Multiple Lie Derivatives and Forests

Florent Hivert, Nefton Pali

Multiple Lie Derivatives and Forests

Florent HIVERT and Nefton PALI

Abstract

We obtain a complete time expansion of the pull-back operator generated by a real analytic flow of real analytic automorphisms acting on analytic tensor sections of a manifold. Our expansion is given in terms of multiple Lie derivatives. Motivated by this expansion, we provide a rather simple and explicit estimate for higher order covariant derivatives of multiple Lie derivatives acting on smooth endomorphism sections of the tangent bundle of a manifold. We assume the covariant derivative to be torsion free. The estimate is given in terms of Dyck polynomials. The proof uses a new result on the combinatorics of rooted labeled ordered forests and Dyck polynomials.

Contents

1 Notations and motivation

In this paper the products over ordered sets of indices are always considered from the left to the right with respect to the order structure of the set. For any integer $k \geqslant 1$, we set $[k] := \{1, \ldots, k\}$ and we denote by $\lambda \vDash k$ any element

$\lambda \equiv (\lambda_1, \ldots, \lambda_{l_\lambda}) \in \mathbb{N}_{>0}^{l_\lambda}$ such that $\sum_{j=1}^{l_\lambda} \lambda_j = k$. We denote by $\mathbb{N}^l(p)$ the set of $P \equiv (p_1, \ldots, p_l) \in \mathbb{N}^l$ such that $\sum_{j=1}^{l} p_j = p$.

We start now with a general remark. Let $(\varphi_t)_{t\in(-\varepsilon,\varepsilon)}$ be a real analytic family of real analytic automorphisms of a real analytic manifold X with $\varphi_0 = \mathrm{id}_X$ and let

$$\xi\,(t) := \dot{\varphi}_t \circ \varphi_t^{-1} = \sum_{k\geqslant 0} \xi_k t^k \,. \tag{1.1}$$

The pull back operator φ_t^* acting on real analytic sections of the tensor bundles $(T_X^*)^{\otimes p} \otimes T_X^{\otimes q}$, with $p, q \geqslant 0$, can be expanded as

$$\varphi_t^* = \mathbb{I} + \sum_{k\geqslant 1} t^k \sum_{\lambda \vDash k} \prod_{j=1}^{l_\lambda} |\lambda|_j^{-1} L_{\xi_{\lambda_j - 1}} \,, \tag{1.2}$$

where $|\lambda|_j := \sum_{r=1}^{j} \lambda_r$. Of course we can relax the analyticity assumption to smooth on the ground variable $x \in X$, but for our future applications we will need to stay in the real analytic category.

Let ∇ be the extension to smooth sections of tensor bundles $(T_X^*)^{\otimes p} \otimes T_X^{\otimes q}$ of any given connection. We remind that the higher order covariant derivative operators ∇^h are defined inductively by:

$$\nabla^{h+1}_{\xi_0 \otimes \cdots \otimes \xi_h} \;:=\; \nabla_{\xi_0} \nabla^h_{\xi_1 \otimes \cdots \otimes \xi_h} - \sum_{p=1}^{h} \nabla^h_{\xi_1 \otimes \cdots \otimes \nabla_{\xi_0} \xi_p \otimes \cdots \otimes \xi_h} \,, \tag{1.3}$$

for any smooth vector fields $\xi_0, \ldots, \xi_h$.

We equip now the Sobolev space $H^r(X, (T_X^*)^{\otimes q} \otimes T_X^{\otimes p})$ with the Sobolev norm $\|\cdot\|_r$ obtained using the covariant derivatives and the pointwise max norm on multilinear forms with respect to a smooth Riemannian metric g. This norm is equivalent to the usual Sobolev norm defined by means of partitions of unity. When $q = p$, we remind that the space $H^r(X, (T_X^*)^{\otimes p} \otimes T_X^{\otimes p})$ is an algebra for $r \in \mathbb{N}$ sufficiently big and for such r hold the inequality $\|uv\|_r \leqslant C_r \|u\|_r \|v\|_r$, for some constant $C_r > 0$. From now on we fix such an r. Sobolev norms are quite natural for Hardy spaces. In any case the estimate in the main theorem below holds with respect to any algebra norm.

A case of major interest is when the pull back operator acts on analytic endomorphism sections of the tangent bundle. Indeed this is the case when we consider a complex structure J over a complex manifold X and we wish to study the dynamics of the flow $\varphi_t^* J$. As explained in [P-S, Pal2, Pal3], among others, this is a central problem in complex differential geometry related with a strong version of the Hamilton-Tian conjecture (See [Pal3]). For the applications it is very important to have an explicit and simple estimate of the multiple Lie derivatives that appear in the expansion (1.2) and their higher order covariant derivatives. This is provided by the following result, which is our main theorem.

Theorem 1. (**MainTheorem**) *Let h be an integer. Let ∇ be the extension to tensor sections of any torsion free connection and let A be a smooth endomorphism section of the tangent bundle. Then for any family of smooth vector fields $(\xi_j)_{j=1}^k$ the estimate holds*

$$\begin{aligned}\frac{1}{h!}\left\|\nabla^h\left(\prod_{j=1}^k L_{\xi_j}\right)A\right\|_r \\ \leqslant\ C_r^k \sum_{\substack{H\in\mathbb{N}^{k+1}(h)\\ P\in\mathrm{Dyck}(k)}} C_P \frac{1}{h_{k+1}!}\|\nabla^{h_{k+1}+D_{P,k}}A\|_r \prod_{j=1}^k \frac{1}{h_j!}\|\nabla^{h_j+p_j}\xi_j\|_r\,,\end{aligned}$$

where

$$\begin{aligned}\mathbb{N}^{k+1}(h) &:= \left\{H\equiv(h_1,\ldots,h_{k+1})\in\mathbb{N}^{k+1}\ \middle|\ \sum_{i=1}^{k+1}h_i=h\right\}\,,\\ \mathrm{Dyck}(k) &:= \left\{P\equiv(p_1,\ldots,p_k)\in\mathbb{N}^k\ \middle|\ \sum_{1\leqslant r\leqslant j}p_r\leqslant j, \forall j\in[k]\right\}\,,\\ D_{P,j} &:= j-\sum_{1\leqslant r\leqslant j}p_r\,,\\ C_P &:= \prod_{j=1}^k\left[2\binom{D_{P,j-1}}{p_j-1}+\binom{D_{P,j-1}}{p_j}\right]\\ &= \prod_{\substack{1\leqslant j\leqslant k\\ p_j\neq 0}}\left(2+\frac{D_{P,j}}{p_j}\right)\binom{D_{P,j-1}}{p_j-1}\,,\end{aligned}$$

with the convention that $\binom{m}{n}=0$ in $n\notin\{0,1,\ldots,m\}$.

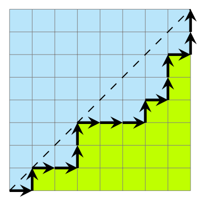
The elements P of Dyck(k) are called *Dyck vectors* of length k. Each Dyck vector is associated to a *Dyck monomial* $X^P = X_1^{p_1}\cdots X_k^{p_k}$. We give below their full list for $k=1,2,3$ together with the associated C_P coefficient

Dyck(1) : (0) 1 (1) 2

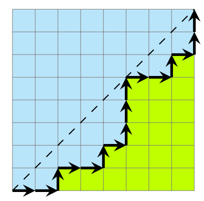
Dyck(2) : (00) 1 (01) 3 (02) 2 (10) 2 (11) 4

Dyck(3) :							
	(000) 1	(001) 4	(002) 5	(003) 2	(010) 3	(011) 9	(012) 6
	(020) 2	(021) 4	(100) 2	(101) 6	(102) 4	(110) 4	(111) 8

It easy to see that the cardinality $|\,\mathrm{Dyck}(k)|$ is the $k+1$-th Catalan number C_{k+1} where $C_k=\frac{1}{k+1}\binom{2k}{k}$. Indeed C_k is known (see [OEIS] sequence A000108) to be the number of so called Dyck paths, that is lattice paths on the grid $\mathbb{N}\times\mathbb{N}$, starting from $(0,0)$ ending at (k,k) with only North and East steps and staying under the diagonal. As illustrated below, such a path can be bijectively encoded by the lengths of the vertical segments, omitting the last one. Requiring that the path stay below the diagonal is equivalent to the condition $\sum_{1 \leqslant r \leqslant j} p_r \leqslant j$



1 0 2 0 0 1 2



0 1 0 1 3 0 1

Example 1. We now illustrate the computation of C_P. Let $P = (0, 1, 0, 1, 3, 0, 1)$. Then the value of $D_{P,j}$ are given by the following array:

j	0	1	2	3	4	5	6	7
p_j		0	1	0	1	3	0	1
$D_{P,j}$	0	1	1	2	2	0	1	1

So that

$$
\begin{aligned}
C_p = {} & \left[2\binom{0}{-1} + \binom{0}{0}\right] \left[2\binom{1}{0} + \binom{1}{1}\right] \left[2\binom{1}{-1} + \binom{1}{0}\right] \times \\
& \times \left[2\binom{2}{0} + \binom{2}{1}\right] \left[2\binom{2}{2} + \binom{2}{3}\right] \left[2\binom{0}{-1} + \binom{0}{0}\right] \left[2\binom{1}{0} + \binom{1}{1}\right] \\
& = 72\,.
\end{aligned}
$$

2 Proof of the expansion formula (1.2)

We remind first (see for instance lemma 27 in the sub-section 19.4 on page 916 of [Pal1]) the well known derivation rule

$$
\frac{d}{dt}\left(\varphi_t^* \alpha_t\right) \;=\; \varphi_t^* \left(\frac{d}{dt}\alpha_t + L_{\xi_t}\alpha_t\right), \tag{2.1}
$$

for any curve $t \longmapsto \alpha_t \in (T_X^*)^{\otimes p} \otimes T_X^{\otimes q}$. What we need to prove in order to obtain (1.2) is the formula

$$
\frac{1}{k!}\frac{d^k}{dt^k}_{|t=0}\left(\varphi_t^* \alpha\right) \;=\; \sum_{\lambda \vDash k} \left[\prod_{j=1}^{l_\lambda} L_{|\lambda|_j^{-1}\xi_{\lambda_j - 1}}\right] \alpha,
$$

with $\alpha \in (T_X^*)^{\otimes p} \otimes T_X^{\otimes q}$, that we rewrite under the form

$$\begin{aligned}
\frac{d^k}{dt^k}_{|t=0} (\varphi_t^* \alpha) &= \sum_{\lambda \vDash k} C_\lambda \left[\prod_{j=1}^{l_\lambda} L_{\xi_0^{(\lambda_j - 1)}} \right] \alpha \,, \\
C_\lambda &= \frac{|\lambda|!}{\prod_{j=1}^{r} [(\lambda_j - 1)! \; |\lambda|_j]} \,, \\
\xi_t^{(k)} &:= \frac{d^k \xi_t}{dt^k} \,,
\end{aligned}$$

with the convention $0! = 1$. We will prove the more general formula

$$\frac{d^k}{dt^k} \varphi_t^* = \sum_{\lambda \vDash k} C_\lambda \varphi_t^* \prod_{j=1}^{l_\lambda} L_{\xi_t^{(\lambda_j - 1)}} \,,$$

by induction. (Obviously the above formula is true for $k = 1$.) Taking one more derivative we obtain thanks to (2.1)

$$\frac{d^{k+1}}{dt^{k+1}} \varphi_t^* = \sum_{\lambda \vDash k} C_\lambda \varphi_t^* \left[\sum_{s=1}^{l_\lambda} \prod_{j=1}^{l_\lambda} L_{\xi_t^{(\lambda_j + \delta_{j,s} - 1)}} + L_{\xi_t} \prod_{j=1}^{l_\lambda} L_{\xi_t^{(\lambda_j - 1)}} \right] .$$

If we identify formally the product $\varphi_t^* \prod_{j=1}^{l_\lambda} L_{\xi_t^{(\lambda_j - 1)}}$ with the composition λ then the previous sum corresponds to the formal sum of compositions

$$S_k := \sum_{\lambda \vDash k} C_\lambda \left[\lambda' + (1, \lambda) \right] , \tag{2.2}$$

where

$$\begin{aligned}
\lambda' &:= \sum_{j=1}^{l_\lambda} (\lambda_1, \ldots, \lambda_j + 1, \ldots, \lambda_{l_\lambda}) \,, & (2.3) \\
(1, \lambda) &:= (1, \lambda_1, \ldots, \lambda_{l_\lambda}) \,. & (2.4)
\end{aligned}$$

We observe that the operation which associates to any composition λ the components of the formal sum $\lambda' + (1, \lambda)$, generates all the compositions of $k + 1$. For any $\lambda \vDash k$ and $\Lambda \vDash k + 1$ let us write $\lambda \to \Lambda$ if $\Lambda = (\lambda_1, \ldots, \lambda_j + 1, \ldots, \lambda_{l_\lambda})$ for some j or $\Lambda = (1, \lambda)$. Then Equation 2.2 rewrites as

$$S_k = \sum_{\lambda \vDash k} C_\lambda \sum_{\substack{\Lambda \vDash k+1 \\ \lambda \to \Lambda}} \Lambda \,. \tag{2.5}$$

Exchanging the two sums yield

$$S_k = \sum_{\Lambda \vDash k+1} \left(\sum_{\substack{\lambda \vDash k \\ \lambda \to \Lambda}} C_\lambda \right) \Lambda \,. \tag{2.6}$$

We observe that $\Lambda \vDash k+1$ being fixed, the $\lambda \vDash k$ such that $\lambda \to \Lambda$ are obtained either by removing 1 in front of Λ (if Λ starts with 1) or by decreasing any component of Λ greater that 2.

The conclusion of the induction will follow from the equality $C_\Lambda = C'_\Lambda$, which rewrites in a more explicit form as

$$\frac{|\Lambda|!}{\prod_{j=1}^{l_\Lambda} \left[(\Lambda_j - 1)! \; |\Lambda|_j\right]} \;=\; \sum_{s=1}^{l_\Lambda} \frac{(|\Lambda|-1)!}{\prod_{j=1}^{l_\Lambda} \left[(\Lambda_j - \delta_{s,j} - 1)_+! \; \sum_{t=1}^{j} (\Lambda_t - \delta_{s,t})\right]},$$

where $(a)_+ := \max\{a, 0\}$. Symplifing the common factor $(|\Lambda|-1)!$, rearranging and setting $l := l_\Lambda$ we infer that the previous equality is equivalent to

$$\sum_{s=1}^{l} \Lambda_s \;=\; \sum_{s=1}^{l} \prod_{j=1}^{l} \frac{(\Lambda_j - 1)! \; \sum_{t=1}^{j} \Lambda_t}{\left[(\Lambda_j - \delta_{s,j} - 1)_+! \; \sum_{t=1}^{j} (\Lambda_t - \delta_{s,t})\right]} \,.$$

The latter rewrites in a simpler way as

$$\sum_{s=1}^{l} \Lambda_s = \sum_{s=1}^{l} (\Lambda_s - 1) \prod_{j=s}^{l} \frac{\sum_{t=1}^{j} \Lambda_t}{\sum_{t=1}^{j} \Lambda_t - 1} \,. \tag{2.7}$$

We show (2.7) by induction on l. The equality (2.7) is obvious for $l = 1$. We decompose the sum

$$\begin{aligned}
&\sum_{s=1}^{l+1} (\Lambda_s - 1) \prod_{j=s}^{l+1} \frac{\sum_{t=1}^{j} \Lambda_t}{\sum_{t=1}^{j} \Lambda_t - 1} \\
&= \; (\Lambda_{l+1} - 1) \frac{\sum_{t=1}^{l+1} \Lambda_t}{\sum_{t=1}^{l+1} \Lambda_t - 1} \\
&+ \; \frac{\sum_{t=1}^{l+1} \Lambda_t}{\sum_{t=1}^{l+1} \Lambda_t - 1} \sum_{s=1}^{l} (\Lambda_s - 1) \prod_{j=s}^{l} \frac{\sum_{t=1}^{j} \Lambda_t}{\sum_{t=1}^{j} \Lambda_t - 1} \\
&= \; (\Lambda_{l+1} - 1) \frac{\sum_{t=1}^{l+1} \Lambda_t}{\sum_{t=1}^{l+1} \Lambda_t - 1} + \frac{\sum_{t=1}^{l+1} \Lambda_t}{\sum_{t=1}^{l+1} \Lambda_t - 1} \sum_{s=1}^{l} \Lambda_s \,,
\end{aligned}$$

by the inductive assumption. We conclude

$$\begin{aligned}
\sum_{s=1}^{l+1} (\Lambda_s - 1) \prod_{j=s}^{l+1} \frac{\sum_{t=1}^{j} \Lambda_t}{\sum_{t=1}^{j} \Lambda_t - 1} \;&=\; \left(\sum_{t=1}^{l+1} \Lambda_t - 1 \right) \frac{\sum_{t=1}^{l+1} \Lambda_t}{\sum_{t=1}^{l+1} \Lambda_t - 1} \\
&=\; \sum_{t=1}^{l+1} \Lambda_t \,,
\end{aligned}$$

which is the equality (2.7) for $l+1$.

Combinatorial proof. We give now a second more combinatorial proof of the formula giving the C_λ. Recall that we encoded the product

$$\varphi_t^* \prod_{j=1}^{l_\lambda} L_{\xi_t^{(\lambda_j-1)}}\,, \tag{2.8}$$

with the composition λ. We denote by D the linear operator acting on formal linear combinations of compositions defined by

$$D(\lambda) := \lambda' + (1,\lambda) = \sum_{\lambda\to\Lambda} \Lambda\,, \tag{2.9}$$

where we defined the relation $\to$ by $\lambda \to \Lambda$ if $\Lambda = (\lambda_1, \dots, \lambda_j + 1, \dots, \lambda_{l_\lambda})$ for some j or $\Lambda = (1, \lambda)$. Then

$$\frac{d}{dt}\left(\varphi_t^* \prod_{j=1}^{l_\lambda} L_{\xi_t^{(\lambda_j-1)}}\right)$$

is encoded by $D(\lambda)$. So that to compute $\frac{d^k}{dt^k}\varphi_t^*$ we need to compute the coefficients c_k of the expansion of $D^k(()) = \sum_{\lambda\vDash k} c_\lambda \lambda$ where $()$ denotes the empty composition of length 0. The relation $\to$ is illustrated in Figure 1, together with the coefficient c_λ.

By definition of D the coefficient c_Λ of each node Λ is the sum of the coefficients of its antecedents by the relation $\lambda \to \Lambda$. As a consequence it is equal to the number of paths from $()$ to Λ, that is sequences

$$() = \lambda^0 \to \lambda^1 \to \lambda^2 \to \cdots \to \lambda^k = \Lambda$$

such that the relation $\lambda^i \to \lambda^{i+1}$ holds for any i such that $0 \le i < k$.

The striking observation is that the number of such paths starting from $()$ to any composition Λ of sum k is equal to the number of partitions of the set $[k] = \{1, \dots, k\}$. Recall that a partition of a set S is a set $\Pi = \{\Pi_1, \dots, \Pi_r\}$ of non empty disjoints sets Π_i whose union is S. The number of partitions of $[k]$ is know as the k-th Bell numbers. The first values are

$$1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147,$$

and one can check on Figure 1 that the sum of the coefficients of the composition of sum 4 is indeed 15. So we can expect that there is a bijection between the set of paths from $()$ to Λ and the set of partitions Π of k verifying certain constraints depending on Λ.

We now describe such a constraint. First of all, we need a unique compact way to write a partition $\Pi = \{\Pi_1, \dots, \Pi_r\}$ of k. So we write the elements of each Π_i in the natural order, separating the Π_i by vertical bars $|$ and sorting the Π_i among themselves according to their *largest* element. We call this ordering $(\Pi_1, \dots, \Pi_r)$ the *normal ordering.* For example $\{\{6\}, \{1\}, \{7, 2, 4\}, \{5, 3\}\}$ which is a partition of $[7]$ is rather written in the order $\{\{1\}, \{3, 5\}, \{6\}, \{2, 4, 7\}\}$ in the compact way $1|35|6|247$.

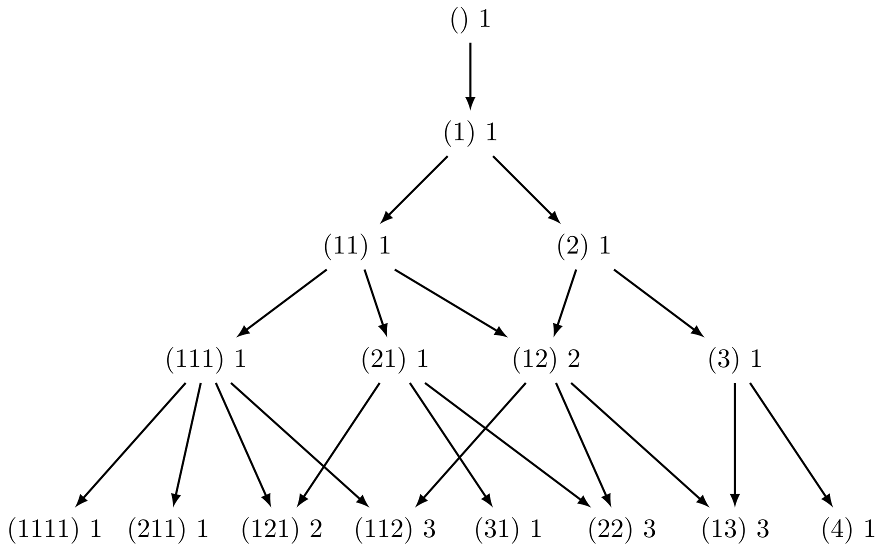


Figure 1: The relation $\lambda \to \Lambda$ and the coefficient c_λ

Definition 1. *Given a partition* Π *of* k *with normal ordering* $(\Pi_1, \ldots, \Pi_r)$*, we call the* shape *of* Π *and denote* $\mathrm{sh}(\Pi)$ *the composition* $(|\Pi_1|, \ldots, |\Pi_r|)$*.*

For example $\mathrm{sh}(1|35|6|247) = (1, 2, 1, 3)$. Then we claim that

Proposition 1. *For any composition* $\Lambda \vDash k$*, the coefficient* c_Λ *is the number of partitions* Π *such that* $\mathrm{sh}(\Pi) = \Lambda$*.*

Before going to the proof we need some extra combinatorial ingredients. Given a partition Π of $k > 0$ we denote Π^- the partition of $k-1$ obtained by decreasing by 1 all the numbers in the elements of Π, removing the obtained 0 and its set as well if it is a singleton. We moreover define $\Pi^{(n)}$ by $\Pi^{(0)} = \Pi$ and $\Pi^{(n)} = (\Pi^{(n-1)})^-$. For example $1|35|6|247^- = 24|5|136$ and $345|26|17^- = 234|15|6$. Here is the sequence $1|35|6|247^{(n)}$ for $n = 0, \ldots 7$, where the second row gives the shapes:

$$\begin{array}{cccccccc} 1|35|6|247 & 24|5|136 & 13|4|25 & 2|3|14 & 1|2|3 & 1|2 & 1 & \emptyset \\ (1,2,1,3) & (2,1,3) & (2,1,2) & (1,1,2) & (1,1,1) & (1,1) & (1) & () \end{array}$$

We notice the following obvious lemma which we just illustrated.

Lemma 1. *For all* Π *partition of* $k > 0$*,* $\mathrm{sh}(\Pi^-) \to \mathrm{sh}(\Pi)$*.*

We now turn to the proof of Proposition 1.

Proof. Let's denote $Q_\Lambda := \{\Pi \mid \mathrm{sh}(\Pi) = \Lambda\}$ and P_Λ the set of paths

$$() = \lambda^0 \to \lambda^1 \to \lambda^2 \to \cdots \to \lambda^k = \Lambda$$

from $()$ to Λ. To prove that $|Q_\Lambda| = |P_\Lambda|$ we define a bijection $F : Q_\Lambda \to P_\Lambda$ by

$$F(\Pi) = (\mathrm{sh}(\Pi^{(k)}), \mathrm{sh}(\Pi^{(k-1)}), \dots, \mathrm{sh}(\Pi^{(1)}), \mathrm{sh}(\Pi^{(0)}))\,. \tag{2.10}$$

Observe that $F(\Pi)$ is obtained by appending $\mathrm{sh}(\Pi)$ to $F(\Pi^-)$. Thanks to Lemma 1, this prove that $F(\Pi) \in P_\Lambda$. We now need to show that F is a bijection, that is, given a path

$$p = (() = \lambda^0 \to \lambda^1 \to \lambda^2 \to \cdots \to \lambda^k = \Lambda)\,,$$

we need to show that there is a unique partition Θ such that $F(\Theta) = p$. We proceed by induction on k. First, for $k = 0$, we observe that there is only one partition of shape $()$, namely the empty partition $\emptyset$. Now suppose that $\Pi = (\Pi_1, \dots, \Pi_r)$ is the unique partition such that $F(\Pi) = (\lambda^0, \dots, \lambda^k)$. We only need to show that there is a unique partition Θ such that $\Theta^- = \Pi$ and $\mathrm{sh}(\Theta) = \lambda^{k+1}$. Recall that if $\lambda^k \to \lambda^{k+1}$, there are two possibilities:

- Either $\lambda^{k+1} = (1, \lambda^k)$, in this case, the only possible Θ is

  $$\Theta = (\{1\}, \Pi_1 + 1, \dots, \Pi_r + 1)\,,$$

  where for any set S of integers, $S + 1 := \{i + 1 \mid i \in S\}$.
- Or writing $\lambda^k = (\lambda^k_1, \dots, \lambda^k_l)$ there exists $j \le l$ such that

  $$\lambda^{k+1} = (\lambda^k_1, \dots, \lambda^k_j + 1, \dots, \lambda^k_l)\,.$$

  Since $\lambda^k = \mathrm{sh}(\Pi)$, it makes sense to define

  $$\Theta = (\Pi_1 + 1, \dots, \{1\} \cup (\Pi_j + 1), \dots \Pi_r + 1)\,.$$

  and again it is the only possibility.

This conclude the proof by induction on k. □

To finish the combinatorial proof of Formula (1.2), we still need to prove the following:

Proposition 2. *For any composition $\lambda \vDash k$, the number of partitions Π of $[k]$ of shape λ is given by*

$$c_\lambda \;=\; \frac{|\lambda|!}{\prod_{j=1}^{r}\left[(\lambda_j - 1)\,!\;|\lambda|_j\right]}\,. \tag{2.11}$$

Proof. We proceed by induction on the length r of any composition λ of any sum k. If $r = 0$, then $\lambda = ()$, and the denominator product is empty so that $c_\lambda = 1$ which is correct since the only partition is the empty one. Now to choose a partition $\Pi = \{\Pi_1, \dots, \Pi_{r+1}\}$ of shape $\lambda = (\lambda_1, \dots, \lambda_{r+1})$ of k, we first need to choose the elements which belongs to Π_{r+1} (which must contain at least k due to the normal ordering). To get the correct shape, there must be $\lambda_{r+1} - 1$ elements different from k in Π_{r+1} that must be chosen in $[k-1] = [|\lambda| - 1]$. So the number of such choices is

$$\begin{aligned}\binom{|\lambda|-1}{\lambda_{r+1}-1} &= \frac{(|\lambda|-1)!}{(\lambda_{r+1}-1)!(\lambda_1+\cdots+\lambda_r)!} \\ &= \frac{|\lambda|!}{(\lambda_{r+1}-1)!|\lambda|_{r+1}|\lambda|_r!}\,,\end{aligned}$$

since $|\lambda| = |\lambda|_{r+1}$. We need now to choose a partition Θ of shape $\mu := (\lambda_1, \dots, \lambda_r)$ of the remaining numbers. By naturally renumbering them, there are as many choices for Θ as partitions of $[|\lambda|_r]$ of shape $(\lambda_1, \dots, \lambda_r) = \mu$. By induction they are c_μ of them. We therefore obtain

$$\frac{|\lambda|!}{(\lambda_{r+1}-1)!|\lambda|_{r+1}|\lambda|_r!}\,\frac{|\lambda|_r!}{\prod_{j=1}^{r}\Big[(\lambda_j-1)\,!\;|\lambda|_j\Big]}\,, \tag{2.12}$$

which simplifies to the announced result. $\square$

Remark 1. The shape map sh and the coefficients c_λ have a nice algebraic interpretation in terms of combinatorial Hopf algebras [HNT2]. Indeed set partition index the monomial basis $(\mathbf{M}_\Pi)$ of the algebra **WSym** of symmetric function in non-commutative variables. Then the operation $\Pi \mapsto \Pi^-$, is encoded in the non-commutative product of **WSym** as

$$\mathbf{M}_{\{\{1\}\}}\mathbf{M}_\Pi = \sum_{\Theta \;|\; \Pi = \Theta^-} \mathbf{M}_\Theta\,. \tag{2.13}$$

Then [HNT2, Section 3.7] consider a quotient of **WSym** by the so-called stalactic congruence. To match our setting we need the right sided stalactic congruence defined by

$$a\ w\ a \equiv w\ a\ a \tag{2.14}$$

for all $a \in A$ and $w \in A^*$. This quotient amounts to identify $\mathbf{M}_\Pi$ and $\mathbf{M}_\Theta$ if and only if $\mathrm{sh}(\Pi) = \mathrm{sh}(\Theta)$ leading naturally to a base (N_λ) of the quotient. As a consequence our c_λ are nothing but the coefficients of the expansion

$$\frac{1}{1-N_{(1)}} = \sum_\lambda c_\lambda N_\lambda\,. \tag{2.15}$$

3 Multiple covariant derivatives and trees

Let X be a smooth manifold and let ∇ be a covariant derivative operator acting on the smooth sections of the tangent bundle T_X. We will still denote by ∇ its natural extension over tensors. For any subset $S \subset \mathbb{N}_{>0}$ we consider a family of vector fields $(\xi_p)_{p\in S}$ and a smooth section A of the tensor bundle $(T^*_X)^{\otimes q} \otimes T_X^{\otimes p}$.

It is known since Cayley [Cay] that trees are the right tool to manipulate nested iterated derivative (see also [Man, HNT1]). He actually invented the very notion of tree for that exact purpose. In this paper we will have to deal with expression such as

$$\nabla^3_{\nabla^1_{\xi_2}\xi_3 \otimes \xi_5 \otimes \nabla^2_{\xi_1\otimes\xi_4}\xi_6} A \;\equiv\; \big((\xi_2 \neg \nabla^1 \xi_3) \otimes \xi_5 \otimes \big((\xi_1 \otimes \xi_4) \neg \nabla^2 \xi_6\big)\big) \neg \nabla^3 A\,.$$

We will manipulate them using trees. For example the previous expression is much easier to read if written as in the left of Figure 2. Moreover, since there is a lot of redundant information such as the ∇^i and the ξ, we will reduce it to the right of Figure 2. We remark that contrary to nature we picture trees growing from top to bottom.

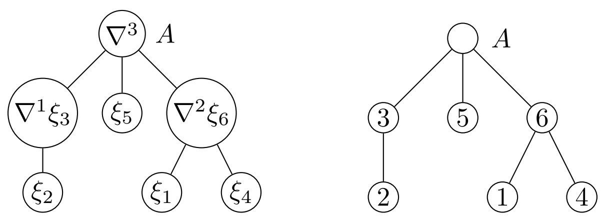


Figure 2: A nested derivative and its corresponding tree

We remark that Cayley used non-planar trees because of the pre-Lie [Man] structure of the Lie algebra of vector fields on an affine space endowed with its canonical flat torsion-free connection. In our setting, due to the possible non-flatness of the connection, the trees and forests considered here should be planar, i.e. the order of the branches matter. However, by construction, in all the trees considered in this paper, the labels are always increasing from left to right so that we can forget about the order.

We now define formally the kinds of trees we need in this paper. We remark that, as depicted in Figure 2, there are no repeated labels in our trees so that we don't have to distinguish between a node and its label. If we orient the edge bottom-up, such a tree is just a graph of a partial function which is loopless. Moreover, for reasons that will become apparent later, our trees have some extra order requirement which ensure the loopless property.

Definition 2. *Let S be a finite totally ordered set. A (rooted, labeled)* strictly decreasing forest *F on S is a partially defined function $F : S \to S$ such that for any ν where F is defined $F(\nu) > \nu$ holds.*

Elements of S *are called* nodes *of* F*. Nodes where* F *is not defined are called* roots *of* F*. A forest* T *with only one root is called a* tree*, in this case we denote the root* $\rho(T)$*.*

The node $F(\nu)$ *is called the* father *of* ν*. The preimages of* $\mu \in F^{-1}(\nu)$ *by* F *are called the* children *of* ν *and we denote their set* $\mathrm{Child}_F(\nu)$ *or even* $\mathrm{Child}(\nu)$ *if* F *is clear from the context and their number by* $\ell_F(\nu)$ *or* $\ell(\nu)$*. Finally, When depicting a tree, we always draw the children in* increasing order from left to right*.*

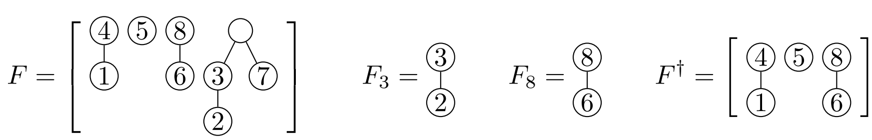
Note that the condition ensures that a non-empty forest must have at least one root, namely $\max S$.

We now fix some more terminology and notations: we often write the forest F for its underlying set S such that $\nu \in F$ means $\nu \in S$. For a tree T, we also write $\nu \in T^*$ for $\nu \in T \setminus \{\rho(T)\}$.

We say that ν is a *brother* of μ, if they have the same father. Also ν is a *descendant* (resp. a *strict descendant*) of μ if there is a $i \geq 0$ (resp. $i > 0$) such that $\mu = F^i(\nu)$. Given $\mu \in S$, the set of descendants of μ determines a natural tree called the *subtree* of F rooted at μ and denoted F_μ. A tree T of a forest F is a subtree whose root $\rho(T)$ is also a root of the forest F. We denote their set with $\mathrm{Tree}(F)$. Of course there are as many trees in a forest as roots, their number is denoted by l_F. We denote $F^\dagger$ the forest obtained from F by removing the tree rooted at its maximal element.

In this paper, most of the forests and trees will have their set of nodes contained in $\mathbb{N}_{>0} \cup \{\circ\}$. When $\circ$ is present it is the largest element and thus a root. When a root is $\circ$, we don't draw it and simply draw an empty node in the picture. For $S \subset \mathbb{N}_{>0}$, we denote by Tree_S, the set of trees with set of nodes $S^\circ := S \cup \{\circ\}$.

Example 2. We show below a forest F. The tree F_3 is a subtree of F, and F_8 is both a subtree and a tree of F, that is $F_8 \in \mathrm{Tree}(F)$. We also examplify $F^\dagger$.



It is well known that the number of forests with set of node any given set of cardinality k is $k!$ (see. [OEIS] sequence A000142). A bijection is given in [HNT1, Section 3.2].

Definition 3. *For any subset* $S \subset \mathbb{N}$ *we consider a family of vector fields* $\xi_\bullet \equiv (\xi_j)_{j \in S}$ *and a smooth section* A *of the tensor bundle* $(T_X^*)^{\otimes q} \otimes T_X^{\otimes h}$*, with* $q, h \geqslant 0$*. For any tree* T *with set of nodes contained in* S° *we define the nested derivative* $\nabla^T_{\xi_\bullet} \xi_{\rho(T)}$ *of* $\xi_{\rho(T)}$*, with* $\xi_\circ := A$ *by the inductive formula*

$$\nabla^T_{\xi_\bullet} \xi_{\rho(T)} \;:=\; \left(\bigotimes_{\nu \in \mathrm{Child}(\rho(T))} \nabla^{T_\nu}_{\xi_\bullet} \xi_{\rho(T_\nu)} \right) \neg \nabla^{l_{\rho(T)}} \xi_{\rho(T)} \,.$$

In particular if S° is the set of nodes of T, then

$$
\begin{aligned}
\nabla^T_{\xi_\bullet} A &= \left(\bigotimes_{\nu \in \mathrm{Child}(\rho(T))} \nabla^{T_\nu}_{\xi_\bullet} \xi_\nu \right) \neg \nabla^{l_{\rho(T)}} A \,, \\
\nabla^{T_\nu}_{\xi_\bullet} \xi_\nu &= \left(\bigotimes_{n \in \mathrm{Child}(\nu)} \nabla^{T_n}_{\xi_\bullet} \xi_n \right) \neg \nabla^{l_\nu} \xi_\nu \,.
\end{aligned}
$$

See Figure 2 for an example.

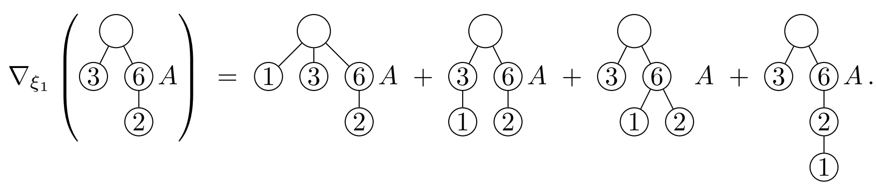
If we now apply recursively the chain rule

$$
\nabla_\xi \nabla^k_{\Xi_1 \otimes \cdots \otimes \Xi_k} A = \nabla^{k+1}_{\xi \otimes \Xi_1 \otimes \cdots \otimes \Xi_k} A + \sum_{j=1}^{k} \nabla^k_{\Xi_1 \otimes \cdots \otimes \nabla_\xi \Xi_j \otimes \cdots \otimes \Xi_k} A \tag{3.1}
$$

to a tree, it writes as

$$
\nabla_{\xi_j} \nabla^T_{\xi_\bullet} A = \sum_U \nabla^U_{\xi_\bullet} A \,, \tag{3.2}
$$

where the sum goes along the set of trees U obtained by grafting j to the left of any nodes of T. As a consequence, there are as many terms in this sum as nodes of T. We give here an example where a tree T stands for $\nabla^T_{\xi_\bullet}$:



Note that if T is strictly decreasing ordered and if i is smaller than any nodes of T then all the trees appearing in this sum are strictly decreasing ordered.

Applying iteratively this rule, since there is a unique way to get a strictly decreasing tree adding nodes one by one in the decreasing order, we get the multiple covariant derivative of A:

$$
\left(\prod_{j \in S} \nabla_{\xi_j} \right) A = \sum_{T \in \mathrm{Tree}_S} \nabla^T_{\xi_\bullet} A \,. \tag{3.3}
$$

With respect to a Sobolev norm $\| \cdot \|_r$ we infer the inequality

$$
\left\| \left(\prod_{j \in S} \nabla_{\xi_j} \right) A \right\|_r \leqslant C_r^{|S|} \sum_{T \in \mathrm{Tree}_S} \left\| \nabla^{\ell(\rho(T))} A \right\|_r \cdot \prod_{\nu \in T^*} \| \nabla^{\ell(\nu)} \xi_\nu \|_r \,. \tag{3.4}
$$

We notice indeed the identity $|S| = \sum_{\nu \in T} \ell(\nu)$ for any $T \in \mathrm{Tree}_S$.

Remark 2. The computation made here are very reminiscent to prelie computation. Indeed, it is well know that given a flat torsion-free connection and ∇ its associated covariant derivative, the bilinear operator $X \triangleright Y := \nabla_X Y$ endows the space of vector fields with a left pre-Lie algebra structure (see [Man, Proposition 3.1]). However in our case, the connection is not flat, so that we can't apply the pre-Lie calculus. Still, we can use trees by ensuring that when computing $\nabla_X Y$, the field X is always a single leaf and not a proper tree.

4 Multiple Lie derivatives of endomorphism sections of the tangent bundle

We notice now that for any smooth endomorphism section A of T_X and any torsion free connection ∇, the following identity holds:

$$L_\xi A \;=\; \nabla_\xi A + [A, \nabla\xi]\,.$$

We consider the operator $\mathrm{ad}\,(A) := [A, \bullet]$ acting on endomorphism sections of T_X. Then the previous identity rewrites also as $L_\xi = \nabla_\xi - \mathrm{ad}\,(\nabla\xi)$ over the space of smooth endomorphism sections of T_X. In order to generalize this fomula to multiple Lie derivatives we need to introduce a few notations. We denote by $\mathcal{P}_k$ the set of partitions of the set $[k]$. We notice that for any $P \in \mathcal{P}_k$ there exists a unique $p \in P$ such that $\max p = k$. We denote p_k such p. We denote by $P^* := P \smallsetminus \{p_k\}$. Moreover for any $p \in P$ we denote $p^* := p \smallsetminus \{\max p\}$. Given a family of smooth vector fields $(\xi_j)_{j=1}^k$ and a subset $S \subset [k]$ we denote by

$$\nabla_{\xi_\bullet}^S \;:=\; \prod_{j\in S} \nabla_{\xi_j}\,,$$

where the product is taken in the increasing order from the left to the right. This notation will only be used in this section.

Lemma 2. *Let ∇ be the extension to tensor sections of any torsion free connection. Then for any family of smooth vector fields $(\xi_j)_{j=1}^k$ the formula*

$$\prod_{j=1}^k L_{\xi_j} = \sum_{P\in\mathcal{P}_{k+1}} (-1)^{|P|-1} \left[\prod_{p\in P^*} \mathrm{ad}\left(\nabla_{\xi_\bullet}^{p^*} \nabla\xi_{\max p}\right)\right] \nabla_{\xi_\bullet}^{p_{k+1}^*}\,, \tag{4.1}$$

holds over the space of smooth endomorphism sections of T_X. The product on the right hand side is taken in the increasing order provided by the max of the elements of P, from the left to the right.

Before giving the proof, we provide an example of a summand in the right-hand-side sum. We pick for $k = 8$ the set partition

$$P = \{\{2,3\},\{1,4,6\},\{7\},\{5,8,9\}\}\,.$$

The associated summand is

$$(-1)^3\,\mathrm{ad}\,(\nabla_{\xi_2}\nabla\xi_3)\,\mathrm{ad}\,(\nabla_{\xi_1}\nabla_{\xi_4}\nabla\xi_6)\,\mathrm{ad}\,(\nabla\xi_7)\,\nabla_{\xi_5}\nabla_{\xi_8}\,.$$

Proof. We show the formula (4.1) by induction on k. By the inductive assumption

$$\prod_{j=2}^{k+1} L_{\xi_j} = \sum_{P\in\mathcal{P}_{2,k+2}} (-1)^{|P|-1} \left[\prod_{p\in P^*} \operatorname{ad}\left(\nabla_{\xi_\bullet}^{p^*}\nabla\xi_{\max p}\right)\right] \nabla_{\xi_\bullet}^{p^*_{k+2}},$$

where $\mathcal{P}_{2,k+2}$ denotes the set of partitions of the set $\{2,\ldots,k+2\}$. For any $P\in\mathcal{P}_{2,k+2}$ we denote by $\mathcal{P}_P\subset\mathcal{P}_{k+2}$ the subset of partitions obtained by adding 1 to one of the parts $p\in P$. We denote by $P_1 := \{\{1\}\}\cup P$. Then

$$\begin{aligned}
\nabla_{\xi_1}\prod_{j=2}^{k+1} L_{\xi_j} &= \sum_{P\in\mathcal{P}_{2,k+2}} (-1)^{|P|-1}\nabla_{\xi_1} \left[\prod_{p\in P^*} \operatorname{ad}\left(\nabla_{\xi_\bullet}^{p^*}\nabla\xi_{\max p}\right)\right] \nabla_{\xi_\bullet}^{p^*_{k+2}}\\
&= \sum_{P\in\mathcal{P}_{2,k+2}} (-1)^{|P|-1} \sum_{P'\in\mathcal{P}_P} \left[\prod_{p'\in P'^*} \operatorname{ad}\left(\nabla_{\xi_\bullet}^{p'^*}\nabla\xi_{\max p'}\right)\right] \nabla_{\xi_\bullet}^{p'^*_{k+2}}\\
&= \sum_{P\in\mathcal{P}_{2,k+2}} \sum_{P'\in\mathcal{P}_P} (-1)^{|P'|-1} \left[\prod_{p'\in P'^*} \operatorname{ad}\left(\nabla_{\xi_\bullet}^{p'^*}\nabla\xi_{\max p'}\right)\right] \nabla_{\xi_\bullet}^{p'^*_{k+2}},
\end{aligned}$$

and

$$\begin{aligned}
&-\operatorname{ad}(\nabla\xi_1)\prod_{j=2}^{k+1} L_{\xi_j}\\
&= \sum_{P\in\mathcal{P}_{2,k+2}} (-1)^{|P|}\operatorname{ad}(\nabla\xi_1) \left[\prod_{p\in P^*} \operatorname{ad}\left(\nabla_{\xi_\bullet}^{p^*}\nabla\xi_{\max p}\right)\right] \nabla_{\xi_\bullet}^{p^*_{k+2}}\\
&= \sum_{P\in\mathcal{P}_{2,k+2}} (-1)^{|P_1|-1} \left[\prod_{p_1\in P_1^*} \operatorname{ad}\left(\nabla_{\xi_\bullet}^{p_1^*}\nabla\xi_{\max p_1}\right)\right] \nabla_{\xi_\bullet}^{p^*_{1,k+2}}.
\end{aligned}$$

The conclusion

$$\prod_{j=1}^{k+1} L_{\xi_j} = \sum_{P\in\mathcal{P}_{k+2}} (-1)^{|P|-1} \left[\prod_{p\in P^*} \operatorname{ad}\left(\nabla_{\xi_\bullet}^{p^*}\nabla\xi_{\max p}\right)\right] \nabla_{\xi_\bullet}^{p^*_{k+2}},$$

follows from the fact that

$$\mathcal{P}_{k+2} = \left[\bigsqcup_{P\in\mathcal{P}_{2,k+2}} \mathcal{P}_P\right] \bigsqcup \left[\bigsqcup_{P\in\mathcal{P}_{2,k+2}} \{P_1\}\right]. \qquad \square$$

Let $\mathcal{F}_{k^\circ}$ be the set of forest over $[k]\cup\{\circ\}$. Recall that, given a forest $F\in\mathcal{F}_{k^\circ}$ we denote F_ν the subtree rooted at ν. We also denote by l_F the number of trees of F. Recall also that $F^\dagger$ is the forest obtained by truncating from F the tree $F_\circ$. Then Formula (4.1) combined with (3.3) implies

$$\prod_{j=1}^{k} L_{\xi_j} \;=\; \sum_{F\in\mathcal{F}_{k^\circ}} (-1)^{l_F-1} \left[\prod_{T\in F^\dagger} \operatorname{ad}\left(\nabla^{T}_{\xi_\bullet}\nabla\xi_{\rho(T)}\right)\right] \nabla^{F_\circ}_{\xi_\bullet}\,, \tag{4.2}$$

where the product is taken in the increasing order provided by the labels of the roots of $F^\dagger$, from the left to the right. Recall that for a tree T, we write $\nu\in T^*$ for $\nu\in T\setminus\{\rho(T)\}$. Then formula (4.2) and the inequality (3.4) imply the estimate

$$\begin{aligned}\left\|\left(\prod_{j=1}^{k} L_{\xi_j}\right) A\right\|_r \;&\leqslant\; C_r^k \sum_{F\in\mathcal{F}_{k^\circ}} 2^{l_F-1}\left(\left\|\nabla^{\ell(\circ)}A\right\|_r \cdot \prod_{\nu\in F_\circ^*}\|\nabla^{\ell(\nu)}\xi_\nu\|_r\right)\times\\ &\times \prod_{T\in F^\dagger}\left(\left\|\nabla^{1+\ell(\rho(T))}\xi_{\rho(T)}\right\|_r\cdot\prod_{\nu\in T^*}\|\nabla^{\ell(\nu)}\xi_\nu\|_r\right),\end{aligned} \tag{4.3}$$

where the power of two comes from the following estimate of the ad operator

$$\|\operatorname{ad}(A)B\|_r \;\leqslant\; 2\,C_r\,\|A\|_r\,\|B\|_r\,, \tag{4.4}$$

which is applied for each $T\in F^\dagger$ in the product of Equation 4.2.

5 Forests and Dyck polynomials

We define the monomial X_T associated to a tree T as

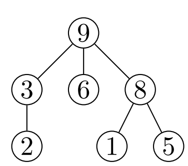
$$X_T \;:=\; \prod_{\nu\in T} X_\nu^{\ell(\nu)+\delta_{\rho(T),\nu}}\,,$$

and the root-truncated monomial X_T^* as

$$X_T^* \;:=\; \prod_{\nu\in T^*} X_\nu^{\ell(\nu)}\,.$$

Example 3. Considering the following tree

T =



,

one finds $X_T = X_1^0X_2^0X_3^1X_5^0X_6^0X_8^2X_9^4$ and $X_T^* = X_1^0X_2^0X_3^1X_5^0X_6^0X_8^2$.

Let $B^l := \|\nabla^l A\|_r$ and $X_\nu^{\ell(\nu)} := \|\nabla^{\ell(\nu)} \xi_\nu\|_r$ in the estimate (4.3). The reader has to be careful that $X_\nu^0 = \|\xi_\nu\| \neq 1$. Then the sum on the right hand side of (4.3) rewrites as

$$\Sigma_k \ := \sum_{F \in \mathcal{F}_{k^\circ}} 2^{l_F - 1} B^{\ell_F(\circ)} X_{F_\circ}^* \prod_{T \in F^\dagger} X_T \,.$$

With these notations we state the following theorem.

Theorem 2. *The sum Σ_k satisfies the polynomial expression*

$$\Sigma_k \ \equiv \ \Sigma\left(X_1, \ldots, X_k\right) = \sum_{P \in \mathrm{Dyck}(k)} C_P B^{D_{P,k}} X_1^{p_1} \cdots X_k^{p_k} \,.$$

Example 4. We give here the first few values of Σ_k.

$$\begin{aligned}
\Sigma_1 &= B\, X^{(0)} + 2\, X^{(1)} \,, \\
\Sigma_2 &= B^2\, X^{(0,0)} + 2\, B\, X^{(1,0)} + 3\, B\, X^{(0,1)} + 4\, X^{(1,1)} + 2\ X^{(0,2)} \,, \\
\Sigma_3 &= B^3\, X^{(0,0,0)} + 2\, B^2\, X^{(1,0,0)} + 3\, B^2\, X^{(0,2,0)} + 4\, B\, X^{(1,1,0)} + 2\, B\, X^{(0,1,0)} \\
&\quad + 4\, B^2\, X^{(0,0,1)} + 6\, B\, X^{(1,0,1)} + 9\, B\, X^{(0,1,1)} + 8\, X^{(1,1,1)} + 4\, X^{(0,2,1)} + \\
&\quad + 5\, B\, X^{(0,0,2)} + 4\, X^{(1,0,2)} + 6\, X^{(0,1,2)} + 2\, X^{(0,0,3)} \,.
\end{aligned}$$

From theorem 2 we infer the estimate

$$\left\| \left(\prod_{j=1}^k L_{\xi_j} \right) A \right\|_r \leqslant C_r^k \sum_{P \in \mathrm{Dyck}(k)} C_P \|\nabla^{D_{P,k}} A\|_r \prod_{j=1}^k \|\nabla^{p_j} \xi_j\|_r \,. \tag{5.1}$$

We need a few preliminaries in order to show theorem 2. Recall that a Dyck vector of length k is a sequence $P \equiv (p_1, \ldots, p_k) \in \mathbb{N}^k$ such that

$$\sum_{1 \leqslant r \leqslant j} p_r \leqslant j \qquad \text{for all } j \in [k] \,. \tag{5.2}$$

We denote their set by $\mathrm{Dyck}(k)$. A monomial $X^P = X_1^{p_1} \cdots X_k^{p_k}$ is *Dyck monomial* if P is a Dyck vector.

Recall also that for any $F \in \mathcal{F}_{k^\circ}$ we denote by F_ν the tree rooted at ν and by $F^\dagger$ the forest $F \smallsetminus \{F_\circ\}$. For any forest $F \in \mathcal{F}_{k^\circ}$ we define

$$X_F \ := \ X_{F_\circ}^* \prod_{T \in F^\dagger} X_T \,.$$

See Example 6 below. We observe that X_F is always a Dyck monomial X^P. Indeed let

$$p_j = \ell_F(j) + \delta_{j,\mathrm{Root}(F)} \,.$$

for all $j = 1, \ldots, k$ where $\delta_{j,\mathrm{Root}(F)}$ is 1 if j is a root of F and 0 otherwise. Then Condition (5.2) follows from the obvious identity

$$\begin{aligned}\sum_{1 \leqslant r \leqslant j} p_r \;&=\; |\{\nu \in F \cap [j] \mid \nu \text{ child in } F \cap [j]\}| \\ &\quad+\; |\{\nu \in F \cap [j] \mid \nu \text{ root in } F\}|\,,\end{aligned}$$

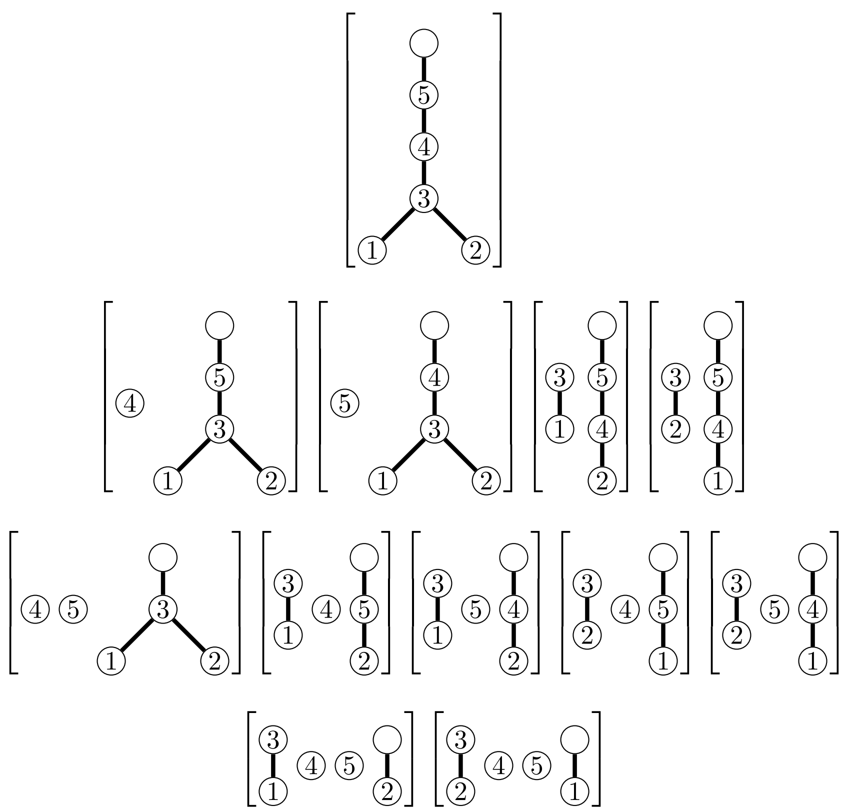
and the fact that

$$\{\nu \in F \cap [j] \mid \nu \text{ child in } F \cap [j]\} \;\sqcup\; \{\nu \in F \cap [j] \mid \nu \text{ root in } F\} \;\subseteq\; [j]\,.$$

For any $P \in \mathrm{Dyck}(k)$, we define the set

$$\mathcal{F}_P \;:=\; \left\{F \in \mathcal{F}_{k^\circ} \mid X_F = X^P\right\}.$$

Example 5. We fix $P = (0, 0, 2, 1, 1)$. The following picture show all the forests such that $X_F = X^P$, sorted according to their length.



As a result we get that

$$C_{(0,0,2,1,1)} = 1 + 4 \cdot 2 + 5 \cdot 2^2 + 2 \cdot 2^3 = 45\,.$$

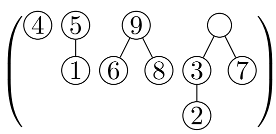
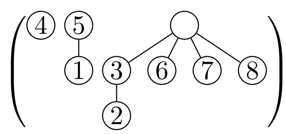
This agrees with

$$C_{(0,0,2,1,1)} = \left[2\binom{0}{-1} + \binom{0}{0}\right]\left[2\binom{1}{-1} + \binom{1}{0}\right] \times$$
$$\times \left[2\binom{2}{1} + \binom{2}{2}\right]\left[2\binom{1}{0} + \binom{1}{1}\right]\left[2\binom{1}{0} + \binom{1}{1}\right].$$

Definition 4. *For any $F \in \mathcal{F}_{k^\circ}$, $k > 0$ we denote by $F' \in \mathcal{F}_k$ the forest obtained by pruning the children of the node labeled $\circ$ and grafting them, to the node labeled k. Of course, the old and new children of k are shuffled to draw them in increasing order. Finally by replacing k by $\circ$ in F', we consider that F' actually belongs to $\mathcal{F}_{k-1^\circ}$.*

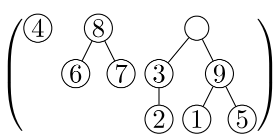
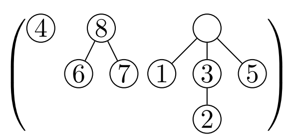
Example 6. It will becomes apparent in the following proofs that there are two different cases, whether k is a child of $\circ$ or not.

- We start by a case where $k = 9$ is a not child of $\circ$ and thus a root of F. We show below a forest F together with its associated F'.

  $F =$  $F' =$ 

  One can check that their associated monomials are $X_F = X^{(0,0,1,1,2,0,0,0,3)}$ and $X_{F'} = X^{(0,0,1,1,2,0,0,0)}$.

- We show now a forest F where $k = 9$ is a child of $\circ$ together with its associated F'.

  $F =$  $F' =$ 

  One can check that their associated monomials are $X_F = X^{(0,0,1,1,0,0,0,3,2)}$ and $X_{F'} = X^{(0,0,1,1,0,0,0,3)}$.

Lemma 3. *For any $F \in \mathcal{F}_{k^\circ}$, with $k \geqslant 0$ the identity $X_{F'} = X_{F|X_k=1}$ holds.*

Proof. By definition of F' we infer the identity $\ell_{F'}(j) = \ell_F(j)$ for any node labeled $j < k$. Moreover the node labeled $j < k$ is a root in F' if and only if it is a root in F. $\square$

For $P = (p_1, \ldots, p_k)$, we denote by $\deg\left(X^P\right)$ the sum $p_1 + \cdots + p_k$.

Lemma 4. *For any $F \in \mathcal{F}_{k^\circ}$, with $k \geqslant 0$ the identity $\ell_F(\circ) = k - \deg(X_F)$ holds. In particular $\ell_F(\circ)$ depends only on X_F and not on F.*

Proof. We prove this by induction on k. If $k = 0$, there is only one forest $F \in \mathcal{F}_{\{\circ\}}$ whose monomial is $X_F = 1$. We consider now a forest $F \in \mathcal{F}_{k^\circ}$ with $X_F = X_1^{p_1} \cdots X_k^{p_k}$. We reccal that the power p_j of a monomial $X_j^{p_j}$ satisfies: $p_j = \ell_F(j) + 1$ if $j = \rho(T)$ for some $T \in F$ and $p_j = \ell_F(j)$ if $j \neq \rho(T)$ for all $T \in F$. By the inductive assumption the statement hold for F'. There are two case, whether k is a child of $\circ$ or a root in F.

- If k is not a child of $\circ$, i.e. a root, then $\ell_{F'}(k) = \ell_F(k) + \ell_F(\circ)$. Therefore

$$\begin{aligned}\ell_F(\circ) &= \ell_{F'}(k) - \ell_F(k) \\ &= k - 1 - \deg(X_{F'}) - (p_k - 1) \\ &= k - \deg(X_F)\ .\end{aligned}$$

  The last equality follows from Lemma 3.

- If k is a child of $\circ$, by definition of F', the children of k in F' are the union of those of k in F and those of $\circ$ exept k. Thus $\ell_{F'}(k) = \ell_F(k) + \ell_F(\circ) - 1$. We infer

$$\begin{aligned}\ell_F(\circ) &= 1 + \ell_{F'}(k) - \ell_F(k) \\ &= 1 + (k - 1 - \deg(X_{F'})) - p_k \\ &= k - \deg(X_F)\ .\end{aligned}$$

  The last equality follows from Lemma 3. □

Proof of theorem 2. For any $P \in \mathrm{Dyck}(k)$, define

$$C'_P \ := \ \sum_{F \in \mathcal{F}_P} 2^{l_F - 1}\,.$$

Then

$$\Sigma_k \ = \ \sum_{F \in \mathcal{F}_{k^\circ}} 2^{l_F - 1} B^{\ell_F(\circ)} X_F \ = \ \sum_{P \in \mathrm{Dyck}(k)} C'_P B^{D_{P,k}} X^P,$$

thanks to lemma 4. Thus our goal is to show the equality

$$C'_P \ = \ \prod_{j=1}^{k} \left[2\binom{D_{P,j-1}}{p_j - 1} + \binom{D_{P,j-1}}{p_j} \right] \ = \ C_P\,. \tag{5.3}$$

We proceed by induction on k. We decompose $\mathcal{F}_P$ as a disjoint union as

$$\mathcal{F}_P \ = \ \bigsqcup_{H \in \mathcal{F}_{p_1, \dots p_{k-1}}} \mathcal{F}_P(H)\,,$$

where

$$\mathcal{F}_P(H) \ := \ \{F \in \mathcal{F}_P \mid F' = H\}\ .$$

As a consequence,

$$C'_P = \sum_{H \in \mathcal{F}_{p_1,\dots p_{k-1}}} \sum_{F \in \mathcal{F}_P(H)} 2^{l_F - 1} . \tag{5.4}$$

Thanks to Lemma 4, we infer $\ell_H(k) = k - 1 - \deg(X_H)$. By Lemma 3, we have $\deg(X_H) = p_1 + \cdots + p_{k-1}$ so that

$$\ell_H(k) = D_{P,k-1} . \tag{5.5}$$

We now distinguish two cases, whether k is a root of $F \in \mathcal{F}_P(H)$ or not:

- The number of $F \in \mathcal{F}_P(H)$ such that k is a root in F is given by

  $$\binom{D_{P,k-1}}{p_k - 1} .$$

  Indeed for fixed H, the equality

  $$F \cap [k-1] = H \cap [k-1] , \tag{5.6}$$

  shows that the only freedom of the forests $F \in \mathcal{F}_P(H)$ which satisfy (5.6) is in the choice of the the $\ell_F(k) = p_k - 1$ children of k in F among the $\ell_H(k) = D_{P,k-1}$ children of k in H. These children are given by the union of the children of $\circ$ and k in F. The case $p_k = 0$ does not occur since k is a root in F. This is consistent with the convention $\binom{a}{-1} = 0$.

  Using the fact that, in this case, $l_F = l_H + 1$, one concludes that for any fixed $H \in \mathcal{F}_{p_1,\dots p_{k-1}}$, one has

  $$\sum_{\substack{F \in \mathcal{F}_P(H) \\ k \text{ is a root of } F}} 2^{l_F - 1} \;=\; 2 \cdot 2^{l_H - 1} \binom{D_{P,k-1}}{p_k - 1} . \tag{5.7}$$

- The number of $F \in \mathcal{F}_P(H)$ such that k is not a root in F is given by

  $$\binom{D_{P,k-1}}{p_k} .$$

  The reason is the same as before. We notice that the definition of Dyck vector allows the case were $p_k = D_{P,k-1} + 1$ (this is the case in Example 1 for $k = 5$). But k has $D_{P,k-1}$ children in F' thanks to (5.5). Therefore it cannot have p_k children in F. This is consistent with the convention $\binom{a}{a+1} = 0$.

  Using the fact that, in this case, $l_F = l_H$, one concludes that for any fixed $H \in \mathcal{F}_{p_1,\dots p_{k-1}}$, one has

  $$\sum_{\substack{F \in \mathcal{F}_P(H) \\ k \text{ is not a root of } F}} 2^{l_F - 1} \;=\; 2^{l_H - 1} \binom{D_{P,k-1}}{p_k} . \tag{5.8}$$

Combining Equation (5.4) with the two identities (5.7) and (5.8) we obtain

$$\begin{aligned} C'_P &= \sum_{H\in\mathcal{F}_{p_1,\dots p_{k-1}}} \left[2\binom{D_{P,k-1}}{p_k-1}+\binom{D_{P,k-1}}{p_k}\right]2^{l_H-1} \\ &= C'_{p_1,\dots p_{k-1}}\left[2\binom{D_{P,k-1}}{p_k-1}+\binom{D_{P,k-1}}{p_k}\right]. \end{aligned}$$

We finally infer the required identity $C'_P = C_P$ by induction on k. □

Remark 3. We notice the formula

$$C_P = \prod_{\substack{1\leqslant j\leqslant k\\ p_j\neq 0}} \left(2+\frac{D_{P,j}}{p_j}\right)\binom{D_{P,j-1}}{p_j-1}.$$

Indeed

$$C_P = \prod_{\substack{1\leqslant j\leqslant k\\ p_j\neq 0}} \left[2\binom{D_{P,j-1}}{p_j-1}+\binom{D_{P,j-1}}{p_j}\right],$$

and

$$\binom{D_{P,j-1}}{p_j} = \frac{D_{P,j-1}-p_j+1}{p_j}\binom{D_{P,j-1}}{p_j-1}.$$

Then the conclusion follows from the identity $D_{P,j} = D_{P,j-1} - p_j + 1$.

6 Higher order covariant derivatives of tensors

In sequel, we denote for any $S \subset \mathbb{N}_{>0}$

$$\nabla^S_{\xi_\bullet} := \left(\bigotimes_{p\in S}\xi_p\right)\neg\nabla^{|S|}.$$

The reader should not confuse this with $\nabla^T_{\xi_\bullet}$ which is used when T is a tree.

We set $\mathrm{Map}(h,l) := \{\mu : [h] \longrightarrow [l]\}$. Let A_j be smooth sections of the bundle $(T^*_X)^{\otimes q_j}\otimes T_X$, $j = 1,\dots,l$. There are many situations in which the notion of product $\prod_{j=1}^l A_j$ is well defined. This is the case for instance when:

1. $q_j = 1$ for all $j = 1,\dots,l-1$. In this case the product is just a composition $A_1\circ A_2\circ\cdots\circ A_l(\xi_1\otimes\cdots\otimes\xi_{q_l})$ of endomorphisms with a q_l-linear map giving a q_l-linear map.

2. $q_1 = l-1$ and $q_j = 0,1$, for $j\geqslant 2$. In this second case, the product is the application of a $(l-1)$-linear map $A_1(A_2[\xi_2]\otimes A_3[\xi_3]\otimes\cdots\otimes A_l[\xi_l])$ to either vector fields A_j when $q_j = 0$ or the value $A_j(\xi_l)$ of the linear map A_j when $q_j = 1$. The bracket around the $[\xi_j]$ means that they are only present if $q_j = 1$. The result is a $q_2+q_3+\cdots+q_l$-linear map.

In all these cases the following lemma hold.

Lemma 5. *Let A_j be smooth sections of $(T_X^*)^{\otimes q_j} \otimes T_X$, $j = 1, \dots, l$ such that the formal product $\prod_{j=1}^l A_j$ is well defined and let $(\xi_p)_{p=1}^h$ be a family of vector fields over X. Then the h-order covariant derivative satisfies the general Leibniz identity*

$$\left(\bigotimes_{p=1}^h \xi_p\right) \neg \nabla^h \left(\prod_{j=1}^l A_j\right) \;=\; \sum_{\mu \in \mathrm{Map}(h,l)} \left(\prod_{j=1}^l \nabla_{\xi_\bullet}^{\mu^{-1}(j)} A_j\right).$$

Proof. We proceed by induction. We remind first the inductive definition of higher order covariant derivative:

$$\nabla^{h+1}_{\xi_0 \otimes \cdots \otimes \xi_h} \;:=\; \nabla_{\xi_0} \nabla^h_{\xi_1 \otimes \cdots \otimes \xi_h} - \sum_{p=1}^h \nabla^h_{\xi_1 \otimes \cdots \otimes \nabla_{\xi_0} \xi_p \otimes \cdots \otimes \xi_h}.$$

Taking a covariant derivative of the inductive assumption we infer

$$\begin{aligned} &\nabla_{\xi_0} \left[\left(\bigotimes_{p=1}^h \xi_p\right) \neg \nabla^h \left(\prod_{j=1}^l A_j\right)\right] \\ &\qquad = \sum_{j=1}^l \left(\sum_{\mu \in \mathrm{Map}(h,l)} \nabla_{\xi_\bullet}^{\mu^{-1}(1)} A_1 \cdots \nabla_{\xi_0} \nabla_{\xi_\bullet}^{\mu^{-1}(j)} A_j \cdots \nabla_{\xi_\bullet}^{\mu^{-1}(l)} A_l\right). \end{aligned}$$

Thanks to the tensorial nature of the multi-covariant derivative, we can assume $\nabla_{\xi_0} \xi_p\,(x) = 0$, $p = 1, \dots, k$ at some arbitrary point x. Then

$$\begin{aligned} &\left(\bigotimes_{p=0}^h \xi_p\right) \neg \nabla^{h+1} \left(\prod_{j=1}^l A_j\right) \\ &\qquad = \sum_{j=1}^l \left(\sum_{\mu \in \mathrm{Map}(h,l)} \nabla_{\xi_\bullet}^{\mu^{-1}(1)} A_1 \cdots \nabla_{\xi_\bullet}^{\{0\} \cup \mu^{-1}(j)} A_j \cdots \nabla_{\xi_\bullet}^{\mu^{-1}(l)} A_l\right). \end{aligned}$$

The conclusion follows from the observation that

$$\mathrm{Map}(\{0, \dots, h\}, l) = \{\mu_j \mid \mu \in \mathrm{Map}(h,l), j \in [l]\}, \tag{6.1}$$

where the map $\mu_j : \{0, \dots, h\} \longrightarrow [l]$ is defined as by $\mu_j(0) = j$ and $\mu_j(i) = \mu(i)$ for $i \neq 0$. $\square$

Corollary 1. *If ξ is a vector field over X then*

$$\xi^{\otimes h} \neg \frac{1}{h!} \nabla^h \left(\prod_{j=1}^l A_j\right) \;=\; \sum_{H \in \mathbb{N}^l(h)} \prod_{j=1}^l \left(\xi^{\otimes h_j} \neg \frac{1}{h_j!} \nabla^{h_j} A_j\right).$$

We infer the inequality with respect to the pointwise max norm on multilinear forms

$$\frac{1}{h!}\left\|\nabla^h\left(\prod_{j=1}^{l} A_j\right)\right\| \leqslant \sum_{H\in\mathbb{N}^l(h)} \prod_{j=1}^{l} \frac{1}{h_j!}\|\nabla^{h_j} A_j\|\,.$$

The previous pointwise inequality leads to the global estimate

$$\frac{1}{h!}\left\|\nabla^h\left(\prod_{j=1}^{l} A_j\right)\right\|_r \leqslant C_r^{l-1} \sum_{H\in\mathbb{N}^l(h)} \prod_{j=1}^{l} \frac{1}{h_j!}\|\nabla^{h_j} A_j\|_r\,. \tag{6.2}$$

7 Proof of the theorem 1

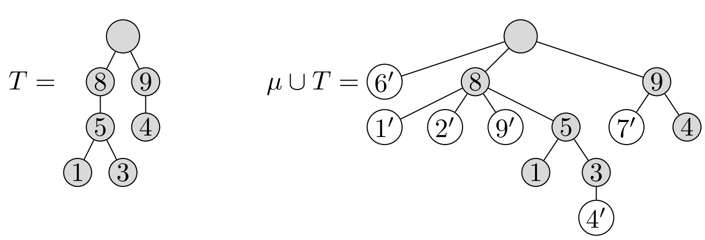
In this final section, we need to consider trees on the union of two families of vector fields $\eta_\bullet, \xi_\bullet$. This means that our label set for the trees will be a subset of two copies of $\mathbb{N}_{\geq 1}$ together with the usual empty root $\circ$. To distinguish the two copies, we write them $\mathbb{N}'_{\geq 1} = \{1', 2', \dots\}$ and $\mathbb{N}_{\geq 1} = \{1, 2, \dots\}$. Recall that the *ordered sum* $S + T$ of two totally ordered sets $(S, \leq_S)$ and $(T, \leq_T)$ is the disjoint union $S + T := S \sqcup T$ together with the order $\leq_{S+T}$ which keeps the relative order of the sets and such that all the elements of S are smaller than the element of T. Formally,

$$x \leq_{S+T} y \quad\iff\quad \begin{cases} x \leq_S y \text{ if } x, y \in S \text{ or} \\ x \leq_T y \text{ if } x, y \in T \text{ or} \\ x \in S \text{ and } y \in T. \end{cases}$$

Definition 5. *Let $S' \subset \mathbb{N}'_{\geq 1}$ and $S \subset \mathbb{N}_{\geq 1} \cup \{\circ\}$. Let $\mu \in \mathrm{Map}(S', S)$ and F a forest on S. We denote $\mu \cup F$ the forest on $S' + S$ with father function f defined by $f(i') = \mu(i')$ for $i' \in S'$ and $f(i) = F(i)$ for $i \in S$.*

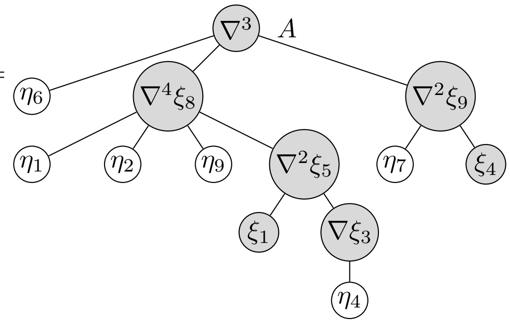
We remark that the roots of $\mu \cup F$ are the same as the roots of F so that if F is actually a tree T then $\mu \cup T$ is also a tree.

Example 7. Let $S = \{1, 3, 4, 5, 8, 9, \circ\}$ and $S' = \{1', 2', 4', 6', 7', 9'\}$. Consider the map $\mu = \begin{cases} 1'\ 2'\ 4'\ 6'\ 7'\ 9' \\ 8\ \ 8\ \ 3\ \ \circ\ \ 9\ \ 8 \end{cases}$. The picture below shows some tree T on S together with the associated $\mu \cup T$ tree:



Expanding the associated nested derivative as in Definition 3 gives:

$$\nabla^{\mu\cup T}_{\eta_\bullet,\xi_\bullet} A =$$



We remark that for any node ν of T

$$\ell_{\mu\cup T}(\nu) = \left|\mu^{-1}(\nu)\right| + \ell_T(\nu)\,.$$

Moreover according to Definition 3, $\nabla^{\mu\cup T}_{\eta_\bullet,\xi_\bullet} A$ is given by

$$\nabla^{\mu\cup T}_{\eta_\bullet,\xi_\bullet} A \;=\; \left[\left(\bigotimes_{p\in\mu^{-1}(\circ)} \eta_p\right)\otimes\left(\bigotimes_{\nu\in\mathrm{Child}(\circ)} \nabla^{\mu\cup T_\nu}_{\eta_\bullet,\xi_\bullet}\xi_\nu\right)\right]\neg\nabla^{\ell_{\mu\cup T}(\circ)} A\,,$$

with

$$\nabla^{\mu\cup T_\nu}_{\eta_\bullet,\xi_\bullet}\xi_\nu \;=\; \left[\left(\bigotimes_{p\in\mu^{-1}(\nu)} \eta_p\right)\otimes\left(\bigotimes_{n\in\mathrm{Child}(\nu)} \nabla^{\mu\cup T_n}_{\eta_\bullet,\xi_\bullet}\xi_n\right)\right]\neg\nabla^{\ell_{\mu\cup T}(\nu)}\xi_\nu\,,$$

and so on, with $\nabla^{\mu\cup\emptyset}_{\eta_\bullet,\xi_\bullet} := \mathbb{I}$ and the abuse of notation $\mu\cup H := \mu|_{\mu^{-1}(H)}\cup H$ for any subtree H of T.

Proof of theorem 1. We apply recursively Lemma 5 in case 2 with

- $A_1 := \nabla^{l_{\rho(T)}} A$,
- $A_i := \nabla^{T_\nu}_{\xi_\bullet}\xi_\nu$, if $i > 1$, where ν is the i-th child of $\rho(T)$.

We get

$$\nabla^{S'}_{\eta_\bullet}\left(\nabla^T_{\xi_\bullet} A\right) \;\equiv\; \left(\bigotimes_{p\in S'}\eta_p\right)\neg\nabla^{|S'|}\left(\nabla^T_{\xi_\bullet} A\right) \;=\; \sum_{\mu\in\mathrm{Map}(S',T)} \nabla^{\mu\cup T}_{\eta_\bullet,\xi_\bullet} A\,. \qquad (7.1)$$

By formula (4.2) and linearity, we get

$$\left(\bigotimes_{p=1}^{h}\eta_p\right)\neg\nabla^h\left[\left(\prod_{j=1}^{k} L_{\xi_j}\right) A\right]$$
$$= \sum_{F\in\mathcal{F}_{k^\circ}} (-1)^{l_F-1}\left(\bigotimes_{p=1}^{h}\eta_p\right)\neg\nabla^h\left[\left(\prod_{T\in F^\dagger}\mathrm{ad}\left(\nabla^T_{\xi_\bullet}\nabla\xi_{\rho(T)}\right)\right)\nabla^{F_\circ}_{\xi_\bullet} A\right].$$

We use now Lemma 5 in case 1, writing $F = \{T_1, \dots, T_{l_F-1}, T_{l_F} = F_\circ\}$ in the increasing order provided by the labels of the roots of F, from the left to the right with

- $A_i := \operatorname{ad}\left(\nabla^{T_i}_{\xi_\bullet}\nabla\xi_{\rho(T_i)}\right)$ for $i = 1, \dots, T_{l_F-1}$,
- $A_{l_F} := \nabla^{F_\circ}_{\xi_\bullet} A$.

One obtains, for a fixed forest F,

$$
\begin{aligned}
G_F &:= \left(\bigotimes_{p=1}^{h} \eta_p\right) \neg \nabla^h \left[\left(\prod_{T\in F^\dagger} \operatorname{ad}\left(\nabla^{T}_{\xi_\bullet}\nabla\xi_{\rho(T)}\right)\right) \nabla^{F_\circ}_{\xi_\bullet} A\right] \\
&= \sum_{\mu\in \operatorname{Map}(h,l_F)} \left[\prod_{j=1}^{l_F-1} \operatorname{ad}\left(\nabla^{\mu^{-1}(j)}_{\eta_\bullet}\nabla^{T_j}_{\xi_\bullet}\nabla\xi_{\rho(T_j)}\right)\right] \nabla^{\mu^{-1}(l_F)}_{\eta_\bullet}\nabla^{F_\circ}_{\xi_\bullet} A\,.
\end{aligned}
$$

Applying Equation 7.1 to each T_j, we obtain

$$
\begin{aligned}
G_F &= \sum_{\mu\in \operatorname{Map}(h,l_F)} \left[\prod_{j=1}^{l_F-1} \operatorname{ad}\left(\sum_{\beta_j\in \operatorname{Map}(\mu^{-1}(j),T_j)} \nabla^{\beta_j\cup T_j}_{\eta_\bullet,\xi_\bullet}\nabla\xi_{\rho(T_j)}\right)\right] \times \\
&\quad \times \left(\sum_{\beta_{l_F}\in \operatorname{Map}(\mu^{-1}(l_F),F_\circ)} \nabla^{\beta_{l_F}\cup F_\circ}_{\eta_\bullet,\xi_\bullet} A\right)\,.
\end{aligned}
$$

We now recombine the maps $\mu, (\beta_j)_{j=1..l_F}$ into a single map $\alpha \in \operatorname{Map}(h, k^\circ)$ by setting $\alpha(p) := \beta_{\mu(p)}(p)$. Each $\alpha \in \operatorname{Map}(h, k^\circ)$ is obtained exactly once from a pair $(\mu, (\beta_j)_{j=1..l_F})$. Then

$$
\begin{aligned}
&\left(\bigotimes_{p=1}^{h} \eta_p\right) \neg \nabla^h \left[\left(\prod_{j=1}^{k} L_{\xi_j}\right) A\right] \\
&\qquad = \sum_{\substack{F\in\mathcal{F}_{k^\circ} \\ \alpha\in \operatorname{Map}(h,k^\circ)}} (-1)^{l_F-1} \left[\prod_{T\in F^\dagger} \operatorname{ad}\left(\nabla^{\alpha\cup T}_{\eta_\bullet,\xi_\bullet}\nabla\xi_{\rho(T)}\right)\right] \nabla^{\alpha\cup F_\circ}_{\eta_\bullet,\xi_\bullet} A\,,
\end{aligned}
$$

where we again used the abuse of notation $\alpha\cup T := \alpha|_{\alpha^{-1}(T)} \cup T$ for any subtree T of F.

In the case $\eta = \eta_p$ for all p, we obtain as for the inequality (6.2)

$$
\begin{aligned}
&\frac{1}{h!}\left\|\nabla^h\left(\prod_{j=1}^k L_{\xi_j}\right)A\right\|_r\\
&\leqslant C_r^k \sum_{\substack{F\in\mathcal{F}_{k^\circ}\\ H\in\mathbb{N}^{k+1}(h)}} 2^{l_F-1}\times\\
&\times\left(\frac{1}{h_{k+1}!}\left\|\nabla^{h_{k+1}+\ell(\circ)}A\right\|_r\cdot\prod_{\nu\in F_\circ^*}\frac{1}{h_\nu!}\|\nabla^{h_\nu+\ell(\nu)}\xi_\nu\|_r\right)\times\\
&\times\prod_{T\in F^\dagger}\left(\frac{1}{h_{\rho(T)}!}\left\|\nabla^{h_{\rho(T)}+1+\ell(\rho(T))}\xi_{\rho(T)}\right\|_r\cdot\prod_{\nu\in T^*}\frac{1}{h_\nu!}\|\nabla^{h_\nu+\ell(\nu)}\xi_\nu\|_r\right). \qquad (7.2)
\end{aligned}
$$

For any fixed $H \in \mathbb{N}^{k+1}(h)$ we consider the terms of the sum in (7.2) and we set $B^l := \frac{1}{h_{k+1}!}\|\nabla^{h_{k+1}+l}A\|_r$ and $X_\nu^l := \frac{1}{h_\nu!}\|\nabla^{h_\nu+l}\xi_\nu\|_r$. Then the estimate in the statement of theorem 1 follows from theorem 2. □

Aknowledgment

This research was driven by computer exploration, using the open-source mathematical software Sagemath [Sage] and its algebraic combinatorics features developed by the Sage-Combinat community [SC].

References

[Cay] Cayley, A., *On the theory of the analytical forms called trees*, Phil. Mag., 13 (1857), 172-6.

[HNT1] Hivert, F., Novelli, J.-C. and Thibon, J.-Y., *Multivariate generalizations of the Foata-Schützenberger equidistribution*, DMTCS Proceedings, AG, (2006), 289–300.

[HNT2] Hivert, F., Novelli, J.-C. and Thibon, J.-Y., *Commutative combinatorial Hopf algebras*, J. Algebr Comb, 28 , (2008) 65–95.

[Man] Manchon, D., *A short survey on pre-Lie algebras*, E. Schrödinger Institut Lectures in Math. Phys., Eur. Math. Soc, A. Carey Ed. (2011).

[Pal1] Pali, N., *The Soliton-Kähler-Ricci Flow over Fano Manifolds*, New York J. Math. 20, (2014), 845-919.

[Pal2] Pali, N., *Variational stability of Kähler-Ricci solitons*, Advances in Mathematics 290, (2016), 15-35.

[Pal3] Pali, N., *The Soliton-Ricci Flow vith variable volume forms*, Complex Manifolds, Vol 3, Issue 1, (2016), 41-144.

[P-S] Phong, D.H., Sturm, J., *On stability and the convergence of the Kähler-Ricci flow*, arXiv:0412185v1, (2004), J. Differential Geom. 72 (2006), no. 1, 149–168.

[Sage] Stein, W et al., *Sage Mathematics Software*, The Sage Development Team. 2018 http://www.sagemath.org.

[SC] The Sage-Combinat community, *Sage-Combinat: enhancing Sage as a toolbox for computer exploration in algebraic combinatorics*, 2008, http://combinat.sagemath.org.

[OEIS] Sloane, N. J. A., *The On-Line Encyclopedia of Integer Sequences*, http://oeis.org.

Florent Hivert
Laboratoire de Recherche en Informatique, Université Paris Sud,
Bâtiment 650 F91405 Orsay, France
E-mail: florent.hivert@lri.fr

Nefton Pali
Université Paris Sud, Département de Mathématiques
Bâtiment 307 F91405 Orsay, France
E-mail: nefton.pali@math.u-psud.fr