mathcal{I}(\delta, (J-R)P)(J-R) \in \mathbb{R}^{2n \times 2n}$ satisfying the equality

$$\delta Q(\delta)P = 2(e^{\delta(J-R)P} - I)(e^{\delta(J-R)P} + I)^{-1}. \quad (23)$$

Again, one gets

$$H(x_{k+1}) - H(x_k) = \delta(\bar{\nabla}H)^\top Q(\delta)\bar{\nabla}H$$
$$= -\int_{k\delta}^{(k+1)\delta}(\nabla H(x(\tau)))^\top R\nabla H(x(\tau))\mathrm{d}\tau < 0$$

so concluding negativity of $Q(\delta)$ for all $\delta \in]0, T[$. Splitting $Q(\delta)$ into a skew-symmetric (conservative) and a symmetric (dissipative) part respectively, the following holds.

Theorem 4.2. Given a dissipative Hamiltonian dynamics (19), then for any $\delta \in]0, T[$ its equivalent sampled model (20) admits a dissipative discrete Hamiltonian form

$$x_{k+1} = x_k + \delta(J(\delta) - R(\delta))\bar{\nabla}H(x_k, x_{k+1}) \quad (24)$$

possessing an asymptotically stable equilibrium in $x_e = 0$ with matrices

$$J(\delta) = \frac{1}{2}(\mathcal{Q}(\delta) - \mathcal{Q}^\top(\delta)), \quad R(\delta) = -\frac{1}{2}(\mathcal{Q}(\delta) + \mathcal{Q}^\top(\delta)) \quad (25)$$

respectively skew-symmetric and symmetric positive semi-definite, and $Q(\delta) \in \mathbb{R}^{2n \times 2n}$ defined in (23).

Proof. Setting $Q(\delta) = \mathcal{I}(\delta, (J-R)P)(J-R)$, one immediately deduces (23) by definition of the $\mathcal{I}(\delta, X)$ matrix which satisfies (12). Accordingly, one computes

$$Q(\delta) = \sum_{p \geq 0} \sum_{j_0 \geq 0, j_1, \ldots, j_p \geq 1} \frac{(-1)^p (\delta(J-R)P)^{\sum_{i=0}^{p} j_i}}{2^p (j_0 + 1)! j_1! \ldots j_p!}(J-R)$$

$$= \left(I - \frac{\delta^2}{3!2}((J-R)P)^2 + \frac{\delta^4}{5!}((J-R)P)^4\right)(J-R) + O(\delta^6).$$

By construction, the matrix $J(\delta)$ defined in (25) is skew symmetric and $R(\delta)$ is symmetric. Moreover, since

$$H(x_{k+1}) - H(x_k) = \delta(\bar{\nabla}H)^\top \mathcal{Q}(\delta)\bar{\nabla}H$$
$$= -\delta(\bar{\nabla}H)^\top R(\delta)\bar{\nabla}H$$
$$= -\int_{k\delta}^{(k+1)\delta}(\nabla H)^\top R\nabla H\mathrm{d}\tau < 0,$$

one concludes positivity of $R(\delta)$ from positivity of R. $\square$

Theorem 4.1 and Theorem 4.2 show that the dynamics (24) (equivalently (21)) preserves the Hamiltonian representation with respect to the same energy function $H(\cdot)$ with modified skew-symmetric $J(\delta)$ and symmetric positive-definite $R(\delta)$ matrices described by their series expansions in δ so as to verify

$$\Delta H = -\delta(\bar{\nabla}H(x_k, x_{k+1}))^\top R(\delta)\bar{\nabla}H(x_k, x_{k+1})$$

with $\Delta H = H(x_{k+1}) - H(x_k)$, (equivalently $H(x_{k+1}) = H(x_k)$ when $R = 0$).

An example. Let the simple continuous-time second order Hamiltonian dynamics

$$\dot{x} = (J - R)\nabla H(x) = \begin{cases} \dot{q} = p \\ \dot{p} = -q - \alpha p \end{cases} \quad (26)$$

with quadratic Hamiltonian function $H(q, p) = \frac{1}{2}q^2 + \frac{1}{2}p^2$ and positive dissipation coefficient α. According to the current literature the sampled Hamiltonian dynamics associated to (26) is assumed of the form

$$\begin{pmatrix} q_{k+1} \\ p_{k+1} \end{pmatrix} = \begin{pmatrix} q_k \\ p_k \end{pmatrix} + \delta \begin{pmatrix} 0 & 1 \\ -1 & -\alpha \end{pmatrix} \bar{\nabla}H(q_k, p_k, q_{k+1}, p_{k+1}). \quad (27)$$

When substituting $\bar{\nabla}H = \frac{1}{2}[q_k + q_{k+1}, p_k + p_{k+1}]^\top$ into (27), one gets

$$\begin{pmatrix} q_{k+1} \\ p_{k+1} \end{pmatrix} = \begin{pmatrix} \dfrac{4 - \delta^2 + 2\delta\alpha}{\delta^2 + 2\delta\alpha + 4} & \dfrac{4\delta}{\delta^2 + 2\delta\alpha + 4} \\ -\dfrac{4\delta}{\delta^2 + 2\delta\alpha + 4} & \dfrac{4 - \delta^2 - 2\delta\alpha}{\delta^2 + 2\delta\alpha + 4} \end{pmatrix} \begin{pmatrix} q_k \\ p_k \end{pmatrix}$$

which differs from the equivalent sampled data model

$$\begin{pmatrix} q_{k+1} \\ p_{k+1} \end{pmatrix} = e^{\delta \begin{pmatrix} 0 & 1 \\ -1 & -\alpha \end{pmatrix}} \begin{pmatrix} q_k \\ p_k \end{pmatrix}$$

which admits a sampled Hamiltonian representation of the form (24).

Figures 5.1 and 5.2 depict the trajectories and the evolutions of $H(\cdot)$ from the initial state $q = p = 2$ under
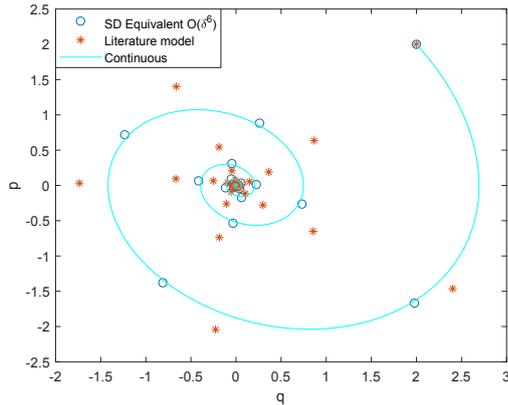
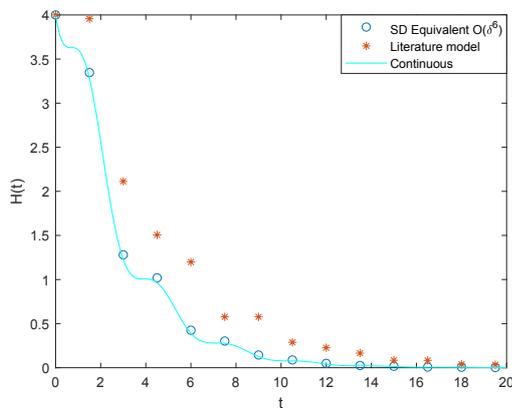Fig. 5.1. Evolution of the system with $\alpha = 0.4, \delta = 1.5$.

Fig. 5.2. Evolution of the energy with $\alpha = 0.4, \delta = 1.5$.

continuous-time (blue), literature sampled (sampled-red) and proposed sampled (sampled-blue) Hamiltonian representations. It is worthy to note the performances of the presently proposed model.

5. CONCLUSIONS

The paper describes discrete gradient and Hamiltonian dynamics in terms of the discrete gradient of a certain real-valued "energy-like" function. It is shown that discrete gradient or Hamiltonian representations can be recovered under exact sampling with respect to the same energy function and modified interconnection and dissipation matrices. The proposed method holds both the objectives of satisfying under sampling the energetic properties which characterize Hamiltonian dynamics and that of matching the state evolutions at the sampling instants. In this sense, the proposed method outperforms the discrete-time forms currently set in the related literature. Work is in progress regarding the extension to the general nonlinear context and how to exploit these forms for control purpose as in Ortega and Garcia-Canseco (2004) and following the lines of a preliminary study in Tiefensee et al. (2010). Finally, it is intriguing to note that in the conservative case, the sampled data Hamiltonian representation can be interpreted as resulting either from the same energy function with δ-dependent interconnection matrix or from a δ-depend energy function and the same interconnection. This suggests that a similar dichotomy should be present in discrete Hamiltonian modeling as in (Talasila et al., 2006).

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