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1 Absence of geometrical regime for impedance-type elastic scatterers

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4 **Abstract**

5 In wave physics, the geometrical limit is defined as a propagation regime where the scattering
6 cross-section σ of an object becomes independent of its internal structure and tends to twice its
7 geometrical cross-section σ_g , a result which is particularly well documented in optics. Following
8 the classification of Wu and Aki (1985b), we study the high-frequency scattering limit for velocity-
9 type and impedance-type elastic perturbations. While velocity-type scatterers do follow the
10 geometrical limit $\sigma \rightarrow 2\sigma_g$, the scattering cross-section of impedance-type scatterers depends on
11 both their density and elastic properties at all frequencies. These results are illustrated with the
12 example of a spherical inclusion exhibiting a small contrast of properties with its environment.
13 We derive simple asymptotic formulas that show good agreement with exact solutions of the
14 boundary value problem. Our results confirm the distinct behavior of velocity-type vs impedance
15 type perturbations at all frequencies.

16 **Introduction**

17 In their seminal paper on the scattering of seismic waves by elastic inclusions, Wu and Aki (1985b)
18 introduced the concepts of “velocity-type” and “impedance-type” perturbations. The former (resp.
19 the latter) refers to an object exhibiting a contrast of velocity (resp. impedance) but no contrast
20 of impedance (resp. velocity) with its environment. Wu and Aki (1985b) demonstrated that any
21 general elastic perturbation may be decomposed into a sum of “velocity-type” and “impedance-
22 type” perturbation in a unique way. From a seismological point of view, this decomposition is

23 particularly meaningful. Indeed, it may be shown in the long wavelength limit that the radiation
 24 pattern of impedance-type perturbations displays large lobes of back-scattering, whereas velocity-
 25 type perturbations tend to scatter energy around the forward direction. Note, however, that in
 26 isotropic elastic media (and in the long wavelength limit) there is no conversion scattering (S to P
 27 or P to S) in the exact forward and backward directions due to the symmetry of the elastic tensor.
 28 As a consequence of the properties outlined above, one may expect impedance-type perturbations
 29 to play an important role in direct wave attenuation and coda wave generation (Sato, 1984; Wu
 30 and Aki, 1985a), whereas velocity-type perturbations participate to phenomena associated with
 31 forward scattering such as envelope broadening (Sato, 1989).

32 The separation of elastic fluctuations into velocity-type and impedance-type perturbations is also
 33 relevant to the theory of attenuation through discrete disordered media. Kawahara (2011) studied
 34 theoretically the case where weak contrast inclusions are dispersed in a homogenous matrix. He
 35 compared the classical theory of Foldy (1945) with a continuous approximation where the random
 36 medium is described with the aid of a 2-point correlation function. In this approximation, the
 37 scatterers are no longer represented as discrete object but rather as fluctuations of a continuum
 38 (see Torquato, 2002, for further details). On the one hand, Foldy's theory is based on the *exact*
 39 solution of the scattering problem for a single inclusion and is expected to be valid when the number
 40 density of scatterers is low (Foldy, 1945). On the other hand, the equivalent continuum description
 41 is expected to be valid for sufficiently weak perturbations only. In the case of spherical inclusions,
 42 Kawahara (2011) calculated attenuation using the results of Korneev and Johnson (1996) in the
 43 framework of Foldy's theory and the Born approximation in the continuum model. The applicability
 44 of Born approximation (also called the Rayleigh-Gans approximation in optics) is usually limited to
 45 sufficiently low frequency (Van de Hulst, 1981). In the case of velocity-type perturbations Kawahara
 46 (2011) indeed found good agreement between Foldy's theory and the continuum approximation at
 47 low frequency, and noted the divergence of Born approximation in the high-frequency limit. In the
 48 case of impedance-type perturbations, however, the author pointed out that the two theories agree
 49 at all frequencies. He attributed the success of Born approximation to the absence of phase-shift
 50 inside the scatterers in the case of pure impedance-type perturbations.

51 Calvet and Margerin (2018) considered the related problem of a symmetric two-phase elastic
 52 random medium. When the volume fractions are equal, the term "symmetric" implies that the

Figure 1: Position of the problem. We consider a plane elastic (P or S) wave incident on a sphere of radius R . The incidence angle of a given ray is denoted by j . The subscript 0 refers to the properties of the matrix with ρ the density, α the P -wave velocity and β the S -wave velocity.

physical properties of the medium are invariant upon inversion of the phases (Torquato, 2002). These authors adopted a continuum description and computed the dispersion and attenuation of elastic waves using a spectral function technique. In this approach, the propagating modes are characterized by measuring both the location of the maximum and the width of the imaginary part of Green's function in the frequency-wavenumber domain. This method is thought to be valid for sufficiently weak perturbations -including in the high-frequency limit-. Similar to the finding of Kawahara (2011), Calvet and Margerin (2018) note that in the case of impedance-type perturbations, calculations of the attenuation based on the Born approximation yield the same result as the spectral function approach (see the erratum to the original article). Furthermore, these authors found that in the high-frequency limit, attenuation does not tend to the usual geometrical limit $l = a$, with l the attenuation length and a the correlation distance of the medium. Note that in the geometrical limit, attenuation is totally independent of the internal properties of the scatterers, a result which is well documented in optics (Van de Hulst, 1981). In sharp contrast, Calvet and Margerin (2018) found on the basis of Born approximation that attenuation by impedance-type perturbations depends on the contrast of elastic properties in the 2-phase medium at all frequencies. This note can be considered as an outlook on the two papers just discussed.

Theory and numerical results

In order to confirm or infirm the conclusions of Kawahara (2011) and Calvet and Margerin (2018), we reexamine the problem of high-frequency scattering by spherical inclusions (see Figure 1). Our

Figure 2: Scattering cross-section (normalized by the geometric cross-section) of P waves (a) and S waves (b) as a function of non-dimensional frequency $k_{p,s}a$ for a velocity-type perturbation equal to -5% . Solid line: series solution of the BVP, dashed line: Ray theory.

objective is 2-fold: evaluate the validity of Born approximation for impedance-type perturbations and provide an independent confirmation of the absence of geometrical regime in this case. To carry out this task, the outcome of three theoretical approaches will be compared: (1) The formal series solution of the boundary value problem (B.V.P.) for spheres as derived by Korneev and Johnson (1996); (2) A high-frequency approximation which couples ray theory with a classical representation theorem for the scattered field; (3) the Born approximation for scattered elastic wavefields. We begin with a brief summary of known results for velocity-type perturbations and proceed with impedance type perturbations next. In numerical applications, we consider the case where the matrix is a Poisson solid.

Velocity-type perturbations

Although velocity-type perturbations have previously been treated by Korneev and Johnson (1996) using the formal series solution of the BVP, we propose an alternative derivation based on a method developed in optics by Van de Hulst (1981). In fact, the approach bears many similarities with the screen method which has been widely used in seismology (e.g. Stoffa et al., 1990; Wu, 1994). It is equally applicable to velocity-type and impedance-type perturbations and allows one to understand the physical origin of the difference between the two cases. We show in Figure 2 and 3 the high-frequency behavior of the P and S scattering cross-sections for -5% velocity-type and impedance-type perturbations, respectively. The scattering response of the two types of inclusions is strikingly different. In the case of a velocity-type perturbation, the similarity between P and S wave scattering

behavior is worth noting and can be explained with the aid of ray theory (which is shown by dashed lines in Figures 2 and 3). To obtain an asymptotic high-frequency approximation, we employ the method of Van de Hulst (1981) which consists in computing the forward scattered field using a combination of Huygens principle and ray theory. The main steps of the method are outlined hereafter.

Our starting point is a representation theorem for scattered elastic wavefields (Pao and Varatharajulu, 1976):

$$\mathbf{u}^{sc}(\mathbf{x}) = \int_S (\mathbf{u}^{sc}(\mathbf{x}') \cdot (\mathbf{n}' \cdot \boldsymbol{\Sigma}^0(\mathbf{x}, \mathbf{x}')) - t^{sc}(\mathbf{x}') \cdot \mathbf{G}^0(\mathbf{x}, \mathbf{x}')) dS' \quad (1)$$

For clarity, we recall the physical quantities appearing in Eq. (1). $\mathbf{u}^{sc}$: the scattered wavefield, $\mathbf{t}^{sc}$ the scattered traction field, $\boldsymbol{\Sigma}^0(\mathbf{x}, \mathbf{x}')$: stress Green tensor in the matrix for source at $\mathbf{x}'$ and receiver at $\mathbf{x}$, $\mathbf{G}^0(\mathbf{x}, \mathbf{x}')$: displacement Green tensor in the matrix, S : surface enclosing the scatterer, $\mathbf{n}'$: inward normal to S . For our problem, it is convenient to let S coincide with the plane tangent to the sphere at the exit point of the central ray (see Figure 1). To close this surface, we may add a large hemisphere of radius r and let $r \rightarrow \infty$. However, there will be no contribution from this surface thanks to the radiation conditions obeyed by the tensors $\boldsymbol{\Sigma}^0$ and $\mathbf{G}^0$. Our next step is to approximate the scattered wavefield $\mathbf{u}^{sc}$ on the exit plane S . We consider the scattered P waves first. For sufficiently weak perturbations, we may neglect the bending of the rays at the interface between the scatterer and the matrix. In this approximation, the main effect of the scatterer on the total transmitted P -wavefield $\mathbf{u}_p$ is to modify its phase according to:

$$\mathbf{u}_p(r, z) = \hat{\mathbf{z}} e^{i\omega z/\alpha_0 + i\delta\phi(r)} \quad z \geq R \quad (2)$$

where ω denotes the circular frequency, $\delta\phi(r)$ is the phase-shift induced by the presence of the scatterer, $\hat{\mathbf{z}}$ is the propagation direction (and polarization vector) of the incident P waves, R is the radius of the sphere and r is the transverse distance to the central ray and α_0 is the wavespeed in the background medium. For simplicity, we set up an (x, y, z) cartesian coordinate system whose origin coincides with the sphere center. From Figure 1, we find $\delta\phi(r) = 2\omega(1/\alpha - 1/\alpha_0)R \cos j$, where j is the incidence angle of the rays hitting the sphere at a transverse distance $r = \sqrt{x^2 + y^2} = R \sin j$. Using the relation $\mathbf{u}_p = \mathbf{u}_p^s + \mathbf{u}_p^i$ (the superscript i stands for the incident field), we conclude that

Figure 3: Scattering cross-section (normalized by the geometric cross-section) of P waves (a) and S waves (b) as a function of non-dimensional frequency $k_{p,s}a$ for an impedance type perturbation equal to -5% . Solid line: series solution of the BVP, dashed line: Born (Rayleigh-Gans) approximation, dotted line: ray theory.

right after passage through the sphere, the ray approximation of the scattered field is given by:

$$\begin{aligned} \mathbf{u}_p^{sc}(r, z) &= \hat{\mathbf{z}} e^{i\omega z/\alpha_0} (e^{i2\omega(1/\alpha - 1/\alpha_0)R \cos j} - 1) & r = R \sin j \leq R, \quad z \geq R \\ \mathbf{u}_p^{sc}(r, z) &= \mathbf{0} & r > R \end{aligned} \quad (3)$$

Applying Hooke's law, it is readily shown that the scattered traction is simply proportional to the scattered displacements:

$$\mathbf{t}_p^{sc}(r, z) = \frac{i(\lambda_0 + 2\mu_0)\omega}{\alpha_0} \mathbf{u}_p^{sc}(r, z) \quad (4)$$

We may now inject the ray approximation (3)-(4) in the representation theorem (1) to calculate the scattered wavefield in the far-field. Actually, we only need the amplitude of the P wave scattered in the forward direction thanks to the optical theorem (Varatharajulu, 1977; Dassios et al., 1987; Margerin and Sato, 2011). Physically, this theorem expresses that the interference of the forward-scattered wave with the incident plane wave entails a reduction of the total amplitude in the far-field, thereby ensuring energy conservation. For a unit amplitude incident wave, the forward-scattered P -wave may be expressed in the far-field as:

$$\mathbf{u}_p^s(r=0, z\hat{\mathbf{z}}) = \hat{\mathbf{z}} \frac{f^p(\hat{\mathbf{z}}) e^{i\omega z/\alpha_0}}{z}, \quad (5)$$

where $f^p(\hat{\mathbf{z}})$ is the scattering amplitude in the forward direction. Mathematically, the optical

128 theorem may be written as:

$$\sigma_p = \frac{4\pi\alpha_0}{\omega} \text{Im} f^p(\hat{\mathbf{z}}), \quad (6)$$

129 where σ_p is the P -wave scattering cross-section and Im denotes the imaginary part of a complex
 130 number. Using the expression of the free space Green's functions found in Aki and Richards (2002, p.
 131 72) and after some straightforward algebra we obtain from Eq. (1) and (3) the following expression
 132 for the forward scattering amplitude:

$$f^p(\hat{\mathbf{z}}) = \frac{i\omega R^2}{2\pi\alpha_0} \int_0^{2\pi} d\varphi \int_0^{\pi/2} (1 - e^{i2\omega(1/\alpha - 1/\alpha_0)R \cos j}) \sin j \cos j dj \quad (7)$$

133 The symbol φ denotes the azimuthal angle with respect to the incident direction and the reader is
 134 reminded that j is the incidence angle of a given ray on the spherical surface. Formula (7) is valid
 135 in the limit $\omega R^2/\alpha_0 z \rightarrow 0$, which allows us to neglect the phase difference between the scattered
 136 wavelets upon propagation from points (x, y, R) of the surface S to the detection point $(0, 0, z)$
 137 located in the far-field in the forward direction. Likewise, to lowest order in the small quantity R/z ,
 138 the geometrical spreading and polarization vector of the scattered P wavelets may be approximated
 139 by $1/z$ and $\hat{\mathbf{z}}$, respectively.

140 Van de Hulst (1981) discusses the physical interpretation of formula (7) in details. The first term
 141 in the parenthesis is completely independent of the properties of the scatterer and quantifies the
 142 role of the waves that are diffracted by the object. This contribution also shows up in the problem
 143 of diffraction by screens in optics. Note that in this work the term “diffraction” is employed in
 144 the restrictive sense of “Fraunhofer” diffraction as defined in optics. Diffraction phenomena that
 145 go beyond our simple ray approximation do occur in the vicinity of the scatterer. For example,
 146 the sharp discontinuity of u^{sc} postulated in Eq. (3) is presumably invalid in a region whose width
 147 scales with the wavelength, and which is located at the edge of the surface S . As a consequence,
 148 our treatment is appropriate at sufficiently high frequency only. The second term of the integrand
 149 of Eq. (7) represents the effect of the waves that are directly transmitted through the object in the
 150 forward direction. Eq. (7) shows that it is the interference of diffracted and transmitted waves that
 151 is responsible for the scattering attenuation. The integrals in Eq. (7) can be performed analytically

and upon using the optical theorem (6), we find:

$$\sigma_p = 4\pi R^2 \left(\frac{1}{2} + \frac{1 - \cos \delta\phi}{\delta\phi^2} - \frac{\sin \delta\phi}{\delta\phi} \right) \quad (8)$$

with $\delta\phi = 2\omega(1/\alpha - 1/\alpha_0)R$ is a phase shift due to the scatterer. Formula (8) was previously derived by Korneev and Johnson (1996) using the exact series solution of the problem. In the high-frequency limit $\omega \rightarrow \infty$, one finds the well-known result $\sigma \rightarrow 2\pi R^2$, i.e., the scattering cross-section tends to twice the geometrical cross-section of the scatterer σ_g . From an energy conservation perspective, the factor 2 is explained by the following equal contributions: (1) the rays that are refracted upon crossing the object; (2) the rays that are diffracted by the disruption of the wavefront in the shadow of the scatterer (Van de Hulst, 1981). Under the weak contrast assumption, the result (8) may also be applied to S waves provided some obvious changes are made (see below). The key point is that the polarization of the incident S wave is barely affected by the transmission through the scatterer. It then suffices to replace the P wave speeds with the S wave speeds in the definition of the phase shift, yielding: $\delta\phi = 2\omega(1/\beta - 1/\beta_0)R$. A verification of the applicability of formula (8) for S wave is shown in Figure 2. After the non-dimensionalization $\omega \rightarrow \omega R/\alpha_0$ for P waves, and $\omega \rightarrow \omega R/\beta_0$, the similarity between the frequency dependence of the scattering cross-section of P and S waves for velocity-type scatterers is striking.

Impedance-type perturbations

We now consider the problem of impedance type perturbations. Comparison of Figures 2 and 3 reveals that the level of scattering -at high-frequency- is much weaker for impedance-type than for velocity-type perturbations. Furthermore, it is clear from Figure 3 that the frequency dependence of the scattering cross-section is different for P and S waves. As shown by a Taylor series expansion, formula (8) predicts that the scattering cross-section should be equal to 0 in the case of impedance-type perturbations, which is clearly incorrect. To find a sensible result we must now take into account the change of amplitude of the waves that are transmitted through the scatterer. The modification of formula (7) is straightforward and one obtains:

$$\sigma_p = 4\pi R^2 \int_0^{\pi/2} (1 - T_p^{m \rightarrow s}(j) T_p^{s \rightarrow m}(j)) \sin j \cos j dj, \quad (9)$$

where T_p is the displacement P to P transmission coefficient for plane waves as given by Aki and Richards (e.g. 2002). The superscript $m \rightarrow s$ indicates transmission from the matrix to the sphere (and conversely for $s \rightarrow m$). Figure 3 shows that Formula (9) gives a very good approximation of the exact result for $\omega R/\alpha_0 > 10$. Integral (9) is not easily computed analytically. However, a reasonable result is obtained by taking the acoustic approximation $T_{pp}^{m \rightarrow s} T_{pp}^{s \rightarrow m} \approx 4\rho_0\rho/(\rho + \rho_0)^2$, which yields $\sigma_p \approx 2\pi R^2(\rho - \rho_0)^2/(\rho + \rho_0)^2$. It is worth noting that for weak contrasts, the first term in the parenthesis of Eq. (9) is almost cancelled out by the second. Following our previous interpretation of formula (7), it means that diffraction effects no longer play a role in the scattering attenuation. Physically, this makes sense since the phase front remains perfectly planar upon crossing the scatterer. To confirm our interpretation, we may compute the scattering cross-section by removing, from the incident flux, the part of the energy that is directly transmitted in the forward direction (i.e., the un-scattered radiation). This yields:

$$\sigma_p = 2\pi R^2 \int_0^{\pi/2} (1 - (T_p^{m \rightarrow s}(j) T_p^{s \rightarrow m}(j)))^2 \sin j \cos j dj, \quad (10)$$

where the first term in the parenthesis represents the total flux intercepted by the scatterer and the second quantifies the fraction of the flux that remains in perfect coherence with the incident wave (and is therefore indistinguishable from it). For weak contrasts the formulas (9) and (10) agree remarkably well, a result that may be proven formally by a Taylor expansion.

The case of shear waves requires extra care (see Appendix A for computational details). Indeed, the reflection coefficients of SH and SV waves differ at the interface between two media exhibiting an impedance contrast. In the case of the ray depicted in Figure 1, the SV (resp. SH) component vibrates in (resp. perpendicular to) the plane of the Figure. Attention must also be paid to the fact that S to P conversions make the transmission coefficient complex for $j > \arcsin \beta_0/\alpha_0$. Polarization effects yield the following modifications of formulas (9)-(10):

$$\sigma_s = 4\pi R^2 \text{Re} \int_0^{\pi/2} \left(1 - \frac{T_{sv}^{m \rightarrow s}(j) T_{sv}^{s \rightarrow m}(j)}{2} - \frac{T_{sh}^{m \rightarrow s} T_{sh}^{s \rightarrow m}}{2} \right) \sin j \cos j dj \quad (11)$$

$$\sigma_s = 2\pi R^2 \int_0^{\pi/2} \left(1 - \frac{|T_{sv}^{m \rightarrow s}(j) T_{sv}^{s \rightarrow m}(j)|^2}{2} - \frac{(T_{sh}^{m \rightarrow s} T_{sh}^{s \rightarrow m})^2}{2} \right) \sin j \cos j dj \quad (12)$$

where Re denotes the real part of the integral and T_{sv} is the SV to SV transmission coefficient (and

193 similarly for T_{sh}). The reader is referred to Appendix A for additional comments on the derivation.
194 Formula (11) follows from the optical theorem while formula (12) provides an interpretation in terms
195 of energy conservation. As previously noted in the case of P waves: (1) the diffraction term cancels
196 out because the transmission coefficients are close to 1. (2) As a consequence, the scattered energy
197 may be interpreted as the flux intercepted by the scatterer minus the fraction which is directly
198 transmitted in the forward direction. A comparison between the asymptotic formula (11) and the
199 series solution of the BVP shows good agreement between the two for $\omega R/\beta_0 > 10$.

200 Discussion and conclusion

201 In this section, we examine the conjecture made by Kawahara (2011) and Calvet and Margerin
202 (2018) about the validity of the Born (or Rayleigh-Gans) approximation at high-frequency in the
203 case of impedance-type perturbations. Loosely speaking, the Born approximation discretizes a finite
204 object into elementary volumes of dimension much smaller than the wavelength. The contribution
205 of each volume to the scattering amplitude is computed to first order in parameter heterogeneity
206 and subsequently summed taking into account the phase difference between the secondary sources
207 of scattered waves. The limitations of this approach have been largely discussed in the literature
208 in the case of velocity-type perturbations. Van de Hulst (1981, P. 87) shows that in the Born
209 approximation, the scattering cross-section is typically proportional to $\omega^4 \alpha_0^{-4} R^6 (\alpha/\alpha_0 - 1)^2$. Thus,
210 at high frequency, the Born approximation predicts an increase of the scattered amplitude which
211 is unrealistically large, eventually causing a divergence of the scattering cross-section in the limit
212 $\omega \rightarrow \infty$. It is worth pointing out that the divergence of the scattering amplitude is concentrated
213 around the forward direction and should become noticeable when the phase-shift $|\omega(1/\alpha - 1/\alpha_0)R|$
214 is of order 1 (in the case of incident P waves, say). Wu and Zheng (2014) show that consideration
215 of higher-order terms of the Born series removes the singularity.

216 Using Eq. (11a)-(11d) and Appendix B of Margerin et al. (2000), we have computed numerically
217 the scattering cross-section of P and S waves in the Born (Rayleigh-Gans) approximation for
218 impedance type perturbations. This theory agrees within a few percents with the series solution
219 of the BVP and does a nice job at explaining the oscillations of the scattering cross-section as
220 a function of frequency. It is very likely that this success can be explained by the absence of

Figure 4: Differential scattering cross-section $d\sigma_{pp}/d\Omega$ as a function of scattering angle θ for P to P scattering in the case of -5% impedance-type perturbations at non-dimensional frequency $\omega R/\alpha_0 = 60$. Note the rapid oscillations (solid line) inside the smooth envelope (dashed line).

221 phase-shift inside the velocity-type scatterer, as put forward by Kawahara (2011) and Calvet and
 222 Margerin (2018). It is also worth noting that in the Born approximation, the scattering amplitude
 223 can be factored into the product of a “shape factor” and a term which depends on the perturbation
 224 of the elastic tensor (Wu and Aki, 1985b). Independently of the polarization of the incident and
 225 scattered wave, the latter is exactly zero in the forward direction in the case of impedance-type
 226 scatterer, which cuts-off any divergence of the scattering amplitude as suggested above. For this
 227 reason, Kawahara (2011) has suggested that the conversion scattering cross-section can be well
 228 approximated by the Born approximation at all frequencies, independent of the type of scatterers.
 229 Note, however, that the forward scattering amplitude can never be exactly zero because it is related
 230 to the extinction cross-section by the optical theorem and therefore assumes a finite value (possibly
 231 purely imaginary).

232 In Figure 3, we remark that there is a small but visible offset between the series solution and
 233 the Born approximation. This discrepancy can be traced back to the violation of the boundary
 234 conditions by the latter at the interface between the matrix and scatterer, i.e., the continuity of
 235 tractions is not ensured. As a consequence, the Born approximation predicts that the scattering
 236 cross-section is proportional to $(\rho - \rho_0)^2/\rho_0^2$, whereas ray theory strongly suggests a dependence on
 237 $(\rho - \rho_0)^2/(\rho + \rho_0)^2$. Indeed, if we (empirically) substitute the former with the latter in the Born
 238 approximation (see formulas below), the agreement with the series solution is greatly improved for
 239 $\omega R/c > 10$ (where c equals α_0 or β_0).

240 Having understood the limit of validity of the Born approximation, we may use it to exhibit
 241 the dependence of the scattering cross-section on the wavespeed ratio $\gamma = \beta_0/\alpha_0$. To do so, we
 242 remark that the scattering amplitude is a rapidly varying function of the scattering angle which
 243 oscillates inside a slowly-varying envelope. This property is illustrated in Figure 4 where we show
 244 the differential scattering cross-section for P to P scattering in the case of -5% impedance-type
 245 perturbations at non-dimensional frequency $\omega R/\alpha_0 = 60$. Note that the differential scattering
 246 cross-section, which is proportional to the squared scattering amplitude, has been normalized by
 247 the maximum of its envelope. The oscillations are described with good accuracy with standard
 248 trigonometric functions of $\omega R/c$ so that we expect that the integral over θ is approximately equal
 249 to half the area delimited by the envelope and the horizontal axis. Using Appendix B of Margerin
 250 et al. (2000), the analytical formula for the envelope may be extracted by factoring out all the terms
 251 that depend on frequency that are therefore rapidly oscillating. In the case of P to P scattering,
 252 the slowly-varying envelope shown in Figure 4 is of the form: $(\rho - \rho_0)^2(\alpha_0^2 - 4\beta_0^2 \cos^2 \theta/2)^2/\rho_0^2$.
 253 Integrating over the volume of the scatterer, we find:

$$\sigma_{pp} = \frac{\pi R^2(\rho - \rho_0)^2}{2\rho_0^2} \left(1 - 4\gamma^2 + \frac{16\gamma^4}{3} \right) \quad (13)$$

254 Proceeding similarly for P to S scattering conversions we find:

$$\sigma_{ps} = \frac{\pi R^2(\rho - \rho_0)^2}{\rho_0^2} \left[\gamma \left(\frac{1}{3} - 2\gamma^2 \right) + \frac{(1 - \gamma^2 + 2\gamma^4)}{2} \log \left(\frac{1 + \gamma}{1 - \gamma} \right) \right] \quad (14)$$

255 and:

$$\sigma_{sp} = \frac{\sigma_{ps}}{2\gamma^2} \quad (15)$$

256 where the last formula follows from reciprocity (e.g. Aki, 1992). In the case of S to S conversions,
 257 we find the particularly simple formula:

$$\sigma_{ss} = \frac{5\pi R^2(\rho - \rho_0)^2}{6\rho_0^2} \quad (16)$$

258 To summarize, we have computed the scattering cross-section of an impedance-type object at
 259 high frequency by three methods: the series solution of the BVP, an asymptotic method due to

260 Van de Hulst (1981) based on ray theory, and the popular Born approximation. We found that the
 261 last two methods provide simple and relatively accurate approximations of the exact response. On
 262 the one hand, Born approximation describes well the oscillations of the scattering cross-section as a
 263 function of frequency and exhibits a small but systematic shift with respect to the exact solution.
 264 This slight discrepancy was explained by the violation of the boundary conditions at the surface of
 265 the scatterer. On the other hand, the ray approximation is in close agreement with the exact result
 266 but fails at predicting the oscillations of the scattering cross-section as a function of frequency. Our
 267 study confirms the conjecture of Kawahara (2011) and Calvet and Margerin (2018) that in the case of
 268 impedance-type scatterers, the Born approximation remains approximately valid at high frequency,
 269 in sharp contrast with the case of velocity-type scatterers. Perhaps the most interesting outcome of
 270 this short study is the clear demonstration that there is no geometrical regime for impedance-type
 271 scatterers, in the sense that their scattering response depends on their internal properties at all
 272 frequencies. Finally, we remark that results from Born approximation and ray theory may readily
 273 be extended to convex objects of otherwise arbitrary shape which may be useful to study the effect
 274 of shape preferred orientation on seismic attenuation (Calvet and Margerin, 2016).

275 Data and Resources

276 Figures were produced with the MATHEMATICA software. The bibliography was produced with the
 277 aid of the style file bssa_usc.bst provided by P. Crotwel.

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280 References

- 281 Aki, K. (1992). Scattering conversions P to S versus S to P, *Bull. Seismol. Soc. Am* **82**, 1969–1972.
 282 Aki, K. and P. G. Richards (2002). *Quantitative seismology*. University Science Books.
 283 Calvet, M. and L. Margerin (2012, Erratum 143, 139–141, 2018). Velocity and attenuation of scalar

284 and elastic waves in random media: A spectral function approach, *J. Acous. Soc. Am.* **131**,
285 1843–1862.

286 Calvet, M. and L. Margerin (2016). Impact of grain shape on seismic attenuation and phase velocity
287 in cubic polycrystalline materials, *Wave Motion* **65**, 29–43.

288 Dassios, G., K. Kiriaki, and D. Polyzos (1987). On the scattering amplitudes for elastic waves,
289 *Zeitschrift für Angewandte Mathematik und Physik (ZAMP)* **38**, 856–873.

290 Foldy, L. L. (1945). The Multiple Scattering of Waves. I. General Theory of Isotropic Scattering by
291 Randomly Distributed Scatterers, *Physical Review* **67**, 107–119.

292 Kawahara, J. (2011). Scattering attenuation of elastic waves due to low-contrast inclusions, *Wave*
293 *Motion* **48**, 290–300.

294 Korneev, V. and L. Johnson (1996). Scattering of P and S waves by a spherically symmetric
295 inclusion, *Pure Appl. Geophys.* **147**, 675–718.

296 Margerin, L., M. Campillo, and B. Van Tiggelen (2000). Monte Carlo simulation of multiple
297 scattering of elastic waves, *J. Geophys. Res.* **105**, 7873–7892.

298 Margerin, L. and H. Sato (2011). Generalized optical theorems for the reconstruction of Green’s
299 function of an inhomogeneous elastic medium, *J. Acous. Soc. Am.* **130**, 1–17.

300 Pao, Y.-H. and V. Varatharajulu (1976). Huygens’ principle, radiation conditions, and integral
301 formulas for the scattering of elastic waves, *J. Acous. Soc. Am.* **59**, 1361–1371.

302 Sato, H. (1984). Attenuation and envelope formation of three-component seismograms of small local
303 earthquakes in randomly inhomogeneous lithosphere, *J. Geophys. Res.* **89**, 1221–1241.

304 Sato, H. (1989). Broadening of seismogram envelopes in the randomly inhomogeneous lithosphere
305 based on the parabolic approximation: Southeastern Honshu, Japan, *J. Geophys. Res.* **94**, 17735–
306 17747.

307 Stoffa, P., J. T. Fokkema, R. de Luna Freire, and W. Kessinger (1990). Split-step Fourier migration,
308 *Geophysics* **55**(4), 410–421.

- 309 Torquato, S. (2002). *Random Heterogeneous Materials: Microstructure and Macroscopic Properties*.
310 Springer-Verlag, New York.
- 311 Van de Hulst, H. C. (1981). *Light scattering by small particles*. New York: Dover Publications.
- 312 Varatharajulu, V. (1977). Reciprocity relations and forward amplitude theorems for elastic waves,
313 *J. Math. Phys.* **18**, 537–543.
- 314 Wu, R. and K. Aki (1985a). Elastic wave scattering by a random medium and the small-scale
315 inhomogeneities in the lithosphere, *J. Geophys. Res.* **90**, 10261–10273.
- 316 Wu, R. and K. Aki (1985b). Scattering characteristics of elastic waves by an elastic heterogeneity,
317 *Geophysics* **50**, 582–595.
- 318 Wu, R.-S. (1994). Wide-angle elastic wave one-way propagation in heterogeneous media and an
319 elastic wave complex-screen method, *Journal of Geophysical Research: Solid Earth* **99**(B1),
320 751–766.
- 321 Wu, R.-S. and Y. Zheng (2014). Non-linear partial derivative and its De Wolf approximation for
322 non-linear seismic inversion, *Geophysical Journal International* **196**(3), 1827–1843.

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List of Figure Captions

Figure 1: Position of the problem. We consider a plane elastic (P or S) wave incident on a sphere of radius R . The incidence angle of a given ray is denoted by j . The subscript 0 refers to the properties of the matrix with ρ the density, α the P -wave velocity and β the S -wave velocity.

Figure 2: Scattering cross-section (normalized by the geometric cross-section) of P waves (a) and S waves (b) as a function of non-dimensional frequency $k_{p,s}R$ for a velocity-type perturbation equal to -5% . Solid line: series solution of the BVP, dotted line: Ray theory.

Figure 3: Scattering cross-section (normalized by the geometric cross-section) of P waves (a) and S waves (b) as a function of non-dimensional frequency $k_{p,s}a$ for an impedance-type perturbation equal to -5% . Solid line: series solution of the BVP, dashed line: Born (Rayleigh-Gans) approximation, dotted line: ray theory.

Figure 4: Differential scattering cross-section $d\sigma_{pp}/d\Omega$ as a function of scattering angle θ for P to P scattering in the case of -5% impedance-type perturbations at non-dimensional frequency $\omega R/\alpha_0 = 60$. Note the rapid oscillations (solid line) inside the smooth envelope (dashed line).

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328 Appendix A

This appendix summarizes the main computational steps for the derivation of Eq. (11). Here, we consider the case of an incident unit amplitude S -wave polarized along the x axis (see Figure 1). At the entrance point of the ray inside the sphere $(R \sin j \cos \varphi, R \sin j \sin \varphi, -R \cos j)$, we decompose the wave motion onto local SV and SH components. The reader may readily verify that an appropriate local basis is given by:

$$\widehat{\mathbf{sh}} = \sin \varphi \widehat{\mathbf{x}} - \cos \varphi \widehat{\mathbf{y}} \quad (\text{A1})$$

$$\widehat{\mathbf{sv}} = \cos \varphi \widehat{\mathbf{x}} + \sin \varphi \widehat{\mathbf{y}} \quad (\text{A2})$$

329 Using this basis, the shear wavefield transmitted through the sphere may be written as:

$$\mathbf{u}_s(x = r \cos \varphi, y = r \sin \varphi, z) = \left(A_{sv}(j) \cos \varphi \widehat{\mathbf{sv}} + A_{sh}(j) \sin \varphi \widehat{\mathbf{sh}} \right) e^{i\omega z / \beta_0} \quad (\text{A3})$$

330 where $A_{sh,sv}$ are amplitude terms (possibly complex). Note that representation (A3) is valid for both
 331 velocity-type and impedance-type perturbations: only the interpretation of the amplitudes differs.
 332 In the case of velocity-type perturbations and neglecting the effect of transmission coefficients, $A_{sh,sv}$
 333 are pure phase terms: $A_{sh} = A_{sv} = e^{i2\omega(1/\beta-1/\beta_0)R \cos j}$. In the case of impedance-type perturbations
 334 and using the notations of the main text, one has $A_{sh} = T_{sh}^{m \rightarrow s} T_{sh}^{s \rightarrow m}$ and $A_{sv} = T_{sv}^{m \rightarrow s}(j) T_{sv}^{s \rightarrow m}(i)$.
 335 The former amplitude term is real and independent of the incidence angle of the ray j , while the
 336 latter is possibly complex and depends explicitly on j . Subtracting the incident field from Eq. (A3)
 337 and reverting to the global cartesian basis, the scattered displacement and traction fields on the
 338 exit surface S may be approximated as:

$$\begin{aligned} \mathbf{u}_s^{sc}(r \cos \varphi, r \sin \varphi, z) = & (A_{sv}(j) \cos^2 \varphi + A_{sh}(j) \sin^2 \varphi - 1) e^{i\omega z / \beta_0} \widehat{\mathbf{x}} \\ & + (A_{sv}(j) - A_{sh}(j)) \cos \varphi \sin \varphi e^{i\omega z / \beta_0} \widehat{\mathbf{y}} \end{aligned} \quad (\text{A4})$$

$$339 \quad \mathbf{t}_s^{sc}(r \cos \varphi, r \sin \varphi, z) = i\mu_0\omega \mathbf{u}_s^{sc}(r \cos \varphi, r \sin \varphi, z) / \beta_0 \quad (\text{A5})$$

340 Eq. (A5) is straightforwardly deduced from Eq. (A4) by application of Hooke's law.

341 The other ingredients that are needed for the calculations are far-field approximations of the dis-

placement and traction Green's functions in the far-field of the surface S . Consider a point $P(x, y, R)$ of S and a detection point in the forward direction $D(0, 0, z)$. The leading term of the transverse part of Green's tensor is given by:

$$G_{nk}^0(D, P) = \frac{e^{i\omega(z-R)/\beta_0}}{4\pi\rho_0\beta_0^2 z} (\delta_{nk} - \hat{z}_n \hat{z}_k) \quad (\text{A6})$$

As discussed in the body of the text, this approximation is valid only when the condition $z \gg \omega R^2/\beta_0$ is met for the phase. For the amplitude and polarization vectors, the much less stringent condition $R \ll |z|$ suffices. To the same level of approximations, the partial derivatives of the transverse part of G with respect to the coordinates of the source point P are given by:

$$G_{nk,l}^0(D, P) = \frac{-i\omega e^{i\omega(z-R)/\beta_0}}{4\pi\rho_0\beta_0^3 z} (\delta_{nk} - \hat{z}_n \hat{z}_k) \hat{z}_l \quad (\text{A7})$$

We are now in a position to calculate the traction term $\mathbf{n}' \cdot \Sigma(\mathbf{x}, \mathbf{x}')$ that appears in Eq. (1). Applying Hooke's law and using component notation, we find:

$$\hat{z}_m c_{mpkl}^0 G_{nk,l}^0(D, P) = -i\mu_0 \omega G_{np}^0(D, P) / \beta_0 \quad (\text{A8})$$

Examination of Eq. (A4), (A5), (A6) and (A8) reveals that the two terms in the integrand of Eq. (7) are equal which yields:

$$\begin{aligned} \mathbf{u}_s^{sc}(0, 0, z\hat{\mathbf{z}}) = & \frac{-i\omega e^{i\omega z/\beta_0}}{2\pi\beta_0 z} \int_0^{2\pi} d\varphi \int_0^{\pi/2} dj [(A_{sv}(j) \cos^2 \varphi + A_{sh}(j) \sin^2 \varphi - 1) \hat{\mathbf{x}} \\ & + (A_{sv}(j) - A_{sh}(j)) \cos \varphi \sin \varphi \hat{\mathbf{y}}] \end{aligned} \quad (\text{A9})$$

The integrals over φ are easy to perform and yield the scattered shear wave in the forward direction:

$$\mathbf{u}_s^{sc}(0, 0, z\hat{\mathbf{z}}) = \frac{i\omega e^{i\omega z/\beta_0}}{z\beta_0} \int_0^{\pi/2} (1 - A_{sv}(j)/2 - A_{sh}(j)/2) dj \hat{\mathbf{x}} \quad (\text{A10})$$

Note that the $\hat{\mathbf{y}}$ component of the scattered field vanishes as required by symmetry considerations. Upon substitution of $A_{sv,sh}(j)$ with $T_{sv,sh}^{m \rightarrow s}(j) T_{sv,sh}^{s \rightarrow m}(i)$ and application of the optical theorem, the result (11) follows.