Technical Efficiency in Firm Games with Constant Returns to Scale and α-Returns to Scale

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Abstract

Under a technology based on the generalized mean of inputs and outputs with constant returns to scale (CRS), the firms have incentive to merge (in a firm game) in order to improve their technical efficiency. A directional complementarity property in inputs and in outputs is introduced. It is shown that the core of the firm game is non-void whenever the aggregate technology of each coalition exhibits complementarity in outputs and CRS. In the case of α-returns to scale, the firms have incentive to merge (improvement of technical efficiency) when there are both directional complementarity in inputs and in outputs.

Keywords: Cooperative games; Complementarity; Directional Distance function; Returns to scale; Technical efficiency.

JEL Codes: D21, D24.

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1 Introduction

This last decade, production games have been introduced in the literature in order to study the formation of coalitions and its impact on productivity measurement. Through the prism of Data Envelopment Analysis (DEA) and cooperative games, Lozano (2012) shows that firms may share some information (data) about inputs and outputs of their processing units in order to improve their benefits. Lozano (2013) studies DEA production games in which different firms have the possibility to merge either with their own technology or with a joint coalitional technology. The same idea is studied in Briec and Mussard (2014) in which firm games may help to find some coalitional technologies associated with an improvement of allocative efficiency and a decline of technical efficiency.

Following Chambers, Chung and Färe (1996, 1998), technical efficiency may be measured by the directional distance function.[1] Peyrache (2013) shows that this distance function has appealing interpretations related to the chosen direction. It allows the firm physical output loss to be measured in terms of the numeraire (output orientation), and it can also be interpreted as the firm physical input waste (input orientation). Recently, Ravelojaona (2019) investigates the employ of a generalized directional distance function based on the generalized mean over inputs and outputs in order to distort the usual technologies in such a way that it becomes possible to measure technical efficiency with non-linear technologies. However, no connection has been proposed with firm games, that is, to investigate the measurement of technical efficiency with aggregate technologies, especially for technologies with constant returns to scale (CRS) and α-returns to scale.

In this paper, the aggregation of technology sets is generalized thanks to an aggregator inspired from Ben-Tal (1977) and Hardy, Littlewood and Pólya (1934), who characterize the generalized mean studied by Ravelojaona (2019). We introduce firm games in which several aggregate technologies (also termed *coalitional technologies* or *joint technologies*) are designed with the aid of the generalized mean aggregator. First, all firms may have a technology with constant returns to scale, see *e.g.* Li and Ng (2001) for CRS technologies of groups of firms. Second, all firms may have a technology with α-returns to scale and may decide to merge in order to create a new aggregate technology. In both cases, there may be either no technical efficiency variation, improvement or decline of technical efficiency due to cooperation (for all possible merging firms). Those three cases are captured by the aggregation bias, *i.e.* the difference between the inefficiency of the firm coalition and the (generalized mean) of the firm inefficiencies. It is shown that the aggregation bias may be either negative (cooperation improves technical efficiency), positive (decline of technical efficiency), or null.

[1] See also Chambers and Färe (1998) for the duality theory.

CRS technologies are associated with a new concept of *directional complementarity* in inputs and in outputs issued from the directional distance function. The directional complementarity in inputs postulates that an increase in input i implies an improvement of technical efficiency due to the role of all inputs but i in producing a given output vector. In the case where all firms have CRS technologies with directional input complementarity and employ the same technique, the aggregation bias is null, that is, the mean of the firms inefficiencies over each coalition corresponds to the inefficiency of the coalition computed on the coalitional technology, implying no incentive to merge (the core of the game is empty). On the contrary, the directional output complementarity postulates that an decrease in output j implies an improvement of technical efficiency due to the role of all outputs but j in using a given input vector. In this case, a null aggregation bias is recorded, but it is not incompatible with the incentive to merge (the core interior is non-void) because each coalition aims at maximizing its outputs. These results generalize the approach suggested by Li (1995) with regard to the characterization of CRS technologies as convex cones, which become generalized convex cones in the context of firm games. The firms may also merge when their technologies all exhibit α-returns to scale. In this firm game, the core interior of the game is not trivially derived. It is shown, for each possible coalition, that the aggregate technology embodied with α-returns to scale is characterized by both directional output and input complementarity and by subadditive games, therefore the technical efficiency of each coalition increases with cooperation. Moreover, in this case, the aggregation bias is negative: the inefficiency of the firm coalition is lower than the generalized mean of the firm inefficiencies.

The paper is organized as follows. Section 2 sets the notations. Section 3 defines the generalized mean and the firm games. Section 4 introduces directional complementarity. Sections 5 and 6 present the results about CRS aggregate technologies and α-returns to scale aggregate technologies, respectively, and the (non)-existence of the core of the firm game. Section 7 concludes.

2 Setup

Notations and definitions. The set of positive integers is denoted $\mathbb{N}_+$, the non-negative part of the real line is $\mathbb{R}_+$ and $\mathbb{R}_{++}$ is its positive part (with $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$ its n-dimensional representation). The set $\{1,\dots,\ell\}$ is denoted $[\ell]$ for any integer ℓ in $\mathbb{N}_+$. The interior of set E is denoted $\overset{\circ}{E}$. Let $\alpha \in \mathbb{R} \setminus \{0\}$ and $\phi_\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ be a real and bijective power map defined as follows:

$$\phi_\alpha(\theta) = \begin{cases} \theta^\alpha & \text{if } \theta > 0 \\ -|\theta|^\alpha & \text{if } \theta < 0 \\ 0 & \text{if } \theta = 0. \end{cases}$$

The power function over $\mathbb{R}^d$, is defined by the map $\Phi_\alpha : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, such that for any vector $z = (z_1, \ldots, z_d) \in \mathbb{R}^d$:

$$\Phi_\alpha(z) = \big(\phi_\alpha(z_1), \ldots, \phi_\alpha(z_d)\big).$$

Definition 2.1 – Generalized Mean:
(i) *For $u = (u_1, \ldots, u_d) \in \mathbb{R}^d$, the generalized mean over $\mathbb{R}_+$ is expressed as:*

$$\phi_\alpha^{-1}\big(\phi_\alpha(u_1) + \cdots + \phi_\alpha(u_d)\big) = u_1 \overset{\alpha}{+} \cdots \overset{\alpha}{+} u_d := \sum_{k\in[\ell]}^{\phi_\alpha} u_k$$

(ii) *For $z^1, \ldots, z^\ell \in \mathbb{R}^d$, the generalized mean over $\mathbb{R}^d_+$ is expressed as:*

$$\Phi_\alpha^{-1}\big(\Phi_\alpha(z^1) + \cdots + \Phi_\alpha(z^\ell)\big) = z^1 \overset{\alpha}{+} \cdots \overset{\alpha}{+} z^\ell := \sum_{k\in[\ell]}^{\Phi_\alpha} z^k.$$

(iii) *The limits of the generalized mean over $\mathbb{R}^d_{++}$ are expressed, for all $z^k \in \mathbb{R}^d_{++}$, as follows:*[2]

$$\lim_{\alpha\to-\infty} \sum_{k\in[\ell]}^{\Phi_\alpha} x^k = \left(\bigwedge_{k\in[\ell]} x_1^k, \ldots, \bigwedge_{k\in[\ell]} x_d^k\right) := \left(\min_{k\in[\ell]} x_1^k, \ldots, \min_{k\in[\ell]} x_d^k\right)$$

$$\lim_{\alpha\to+\infty} \sum_{k\in[\ell]}^{\Phi_\alpha} x^k = \left(\bigvee_{k\in[\ell]} x_1^k, \ldots, \bigvee_{k\in[\ell]} x_d^k\right) := \left(\max_{k\in[\ell]} x_1^k, \ldots, \max_{k\in[\ell]} x_d^k\right)$$

$$\lim_{\alpha\to 0} \sum_{k\in[\ell]}^{\Phi_\alpha} x^k = \left(\prod_{k\in[\ell]} x_1^k, \ldots, \prod_{k\in[\ell]} x_d^k\right).$$

The generalized mean has been characterized and analyzed by Hardy, Littlewood and Pólya (1934). It has been employed in welfare economics and economic inequality by Atkinson (1970), and characterized by Blackorby *et al.* (1981) among others, in order to aggregate utility or incomes for the specification of social welfare functions and inequality measures.[3]

The firms use inputs and produce outputs. Let $x \in \mathbb{R}^n_+$ and $y \in \mathbb{R}^m_+$ be the input and output vectors, respectively. The n-dimensional [m-dimensional] vector of zeros is 0_n [0_m], the same thing for vectors of ones 1_n [1_m], and finally $\geq$ [$\leq$] denotes inequalities over scalars and $\geqslant$ [$\leqslant$] over vectors.
Assumptions. The technology T of the firms might satisfy the following standard assumptions outlined in the literature of productivity measurement.
1. There is no free lunch:

$$\big[(0_n, 0_m) \in T, (0_n, y) \in T\big] \implies \big[y = 0_m\big]. \quad \text{(T1)}$$

[2]See Briec (2015) for the operator of the generalized mean over $\mathbb{R}^d$ and its limits.
[3]See also Ben-Tal (1977) for the notion of generalized convex functions.

2. The set $A(x)$ of dominating observations is bounded, i.e., infinite outputs cannot be obtained from a finite input vector:

$$A(x) := \{(u, y) \in T : u \leqslant x\}, \forall x \in \mathbb{R}^n_+. \tag{T2}$$

3. The set T is closed, i.e., *for all sequences $(x_s, y_s)_s$ of T,*

$$\big[\lim(x_s, y_s) = (x, y) \in \mathbb{R}^{n+m}_+\big] \implies \big[(x, y) \in T\big]. \tag{T3}$$

4. Fewer outputs can always be produced with more inputs:

$$\big[\forall (x, y) \in T, (x, -y) \leqslant (u, -v)\big] \implies \big[(u, v) \in T\big]. \tag{T4}$$

5. The technology exhibits Constant Returns to Scale (CRS):

$$\big[\forall \lambda \geq 0, (x, y) \in T\big] \implies \big[(\lambda x, \lambda y) \in T\big]. \tag{T5}$$

6. The technology exhibits α-returns to scale, if for all $(x, y) \in \mathbb{R}^{n+m}_{++}$:

$$\big[\forall \lambda > 0, (x, y) \in T\big] \implies \big[\forall \alpha \in \mathbb{R}, (\lambda x, \lambda^\alpha y) \in T\big]. \tag{T6}$$

Given a production set one can define an input correspondence of firm k by $L^k : \mathbb{R}^m_+ \longrightarrow 2^{\mathbb{R}^n_+}$ and the output correspondence of firm k as $P^k : \mathbb{R}^n_+ \longrightarrow 2^{\mathbb{R}^m_+}$ such that the technology of firm k in N is expressed as follows:

$$T^k = \big\{(x^k, y^k) \in \mathbb{R}^{n+m}_+ : x^k \in L^k(y^k)\big\} = \big\{(x^k, y^k) \in \mathbb{R}^{n+m}_+ : y^k \in P^k(x^k)\big\},$$

where the couple (x^k, y^k) denotes the input-output vector of firm k. The technical (in)efficiency of firm k is measured by distance functions. The directional distance function (Chambers, Chung and Färe, 1996, 1998) of firm k, $D_{T^k} : \mathbb{R}^{n+m}_+ \times -\mathbb{R}^n_+ \times \mathbb{R}^m_+ \longrightarrow \mathbb{R}_+$, is given by:

$$D_{T^k}(x^k, y^k; g) = \sup_\delta \big\{\delta \in \mathbb{R} : (x^k - \delta g_i, y^k + \delta g_o) \in T^k\big\}.$$

The less (the more) the distance of the couple (x^k, y^k) to the technology frontier of T^k, the more the (in)efficiency is.

The set of firms (players) is fixed and finite, it is given by $N := \{1, \dots, |N|\}$, where $|N| \geq 2$. The subsets of the grand coalition N are denoted by $\mathcal{S}$. A transferable utility game, *i.e.* a TU-game, is a pair (N, v), where v is defined as $v : 2^N \to \mathbb{R}_+$ such that $v(\emptyset) := 0$. The set of all maps v is denoted Γ, such that $v(\mathcal{S})$ provides the worth of coalition $\mathcal{S}$. A valued solution $\varphi(v)$ is the *payoff* vector of the TU-game (N, v) that is a n-dimensional real vector that represents what the firms could take benefit from cooperation. The function φ is called a value or an allocation.

3 Firm games and aggregate technologies

In this Section, we define firm games with aggregate technologies issued from the generalized mean. Moreover Farrell's structural and industrial technical efficiencies are introduced in order to define the bias of aggregation.

3.1 Firm games defined on generalized means

In production theory, Li and Ng (2001) and subsequently by Briec and Mussard (2014), introduced the aggregate technology. In their frameworks, the technology of the grand coalition is the standard sum of input and output vectors. Following this specification, the technology of any given coalition $\mathcal{S} \subseteq N$ is the standard sum of the technologies T^k of each firm $k \in \mathcal{S}$:

$$T^{\mathcal{S}} := \sum_{k\in\mathcal{S}} T^k = \left\{ \left(\sum_{k\in\mathcal{S}} x^k, \sum_{k\in\mathcal{S}} y^k \right) : (x^k, y^k) \in T^k \right\}.$$

This specification is dropped in order to investigate aggregate technologies in firm games based on the generalized mean.

Definition 3.1 – **Firm Game:** *A firm game defined on the generalized mean operator is a triplet (N, v, Φ_α) with $\alpha \in \mathbb{R} \setminus \{0\}$ such that the resulting aggregate technology of coalition $\mathcal{S} \subseteq N$ after cooperation is:*

$$T^{\mathcal{S}}_\alpha := \sum_{k\in\mathcal{S}}^{\Phi_\alpha} T^k = \left\{ \left(\sum_{k\in\mathcal{S}}^{\Phi_\alpha} x^k, \sum_{k\in\mathcal{S}}^{\Phi_\alpha} y^k \right) : (x^k, y^k) \in T^k \right\}.$$

The game $v : 2^N \to \mathbb{R}$ provides the technical efficiency of all possible coalitions $\mathcal{S} \subseteq N$ with aggregate technology $T^{\mathcal{S}}_\alpha$, for which the characteristic function is $D_{T^{\mathcal{S}}_\alpha}$:

$$v(\mathcal{S}) \equiv D_{T^{\mathcal{S}}_\alpha} \left(\sum_{k\in\mathcal{S}}^{\Phi_\alpha} (x^k, y^k); g \right) \geq 0, \ \textit{with } v(\emptyset) = 0.$$

The greater $D_{T^{\mathcal{S}}_\alpha}$ is, the more the technical inefficiency of coalition $\mathcal{S} \subseteq N$ is. The technical inefficiency of a group of firms is computed on the basis of the aggregate technology inherent to the *firm game* by which each firm may decide (or not) to cooperate with other firms in order to create a greater technical efficiency at the sectoral level. The measure $D_{T^{\mathcal{S}}_\alpha}$ may be compared with individual technical efficiencies D_{T^k} to judge whether the cooperation improves the technical efficiency of coalition $\mathcal{S}$.

3.2 Farrell's technical efficiency measurement and aggregation bias

Following Farrell (1957), the technical efficiency may be computed at the sectoral level following two types of indices, based on the simple sum of input and output vectors. Since the technologies are defined on the generalized mean, we specify Farrell's definitions as follows. A firm game (N, v, Φ_α) yields the *structural technical inefficiency*: $D_{T^{\mathcal{S}}_\alpha}\Big(\sum_{k\in\mathcal{S}}^{\Phi_\alpha} (x^k, y^k); g \Big)$. The structural technical inefficiency $D_{T^{\mathcal{S}}_\alpha}$ of coalition $\mathcal{S}$ based on the generalized mean of inputs x^k and output y^k of firms k in $\mathcal{S}$ corresponds to the technical efficiency of the group issued from the aggregate technology $T_\alpha^{\mathcal{S}}$, that is, the firms k in coalition $\mathcal{S}$ merge. *Industrial technical inefficiency* is defined as $\sum_{k\in\mathcal{S}}^{\phi_\alpha} D_{T^k}(x^k, y^k; g)$. The industrial technical inefficiency is the generalized mean of the firms' inefficiencies. In some cases, both measures of aggregate efficiency do not always coincide, that is,

$$\sum_{k\in\mathcal{S}}^{\phi_\alpha} D_{T^k}(x^k, y^k; g) \lesseqgtr D_{T_\alpha^{\mathcal{S}}}\left(\sum_{k\in\mathcal{S}}^{\Phi_\alpha}(x^k, y^k); g\right),\ \mathcal{S} \subseteq N.$$

(i) $\sum_{k\in\mathcal{S}}^{\phi_\alpha} D_{T^k}(x^k, y^k; g) > D_{T_\alpha^{\mathcal{S}}}\Big(\sum_{k\in\mathcal{S}}^{\Phi_\alpha}(x^k, y^k); g\Big)$: The technical efficiency of coalition $\mathcal{S}$ measured on the joint technology $T_\alpha^{\mathcal{S}}$ is greater than the mean of technical efficiency issued from the individual technologies T^k.

(ii) $\sum_{k\in\mathcal{S}}^{\phi_\alpha} D_{T^k}(x^k, y^k; g) < D_{T_\alpha^{\mathcal{S}}}\Big(\sum_{k\in\mathcal{S}}^{\Phi_\alpha}(x^k, y^k); g\Big)$: The joint technology $T_\alpha^{\mathcal{S}}$ does not provide a greater technical efficiency than the mean of the individual technical efficiencies.

(iii) $\sum_{k\in\mathcal{S}}^{\phi_\alpha} D_{T^k}(x^k, y^k; g) = D_{T_\alpha^{\mathcal{S}}}\Big(\sum_{k\in\mathcal{S}}^{\Phi_\alpha}(x^k, y^k); g\Big)$: The (generalized) mean of the technical efficiencies of the firms is equivalent to the technical efficiency they can expect with a new coalitional technology issued from cooperation.

The difference between the structural and the industrial technical efficiency is said the *aggregation bias*.

Definition 3.2 – Aggregation Bias: *For all firm games (N, v, Φ_α), the aggregation bias of coalition $\mathcal{S} \subseteq N$ of size $|\mathcal{S}| \geq 2$ is defined as follows:*

$$AB_\alpha(\mathcal{S}; g) := D_{T_\alpha^{\mathcal{S}}}\left(\sum_{k\in\mathcal{S}}^{\Phi_\alpha}(x^k, y^k); g\right) - \sum_{k\in\mathcal{S}}^{\phi_\alpha} D_{T^k}(x^k, y^k; g).$$

Briec and Mussard (2014) show, under assumptions (T1)-(T4), that the aggregation bias may be positive or negative. In particular, the aggregation bias of allocative efficiency is non-positive, meaning that the aggregate technology based on the simple sum of inputs and outputs vectors yields a greater allocative efficiency than the sum of the individual efficiencies. On the other hand, for these aggregate technologies, the aggregation bias of technical efficiency is non-negative.[4]

We show below, without invoking any restriction on the firm game under the usual assumptions (T1)-(T4), that the aggregation bias is always non-negative. This result parallels that of Briec and Mussard (2014) in which the aggregate technology is issued from the simple sum of inputs and outputs.

[4]See also Briec *et al.* (2003) for the special case of null aggregation bias.

Proposition 3.1 (non-negative bias) *Suppose a firm game* (N, v, Φ_α) *such that each firm's technology* T^k *of coalition* $\mathcal{S}$ *satisfies* (T1)-(T4) *for all* $k \in \mathcal{S}$. *Then, the aggregation bias in the direction of* g *is non-negative:*

$$AB_\alpha(\mathcal{S}; g) = D_{T^{\mathcal{S}}_\alpha}\left(\sum_{k\in\mathcal{S}}^{\Phi_\alpha}(x^k, y^k); g\right) - \sum_{k\in\mathcal{S}}^{\phi_\alpha} D_{T^k}(x^k, y^k; g) \geq 0, \ \forall \mathcal{S} \subseteq N.$$

Proof:

Let us begin with some claims.

Claim 3.1 *For all* $\alpha \in \mathbb{R} \setminus \{0\}$ *and* $a, b, c > 0$, *the generalized mean* $\overset{\alpha}{+}$ *is associative:*

$$(a \overset{\alpha}{+} -b) \overset{\alpha}{+} c = -b \overset{\alpha}{+} (a \overset{\alpha}{+} c).$$

Claim 3.2 *The directional distance function is translatable. For all* $\gamma > 0$*:*

$$D_{T^k}(x^k - \gamma g_i, y^k + \gamma g_i; g) = D_{T^k}(x^k, y^k; g) - \gamma.$$

Following the previous claims, we have for all $k \in \mathcal{S}$ and all $\alpha \neq 0$:

$$\left(x^k \overset{\alpha}{+} -D_{T^k}(x^k, y^k; g)g_i, y^k \overset{\alpha}{+} D_{T^k}(x^k, y^k; g)g_o\right) \in T^k$$
$$\iff D_{T^k}\left(\left(x^k \overset{\alpha}{+} -D_{T^k}(x^k, y^k; g)g_i, y^k \overset{\alpha}{+} D_{T^k}(x^k, y^k; g)g_o\right); g\right) \geq 0.$$

By the definition of the firm game, we have for all $\mathcal{S} \subseteq N$:

$$\left(\sum_{k\in\mathcal{S}}^{\Phi_\alpha}\left(x^k \overset{\alpha}{+} -D_{T^k}(x^k, y^k; g)g_i\right), \sum_{k\in\mathcal{S}}^{\Phi_\alpha}\left(y^k \overset{\alpha}{+} D_{T^k}(x^k, y^k; g)g_o\right)\right) \in \sum_{k\in\mathcal{S}}^{\Phi_\alpha} T^k = T^{\mathcal{S}}_\alpha$$
$$\iff D_{T^{\mathcal{S}}_\alpha}\left(\sum_{k\in\mathcal{S}}^{\Phi_\alpha} x^k \overset{\alpha}{+} - \sum_{k\in\mathcal{S}}^{\phi_\alpha} D_{T^k}(x^k, y^k; g)g_i, \sum_{k\in\mathcal{S}} y^k \overset{\alpha}{+} \sum_{k\in\mathcal{S}}^{\phi_\alpha} D_{T^k}(x^k, y^k; g)g_o; g\right) \geq 0$$
$$\iff D_{T^{\mathcal{S}}_\alpha}\left(\sum_{k\in\mathcal{S}}^{\Phi_\alpha} x^k, \sum_{k\in\mathcal{S}}^{\Phi_\alpha} y^k; g\right) \overset{\alpha}{+} - \sum_{k\in\mathcal{S}}^{\phi_\alpha} D_{T^k}\left(x^k, y^k; g\right) \geq 0$$
$$\iff D_{T^{\mathcal{S}}_\alpha}\left(\sum_{k\in\mathcal{S}}^{\Phi_\alpha} x^k, \sum_{k\in\mathcal{S}}^{\Phi_\alpha} y^k; g\right) \geq \sum_{k\in\mathcal{S}}^{\phi_\alpha} D_{T^k}\left(x^k, y^k; g\right)$$
$$\iff AB_\alpha(\mathcal{S}; g) \geq 0.$$

■

The previous result outlines, under (T1)-(T4), that the aggregate technology $T^{\mathcal{S}}_\alpha$ issued from the firm game provides a greater inefficiency than the current average of the individual technical inefficiencies. We will see in Section 5 that the sign of the aggregation bias may change with respect to complementarity assumptions.

4 Directional complementarity

In this Section we introduce complementarity properties derived from the directional distance function, the so-called directional complementarity. The literature outlines two standard definitions of complementarity:

(i) *Input complementarity:* An aggregate technology with input complementarity requires that the demand of input i decreases as the price of input j increases (*e.g.* the case of Leontief production functions).

(ii) *Output complementarity:* An aggregate technology with output complementarity requires that the supply of output ℓ decreases as the price of output h decreases (*e.g.* the case of Kohli's (1983) technology).

We propose below, on the one hand, a new definition of input complementarity based on the directional distance function without imposing price effects. This definition postulates that an increase in input i implies an improvement of technical efficiency due to the role of all inputs but i in producing a given output vector. Let $g = (h, k) \in \mathbb{R}^n_+ \times \mathbb{R}^m_+$ and let us denote $I_+(h) = \{i \in [n] : h_i > 0\}$. Let $\{e_1, \cdots, e_n\}$ and $\{f_1, \cdots, f_m\}$ denote the canonical basis of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. In the following we say that the direction is *non-canonical* if $|I_+(h)| \geq 2$.

Definition 4.1 – Directional input complementarity: *Let $h \in \mathbb{R}^n_+$ be a non-canonical input directional vector. For all $i \in [n]$ let us denote $h_{-i} = \sum_{r \in [n] \setminus \{i\}} h_r e_r$. For all $y \in \mathbb{R}^m_+$, we say that the input set satisfies a complementarity assumption in the direction of h if for all $x \in L(y)$, all $\delta > 0$ and any $i \in [n]$*

$$D_T(x + \delta e_i, y; h_{-i}, 0_m) \leq D_T(x, y; h_{-i}, 0_m).$$

If for all non-canonical direction vectors h, $L(y)$ satisfies a complementarity assumption in the direction of h, then we say that it satisfies a directional input complementarity assumption.

We also propose, on the other hand, a new definition of output complementarity without taking recourse to price effects. It postulates that an decrease in output j implies an improvement of technical efficiency due to the role of all outputs but j in using a given input vector.

Definition 4.2 – Directional output complementarity: *Let $k \in \mathbb{R}^m_+$ be a non-canonical output directional vector. For all $j \in [m]$ let us denote $k_{-j} = \sum_{s \in [m] \setminus \{j\}} k_s f_s$. For all $x \in \mathbb{R}^n_+$, we say that the output set $P(x)$ satisfies a complementarity assumption in the direction of k if for all $y \in P(x)$, all $\delta > 0$ and any $j \in [m]$*

$$D_T(x, y - \delta f_j; 0_n, k_{-j}) \leq D_T(x, y; 0_n, k_{-j}).$$

If for all non-canonical direction vectors k, $P(x)$ satisfies a complementarity assumption in the direction of k, then we say that it satisfies a directional output complementarity assumption.

Input and output directional complementarity assumptions are depicted in Figure 1 below.

Figure 1. Input and Output Directional Complementarity

On the one hand, technologies with directional input complementarity are characterized with the aid of assumptions (T3) and (T4). Notice that for all output vectors y, if $L(y)$ has an infimum element it is denoted $\inf\{x : x \in L(y)\}$.

Proposition 4.1 (characterization of directional input complementarity) *Suppose that the input set is closed* (T3) *and satisfies a free disposal assumption* (T4). *Then the input set satisfies a directional complementarity assumption in input if and only if,*

$$L(y) = \inf\{x : x \in L(y)\} + \mathbb{R}^n_+.$$

Proof:
Suppose that $L(y) = \inf\{x : x \in L(y)\} + \mathbb{R}^n_+$. Let us denote $\underline{\mathrm{x}} = \inf\{x : x \in L(y)\}$. Then for all $i \in [n]$,

$$D_T(x + \delta e_i, y; h_{-i}, 0_m) = \min_{k \in [n] \backslash \{i\}} \left\{ \frac{x_k + \delta - (\underline{\mathrm{x}}_k + \delta)}{h_k} \right\}.$$

For any $i \in [n]$, the result is independent of i. Therefore for all $\delta > 0$,

$$D_T(x + \delta e_i, y; h_{-i}, 0_m) = \min_{k \in [n] \backslash \{i\}} \left\{ \frac{x_k - \underline{\mathrm{x}}_k}{h_k} \right\} = D_T(x, y; h_{-i}, 0_m),$$

which proves complementarity. Conversely, suppose the complementarity assumption in input for all positive input directions and let us prove that $L(y)$ has a minimum point. Suppose that this is not true and let us show a contradiction. In such a case one can find two points x' and x'' lying on the boundary of $L(y)$ such that $x' \wedge x'' \notin L(y)$. This implies that there are $i' \neq i''$ such that $x'_{i''} < x''_{i''}$ and $x'_{i'} > x''_{i'}$. By hypothesis $\delta = (x''_{i''} - x'_{i''}) > 0$, therefore

$$D_T\big(x' + (x''_{i''} - x'_{i''})e_{i''}, y; e_{i'}, 0_m\big) = x'_{i'} - x''_{i'} > x_{i'} - (x' \wedge x'')_{i'}.$$

However, since

$$D_T\big(x' + (x''_{i''} - x'_{i''})e_{i''}, y; e_{i'}, 0_m\big) = x_{i'} - (x' \wedge x'')_{i'},$$

we have a contradiction, which ends the proof. ■

On the other hand, technologies with directional output complementarity are also characterized with the aid of assumptions (T3) and (T4). For all input vectors x, if $P(x)$ has a supremum element then it is denoted $\sup\{y : y \in P(x)\}$.

Proposition 4.2 (characterization of directional output complementarity) *Suppose that the output set is closed $(T3)$ and satisfies a free disposal assumption $(T4)$. Then the output set satisfies a directional complementarity assumption in output if and only if,*

$$P(y) = (\sup\{y : y \in P(x)\} - \mathbb{R}^m_+) \cap \mathbb{R}^m_+.$$

Proof:
Mutatis Mutandis in the proof of Proposition 4.1. ■

Both concepts of directional complementarity may be now connected to aggregate CRS technologies.

5 Firm games and CRS technology

In this section we investigate constant returns to scale technologies for coalitions of firms. We first characterize CRS technologies for coalitions of firms by generalizing the concept of arcwise connected cone (Li, 1995) associated with CRS technologies. Then, the game relying on this aggregate technology is characterized. Finally, we prove that the core of the game exists and may be non-empty for technologies exhibiting directional complementarity in outputs.

5.1 Characterizations of the game and CRS technology

Li and Ng (2001) and Li (1995) establish some links between convex technologies and the aggregate technology of a group of firms based on the simple arithmetic mean of the technologies. The aggregation of inputs and outputs is performed by the operator Φ_α in order to generalize Li and Ng's approach. This generalization is issued from the notion of Φ_α-convex technology, which generalizes the usual notion of convex technology.

Definition 5.1 **Φ_α-convex technologies:** *A subset C of $\mathbb{R}^d_{++}$ is Φ_α-convex if for all $z, w \in C$ and all $s, t \in [0,1]$ such that $s \overset{\alpha}{+} t = 1$, we have $sz \overset{\alpha}{+} tw \in C$. An aggregated Φ_α-convex technology is given by:*

$$\Phi_\alpha^{-1}(\Phi_\alpha(T) + \Phi_\alpha(T)) = T \overset{\alpha}{+} T = \{z \overset{\alpha}{+} w : z, w \in T\},\ \alpha \in \mathbb{R} \setminus \{0\}.$$

On this basis, let us characterize the aggregate Φ_α-convex technology for a group of firms.

Lemma 5.1 *A technology T is Φ_α-convex if, and only if,*

$$\beta_1 T \overset{\alpha}{+} \beta_2 T = (\beta_1 \overset{\alpha}{+} \beta_2)T,\ \forall \beta_1, \beta_2 > 0.$$

Proof:
[if]: Let $\beta_1 \overset{\alpha}{+} \beta_2 = 1$ and the "if" part follows.
[only if]: Suppose T is Φ_α-convex. Then for $z^1, z^2 \in T$ we get that $(\beta_1/\beta)z^1 \overset{\alpha}{+} (\beta_2/\beta)z^2 \in T$ with $\beta = \beta_1 \overset{\alpha}{+} \beta_2$. Equivalently, we have $(\beta_1 z^1)^\alpha + (\beta_2 z^2)^\alpha \in \beta^\alpha \Phi_\alpha(T)$. Since Φ_α is bijective this implies that $\beta_1 z^1 \overset{\alpha}{+} \beta_2 z^2 \in \beta T$, and so $\beta_1 T \overset{\alpha}{+} \beta_2 T \subseteq \beta T = (\beta_1 \overset{\alpha}{+} \beta_2)T$. On the other hand, let $z^\alpha \in \Phi_\alpha(T)$, then $(\frac{\beta_1}{\beta} z)^\alpha \in (\frac{\beta_1}{\beta})^\alpha \Phi_\alpha(T)$ and $(\frac{\beta_2}{\beta} z)^\alpha \in (\frac{\beta_2}{\beta})^\alpha \Phi_\alpha(T)$, then $(\frac{\beta_1}{\beta} z)^\alpha + (\frac{\beta_2}{\beta} z)^\alpha \in (\frac{\beta_1}{\beta})^\alpha \Phi_\alpha(T) + (\frac{\beta_2}{\beta})^\alpha \Phi_\alpha(T)$. Multiplying the previous expression by β^α, we get that $\beta_1^\alpha z^\alpha + \beta_2^\alpha z^\alpha \in \beta_1^\alpha \Phi_\alpha(T) + \beta_2^\alpha \Phi_\alpha(T)$. Since Φ_α is bijective this yields $(\beta_1 \overset{\alpha}{+} \beta_2)z \in \beta_1 T \overset{\alpha}{+} \beta_2 T$. Since $\beta = \beta_1 \overset{\alpha}{+} \beta_2$ then $\beta z \in \beta_1 T \overset{\alpha}{+} \beta_2 T$ and so $\beta T \subseteq \beta_1 T \overset{\alpha}{+} \beta_2 T$, thus $\beta T \subseteq \beta_1 T \overset{\alpha}{+} \beta_2 T$, which ends the proof. The reader is referred to Li and Ng (2001) for the standard convex case. ■

By Lemma 5.1, it can be shown that the Φ_α-convex technology of coalition $\mathcal{S}$ is independent of the number of firms inside coalition $\mathcal{S}$ whenever these firms have the same technology. Moreover, if the firms have all the same technology, the aggregation bias remains non-negative.

Proposition 5.1 **(independence)** *Let (N, v, Φ_α) be a firm game such that each firm's technology of coalition $\mathcal{S} \subseteq N$ is given by $\beta_k T^k$ with $T^1 = \cdots = T^{|\mathcal{S}|} = T$, with T^k being Φ_α-convex for all $k \in \mathcal{S}$ and with $\sum_{k \in \mathcal{S}}^{\phi_\alpha} \beta_k = 1$. Then, the following results hold for all $\mathcal{S} \subseteq N$ such that $|\mathcal{S}| \geq 2$ and $\alpha \in \mathbb{R} \setminus \{0\}$.*
(i) *The technology $T_\alpha^{\mathcal{S}}$ of coalition $\mathcal{S}$ is Φ_α-convex: $T_\alpha^{\mathcal{S}} = \sum_{k \in \mathcal{S}}^{\Phi_\alpha} \beta_k T^k = T$.*
(ii) *$AB_\alpha(\mathcal{S}; g) \geq 0$.*

Proof:
(i) and (ii) follow from Lemma 5.1 and Proposition 3.1. ■

Now we can analyze CRS technologies based on the previous results. If constant returns to scale is satisfied by all members of coalition $\mathcal{S}$, then the aggregate technology also satisfies constant returns to scale, however the aggregation bias remains non-negative, consequently the cooperation cannot improve the technical efficiency of coalition $\mathcal{S}$.

Proposition 5.2 (aggregate CRS technology) *Let* (N, v, Φ_α) *be a firm game such that each firm's technology of coalition* $\mathcal{S} \subseteq N$ *given by* T^k *is* Φ_α*-convex and respects* (T5) *for all* $k \in \mathcal{S}$ *such that* $|\mathcal{S}| \geq 2$ *and* $\alpha \in \mathbb{R} \setminus \{0\}$*, then the following results hold for all* $\mathcal{S} \subseteq N$*.*
(i) $T_\alpha^{\mathcal{S}} = \sum_{k \in \mathcal{S}}^{\Phi_\alpha} T^k$ *respects* (T5)*.*
(ii) $AB_\alpha(\mathcal{S}; g) \geq 0$*.*

Proof:
(i) Let $z, w \in T_\alpha^{\mathcal{S}}$, we have to prove that $\lambda z \overset{\alpha}{+} \lambda w \in T_\alpha^{\mathcal{S}}$, with $z = \sum_{k \in \mathcal{S}}^{\Phi_\alpha} z^k$ and $w = \sum_{k \in \mathcal{S}}^{\Phi_\alpha} w^k$. Thus:

$$\lambda z \overset{\alpha}{+} \lambda w = \lambda \sum_{k \in \mathcal{S}}^{\Phi_\alpha} z^k \overset{\alpha}{+} \lambda \sum_{k \in \mathcal{S}}^{\Phi_\alpha} w^k = \sum_{k \in \mathcal{S}}^{\Phi_\alpha} (\lambda z^k \overset{\alpha}{+} \lambda w^k).$$

Since T^k is Φ_α-convex and respects (T5), then

$$z^k \overset{\alpha}{+} w^k \in T^k \implies \lambda z^k \overset{\alpha}{+} \lambda w^k \in T^k.$$

Thus,

$$\sum_{k \in \mathcal{S}}^{\Phi_\alpha} (\lambda z^k \overset{\alpha}{+} \lambda w^k) \in \sum_{k \in \mathcal{S}}^{\Phi_\alpha} T^k = T_\alpha^{\mathcal{S}},$$

and so, the aggregate technology $T_\alpha^{\mathcal{S}}$ respects (T5).
(ii) It follows from Proposition 3.1. ■

A similar result can be established with the aid of arcwise connected cone, introduced by Li (1995), which implies (T5). We first show the relation between arcwise connected cone and Φ_α-convex cone, from which constant returns to scale follow.

Proposition 5.3 (CRS characterization) *Let* (N, v, Φ_α) *be a firm game and let* $T \setminus \{0\} \subseteq \mathbb{R}_+^{n+m}$ *be an arcwise connected cone being* Φ_α*-convex for some* $\alpha \neq 0$*. Then, the following results hold.*
(i) $T \overset{\alpha}{+} T$ *is a* Φ_α*-convex cone.*
(ii) $T \overset{\alpha}{+} T$ *respects* (T5)*.*
(iii) *If in addition the technologies of coalition* $\mathcal{S} \subseteq N$ *are such that* $T^k = T$ *for all* $k \in \mathcal{S}$*, then* $T_\alpha^{\mathcal{S}} = \sum_{k \in \mathcal{S}}^{\Phi_\alpha} T^k = T$ $\forall \mathcal{S} \subseteq N$*.*
(iv) $AB_\alpha(\mathcal{S}; g) \geq 0$ $\forall \mathcal{S} \subseteq N$*.*

Proof:
(i) From Li (1995), if T is an arcwise connected cone, then $T+T$ is a convex cone. By definition,

$$T \overset{\alpha}{+} T = \Phi_\alpha^{-1}(\Phi_\alpha(T) + \Phi_\alpha(T)).$$

Since T is Φ_α-convex, it follows that $\Phi_\alpha(T)$ is convex. From Li (1995), we deduce that $\Phi_\alpha(T)+\Phi_\alpha(T)$ is convex. Hence $\Phi_\alpha^{-1}(\Phi_\alpha(T)+\Phi_\alpha(T)) = T \overset{\alpha}{+} T$ is Φ_α-convex. To prove that $T \overset{\alpha}{+} T$ is a Φ_α-convex cone, we must prove that $\lambda(T \overset{\alpha}{+} T) = T \overset{\alpha}{+} T$ for all $\lambda > 0$:

$$\lambda(T \overset{\alpha}{+} T) = \lambda\big(\Phi_\alpha^{-1}\big(\Phi_\alpha(T)+\Phi_\alpha(T)\big)\big) = \Phi_\alpha^{-1}\big(\phi_\alpha(\lambda)\Phi_\alpha(T)+\phi_\alpha(\lambda)\Phi_\alpha(T)\big)$$
$$= \Phi_\alpha^{-1}\big(\Phi_\alpha(\lambda T)+\Phi_\alpha(\lambda T)\big) = \Phi_\alpha^{-1}\big(\Phi_\alpha(T)+\Phi_\alpha(T)\big) = T \overset{\alpha}{+} T.$$

(ii) Let $z, w \in T \overset{\alpha}{+} T$, we have to prove that $\lambda z \overset{\alpha}{+} \lambda w \in T \overset{\alpha}{+} T$: see the proof of Proposition 5.2 (i).
(iii) From Li and Ng (1995), if T^k is a convex cone then $\sum_{k\in\mathcal{S}} T^k = T$. If T is Φ_α-convex, then $\Phi_\alpha(T)$ is a convex cone. Thus, from Li and Ng (1995):

$$\Phi_\alpha(T) = \sum_{k\in\mathcal{S}} \Phi_\alpha(T^k) \implies T = \Phi_\alpha^{-1}\Big(\sum_{k\in\mathcal{S}} \Phi_\alpha(T^k)\Big) = \sum_{k\in\mathcal{S}}^{\Phi_\alpha} T^k.$$

(iv) It follows from Proposition 3.1. ■

An example of aggregate technology based on the generalized mean and respecting assumption (T5) is the following.

Example 5.1 *An example of CRS technology is for instance the production set, for t_j the jth element of a vector t,*

$$T_\alpha^{\mathcal{S}} = \Big\{(x,y)\in\mathbb{R}_+^{m+n} : x \geqslant \sum_{j\in\mathcal{S}}^{\Phi_\alpha} t_j x^j, y \leqslant \sum_{j\in\mathcal{S}}^{\Phi_\alpha} t_j y^j, t \in \mathbb{R}_+^{|S|}\Big\}.$$

From Boussemart et al. *(2009), it satisfies a constant returns to scale assumption (T5).*

As can be seen in the previous result, the structural technical efficiency remains lower than or equal to the industrial technical efficiency when CRS technologies are invoked for all firms of a given coalition. In order to make both measures of aggregate technical efficiency to be equal, the same technique has to be employed by each firm of the coalition. In this case, it can be proven that a null aggregation bias implies that the characteristic function of the game is a generalized mean of input and outputs vectors.

Proposition 5.4 (game characterization) *Let (N, v, Φ_α) be a firm game and let $T\setminus\{0\}\subseteq\mathbb{R}_+^{n+m}$ be an arcwise connected cone being Φ_α-convex for some $\alpha\neq 0$ and respecting* (T1)-(T4) *such that $T^k = T$ for all $k\in\mathcal{S}$, and such that each firm $k\in\mathcal{S}$ employs the same technique i.e. $x_i^k = \beta_{ij}x_j^k$, $y_\ell^k = \gamma_{\ell p}y_p^k$, $x_i^k = \delta_{ip}y_p^k$ where $\beta_{ij}, \gamma_{\ell p}, \delta_{ip}$ are constant for all $i, j\in\{1,\dots,n\}$ and for all $\ell, p\in\{1,\dots,m\}$. Then, the following results hold:*

(i) $T_\alpha^{\mathcal{S}} = \sum_{k\in\mathcal{S}}^{\Phi_\alpha} T^k = T$ *respects* (T5) *and* $AB_\alpha(\mathcal{S}; g) = 0\ \forall \mathcal{S} \subseteq N$.
(ii) *For* $z^k := (x^k, y^k) \in \mathbb{R}_+^{n+m}$*, the game* v *is given by:*

$$v(\mathcal{S}) \equiv D_{T_\alpha^{\mathcal{S}}}\left(\sum_{k\in\mathcal{S}}^{\Phi_\alpha} z^k; g\right) = \left(\sum_{k\in\mathcal{S}}\sum_{\ell=1}^{n+m} b_\ell (z_\ell^k)^\alpha\right)^{\frac{1}{\alpha}},\ \forall \mathcal{S} \subseteq N$$

where $b_\ell \geq 0$*, for all* $\ell \in \{1, \ldots, n+m\}$*.*

Proof:
(i) Since $x_i^k = \beta_{ij} x_j^k$, $y_\ell^k = \gamma_{\ell p} y_p^k$, $x_i^k = \delta_{ip} y_p^k$ for all $i, j \in \{1, \ldots, n\}$ and for all $\ell, p \in \{1, \ldots, m\}$, then it exists $(t, w) \in T$ such that $(x^k, y^k) = \alpha_k(t, w)$. Therefore,

$$D_{T_\alpha^{\mathcal{S}}}\left(\sum_{k\in\mathcal{S}}^{\Phi_\alpha} (x^k, y^k); g\right) = D_{T_\alpha^{\mathcal{S}}}\left(\sum_{k\in\mathcal{S}}^{\Phi_\alpha} \alpha_k(t, w); g\right) = \sum_{k\in\mathcal{S}}^{\phi_\alpha} \alpha_k D_{T_\alpha^{\mathcal{S}}}(t, w; g).$$

Since $T_\alpha^{\mathcal{S}} = T = T^k$ by Proposition 5.3, then

$$\sum_{k\in\mathcal{S}}^{\phi_\alpha} \alpha_k D_{T_\alpha^{\mathcal{S}}}((t, w); g) = \sum_{k\in\mathcal{S}}^{\phi_\alpha} D_{T_\alpha^{\mathcal{S}}}(\alpha_k(t, w); g) = \sum_{k\in\mathcal{S}}^{\phi_\alpha} D_{T^k}(x^k, y^k; g).$$

Hence,

$$D_{T_\alpha^{\mathcal{S}}}\left(\sum_{k\in\mathcal{S}}^{\Phi_\alpha} (x^k, y^k); g\right) = \sum_{k\in\mathcal{S}}^{\phi_\alpha} D_{T^k}(x^k, y^k; g),$$

and the result follows.
(ii) By result (i), let $AB_\alpha(\mathcal{S}; g) = 0$ for all $\alpha \neq 0$ and let $f_{\mathcal{S}} := D_{T_\alpha^{\mathcal{S}}}$ and $f_k := D_{T^k}$. For all $\mathcal{S} \subseteq N$ such that $|\mathcal{S}| \geq 2$:

$$AB_\alpha(\mathcal{S}; g) = 0 \iff f_{\mathcal{S}}\left(\sum_{k\in\mathcal{S}}^{\Phi_\alpha} (x^k, y^k)\right) = \sum_{k\in\mathcal{S}}^{\phi_\alpha} \left[f_k(x^k, y^k)\right].$$

Thus,

$$f_{\mathcal{S}}\left[{\Phi_\alpha}^{-1}\left(\sum_{k\in\mathcal{S}} \Phi_\alpha(x^k, y^k)\right)\right] = \phi_\alpha^{-1}\left(\sum_{k\in\mathcal{S}} \phi_\alpha\left[f_k(x^k, y^k)\right]\right).$$

Let us denote the vector $z^k := (x^k, y^k) \in \mathbb{R}_+^{n+m}$ such that $u^k := \Phi_\alpha(z^k)$, then:

$$\phi_\alpha\left(f_{\mathcal{S}}\left[\Phi_\alpha^{-1}\left(\sum_{k\in\mathcal{S}} u^k\right)\right]\right) = \sum_{k\in\mathcal{S}} \phi\left[f_k(z^k)\right].$$

Set $\phi_\alpha \circ f_{\mathcal{S}} =: \phi_{\mathcal{S}}$ and $\phi_\alpha \circ f_k =: \phi_k$, for all $k \in \mathcal{S}$. Since $z^k = {\Phi_\alpha}^{-1}(u^k)$, then

$$\phi_{\mathcal{S}}\left[\Phi^{-1}\left(\sum_{k\in\mathcal{S}} u^k\right)\right] = \sum_{k\in\mathcal{S}} \phi_k\left[\Phi^{-1}(u^k)\right].$$

The previous expression is the well-known Pexider's equation of solution (see Aczél, 1966, p.141):

$$\phi_{\mathcal{S}} \circ {\Phi_\alpha}^{-1}\left(\sum_{k\in\mathcal{S}} u^k\right) = \mathbf{c}\cdot\left(\sum_{k\in\mathcal{S}} u^k\right) + \sum_{k\in\mathcal{S}} c_k \ ;$$
$$\phi_k \circ {\Phi_\alpha}^{-1}(u^k) = \mathbf{c}\cdot u^k + c_k \ ,$$

where the vector $\mathbf{c} \in \mathbb{R}_+^{n+m}$ and the constants $c_k \in \mathbb{R}_+$ are set to be non-negative in order to get a well-defined distance function (being non-negative). The solution can be rewritten in a general setting as:

$$\phi_\alpha \circ f_k(z^k) = \mathbf{c}\cdot(z^k)^\alpha + c_k \ , \ \forall k \in \mathcal{S},$$

and,

$$\phi_\alpha \circ f_{\mathcal{S}}\left(\sum_{k\in\mathcal{S}}^{\Phi_\alpha} z^k\right) = \mathbf{c}\cdot\sum_{k\in\mathcal{S}}(z^k)^\alpha + \sum_{k\in\mathcal{S}} c_k \ , \ \forall \mathcal{S} \subseteq N.$$

Consequently, the game is expressed as:

$$v(\mathcal{S}) \equiv f_{\mathcal{S}}\left(\sum_{k\in\mathcal{S}}^{\Phi_\alpha} z^k\right) = \left(\mathbf{c}\cdot\sum_{k\in\mathcal{S}}\left(z^k\right)^\alpha + \sum_{k\in\mathcal{S}} c_k\right)^{\frac{1}{\alpha}} \ , \ \forall \mathcal{S} \subseteq N.$$

Setting $\mathbf{c} := (b_1, \ldots, b_{n+m})$, we get:

$$D_{T_\alpha^{\mathcal{S}}}\left(\sum_{k\in\mathcal{S}}^{\Phi_\alpha} z^k; g\right) = \left(\sum_{k\in\mathcal{S}}\sum_{\ell=1}^{n+m} b_\ell (z_\ell^k)^\alpha + \sum_{k\in\mathcal{S}} c_k\right)^{\frac{1}{\alpha}} \ , \ \forall \mathcal{S} \subseteq N.$$

Following Chambers, Chung and Färe (1996, 1998), the directional distance function is homogeneous of degree 1 in inputs and outputs. For all $\lambda > 0$:

$$D_{T^k}\left(\lambda x^k, \lambda y^k; g\right) = \lambda D_{T^k}\left(x^k, y^k; g\right).$$

This implies $\sum_{k\in\mathcal{S}} c_k = 0$, thus:

$$v(\mathcal{S}) \equiv D_{T_\alpha^{\mathcal{S}}}\left(\sum_{k\in\mathcal{S}}^{\Phi_\alpha} z^k; g\right) = \left(\sum_{k\in\mathcal{S}}\sum_{\ell=1}^{n+m} b_\ell (z_\ell^k)^\alpha\right)^{\frac{1}{\alpha}} \ , \ \forall \mathcal{S} \subseteq N.$$

■

Based on the previous result, it is possible to analyze whether the coalitions lead to stable solutions lying in the core of the game.

5.2 CRS technologies, core and complementarity

In Section 4, technologies with either input or output directional complementarity have been characterized with the minimum use of inputs and the maximum use of outputs, respectively. As a consequence, the coalitional technologies based on the generalized mean (with $\alpha \to \pm\infty$) are welcome to capture these directional complementarity assumptions. These technologies, see Andriamasy *et al.* (2017), are the following:

$$T^{\mathcal{S}}_{+\infty} := \lim_{\alpha\to+\infty} \sum_{k\in\mathcal{S}}^{\Phi_\alpha} T^k = \left\{ (x,y) \in \mathbb{R}^{n+m}_{++} : x \geqslant \bigvee_{k\in\mathcal{S}} t_k x^k, y \leqslant \bigvee_{k\in\mathcal{S}} t_k y^k, t \geqslant 0 \right\}$$

$$T^{\mathcal{S}}_{-\infty} := \lim_{\alpha\to-\infty} \sum_{k\in\mathcal{S}}^{\Phi_\alpha} T^k = \left\{ (x,y) \in \mathbb{R}^{n+m}_{++} : x \geqslant \bigwedge_{k\in\mathcal{S}} t_k x^k, y \leqslant \bigwedge_{k\in\mathcal{S}} t_k y^k, t \geqslant 0 \right\}.$$

The aim of this subsection is to establish a link between the directional complementarity assumption, the core of firm games and CRS Φ_α-convex technologies.

The core of the firm game relies on *individual rationality* and *collective rationality.* Let $\varphi \in \mathbb{R}^n$ be a value inherent to the firm game.

(i) *Individual rationality*: $\varphi_k \leq D_{T^k}\left(x^k, y^k; g\right)$, for all $k \in N$.

(ii) *Collective rationality*: $\sum_{k\in\mathcal{S}} \varphi_k \leq D_{T^{\mathcal{S}}}\left(\sum_{k\in\mathcal{S}}^{\Phi_\alpha}(x^k, y^k); g\right)$, for all $\mathcal{S} \subseteq N$.

The individual rationality means that each firm exhibits less technical inefficiency φ_k than its stand alone technical inefficiency D_{T^k}. The collective rationality means that if each firm inside a coalition $\mathcal{S}$ decides to cooperate with others, then the resulting technical inefficiency is lower than the aggregate inefficiency of the coalition. The core of the firm game is concerned with both rationalities.

Definition 5.2 *The core of the firm game (N, v, Φ_α) is the set of imputations $\varphi \in \mathbb{R}^n_+$ that respect individual and collective rationality:*

$$\mathcal{C}_{D_T} = \left\{ \sum_{k\in\mathcal{S}} \varphi_k \leq v(\mathcal{S}), \forall \mathcal{S} \subset N \right\} \cap \left\{ \sum_{k\in N} \varphi_k = v(N) \right\}$$

We show below that the existence of the core depends on CRS technologies that exhibit directional complementarity in outputs.

Proposition 5.5 (core and complementarity) *Let (N, v, Φ_α) be a firm game and let $T \setminus \{0\} \subseteq \mathbb{R}^{n+m}_{++}$ be an arcwise connected cone being Φ_α-convex respecting* (T1)-(T4) *with $T^k = T$ for all $k \in \mathcal{S}$ and $\alpha \in \mathbb{R} \setminus \{0\}$. Each firm $k \in \mathcal{S}$ employs the same technique i.e. $x^k_i = \beta_{ij} x^k_j$, $y^k_\ell = \gamma_{\ell p} y^k_p$, $x^k_i = \delta_{ip} y^k_p$ where $\beta_{ij}, \gamma_{\ell p}, \delta_{ip}$ are constant for all $i, j \in \{1, \ldots, n\}$ and for all $\ell, p \in$*

$\{1,\ldots,m\}$. *Then, the firm game* (N, v, Φ_α) *with a finite number of firms yields the following results.*

(i) *If* $T^{\mathcal{S}}_\alpha = \sum^{\Phi_\alpha}_{k\in\mathcal{S}} T^k$ *such that* $\alpha \to \infty$ *for all* $\mathcal{S} \subseteq N$*, then:*

(i.a) $T^{\mathcal{S}}_{+\infty}$ *satisfies directional output complementarity for all* $\mathcal{S} \subseteq N$.

(i.b) $\overset{\circ}{\mathcal{C}}_{D_T} \neq \emptyset$.

(ii) *If* $T^{\mathcal{S}}_\alpha = \sum^{\Phi_\alpha}_{k\in\mathcal{S}} T^k$ *such that* $\alpha \to -\infty$ *for all* $\mathcal{S} \subseteq N$*, then:*

(ii.a) $T^{\mathcal{S}}_{-\infty}$ *satisfies directional input complementarity for all* $\mathcal{S} \subseteq N$.

(ii.c) $\overset{\circ}{\mathcal{C}}_{D_T} = \emptyset$.

Proof:

(i.a) It follows from Proposition 4.2.

(i.b): From Shapley (1972), any given convex game provides a solution in the core. Since the game $v(\mathcal{S})$ corresponds to the industrial technical inefficiency of coalition $\mathcal{S}$, it represents a cost. Following the cost-sharing literature, the core is non-empty whenever the game is concave *i.e.*,

$$v(\mathcal{S} \cap \mathcal{R}) + v(\mathcal{S} \cup \mathcal{R}) \leq v(\mathcal{S}) + v(\mathcal{R}),\ \forall \mathcal{S}, \mathcal{R} \subseteq N. \tag{5.1}$$

From the conditions imposed in the Proposition, the results of Proposition 4.2 and 5.4 apply: $\lim_{\alpha\to+\infty} AB_\alpha(\mathcal{S}; g) = 0$ for all $\mathcal{S} \subseteq N$, that is,

$$v(\mathcal{S}) \equiv D_{T^{\mathcal{S}}_\alpha}\left(\bigvee_{k\in\mathcal{S}} (x^k, y^k); g\right) = \max_{k\in\mathcal{S}} D_{T^k}((x^k, y^k); g),\ \forall \mathcal{S} \subseteq N.$$

Then, by Eq.(5.1):

$$\begin{aligned}&\max_{k\in\mathcal{S}\cap\mathcal{R}} D_{T^k}((x^k, y^k); g) + \max_{k\in\mathcal{S}\cup\mathcal{R}} D_{T^k}((x^k, y^k); g)\\ \leq &\max_{k\in\mathcal{S}} D_{T^k}((x^k, y^k); g) + \max_{k\in\mathcal{R}} D_{T^k}((x^k, y^k); g),\ \forall \mathcal{S}, \mathcal{R} \subseteq N.\end{aligned}$$

• First suppose that $\max_{k\in\mathcal{S}\cup\mathcal{R}} D_{T^k}((x^k, y^k); g) = \max_{k\in\mathcal{S}} D_{T^k}((x^k, y^k); g)$, then Eq.(5.1) becomes:

$$\max_{k\in\mathcal{S}\cap\mathcal{R}} D_{T^k}((x^k, y^k); g) \leq \max_{k\in\mathcal{R}} D_{T^k}((x^k, y^k); g),\ \forall \mathcal{S}, \mathcal{R} \subseteq N,$$

and because $\mathcal{S} \cap \mathcal{R} \subseteq \mathcal{R}$, the previous expression is always true.

• Second, suppose that $\max_{k\in\mathcal{S}\cup\mathcal{R}} D_{T^k}((x^k, y^k); g) = \max_{k\in\mathcal{R}} D_{T^k}((x^k, y^k); g)$, then Eq.(5.1) becomes:

$$\max_{k\in\mathcal{S}\cap\mathcal{R}} D_{T^k}((x^k, y^k); g) \leq \max_{k\in\mathcal{S}} D_{T^k}((x^k, y^k); g),\ \forall \mathcal{S}, \mathcal{R} \subseteq N,$$

and because $\mathcal{S} \cap \mathcal{R} \subseteq \mathcal{S}$, the previous expression is always true.

• Third, suppose that $\max_{k\in\mathcal{S}\cup\mathcal{R}} D_{T^k}((x^k, y^k); g) = \max_{k\in\mathcal{S}\cap\mathcal{R}} D_{T^k}((x^k, y^k); g)$, then Eq.(5.1) becomes:

$$\begin{aligned}&\max_{k\in\mathcal{S}\cap\mathcal{R}} D_{T^k}((x^k, y^k); g) + \max_{k\in\mathcal{S}\cup\mathcal{R}} D_{T^k}((x^k, y^k); g)\\ = &\max_{k\in\mathcal{S}} D_{T^k}((x^k, y^k); g) + \max_{k\in\mathcal{R}} D_{T^k}((x^k, y^k); g),\ \forall \mathcal{S}, \mathcal{R} \subseteq N.\end{aligned}$$

Therefore Eq.(5.1) is always respected. By Shapley (1972), the core is non-empty. Since the Shapley value is the barycenter of the core, the result follows.

(ii.a) It follows from Proposition 4.1.
(ii.b): By Proposition 4.1 and 5.4: $\lim_{\alpha\to-\infty} AB_\alpha(\mathcal{S};g) = 0$ for all $\mathcal{S} \subseteq N$, therefore

$$D_{T_\alpha^{\mathcal{S}}}\left(\bigwedge_{k\in\mathcal{S}}(x^k,y^k);g\right) = \min_{k\in\mathcal{S}} D_{T^k}((x^k,y^k);g),\ \forall\mathcal{S}\subseteq N.$$

• First suppose that $\min_{k\in\mathcal{S}\cup\mathcal{R}} D_{T^k}((x^k,y^k);g) = \min_{k\in\mathcal{S}} D_{T^k}((x^k,y^k);g)$, then Eq.(5.1) is violated because,

$$\min_{k\in\mathcal{S}\cap\mathcal{R}} D_{T^k}((x^k,y^k);g) \geq \min_{k\in\mathcal{R}} D_{T^k}((x^k,y^k);g),\ \forall\mathcal{S},\mathcal{R}\subseteq N.$$

• Second, suppose that $\min_{k\in\mathcal{S}\cup\mathcal{R}} D_{T^k}((x^k,y^k);g) = \min_{k\in\mathcal{R}} D_{T^k}((x^k,y^k);g)$, then Eq.(5.1) is violated since,

$$\min_{k\in\mathcal{S}\cap\mathcal{R}} D_{T^k}((x^k,y^k);g) \geq \min_{k\in\mathcal{S}} D_{T^k}((x^k,y^k);g),\ \forall\mathcal{S},\mathcal{R}\subseteq N.$$

• Third, suppose that $\min_{k\in\mathcal{S}\cup\mathcal{R}} D_{T^k}((x^k,y^k);g) = \min_{k\in\mathcal{S}\cap\mathcal{R}} D_{T^k}((x^k,y^k);g)$, then Eq.(5.1) becomes:

$$\begin{aligned}&\min_{k\in\mathcal{S}\cap\mathcal{R}} D_{T^k}((x^k,y^k);g) + \min_{k\in\mathcal{S}\cup\mathcal{R}} D_{T^k}((x^k,y^k);g)\\ =&\min_{k\in\mathcal{S}} D_{T^k}((x^k,y^k);g) + \min_{k\in\mathcal{R}} D_{T^k}((x^k,y^k);g),\ \forall\mathcal{S},\mathcal{R}\subseteq N.\end{aligned}$$

The three cases above provide:

$$\begin{aligned}&\min_{k\in\mathcal{S}\cap\mathcal{R}} D_{T^k}((x^k,y^k);g) + \min_{k\in\mathcal{S}\cup\mathcal{R}} D_{T^k}((x^k,y^k);g)\\ \geq&\min_{k\in\mathcal{S}} D_{T^k}((x^k,y^k);g) + \min_{k\in\mathcal{R}} D_{T^k}((x^k,y^k);g),\ \forall\mathcal{S},\mathcal{R}\subseteq N\end{aligned}$$

that is,

$$v(\mathcal{S}\cap\mathcal{R}) + v(\mathcal{S}\cup\mathcal{R}) \geq v(\mathcal{S}) + v(\mathcal{R}),\ \forall\mathcal{S},\mathcal{R}\subseteq N.$$

Therefore, the core is empty, and this ends the proof. ■

The interpretation of the non-vacuity of the core is related to the algebraic structure of the aggregate technology being a Φ_α-convex cone. Let us take en example with the well-known Shapley (1953) value.

Example 5.2 (Shapley value) *Let us consider a 3-firm game. It is assumed that the technology* $T\setminus\{0\}\subseteq\mathbb{R}^{n+m}_{++}$ *is an arcwise connected cone being* Φ_α*-convex and respecting* (T1)-(T4) *such that* $T^k = T$ *for all* $k\in\mathcal{S}$ *and each firm employs the same technique. Therefore, by Proposition 5.4,*

$AB_{+\infty}(\mathcal{S};g)=0$ for all $\mathcal{S}\subseteq N$. In words, the structural technical inefficiency of coalition $\mathcal{S}$ measured by $v(\mathcal{S})$ is equal to the industrial technical inefficiency of coalition $\mathcal{S}$.

$$v(\mathcal{S})\equiv D_{T_\alpha^{\mathcal{S}}}\left(\lim_{\alpha\to+\infty}\sum_{k\in\mathcal{S}}^{\Phi_\alpha}(x^k,y^k);g\right)=\lim_{\alpha\to+\infty}\sum_{k\in\mathcal{S}}^{\phi_\alpha}D_{T^k}((x^k,y^k);g).$$

For notational convenience, the singleton $\{i\}$ is denoted i for any firm i in N. For the ease of the example, it is assumed that $v(1)\leq v(2)\leq v(3)$. Therefore,

$$v(\mathcal{S})=\max_{k\in\mathcal{S}}D_{T^k}((x^k,y^k);g). \tag{5.2}$$

In words, $v(3)$ is the technical inefficiency of firm 3 and also is the technical inefficiency of the coalitions $\{1,3\}$, $\{2,3\}$ and N. The technical inefficiency of firm 2, $v(2)$ also is that of the coalition $\{1,2\}$. Moreover,

$$v(1)+v(2)+v(3)\leq v(N)=v(3).$$

That is, the coalition of all 3 firms alleviates the burden reached by cumulating the firm's inefficiencies when they do not cooperate. Therefore, each firm's contribution to bear the structural technical inefficiency can be less than its stand-alone technical inefficiency. If so, the allocation is individually rational because every firm has an incentive to cooperate.

Let us show that the core of (N,v,Φ_∞) is non-empty since the Shapley value lies in. The Shapley value defined on the firm game (N,v,Φ_α) is a valued solution $Sh:\Gamma\to\mathbb{R}^n$ expressed as, for all $\alpha\in\mathbb{R}\setminus\{0\}$,

$$Sh_k(v,\Phi_\alpha):=\sum_{\mathcal{S}\subseteq\mathcal{K}\setminus\{k\}}\frac{(n-1-|\mathcal{S}|)!|\mathcal{S}|!}{n!}\left[v(\mathcal{S}\cup k)-v(\mathcal{S})\right],\ k\in N.$$

By Eq.(5.2), the Shapley allocation for every player is as follows.

$$Sh_1(v,\Phi_\infty)=\frac{1}{3}v(1)\ ;\ Sh_2(v,\Phi_\infty)=\frac{1}{2}v(2)-\frac{1}{6}v(1)\ ;$$
$$Sh_3(v,\Phi_\infty)=v(3)-\frac{1}{2}v(2)-\frac{1}{6}v(1).$$

The value is individually rational:

$$Sh_1(v,\Phi_\infty)\leq v(1)\ ;\ Sh_2(v,\Phi_\infty)\leq v(2)\ ;\ Sh_3(v,\Phi_\infty)\leq v(3).$$

The allocation is also collectively rational. If two firms cooperate, both share the burden of the least efficient firm among them. If all firms cooperate, they share, once again, the cost of the least efficient firm among them. Formally, $\sum_{k\in\mathcal{S}}Sh_k\leq v(\mathcal{S})$ for all $\mathcal{S}\subseteq N$. Indeed, if firms 1 and 2 cooperate, they share the burden due to firm 2's technical inefficiency. Therefore, *firm 1 bears less than it would bear for its own stand-alone inefficiency and the same applies for firm 2. The statement holds true for any coalition.*

a) For $\mathcal{S} = \{1, 2\}$*:*

$$Sh_1(v, \Phi_\infty) + Sh_2(v, \Phi_\infty) = \frac{1}{6}v(1) + \frac{1}{2}v(2)$$
$$\Longrightarrow Sh_1(v, \Phi_\infty) + Sh_2(v, \Phi_\infty) \leq v(2).$$

b) For $\mathcal{S} = \{1, 3\}$*:*

$$Sh_1(v, \Phi_\infty) + Sh_3(v, \Phi_\infty) = \frac{1}{6}v(1) + v(3) - \frac{1}{2}v(2)$$
$$\Longrightarrow Sh_1(v, \Phi_\infty) + Sh_3(v, \Phi_\infty) \leq v(3).$$

c) For $\mathcal{S} = \{2, 3\}$*:*

$$Sh_2(v, \Phi_\infty) + Sh_3(v, \Phi_\infty) = v(3) - \frac{1}{3}v(1)$$
$$\Longrightarrow Sh_2(v, \Phi_\infty) + Sh_3(v, \Phi_\infty) \leq v(3).$$

d) The last step to check for collective rationality of the allocation is straightforward since $\sum_{k \in N} Sh_k = v(N)$*:*

$$\begin{aligned}
&Sh_1(v, \Phi_\infty) + Sh_2(v, \Phi_\infty) + Sh_3(v, \Phi_\infty) \\
&= \frac{1}{3}v(1) + \frac{1}{2}v(2) - \frac{1}{6}v(1) + v(3) - \frac{1}{2}v(2) - \frac{1}{6}v(1) \\
&= v(3) \\
&= \max_{k \in N} v(k) \\
&= v(N) \ (\textit{since } AB_\infty(N; g) = 0).
\end{aligned}$$

Based on a)–d), the core of the firm game is non void whenever $\lim_{\alpha \to \infty} AB_\alpha(\mathcal{S}; g) = 0$ *for all* $\mathcal{S} \subseteq N$*.*

When the structural technical inefficiency of the group of firms corresponds to the industrial technical inefficiency ($\lim_{\alpha \to \infty} AB_\alpha(\mathcal{S}; g) = 0$), this means that coalition $\mathcal{S}$ gets the average of technical efficiency of the firms in $\mathcal{S}$. However, the previous result shows that each firm has still some interest to merge with others in order to reduce its technical inefficiency (the core is non-empty). When they merge, the new aggregate technology $T_{\mathcal{S}}$ satisfies directional complementarity in outputs. This result may be generalized to a firm game with α-returns to scale.

6 Technologies with α-Returns to Scale

Let us investigate Constant Elasticity of Substitution (CES)-Constant Elasticity of Transformation (CET) models introduced by Färe *et al.* (1988).

These models have been generalized by Boussemart *et al.* (2009) with α-returns to scale, in which the output part is characterized by a CET expression and the input part by a CES expression.

The coefficient of returns to scale α is decomposed into two parts: $\alpha = q/r > 0$, with $q, r \in \mathbb{R} \setminus \{0\}$. Assume that $A = \{(x_k, y_k) : k \in \mathcal{S}\} \subset \mathbb{R}^{m+n}_{++}$, then the following model (see Andriamasy *et al.*, 2017),

$$T_{\mathcal{S}}^{(q,r)} = \Big\{(x, y) : x \geq \Phi_q^{-1}\big(\sum_{k\in\mathcal{S}} t_k \Phi_q(x_k)\big), y \leq \Phi_r^{-1}\big(\sum_{k\in\mathcal{S}} t_k \Phi_r(y_k)\big), t \geqslant 0\Big\},$$

is a generalization of the CES-CET model since the variable returns to scale constraint $\sum_{k\in\mathcal{S}} t_k = 1$ is dropped. Andriamasy *et al.* (2017) show that this technology satisfies α-returns to scale, that is, for all $\lambda > 0$,

$$(x, y) \in T_{\mathcal{S}}^{(q,r)} \quad \text{implies} \quad (\lambda x, \lambda^\alpha y) \in T_{\mathcal{S}}^{(q,r)}.$$

Recently, Ravelojaona (2019) introduces a directional distance function relevant to technologies $T^{(q,r)}$ with two parameters. In the context of aggregate technologies and firm games, this directional distance function may be rewritten as:

$$v(\mathcal{S}) \equiv D_{T_{\mathcal{S}}^{(q,r)}}\left(\sum_{k\in\mathcal{S}}^{\Phi_q} x^k, \sum_{k\in\mathcal{S}}^{\Phi_r} y^k; g\right).$$

On this basis, because $\alpha = q/r$, the aggregation bias of technical inefficiency becomes:

$$AB_\alpha(\mathcal{S}; g) = D_{T_\alpha^{\mathcal{S}}}\left(\sum_{k\in\mathcal{S}}^{\Phi_q} x^k, \sum_{k\in\mathcal{S}}^{\Phi_r} y^k; g\right) - \sum_{k\in\mathcal{S}}^{\phi_\alpha} D_{T^k}(x^k, y^k; g).$$

The aggregation bias may be either positive, negative or null. As shown in the first proposition below, there are several cases to consider, for instance $q \to -\infty$ and $r \to +\infty$ such that $|q| \gg |r|$. In this case, everything happens as if, in the firm game, all firms merge by minimizing their inputs over each dimension (labor, capital, and so on), and in the same time, they merge by maximizing their outputs. The technology of coalition $\mathcal{S}$ is said to be super-efficient in the sense that it allows input waste to be avoided while increasing the number of outputs (input and output complementarity). Cooperation reduces the aggregate technology inefficiency since the aggregation bias is always non-positive: The structural inefficiency is lower than the inefficiency mean (industrial inefficiency).

On the other hand, the parameter $\alpha \to \infty$ could be decomposed such that $q \to +\infty$ and $r \to -\infty$. In this case, the firms merge however they are sub-efficient, since they are organized in such a way that they employ too much inputs for a few outputs. The joint technology is more inefficient than the industrial inefficiency because the aggregation bias is positive. However, these cases do not tell us the whole story about the possible improvement of technical efficiency due to cooperation between firms. A desirable property of cooperation is that of subadditivity of the game.

Definition 6.1 – Subadditive games: *The firm game is subadditive if, for all $\mathcal{S}, \mathcal{R} \subseteq N$ such that $\mathcal{S} \cap \mathcal{R} = \emptyset$,*

$$v(\mathcal{S} \cup \mathcal{R}) \leq v(\mathcal{S}) + v(\mathcal{R}).$$

Because the game v represents the directional distance function of any given coalition of firms, it is desirable to get subadditive games, in which case the cooperation inherent to coalition $\mathcal{S}$ implies an improvement of technical efficiency for the coalition (or a decrease of technical inefficiency equivalent to a decrease of the distance function). In contrast to this, the distance function is *superadditive* if, for all $\mathcal{S}, \mathcal{R} \subseteq N$ such that $\mathcal{S} \cap \mathcal{R} = \emptyset$,

$$v(\mathcal{S} \cup \mathcal{R}) \geq v(\mathcal{S}) + v(\mathcal{R}).$$

Proposition 6.1 (subadditivity and complementarity) *Let (N, v, Φ_α) be a firm game such that each firm's technology is such that $T^k = T_{\mathcal{S}}^{(q,r)}$ and respects* (T1)-(T4) *for all $k \in \mathcal{S}$ with $\sum_{k\in\mathcal{S}} \Phi_\alpha(T^k) = T_{\mathcal{S}}^{(q,r)}$.*

(i) *If $\alpha \to -\infty$ such that $q \to -\infty$ and $r \to +\infty$ with $|q| \gg |r|$, then:*

- (i.a) *The distance function $D_{T^{(q,r)}}$ is subadditive.*
- (i.b) $\lim_{\alpha\to-\infty} AB_\alpha(\mathcal{S}; g) \leq 0$.
- (i.c) $T_{\mathcal{S}}^{(q,r)}$ *satisfies directional input complementarity for all $\mathcal{S} \subseteq N$.*
- (i.d) $T_{\mathcal{S}}^{(q,r)}$ *satisfies directional output complementarity for all $\mathcal{S} \subseteq N$.*

(ii) *If $\alpha \to -\infty$ such that $q \to +\infty$ and $r \to -\infty$ with $|q| \gg |r|$, then:*

- (ii.a) *The distance function $D_{T^{(q,r)}}$ is superadditive.*
- (ii.b) $\lim_{\alpha\to-\infty} AB_\alpha(\mathcal{S}; g) \geq 0$ *for all $\mathcal{S} \subseteq N$.*

Proof:

(i.a): Take $\mathcal{S}, \mathcal{R} \subset N$, such that $\mathcal{S} \cap \mathcal{R} \neq \emptyset$. Since $T^k = T_{\mathcal{S}}^{(q,r)} = T_{\mathcal{S}\cup\mathcal{R}}^{(q,r)}$, from weak monotonicity and $q \to -\infty$ and $r \to +\infty$, it comes that,

$$D_{T_{\mathcal{S}\cup\mathcal{R}}^{(q,r)}}\left(\bigwedge_{k\in\mathcal{S}\cup\mathcal{R}} x^k, \bigvee_{k\in\mathcal{S}\cup\mathcal{R}} y^k; g\right) \leq D_{T_{\mathcal{S}}^{(q,r)}}\left(\bigwedge_{k\in\mathcal{S}} x^k, \bigvee_{k\in\mathcal{S}} y^k; g\right)$$

and,

$$D_{T_{\mathcal{S}\cup\mathcal{R}}^{(q,r)}}\left(\bigwedge_{k\in\mathcal{S}\cup\mathcal{R}} x^k, \bigvee_{k\in\mathcal{S}\cup\mathcal{R}} y^k; g\right) \leq D_{T_{\mathcal{R}}^{(q,r)}}\left(\bigwedge_{k\in\mathcal{R}} x^k, \bigvee_{k\in\mathcal{R}} y^k; g\right).$$

Then, for all $\mathcal{S}, \mathcal{R} \subset N$ such that $\mathcal{S} \cap \mathcal{R} \neq \emptyset$:

$$2D_{T_{\mathcal{S}\cup\mathcal{R}}^{(q,r)}}\left(\bigwedge_{k\in\mathcal{S}\cup\mathcal{R}} x^k, \bigvee_{k\in\mathcal{S}\cup\mathcal{R}} y^k; g\right) \leq D_{T_{\mathcal{S}}^{(q,r)}}\left(\bigwedge_{k\in\mathcal{S}} x^k, \bigvee_{k\in\mathcal{S}} y^k; g\right) + D_{T_{\mathcal{R}}^{(q,r)}}\left(\bigwedge_{k\in\mathcal{R}} x^k, \bigvee_{k\in\mathcal{R}} y^k; g\right),$$

and the result follows.

(i.b): By (T4), it follows that the directional distance function is weakly monotonic on T^k, that is, $(x^k, y^k), (u, v) \in T^k$ such that $u \leqslant y$ and $v \geqslant x$ imply that $D_{T^k}(u, v; g) \geq D_{T^k}(x^k, y^k; g)$. First, we remark that,

$$D_{T_{\mathcal{S}}^{(q,r)}}\left(\lim_{q\to-\infty}\sum_{k\in\mathcal{S}}^{\Phi_q} x^k, \lim_{r\to+\infty}\sum_{k\in\mathcal{S}}^{\Phi_r} y^k; g\right) = D_{T_{\mathcal{S}}^{(q,r)}}\left(\bigwedge_{k\in\mathcal{S}} x^k, \bigvee_{k\in\mathcal{S}} y^k; g\right)$$

Since $T^k = T_{\mathcal{S}}^{(q,r)}$, from weak monotonicity we have for all $k \in \mathcal{S}$:

$$D_{T_{\mathcal{S}}^{(q,r)}}\left(\bigwedge_{k\in\mathcal{S}} x^k, \bigvee_{k\in\mathcal{S}} y^k; g\right) \leq D_{T^k}\left(x^k, \bigvee_{k\in\mathcal{S}} y^k; g\right) \leq \min_{k\in\mathcal{S}} D_{T^k}\left(x^k, y^k; g\right)$$

Thus,

$$D_{T_{\mathcal{S}}^{(q,r)}}\left(\lim_{q\to-\infty}\sum_{k\in\mathcal{S}}^{\Phi_q} x^k, \lim_{r\to+\infty}\sum_{k\in\mathcal{S}}^{\Phi_r} y^k; g\right) \leq \min_{k\in\mathcal{S}} D_{T^k}(x^k, y^k; g)$$

This is equivalent to:

$$D_{T_{\mathcal{S}}^{(q,r)}}\left(\lim_{q\to-\infty}\sum_{k\in\mathcal{S}}^{\Phi_q} x^k, \lim_{r\to+\infty}\sum_{k\in\mathcal{S}}^{\Phi_r} y^k; g\right) \leq \lim_{\alpha\to-\infty}\sum_{k\in\mathcal{S}}^{\Phi_\alpha} D_{T^k}(x^k, y^k; g)$$

Therefore, $\lim_{\alpha\to-\infty} AB_\alpha(\mathcal{S}; g) \leq 0$.
(i.c) and (i.d) follow from Propositions 4.1 and 4.2.

(ii.a)–(ii.b) can be proven with the same reasoning as in (i.a)–(i.b). ■

Proposition 6.1 provides some simple conditions that support cooperation (subadditive games) when each firm has α-returns to scale technologies. Moreover, technologies with α-returns to scale characterized by subadditive games are compatible with a non-positive aggregation bias. The non-positive sign of the aggregation bias is a tool to select some coalitions that improve technical efficiency when the size of the coalitions increase. Also, when the bias is positive, it is a tool allowing for more inefficiency removal when firms contemplate doing some mergers. Finally, these technologies associated with subadditive games satisfy simultaneously directional complementarity in inputs and in outputs.

7 Conclusion

In this paper it has been shown, at a sectoral level, that it is possible to measure technical efficiency of any given coalition following two ways. The first one is the generalized mean of the technical efficiency of each firm of the coalition and the second one is the structural technical efficiency of a coalition computed with an aggregate technology.

Based on the standard axioms of the literature of productivity measurement, it is first shown that both measures of inefficiency do not necessarily coincide for CRS technologies. However, if the firms of the coalition employ the same technique, both measures of technical inefficiency may coincide. In this respect, the technical inefficiency of each firm (the game) is a generalized mean of inputs and outputs vectors. The case of CRS technologies with firms employing the same technique associated with directional output complementarity provides a non-empty core.

Finally, in the case of α-returns to scale technologies, the aggregation bias may be either negative or positive. Also, even if there is no guarantee to have a solution in the core of the firm game, the games may be subadditive, in other terms, the cooperation between firms improves technical efficiency.

References

[1] Aczél, J. (1966), *Lectures on Functional Equations and their Applications*, Academic Press, New York.

[2] Andriamasy, L., Briec, W. and S. Mussard (2017), On Some Relations Between Several Generalized Convex DEA Models, *Optimization*, 66(4), 547-570.

[3] Atkinson, A.B. (1970), On the Measurement of Inequality, *Journal of Economic Theory*, 2, 244-263.

[4] Ben-Tal, A. (1977), On Generalized Means and Generalized Convex Functions, *Journal of Optimization Theory and Applications*, 21(1), 1-13.

[5] Blackorby, C., Donaldson, D. and M. Auersperg (1981), A New Procedure for the Measurement of Inequality within and among Population Subgroups, *Canadian Journal of Economics*, 14, 665-685.

[6] Briec, W. (2015), Some Remarks on an Idempotent and Non-Associative Convex Structure, *Journal of Convex Analysis*, 22, 259-289.

[7] Briec, W., Dervaux, B., and H. Leleu (2003), Aggregation of Directional Distance Functions and Industrial Efficiency, *Journal of Economics*, 79, 237-261.

[8] Briec, W. and S. Mussard (2014), Efficient Firm Groups: Allocative Efficiency in Cooperative Games, *European Journal of Operational Research*, 239, 286-296.

[9] Chambers, R.G., Chung, Y. and R. Färe (1996), Benefit and Distance Functions, *Journal of Economic Theory*, 70, 407-419.

[10] Chambers, R.G., Chung, Y. and R. Färe (1998), Profit, Directional Distance Functions, and Nerlovian Efficiency, *Journal of Optimization Theory and Applications*, 98, 351-364.

[11] Chambers, R.G and R. Färe (1998), Translation Homotheticity, *Economic Theory*, 11, 629-641.

[12] Färe, R., Grosskopf, S. and S. K. Li (1992), Linear Programming Models for Firm and Industry Performance, *Scandinavian Journal of Economics*, 94, 599-608.

[13] Hardy, G.H., Littlewood, J.E., and G. Pólya (1934), *Inequalities*, Cambridge University Press.

[14] Kohli, U. (1983), Nonjoint Technologies, *Review of Economic Studies*, 50, 209-19.

[15] Li, S.-K. (1995), Relations Between Convexity and Homogeneity in Multioutput Technologies, *Journal of Mathematical Economics*, 24(4), 311-318.

[16] Li, S.-K. and Y.C. Ng (2001), Measuring the Productive Efficiency of a Group of Firms, *International Advances in Economic Research*, 1, 377-390.

[17] Lozano, S. (2012), Information Sharing in DEA: A Cooperative Game Theory Approach, *European Journal of Operational Research*, 222(3), 558-565.

[18] Lozano, S. (2013), DEA Production Games, *European Journal of Operational Research*, 231(2), 405-413.

[19] Peyrache, A. (2013), Industry Structural Inefficiency and Potential Gains from Mergers and Break-ups: A Comprehensive Approach, *European Journal of Operational Research*, 230, 422-430.

[20] Ravelojaona, P. (2019), On Constant Elasticity of Substitution - Constant Elasticity of Transformation Directional Distance Functions, *European Journal of Operational Research*, 272(2), 780-791.

[21] Shapley, L. (1972), Cores of Convex Games, *International Journal of Game Theory*, 1 , 11-26.