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ON SHIFTED PRIMES WITH LARGE PRIME FACTORS AND THEIR PRODUCTS

J. WU

ABSTRACT. In this short note, we give partial answers to two questions on shifted primes with large prime factors, posed by Luca, Menares & Pizarro-Madariaga (2015) and by Yonggao Chen & Fengjuan Chen (2016), respectively.

1. INTRODUCTION

The distribution of shifted primes with large prime factors is an interesting subject in number theory, which has received much attention. It is related to many well known arithmetic problems such as the last Fermat theorem [6], the Brun-Titchmarsh theorems [1], the twin prime conjecture [14], etc. As usual, we use $P^+(n)$ to denote the largest prime factor of n with the convention $P^+(1) = 1$. Recently Luca, Menares and Pizarro-Madariaga [9] considered the following counting function

$$(1.1) \quad T_\theta(x) := |\{p \leq x : P^+(p-1) \geq p^\theta\}|$$

and proved (see [9, Theorem 1])

$$(1.2) \quad T_\theta(x) \geq (1 - \theta)\pi(x)\{1 + O_\theta(E(x))\},$$

where $\pi(x)$ denotes the number of primes $p \leq x$, the implied constant depends on θ only and

$$E(x) := \begin{cases} \frac{\log \log x}{\log x} & \text{for } \frac{1}{4} < \theta < \frac{1}{2}, \\ \frac{1}{(\log x)^{2/3}} & \text{for } \theta = \frac{1}{4}. \end{cases}$$

In [4], Y.-G. Chen and F.-J. Chen improved this result by showing that (1.2) holds for $0 < \theta < \frac{1}{2}$ with a better error term. They also conjectured that

$$(1.3) \quad T_\theta(x) \sim (1 - \rho(1/\theta))\pi(x)$$

as $x \rightarrow \infty$, where $\rho(\theta)$ is the Dickman function. Very recently, Feng & Wu [5] and Liu, Wu & Xi [8] introduced and developed the friable number argument in this problem, and improved density $1 - \theta$ to $1 - 4\rho(1/\theta)$ for $0 < \theta < \Theta \approx 0.3734$, where Θ is the unique solution of the equation $\theta - 4\rho(1/\theta) = 0$.

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Motivated by Billerey & Menares' work [3] on the modularity of reducible mod ℓ Galois representations, Luca, Menares and Pizarro-Madariaga, in the same paper [9], also studied the following counting function

$$(1.4) \quad T_{k,\theta}(x) := \sum_{\substack{p_1 \cdots p_k \leq x \\ P^+(\gcd(p_1-1, \dots, p_k-1)) \geq (p_1 \cdots p_k)^\theta}} 1,$$

which can be regarded as a generalisation of (1.4), by noticing $T_{1,\theta}(x) = T_\theta(x)$. With the help of the Brun-Titchmarsh theorem (see Lemmas 2.1-2.2 below), they proved that for fixed integer $k \geq 2$ and real $\theta \in [1/(2k), 17/(32k))$, inequalities

$$(1.5) \quad \frac{x^{1-(k-1)\theta}}{(\log x)^{k+1}} \ll_k T_{k,\theta}(x) \ll_k \frac{x^{1-\theta(k-1)}}{(\log x)^2} (\log \log x)^{k-1}$$

hold as $x \rightarrow \infty$ (see [9, Theorem 2]), where the implied constants depend on k . The case $\theta = 1/(2k)$ is important for the results from [3]. They also wrote (see [9, page 41]): “We leave as a problem for the reader to determine the exact order of magnitude of $T_{k,\theta}(x)$, or an asymptotic for it.”

The first aim of this short note is to give a partial answer to this question by improving the lower part in (1.5).

Theorem 1. *Let $k \geq 2$ and $\theta \in [1/(2k), 17/(32k))$ be fixed. Then we have*

$$(1.6) \quad T_{k,\theta}(x) \gg_k \frac{x^{1-\theta(k-1)}}{(\log x)^2}$$

for $x \rightarrow \infty$, where the implied constant depends on k .

Remark 1. In view of (1.5) and (1.6), it seems reasonable to think that

$$(1.7) \quad T_{k,\theta}(x) \asymp \frac{x^{1-\theta(k-1)}}{(\log x)^2} (\log \log x)^{k-1}$$

for $x \rightarrow \infty$.

In [4], Y.-G. Chen and F.-J. Chen also posed a question. Defining

$$T'_\theta(x) := |\{p \leq x : P^+(p-1) \geq x^\theta\}|,$$

they wrote (see [4, Remark 3.1]): “Currently, we cannot give a proof of

$$(1.8) \quad T_\theta(x) = T'_\theta(x) + o(\pi(x)).$$

We believe that this is not really difficult.”

The second aim of this short note is to prove (1.8). Our result is as follows.

Theorem 2. *We have*

$$(1.9) \quad T_\theta(x) = T'_\theta(x) + O\left(\pi(x) \frac{\log \log x}{\log x}\right)$$

for $x \rightarrow \infty$. Further assuming the Elliott-Halberstam conjecture, Chen-Chen's conjecture (1.3) holds.

Remark 2. In view of the second assertion of Theorem 2, it seems rather difficult to get an asymptotic formula for $T_{k,\theta}(x)$ with $k \geq 2$ and $0 < \theta < 1$ unconditionally.

2. BRUN-TITCHMARSH INEQUALITIES

In this section, we shall cite two versions of the Brun-Titchmarsh theorem, an individual and an average on the modulus. For $x \geq 2$ and $1 \leq a \leq q$ with $(a, q) = 1$, we define

$$\pi(x; q, a) := \sum_{\substack{p \leq x \\ p \equiv a \pmod{q}}} 1.$$

The first one is due to Montgomery-Vaughan [11].

Lemma 2.1. *Let $x \geq 2$ and $1 \leq a \leq q$ with $(a, q) = 1$. Then we have*

$$(2.1) \quad \pi(x; q, a) \leq \frac{2x}{\varphi(q) \log(x/q)},$$

where $\varphi(q)$ is the Euler totient function.

The following result is Lemma 2.1 of [2], due to Bombierie-Vinogradov for $0 < \theta < \frac{1}{2}$, Baker-Harman [1] for $\frac{1}{2} \leq \theta \leq \frac{13}{25}$ and Mikawa [10] for $\frac{13}{25} \leq \theta \leq \frac{17}{32}$.

Lemma 2.2. *There exist two functions $C_2(\theta) > C_1(\theta) > 0$, defined on the interval $(0, \frac{17}{32})$ such that for each fixed integer $a \in \mathbb{Z}^*$, each fixed real $A > 0$ and all sufficiently large $Q = x^\theta$, the inequalities*

$$(2.2) \quad C_1(\theta) \frac{\pi(x)}{\varphi(q)} \leq \pi(x; q, a) \leq C_2(\theta) \frac{\pi(x)}{\varphi(q)}$$

hold for all primes $q \in (Q, 2Q]$ with at most $O_A(Q(\log Q)^{-A})$ exceptions, where the implied constant depends only on a, A and θ . Moreover, for any fixed $\varepsilon > 0$, these functions can be chosen to satisfy the following properties:

- $C_1(\theta)$ is monotonic decreasing, and $C_2(\theta)$ is monotonic increasing;
- $C_1(\frac{1}{2}) = 1 - \varepsilon$ and $C_2(\frac{1}{2}) = 1 + \varepsilon$.

3. PROOF OF THEOREM 1

Denote by $\mathbb{P}$ the set of all prime numbers. Let $k \geq 2$ and $\theta \in [1/(2k), 17/(32k))$. Put $Q := x^\theta$. According to (2.2) of Lemma 2.2, we have

$$(3.1) \quad C_1(\theta) \frac{\pi(x)}{\varphi(q)} \leq \pi(x; q, a) \leq C_2(\theta) \frac{\pi(x)}{\varphi(q)}$$

for all primes $q \in (Q, 2Q]$ with at most $O_A(Q(\log Q)^{-A})$ exceptions, where the implied constant depends only on A, a, k and θ . We introduce

$$(3.2) \quad \mathcal{Q}_g(x) := \{q \in \mathbb{P} \cap (Q, 2Q] : (3.1) \text{ above holds}\},$$

$$(3.3) \quad \mathcal{Q}_b(x) := (\mathbb{P} \cap (Q, 2Q]) \setminus \mathcal{Q}_g(x).$$

By Lemma 2.2, we have

$$(3.4) \quad |\mathcal{Q}_b(x)| \ll_{A,a,k,\theta} Q(\log Q)^{-A}.$$

Firstly we write

$$T_{k,\theta}(x) \geq \sum_{p_1 \leq x^{1/k}} \cdots \sum_{\substack{p_{k-1} \leq x^{1/k} \\ P^+(\gcd(p_1-1, \dots, p_{k-1}-1)) \geq x^\theta}} \sum_{p_k \leq x/p_1 \cdots p_{k-1}} 1.$$

Putting $q := P^+(\gcd(p_1 - 1, \dots, p_{k-1} - 1))$, we have $p_j \equiv 1 \pmod{q}$ for $1 \leq j \leq k$. Thus

$$T_{k,\theta}(x) \geq \sum_{q \in \mathcal{Q}_g(x)} \sum_{\substack{p_1 \leq x^{1/k} \\ p_1 \equiv 1 \pmod{q}}} \cdots \sum_{\substack{p_{k-1} \leq x^{1/k} \\ p_{k-1} \equiv 1 \pmod{q}}} \sum_{\substack{p_k \leq x/p_1 \cdots p_{k-1} \\ p_k \equiv 1 \pmod{q}}} 1.$$

Applying (3.1) to the inner sum over p_k and noticing that $\varphi(q) = q - 1 \geq q/2$, it follows that

$$T_{k,\theta}(x) \gg_k \sum_{q \in \mathcal{Q}_g(x)} \sum_{\substack{p_1 \leq x^{1/k} \\ p_1 \equiv 1 \pmod{q}}} \cdots \sum_{\substack{p_{k-1} \leq x^{1/k} \\ p_{k-1} \equiv 1 \pmod{q}}} \frac{x}{qp_1 \cdots p_{k-1} \log(x/p_1 \cdots p_{k-1})}.$$

In view of the fact that $x/p_1 \cdots p_{k-1} \geq x^{1/k}$, we can derive that

$$(3.5) \quad T_{k,\theta}(x) \gg_k \frac{x}{\log x} \sum_{q \in \mathcal{Q}_g(x)} \frac{1}{q} \left(\sum_{\substack{p \leq x^{1/k} \\ p \equiv 1 \pmod{q}}} \frac{1}{p} \right)^{k-1}.$$

Clearly, $\pi(t; q, 1) = 0$ for $1 \leq t \leq q$ and $q \geq x^{1/(2k)}$ for $q \in \mathcal{Q}_g(x)$. Let $\delta_{k,\theta} := \frac{1}{2}(\frac{1}{k} + \frac{32}{17}\theta)$. It is easy to verify that

$$(3.6) \quad \theta \in [1/(2k), 17/(32k)) \Rightarrow \textcolor{red}{1/(2k)} < \delta_{k,\theta} < 1/k \quad \text{and} \quad \theta/\delta_{k,\theta} < 17/32.$$

Thus a simple partial integration gives us

$$\begin{aligned} \sum_{\substack{p \leq x^{1/k} \\ p \equiv 1 \pmod{q}}} \frac{1}{p} &= \sum_{\substack{\textcolor{red}{x^{1/(2k)}} \leq p \leq x^{1/k} \\ p \equiv 1 \pmod{q}}} \frac{1}{p} = \int_{\textcolor{red}{x^{1/(2k)}}}^{x^{1/k}} \frac{d\pi(t; q, 1)}{t} \\ &= \frac{\pi(x^{1/k}; q, 1)}{x^{1/k}} + \int_{\textcolor{red}{x^{1/(2k)}}}^{x^{1/k}} \frac{\pi(t; q, 1)}{t^2} dt \\ &\geq \int_{x^{\delta_{k,\theta}}}^{x^{1/k}} \frac{\pi(t; q, 1)}{t^2} dt \end{aligned}$$

for $q \in \mathcal{Q}_g(x)$. In view of the last inequality in (3.6), we can apply the lower bound in (3.1) to derive

$$\sum_{\substack{p \leq x^{1/k} \\ p \equiv 1 \pmod{q}}} \frac{1}{p} \gg_k \int_{x^{\delta_{k,\theta}}}^{x^{1/k}} \frac{dt}{qt \log t} \gg_k \frac{1}{q}.$$

Inserting it into (3.5) and using (3.4) with $A = 2$, we find that

$$\begin{aligned}
T_{k,\theta}(x) &\gg_k \frac{x}{\log x} \sum_{q \in \mathcal{Q}_g(x)} \frac{1}{q^k} \\
&\gg_k \frac{x}{\log x} \left\{ \sum_{q \in \mathbb{P} \cap (Q, 2Q]} \frac{1}{q^k} + O\left(\frac{1}{Q^{k-1}(\log Q)^2}\right) \right\} \\
&\gg_k \frac{x}{\log x} \cdot \frac{1}{Q^{k-1} \log Q} \\
&\gg_k \frac{x^{1-(k-1)\theta}}{(\log x)^2}.
\end{aligned}$$

This completes the proof of Theorem 1.

4. PROOF OF THEOREM 2

It is sufficient to prove that we have

$$(4.1) \quad T_\theta(x) - T'_\theta(x) = \sum_{\substack{p \leq x \\ p^\theta \leq P^+(p-1) < x^\theta}} 1 \ll \pi(x) \frac{\log \log x}{\log x}$$

for $x \rightarrow \infty$. For this, we write

$$(4.2) \quad T_\theta(x) - T'_\theta(x) = \sum_{\substack{p \leq x\delta(x) \\ p^\theta \leq P^+(p-1) < x^\theta}} 1 + \sum_{\substack{x\delta(x) < p \leq x \\ p^\theta \leq P^+(p-1) < x^\theta}} 1 =: S_1 + S_2,$$

where $\delta(x) \rightarrow 0$ is a function to be chosen later.

By the Chebyshev upper bound, we have

$$(4.3) \quad S_1 \leq \sum_{p \leq x\delta(x)} 1 \ll \frac{x\delta(x)}{\log(x\delta(x))}.$$

In order to estimate S_2 , we write $p - 1 = mq$ with $q = P^+(p - 1)$ and $P^+(m) \leq q$. Thus

$$S_2 \leq \sum_{\substack{(x\delta(x))^\theta \leq q < x^\theta \\ q \text{ is prime}}} \sum_{\substack{p \leq x \\ p \equiv 1 \pmod{q}}} 1.$$

By the Brun-Titchmarsh inequality (2.1), we can derive

$$\begin{aligned}
(4.4) \quad S_2 &\leq \frac{4x}{(1-\theta)\log x} \sum_{q \in \mathbb{P} \cap ((x\delta(x))^\theta, x^\theta]} \frac{1}{q} \\
&\ll_\theta \pi(x) \log \left(\frac{\log x}{\log(x\delta(x))} \right) \\
&\ll_\theta \frac{x}{\log x} \frac{|\log \delta(x)|}{\log x}.
\end{aligned}$$

Inserting (4.3)-(4.4) into (4.2) and taking $\delta(x) = (\log \log x)/\log x$, we obtain the required estimate (1.9).

Pomerance [12] conjectured that for $\theta \in (0, 1)$ we have

$$(4.5) \quad \pi(x, x^\theta) := \sum_{\substack{p \leq x \\ P^+(p-1) \leq x^\theta}} 1 \sim \rho(1/\theta)\pi(x)$$

as $x \rightarrow \infty$. This conjecture is still open today. Granville [7] announced that (4.5) follows from the Elliott-Halberstam conjecture without proof. A such proof has been given by Wang [13] very recently. Noticing that $T'_\theta(x) = \pi(x) - \pi(x, x^\theta)$, for $\theta \in (0, 1)$ and $x \rightarrow \infty$ we have

$$T_{1,\theta}(x) = T_\theta(x) \sim T'_\theta(x) \sim (1 - \rho(1/\theta))\pi(x)$$

under the Elliott-Halberstam conjecture.

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