Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

doi:10.3934/xx.xx.xx.xx

pp. **X–XX**

ANALYSIS OF AN ANAEROBIC DIGESTION MODEL IN LANDFILL WITH MORTALITY TERM

S. Ouchtout
Interdisciplinary Laboratory for Natural Resources and Environment
Ibn Tofaïl University
Kénitra, Morocco
and
Laboratory of Mathematics, Informatics and Applications
University of Haute-Alsace
Mulhouse, France

Z. Mghazli
Interdisciplinary Laboratory for Natural Resources and Environment
Ibn Tofaïl University
Kénitra, Morocco

J. Harmand
LBE
Univ. Montpellier, INRA
Narbonne, France

A. Rapaport*
MISTEA
Univ. Montpellier, INRA, Montpellier SupAgro
Montpellier, France

Z. Belhachmi
Laboratory of Mathematics, Informatics and Applications
University of Haute-Alsace
Mulhouse, France

(Communicated by the associate editor name)

Abstract. We study a mathematical model of anaerobic digestion with biomass recirculation, dedicated to landfill problems, and analyze its asymptotic behavior. We show that the global attractor is composed of an infinity of non-hyperbolic equilibria. For non-monotonic growth functions, this set is non connected, which impacts the performances of the bioprocess.

1. **Introduction.** The anaerobic digestion process is a natural biological process of decomposition of organic matter by microorganisms (bacteria) that are activated under anaerobic conditions, that is to say without oxygen. It is characterized by a succession of complex reactions both in parallel and in series. In the long term, the organic matter is transformed into biogas, a mixture mainly composed of methane

2010 *Mathematics Subject Classification.* Primary: 37N25, 93D20; Secondary: 34C11, 34D35.
Key words and phrases. bioprocesses, biogas production, ordinary differential equations, non-hyperbolic equilibrium.

* Corresponding author: Alain Rapaport.

and carbon dioxide. The main stages of this process are hydrolysis, acidogenesis, acetogenesis and methanogenesis. Models such as the "ADM1" allow a detailed description of this process, cf. [14]. However, such complex models are not well suited for mathematical analysis. It is why a number of simpler models have been investigated in the literature over theses last years [8, 11, 23, 12, 21, 22]. When dealing with the digestion of wastewater, it is recognized that the limiting step is the methanoganesis. In such a case, modelling include one-, two- or three-step models. Of particular interest is the model by Bernard et al., 2001 ([6]) which proposes to model the anaerobic digestion process as a two-step process involving both the acidogenesis (using a Monod kinetics) and the methanogenesis processes (using a Haldane kinetics). This model, for which the mathematical analysis has been conducted by Benyahia et al., 2012, cf. [5], is very popular, notably for control purposes, since it remains of moderate complexity while being quite easy to calibrate to predict process behaviour with satisfying performances, [9, 2] or still [15]. It has also been the basis for proposing a systematic way to link simple models to ADM1 predictions, cf. [14]. In this work, we consider the model proposed by M. Rouez [23]. He is considering the anaerobic process in two steps that are Hydrolysis and Methanogenesis. Such description is well suited as long as the acidogenesis is not the limiting step, which is the case for the digestion of solid waste, as considered in this paper. In landfills, there is always a part of the raw material that has no access to oxygen. In addition, we consider here mortality. When a biological process is running in continuous mode, it is the rule rather than the exception not to consider mortality terms, cf. for instance [16] or [17], and such term can have a great influence on the process behaviour, cf. for instance [13]. However, when working in a closed environment - as it is the case in landfills - the mortality can no longer be neglected: part of the mortality of microorganism then returns to the slowly biodegradable material, itself being further hydrolyzed in rapidly biodegradable material. The role of the mortality term and the growth function on the performances of the overall process, notably in terms of biogas production, is of primer importance in landfill applications. The main objective of the present paper is to give deeper insights and predictions of this role with the help of a mathematical model.

The paper is organized as follows. In the next section, model and assumptions are presented and discussed. Then, the dynamics is mathematically analyzed in the following section. In particular, we show that there exists an infinity of equilibria to which solutions converge, depending on initial conditions. Finally, numerical simulations with different value of the death rate and various initial conditions are presented in a dedicated section, before discussions and conclusions are drawn.

2. **Model and assumptions.**

The hydrolysis step of transformation of organic matter of concentration is an important phase in the biodegradation process because it is a substrate preparation step. It is modeled by the first order equation

$$\frac{dX}{dt} = -K_h X \tag{1}$$

where $X(t)$ is the concentration of organic matter at time t and K_h is the hydrolysis constant.

On the other hand, the methanogenesis is the last step in the anaerobic digestion process and leads to the production of biogas. Based on the principle of mass conservation and the fundamental relations of biological kinetic (that are the growth

rate of bacteria and the use rate of the substrate), this step is modeled as follow

$$\begin{cases} \dfrac{dS}{dt} = -\dfrac{1}{Y}\mu(S)B \\ \dfrac{dB}{dt} = (\mu(S) - K_d)\, B \\ \dfrac{d[CO_2]}{dt} = (1-f_2)\dfrac{1-Y}{Y}\mu(S)B \\ \dfrac{d[CH_4]}{dt} = f_2\dfrac{1-Y}{Y}\mu(S)B \end{cases} \tag{2}$$

where $S(t)$ and $B(t)$ represent the concentrations of soluble organic matter and methanogenic biomass at time t. The symbols μ, Y, and K_d are successively the specific growth rate, the rate of use of the substrate and the mortality rate. Parameters f_2 and $(1-f_2)$ are the stoichiometric coefficients, representing the parts of soluble organic matter transformed into carbon dioxide and methane during methanogenesis step, of cumulative concentrations at time t denoted by the variables $[CO_2](t)$ and $[CH_4](t)$.

In the present work, we consider that during the process the death of methanogenic bacteria constitute a substrate (given by a proportion $\alpha K_d B$ of the death biomass, where α is a constant parameter) to the hydrolysis step, which brings a source term in the equation (1). This means that only a fraction of the biomass mortality is used again as a substrate in the methanogenesis step, as described in [6, 4, 10]). This couples the hydrolysis and methanogenesis steps as represented in the model below, which makes the mathematical analysis of the model not straightforward. In addition, we introduce the stoichiometric coefficients f_1 and $(1-f_1)$ to represent the parts of biomass X transformed into organic matter S and carbon dioxide CO_2 during the "hydrolysis / acidogenesis" process. The overall process is depicted on Figure 1.

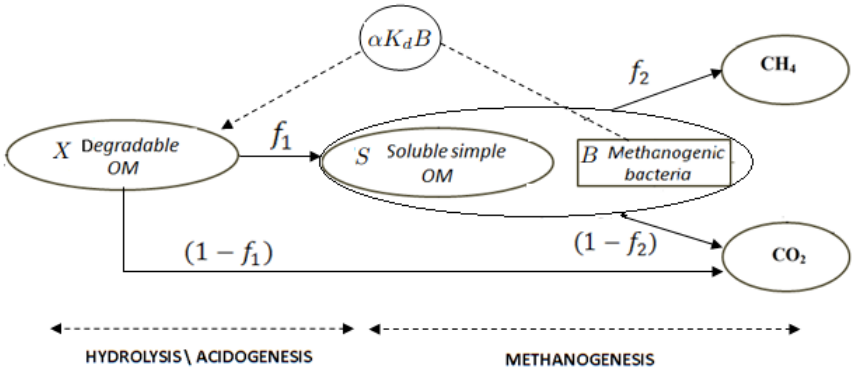


FIGURE 1. Overall scheme of the anaerobic degradation of organic matter

It is modeled by the following dynamic system (that was already proposed by Rouez in [23] but for a specific function μ and coefficient $\alpha = 1$).

$$\begin{cases} \dfrac{dX}{dt} = -K_h X + \alpha K_d B \\ \dfrac{dS}{dt} = f_1 K_h X - \dfrac{1}{Y}\mu(S)B \\ \dfrac{dB}{dt} = (\mu(S) - K_d)\, B \end{cases} \tag{3a}$$

$$\begin{cases} \dfrac{d[CO_2]}{dt} = (1-f_1)K_h X + (1-f_2)\dfrac{1-Y}{Y}\mu(S)B \\ \dfrac{d[CH_4]}{dt} = f_2 \dfrac{1-Y}{Y}\mu(S)B \end{cases} \tag{3b}$$

The total biogas $G(t)$ produced by the process at time t is the sum of carbon dioxide $[CO_2](t)$ and methane $[CH_4](t)$. Notice that in absence of initial substrate S or initial biomass B, the model (2) does not produce any biogas. We shall see that this is not the case for model (3), provided that there is initial matter X.

The specific growth rate function satisfy the following properties.

Hypothesis 1. The function μ is C^1 with $\mu(0) = 0$ and $\mu(S) > 0$ for $S > 0$.

Many functions that satisfy this hypothesis are met in the literature. The most popular ones are (see Figure 2) :

1. the Monod law [18], which is related to a growth saturation or limitation :
$$\mu(S) = \frac{\mu_{max} S}{K_S + S} \tag{4}$$
where μ_{max} is the maximum growth rate and K_S the half-saturation constant.
2. the Haldane law [1], which is also characterized by an inhibition phenomenon for large values of the substrate concentration:
$$\mu(S) = \frac{\mu_m S}{K_S + S + \frac{S^2}{K_I}} \tag{5}$$
where K_I is the inhibition constant.

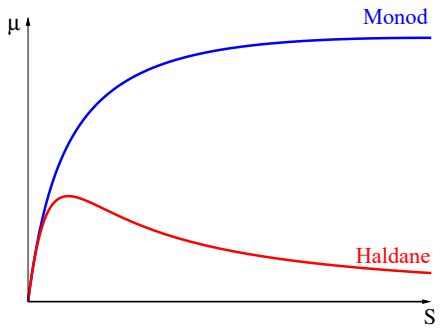


FIGURE 2. Graphs of Monod and Haldane functions

The Haldane expression is often considered to be more appropriate to the anaerobic process. Notice however that the Haldane function can be seen as a generalization of the Monod one on a fixed interval for large values of K_I.

Hypothesis 2. Coefficients α, K_d, Y, f_1 and f_2 fulfill the following conditions.

1. The proportion of nutrient recycling α cannot exceed 1
$$0 < \alpha \leq 1 \tag{6}$$
2. The mortality rate K_d is a positive parameter which is below the maximum growth rate
$$0 < K_d < \max_s \mu(s) \tag{7}$$
3. The rate of use of the substrate is a strictly positive parameter such that
$$0 < Y < 1 \tag{8}$$
4. The stoichiometric coefficients parameters f_1 and f_2 are strictly positive and satisfy
$$0 < f_1 < 1 \quad \text{and} \quad 0 < f_2 < 1 \tag{9}$$

Condition (7) means that the choice of bacteria and operating conditions, which impact the growth and death rates (such as temperature, pH...), are such that the bacterial growth is possible. For convenience, we define the set

$$\mathcal{E} := \{s \in \mathbb{R}_+ \, ; \, \mu(s) \leq K_d\}$$

which has non-empty interior, under Hypotheses 1 and 2. For the Monod expression, one has

$$\mathcal{E} = [0, \lambda] \text{ with } \lambda = \frac{K_d K_S}{\mu_{max} - K_d}$$

and for the Haldane expression, the set $\mathcal{E}$ has two connected components:

$$\mathcal{E} = [0, \lambda^-] \cup [\lambda^+, +\infty) \text{ with } \lambda^{\pm} = \frac{\mu_m - K_d \pm \sqrt{\Delta}}{2K_d/K_I} \tag{10}$$

where

$$\Delta = \mu_m^2 - 2\mu_m.K_d + \left(1 - 4\frac{K_S}{K_I}\right).K_d^2$$

which is positive under Hypothesis 2.

3. Study of the asymptotic behavior.

The dynamics (3) has a cascade structure. We thus begin by the study of the asymptotic behavior of the sub-system (3a).

Proposition 1. *Under Hypotheses 1, 2, for any non-negative vector (X_0, S_0, B_0), the solutions of system (3a) for the initial condition $(X(0), S(0), B(0)) = (X_0, S_0, B_0)$ are non-negative, bounded and verify*

$$\lim_{t \to +\infty} X(t) = \lim_{t \to +\infty} B(t) = 0$$

and

$$\lim_{t \to +\infty} S(t) = S^\star \leq S_0 + \frac{X_0 + \alpha B_0}{\alpha Y}$$

Moreover, when X_0 or S_0 is non null, one has $S^\star > 0$.

Proof. Let us first show that for any initial condition (X_0, S_0, B_0) in $\mathbb{R}^3_+$, the forward solution $(X(t), S(t), B(t))$ remains in the non-negative orthant $\mathbb{R}^3_+$.

- If $B_0 = 0$, then the solution verifies $B(t) = 0$ for any $t > 0$, whatever are X_0, S_0. Therefore, by uniqueness of solutions of the Cauchy problem, a solution $B(\cdot)$ cannot cross the plane $B = 0$ neither reaches 0 in finite time. Then one has $B(t) \geq 0$ whatever is (X_0, S_0, B_0) in $\mathbb{R}^3_+$.
- From equations (3a), any solution with (X_0, S_0, B_0) in $\mathbb{R}^3_+$ verifies $\frac{dX(t)}{dt} \geq -K_h X(t)$ for any $t \geq 0$, from which one deduces the inequality $X(t) \geq X_0 e^{-K_h t}$ for any $t \geq 0$. This proves that the solution $X(t)$ is non-negative for any positive t, and cannot reach 0 in finite time, whatever is (X_0, S_0, B_0) in $\mathbb{R}^3_+$.
- Finally, from equations (3a), any solution with (X_0, S_0, B_0) in $\mathbb{R}^3_+$ verifies $\frac{dS(t)}{dt} \geq -\frac{1}{Y}\mu(S(t))B(t)$ for any $t \geq 0$. By comparison of solutions of scalar ordinary differential equations (see e.g. [24]), one has $S(t) \geq \underline{S}(t)$ for any $t > 0$, where $\underline{S}$ is solution of the differential equation $\frac{d\underline{S}}{dt} = -\frac{1}{Y}\mu(\underline{S})B(t)$ with $\underline{S}(0) = S_0$. As $\mu(0) = 0$, $\underline{S}(t) = 0$ for any $t > 0$ is solution when $S_0 = 0$. By uniqueness of the solutions (the function μ being Lipschitz continuous), a solution $\underline{S}(\cdot)$ cannot cross the plane $\underline{S} = 0$ neither reaches 0 in finite time. We deduce that $S(t)$ is also non-negative for any $t > 0$, and cannot reaches 0 in finite time, whatever is (X_0, S_0, B_0) in $\mathbb{R}^3_+$.

We show now the system (3a) is dissipative. Consider the "storage" function

$$V(t) = X(t) + \alpha B(t) + \alpha Y S(t). \tag{11}$$

From equations (3a), one has

$$\frac{dV}{dt} = K_h(\alpha Y f_1 - 1)X. \tag{12}$$

By Hypothesis 2, one has $\alpha Y f_1 < 1$ and as $X(\cdot)$ is non-negative, we deduce that V is non-increasing. Being bounded from below by 0, $V(t)$ converges to a limit $V_\infty \geq 0$ when t tends to $+\infty$. This implies that the variables $X(t)$, $S(t)$ and $B(t)$ are bounded. Moreover, one has

$$\frac{d^2V}{dt^2} = K_h(\alpha Y f_1 - 1)(-K_h X + \alpha K_d B)$$

from which we deduce that d^2V/dt^2 is bounded and thus dV/dt is uniformly continuous on $\mathbb{R}_+$. By Barbalat's Lemma [3], we obtain that $dV(t)/dt$ converges to 0 when t tends to $+\infty$. Therefore, we obtain

$$\lim_{t\to+\infty} X(t) = 0.$$

Similarly, d^2X/dt^2 is bounded, thus dX/dt uniformly continuous and by Barbalat's Lemma, $dX(t)/dt$ converges to 0 when t tends to $+\infty$, which gives

$$\lim_{t\to+\infty} B(t) = 0.$$

Finally, as $V(t)$ converges to a limit when t tends to $+\infty$, we obtain

$$\lim_{t\to+\infty} S(t) = S^\star = \frac{V_\infty}{\alpha Y}.$$

Let us show that $S^\star$ cannot be equal to 0 when X_0 or S_0 is non null. If not, $S(t)$ has to tends to zero and therefore there exists $T > 0$ such that $\mu(S(t)) \leq f_1 Y \alpha K_d$

for any $t > T$. Then one has

$$\frac{d}{dt}\left(f_1 X(t) + S(t)\right) \geq 0, \quad t > T.$$

When $f_1 X(0) + S(0) > 0$, the variable $X(\cdot)$ or $S(\cdot)$ cannot reaches 0 in finite time, as shown previously, and one has then $f_1 X(t) + S(t) \geq f_1 X(T) + S(T) > 0$ for any $t > T$, which is in contradiction with the fact that $f_1 X(t) + S(t)$ tends to 0 when t tends to $+\infty$.

Finally, from the fact that V is non increasing, we obtain

$$V_\infty = \alpha Y S^\star \leq V(0) = X_0 + \alpha B_0 + \alpha Y S_0 \;\Rightarrow\; S^\star \leq S_0 + \frac{X_0 + \alpha B_0}{\alpha Y}\ .$$

□

Let us now characterize the set of equilibria that can be reached by positive solutions.

Proposition 2. *Under Hypotheses 1, 2, the solution of (3a) for an initial condition in $\mathbb{R}^3_+$ with $X_0 > 0$ and $B_0 > 0$ converges asymptotically to an equilibrium $(0, S^\star, 0)$ where $S^\star > 0$ belongs to $\mathcal{E}$.*

Proof. If there exists a solution of (3a) such that $S(\cdot)$ converges asymptotically to $S^\star > 0$ that does not belong to $\mathcal{E}$, there exists $T > 0$ such that one has

$$\mu(S(t)) - K_d > \eta := \frac{\mu(S^\star) - K_d}{2}, \quad t > T,$$

but then the solution $B(\cdot) > 0$ verifies

$$\frac{dB(t)}{dt} > \eta B(t), \quad t > T.$$

With standard comparison theorem for differential inequalities (see e.g. [24]), we conclude that $B(\cdot)$ cannot converge asymptotically to 0, and thus a contradiction with the results of Proposition 1. □

The system (3a) admits a continuum of equilibria which are not hyperbolic. One cannot conclude about their stability by studying the single linearization of the dynamics. The Center Manifold Theorem [7] could be used but it turns out to be not enough informative for our problem. However, we obtain the following result.

Proposition 3. *Assume that Hypotheses 1 and 2 are fulfilled. For each steady state $E = (0, S^\star, 0)$ with $S^\star \in \operatorname{int}\mathcal{E}$, there exists an invariant two-dimensional manifold $\mathcal{M}$ in $\mathbb{R}^3_+$ such that any solution of (3a) with initial condition in $\mathcal{M}$ converges asymptotically to E.*

Proof. Let us fix $S^\star > 0$ such that $\mu(S^\star) < K_d$. For solutions with $B(t) > 0$, $t \in [0 + \infty)$, consider the variable

$$Z(t) = \frac{f_1 X(t) + (S(t) - S^\star)}{B(t)} + \beta$$

with

$$\beta := \frac{\mu(S^\star)/Y - f_1 \alpha K_d}{\mu(S^\star) - K_d}.$$

Then, a direct computation gives

$$\frac{dZ}{dt} = -\gamma(\mu(S) - \mu(S^\star)) - (\mu(S) - K_d)Z,$$

with

$$\gamma := \frac{K_d(1 - Y f_1 \alpha)}{Y(K_d - \mu(S^\star))} > 0.$$

We can then write the system (3a) in $\mathbb{R}^2_+ \times \mathbb{R}^\star_+$ equivalently on the domain

$$\mathcal{D} := \{(Z, X, B) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^\star_+ \, ; \, S^\star - f_1 X + (Z - \beta)B \geq 0\}$$

as follows

$$\left\{ \begin{array}{rcl} \dfrac{dZ}{dt} & = & -\gamma(g(Z, X, B) - \mu(S^\star)) - (g(Z, X, B) - K_d)Z \\ \dfrac{dX}{dt} & = & -K_h X + \alpha K_d B \\ \dfrac{dB}{dt} & = & (g(Z, X, B) - K_d)B \end{array} \right. \tag{13}$$

with

$$g(Z, X, B) := \mu(S^\star - f_1 X + (Z - \beta)B).$$

One can check that the domain $\mathcal{D}$ is (positively) invariant by the dynamics (13). Moreover, the dynamics (13) is well defined for $B = 0$ and regular on the set $\overline{\mathcal{D}}$ (which is also invariant). Any trajectory $(Z(\cdot), X(\cdot), B(\cdot))$ in $\mathcal{D}$ matches a trajectory $(X(\cdot), S(\cdot), B(\cdot))$ of (3a) where

$$S(t) = S^\star - f_1 X(t) + (Z(t) - \beta)B(t), \quad \forall t \geq 0.$$

However, trajectories of (13) with $B(\cdot) = 0$ does not necessarily match a trajectory of (3a). As $\mu(S^\star) \neq K_d$, one can check that 0 is the only equilibrium of (13) in $\overline{\mathcal{D}}$. The Jacobian matrix at 0 is given by

$$J(0) = \begin{bmatrix} -(\mu(S^\star) - K_d) & \gamma f_1 \mu'(S^\star) & \gamma\beta\mu'(S^\star) \\ 0 & -K_h & \alpha K_d \\ 0 & 0 & \mu(S^\star) - K_d \end{bmatrix}.$$

Its eigenvalues are $-(\mu(S^\star) - K_d)$, $-K_h$, $\mu(S^\star) - K_d$. Under the condition $\mu(S^\star) < K_d$, 0 is thus an hyperbolic equilibrium with a two-dimensional stable manifold $\mathcal{S}$ and a one-dimensional unstable manifold $\mathcal{U}$ (see e.g. [19]). Clearly, one has $\mathcal{U} = \mathbb{R} \times \{0\} \times \{0\}$. Moreover, any solution of (13) with $B(0) = 0$ verifies $B(t) = 0$ for any $t > 0$ and $\lim_{t \to +\infty} X(t) = 0$, which implies that the function

$$h(t) := g(Z(t), X(t), B(t)) = \mu(S^\star - f_1 X(t))$$

converges to $\mu(S^\star)$ when $t \to +\infty$, whatever is $Z(0)$. Therefore, for any $Z(0)$, $X(0)$, there exists $T > 0$ such that

$$K_d - h(t) > \eta := \frac{1}{2}(K_d - \mu(S^\star)) > 0, \quad t > T$$

and then one obtains from (13) that $Z(t)$ tends to infinity when $t \to +\infty$. We deduce that any point $(Z, X, B) \in \mathcal{S} \setminus \{0\}$ has to be such that $B \neq 0$. Finally, we conclude that any trajectory of (13) in the two-dimensional invariant manifold $\mathcal{M} := \mathcal{S} \cap \mathcal{D}$ matches a trajectory of (3a) and converges asymptotically to 0, that is $(X(t), S(t), S(t))$ converges asymptotically to $(0, S^\star, 0)$.

□

Propositions 2 and 3 together allow to state the following result.

Corollary 1. *Under Hypotheses 1 and 2, the set* $\{0\}\times\mathcal{E}\times\{0\}$ *is a forward attractor of the dynamics* (3a).

Finally, let us consider the sub-system (3b).

Proposition 4. *Under Hypotheses 1, 2, for any non-negative vector* (X_0, S_0, B_0)*, the solutions of* (3) *verify*

$$\lim_{t\to+\infty}[CO_2](t) - [CO_2](0) = a(X_0 + \alpha B_0) + b(S_0 - S^\star) \tag{14}$$

$$\lim_{t\to+\infty}[CH_4](t) - [CH_4](0) = c(X_0 + \alpha B_0) + d(S_0 - S^\star) \tag{15}$$

where $S^\star$ *is the asymptotic value of* $S(\cdot)$ *and the coefficients* a*,* b*,* c*,* d *are positive numbers given by the following expressions.*

$$a = \frac{1 - f_1 + f_1(1-f_2)(1-Y)}{1-\alpha Y f_1}, \quad b = \frac{(1-f_1)\alpha Y + (1-f_2)(1-Y)}{1-\alpha Y f_1}$$

$$c = \frac{f_1 f_2 (1-Y)}{1-\alpha Y f_1}, \qquad d = \frac{f_2(1-Y)}{1-\alpha Y f_1}.$$

Proof. Consider again the function V defined in (11). Equation (12) gives

$$V(t) = V(0) + K_h(\alpha Y f_1 - 1)\int_0^t X(\tau)\,d\tau.$$

As $V(\cdot)$ is bounded, we obtain

$$\int_0^t X(\tau)\,d\tau < +\infty. \tag{16}$$

From the first equation of (3a), we obtain also

$$X(t) = X(0) - K_h\int_0^t X(\tau)\,d\tau + \alpha K_d\int_0^t B(\tau)\,d\tau$$

and with (16), we obtain

$$\int_0^t B(\tau)\,d\tau < +\infty. \tag{17}$$

Then, from the last equation of (3a), one has

$$B(t) = B(0) + \int_0^t \mu(S(\tau))B(\tau)\,d\tau - K_d\int_0^t B(\tau)\,d\tau,$$

from which we get with (17)

$$\int_0^t \mu(S(\tau))B(\tau)\,d\tau < +\infty. \tag{18}$$

We conclude, from equations (3b), that $[CO_2](t)$ and $[CH_4](t)$ converges asymptotically to finite values.

The integration of equations (3a) between $t = 0$ and $t = +\infty$ leads to the following system of equations

$$K_h\int_0^{+\infty} X(\tau)\,d\tau - \alpha\int_0^{+\infty}\mu(S(\tau))B(\tau)\,d\tau = X_0 + \alpha B_0$$

$$Y f_1 K_h\int_0^{+\infty} X(\tau)\,d\tau - \int_0^{+\infty}\mu(S(\tau))B(\tau)\,d\tau = Y(S^\star - S_0)$$

whose solution is unique, given by the expressions

$$\int_0^{+\infty} X(\tau)\,d\tau = \frac{X_0 + \alpha Y(S_0 - S^\star) + \alpha B_0}{K_h(1-\alpha Y f_1)}$$

$$\int_0^{+\infty} \mu(S(\tau))B(\tau)\,d\tau = Y\frac{f_1 X_0 + (S_0 - S^\star) + \alpha f_1 B_0}{1-\alpha Y f_1}$$

because $1-\alpha Y f_1 > 0$ under Hypothesis 2, which provide the formulae (14), (15). □

Remark 1. One can check that when $B_0 = 0$, there is no production of methane (as expected) because the trajectory stays on the half line

$$\{(X,S,B)\,;\ f_1 X + S = f_1 X_0 + S_0,\ B = 0\}$$

and one has then $cX_0 + d(S_0 - S^\star) = 0$. The production of dioxide is then equal to $(a+\alpha b)X_0$ whatever is S_0.

When the set $\mathcal{E}$ is not connected, as an union of disjoint intervals $\mathcal{E} = I_1 \cup I_2 \cup \cdots$, the state space can be split into a family $\{\mathcal{B}_i\}$ of attraction basins of the subsets $\{0\} \times I_i \times \{0\}$. These basins conduct the system to different levels of performances. For instance, with the Haldane law, there are two basins $\mathcal{B}^-, \mathcal{B}^+$ leading to equilibria with $S^\star$ in $[0, \lambda^-]$ or in $[\lambda^+, +\infty)$. The separating surface $\mathcal{S}$ of $\mathcal{B}^-$ and $\mathcal{B}^+$ is numerically investigated in the next section.

4. **Application.** Proposition 4 shows that the production of the total biogas G is impacted by the final value $S^\star$ of the remaining soluble matter, which is itself related to the death rate K_d (as $S^\star$ belongs to the set $\mathcal{E}$).

For the Haldane growth function, the process leads either to a relatively large production of biogas (when $S^\star < \lambda^-$) or to a relatively small production of it (when $S^\star > \lambda^+$), depending on the initial condition. The difference between these two situations get larger when the death rate K_d is small, differently to the Monod case which is more robust.

Because the Haldane function is often more realistic (and the Monod function can be seen as a particular case for large values of the parameter K_I), we present here simulations only for the Haldane one, with $\mu_m = 0.3$, $K_S = 160$ and $K_I = 10$ and other model parameters reported in the following Table.

TABLE 1.

K_h	Y	f_1	f_2	α
0.176	0.05	0.7	0.76	0.9

With these parameters, the corresponding Haldane function is plotted in Figure 3.

Let us fix the death rate $K_d = 0.02$. For such value, we obtain $\lambda^- = 12.5$ and $\lambda^+ = 127.4$ (see equations (10)). Consider $S_0 = 0$ and $B_0 = 2$. Depending on the initial condition of X, S converges either towards a value smaller than λ^- or larger than λ^+. Such a behaviour can be seen with five initial conditions of X in Figure 4 ($X_0 = 340$ to $X_0 = 360$ with an increment of 5). As expected, there is a sudden change on the asymptotic value of S when X_0 passes a threshold: for $X_0 = 340$,

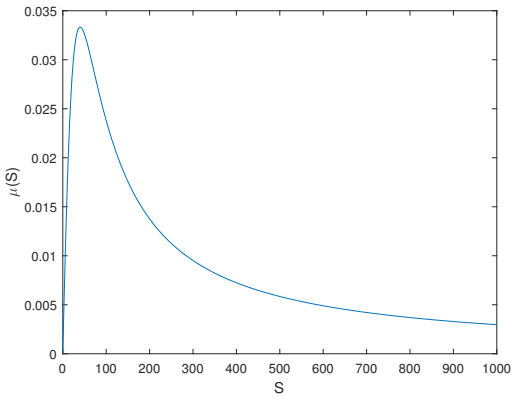


FIGURE 3. Graph of the Haldane function considered in the example

345 and 350, S converges towards values smaller than λ^- while it converges towards values larger than λ^+ for $X_0 = 355$ and 360.

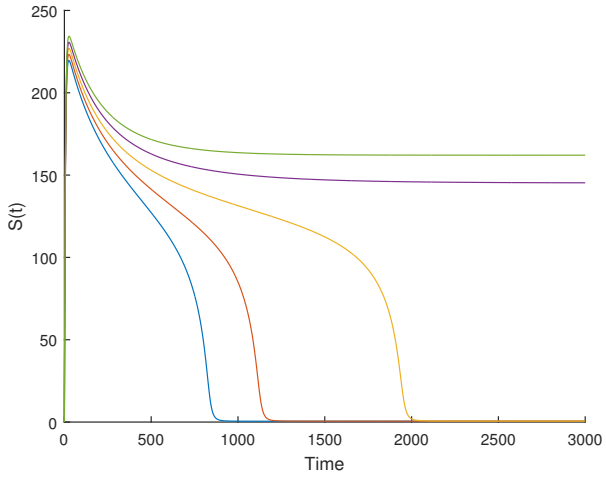


FIGURE 4. Graph of $S(\cdot)$ for initial conditions $X_0 = 340$ to $X_0 = 360$ with a step of 5

As K_d increases, λ^- increases while λ^+ decreases and for very small values of K_d, λ^+ may be very large. In Figure 5, we plotted the value of the threshold on X_0 for which the asymptotic value $S^\star$ is under λ^- or over λ^+ for a range of values of K_d between 0.005 and 0.03. One can observe that it is a decreasing function of the X_0 threshold.

The biogas production can be seen as an increasing function of X_0 as long as S converges under λ^- (since asymptotically there remains a low amount of substrate) and an increasing one as soon as the threshold on X_0 is crossed over (when S converges over λ^+). In Figure 6, we plotted the total production of biogas production

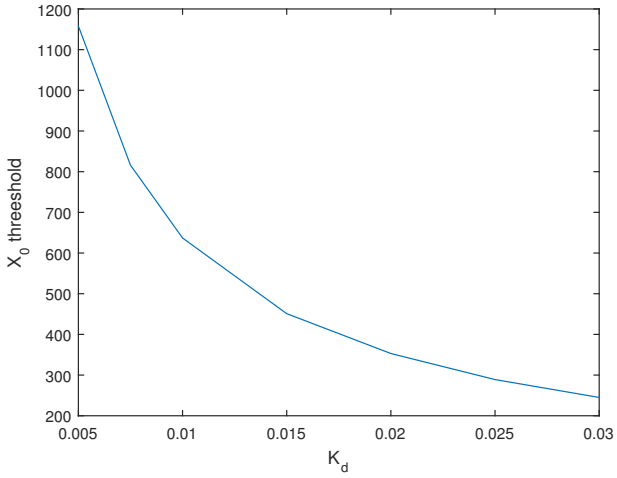


FIGURE 5. Threshold on X_0 as a function of K_d for model parameters indicated in Table 1

for $K_d = 0.02$ as a function of X_0 from 300 to 400 (recall that in this case the threshold value for X_0 is about 357), which shows the discontinuity.

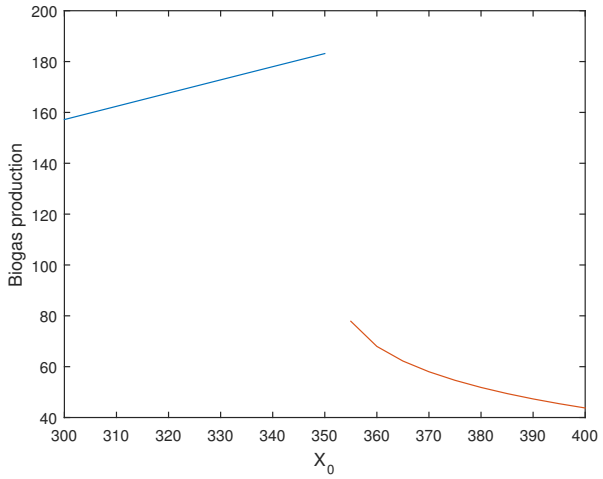


FIGURE 6. Biogas production as a function of X_0 for $K_d = 0.02$

Finally, we have plotted in Figure 7 the values λ^-, λ^+ as functions of K_d, which show the distance between the two attractors. In addition, the biogas productions obtained on both sides of the switching value of X_0, denoted by Biogas$^-$ and Biogas$^+$, have been plotted. As expected, for small values of K_d, the difference between the two extremes values of the biogas productions are high.

This study reveals interesting insights for practitioners.

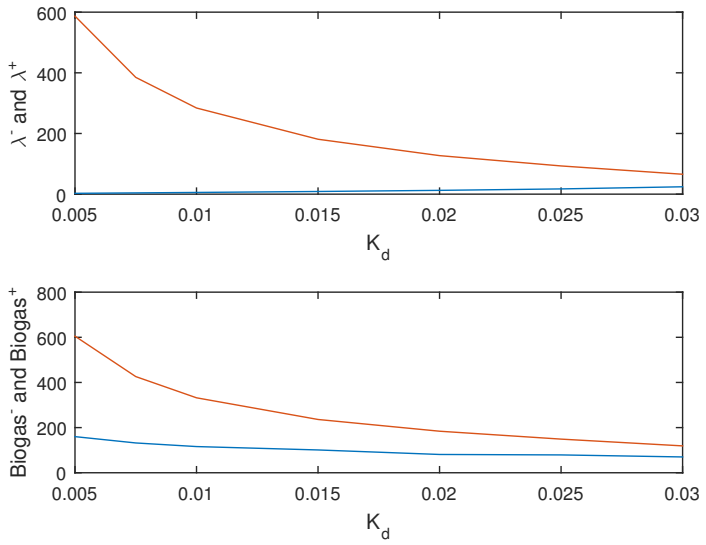


FIGURE 7. λ^-, λ^+ (on top) Biogas$^-$ and Biogas$^+$ (on bottom) as functions of K_d

1. When S_0 is small (or null), considering that an inhibition cannot occur because it concerns only high values of $S^\star$, and adopting a Monod function instead of an Haldane one (or a large value of K_I), could lead to wrong predictions. The biogas production could be poor under high initial concentration of organic matter X_0, because it conducts the system to large asymptotic values of $S^\star$ (see Figures 4 and 6).
2. When the death rate K_d is high (but not too much to fulfill condition (7)), the performances on biogas production are quite sensitive to the initial load of organic matter X_0, because the threshold on X_0 is low (see Figure 5).
3. When the death rate K_d is low, the system has good performances on the condition that the initial load X_0 does exceed the threshold, which is relatively large. However, if it is exceeded, the performances collapse dramatically (see Figure 7).

5. **Conclusion.** With the help of Barbalat's Lemma and the Stable and Unstable Manifolds Theorem, we have shown that each trajectory of the system is bounded and converges to one of the non-hyperbolic equilibria. For non-monotonic growth functions, such as the Haldane function, the global attractor is non-connected. In this case, we have shown that the performances in terms of biogas production is discontinuous with respect to the initial condition.

For the Haldane law, a too high initial load of organic matter could be penalizing because there could be a significant quantity of residual soluble matter, especially when the death rate is small. A strategy could be then to fractionate the load of organic matter over the time, i.e. have a smaller initial load and re-introduce the remaining quantity of matter to be treated later (in one or several times). This leads to a control problem for choosing optimally the proportion of splitting and the

corresponding re-introduction time(s) to obtain the best performances. This could be the matter of a future work. Another possibility is to play with the recirculation rate (in a completely mixed system), as in [20].

Moreover, the introduction of spatial heterogeneity in the model, in terms of a system of reaction diffusion p.d.e. instead of a system of o.d.e. would be a reasonable extension of the present work, taking into consideration that spatial heterogeneity is usually observed in real landfills. One expects to find an infinite number of stationary solutions, as for the o.d.e. model. A relevant question of interest for practitioners would be then to study the effect of the diffusion on the performances of the biogas production, in particular for the case of the Haldane growth function.

Acknowledgments. The authors are gratefully for the French-Moroccan exchange program "PHC TOUBKAL 17/47".

REFERENCES

[1] J. Andrews, *A mathematical model for the continuous culture of microorganisms utilizing inhibitory substrates*, Biotechnology and Bioengineering, 10(6) (1968), 707–723.

[2] J. Arzate, M. Kirstein, F. Ertem, E. Kielhorn, H. Malule, P. Neubauer, M. Cruz-Bournazou and S. Junne, *Anaerobic Digestion Model (AM2) for the Description of Biogas Processes at Dynamic Feedstock Loading Rates*, Chemie Ingenieur Technik, 89(5) (2017) 686–695.

[3] I. Barbalat, *Systèmes d'équations différentielles d'oscillations non linéaires*, Rev. Math. Pures Appl. 4 (1959), 267-270.

[4] G. Bastin and D. Dochain., *On-line estimation and adaptive control of bioreactors, Dynamics of Microbial Competition* Elsevier Science Publishers, New-York (1991).

[5] B. Benyahia, T. Sari, B. Cherki and J. Harmand, *Bifurcation and stability analysis of a two step model for monitoring anaerobic digestion processes*, Journal of Process Control, 22(6) (2012), 1008–1019.

[6] O. Bernard, Z. Hadj-Sadok, D. Dochain, A. Genovesi and J.P. Steyer, *Dynamical model development and parameter identification for an anaerobic wastewater treatment process.* Biotechnology and bioengineering, 75(4) (2001), 424–438.

[7] J. Carr, *Applications of Centre Manifold Theory*, Springer-Verlag (1981).

[8] D. Chenu, *Modélisation des transferts réactifs de masse et de chaleur dans les installations de stockage de déchets ménagers: application aux installations de type bioréacteur.* PhD thesis, Institut National Polytechnique de Toulouse, France (2007).

[9] I. Didi, H. Dib and B. Cherki, *A Luenberger-type observer for the AM2 model*, Journal of Process control, 32 (2015), 117–126.

[10] D. Dochain, *Automatic Control of Bioprocesses Control systems* , John Wiley and Sons (2010).

[11] G. Dollé, O. Duran, N. Feyeux, E. Frénod, M. Giacomini and C. Prud'Homme, *Mathematical modeling and numerical simulation of a bioreactor landfill using Feel++.* ESAIM: Proceedings and Surveys, 55 (2016), 83–110.

[12] R. Fekih-Salem, *Modèles Mathématiques pour la compétition et la coexistence des espèces microbiennes dans un chémostat.* PhD thesis, Univ. Montepllier, France (2013).

[13] J. Harmand, C. Lobry, A. Rapaport and T. Sari, *The Chemostat: mathematical theory of microorganisms cultures*, ISTE Wiley (2017).

[14] S. Hassam, E. Ficara, A. Leva, J. Harmand, *A generic and systematic procedure to derive a simplified model from the Anaerobic Digestion Model* No. 1 (ADM1), Biochemical Engineering Journal 99 (2015), 193–203.

[15] M. Hmissi, J. Harmand, V. Alcaraz-Gonzalez and H. Shayeb, *Evaluation of alkalinity spatial distribution in an up-flow fixed bed anaerobic digester*, Water Science and Technology, 77(4) (2018), 948–959.

[16] S.J. Jang and J. Baglama, *Nutrient-plankton models with nutrient recycling.* Computers and Mathematics with Applications, 49(2-3) (2005), 375–387.

[17] M. Loreau, *Material cycling and the stability of ecosystems.* The American Naturalist, 143(3) (1994), 508–513.

[18] J. Monod, *La technique de la culture continue: Théorie et applications*, Ann. Inst. Pasteur, Lille, 79 (1950), 390–410.
[19] L. Perko, *Differential Equations and Dynamical Systems*, Springer, 3rd ed. (2011).
[20] A. Rapaport, T. Bayen, M. Sebbah, A. Donoso-Bravo and A. Torrico, *Dynamical modelling and optimal control of landfills*, Mathematical Models and Methods in Applied Sciences, 26(5) (2016), 901–929.
[21] A. Rapaport, T. Nidelet, S. El Aida and J. Harmand. *About biomass overyielding of mixed cultures in batch processes*. Prepint hal (2019).
[22] A. Rapaport, T. Nidelet and J. Harmand, *About biomass overyielding of mixed cultures in batch processes*, 8th IFAC Conference on Foundations of Systems Biology in Engineering (FOSBE), Valencia, Spain, 15-18 Oct. (2019).
[23] M. Rouez, *Dégradation anaérobie de déchets solides: Caractérisation, facteurs d'influence et modélisations*. PhD thesis, Institut National des Sciences Appliquées, Lyon, France (2008).
[24] W. Walter, *Ordinary Differential Equations*, Springer (1998).

Received xxxx 20xx; revised xxxx 20xx.

E-mail address: salih.ouchtout@gmail.com
E-mail address: zoubida.mghazli@uit.ac.ma
E-mail address: jerome.harmand@inra.fr
E-mail address: alain.rapaport@inra.fr
E-mail address: zakaria.belhachmi@uha.fr