Convexity Preserving Contraction of Digital Sets[*]

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Abstract. Convexity is one of the useful geometric properties of digital sets in digital image processing. There are various applications which require deforming digital convex sets while preserving their convexity. In this article, we consider the contraction of such digital sets by removing digital points one by one. For this aim, we use some tools of combinatorics on words to detect a set of removable points and to define such convexity-preserving contraction of a digital set as an operation of re-writing its boundary word. In order to chose one of removable points for each contraction step, we present three geometrical strategies, which are related to vertex angle and area changes. We also show experimental results of applying the methods to repair some non-convex digital sets, which are obtained by rotations of convex digital sets.

Keywords: Digital convexity · Digital set contraction · Christoffel words · Lyndon words

1 Introduction

Convexity is one of the useful geometric properties of digital sets in digital image processing. There are various applications which require deforming digital convex sets while preserving their convexity. In this article, we consider the contraction of such digital sets by removing digital points one by one. For this aim, we use some tools of combinatorics on words to detect a set of removable points and to define such convexity-preserving contraction of a digital set as an operation of re-writing its boundary word.

A relation between combinatorics on words [16] and digital geometry [15] has afforded many advantages to both areas and has led to interesting results. Indeed, such a connection realizes through the so-called Freeman coding, introduced by Freeman in 1961 [13], that allows to uniquely determine a 4- or 8-connected finite set of points in the discrete plane by means of its boundary word, i.e. a word over the alphabet of cardinal four $A = \{0, 1, 2, 3\}$. This code is the bridge between these two worlds. In this article, we deal with one of the main notions of convexity for polyominos presenting in literature: digital convexity. During the

last decades, many definitions were given; it started with Minsky and Papert in 1969 [18] and then Hubler (further notions and related studies can be found in [4, 5, 9, 10]). A subset C in a digital image is convex if the straight line segment joining any two pixels P and Q of C lies entirely in C. We also present another notion of digital convexity with a help of the convex hull of C. In this article, we rely on a recent result by Berleck and al. [3], who have defined a set to be digitally convex if and only if the Lyndon factorization of its boundary word is made of Christoffel words. Based on this definition, we show how we can deflate a digitally convex set by choosing correct digital points to remove and preserving the convexity at the same time. We can find some previous studies on inflating digitally convex sets [11]. One of the main contributions of this article is to give three different strategies that allow us to determine an order to follow in removing digital points.

This article is decomposed into four parts. We first introduce digital convexity and give some related notions about two families of words. Then, we show the link between digital convexity and combinatorics on words. In the second part, we do modifications on the border of a digital convex set, and show all the perturbations obtained after removing a digital point. The third part is for presenting three different techniques in choosing a certain order for deflating digital convex sets. The last part is dedicated for showing experimental results, where we apply our methods to a rotated digital convex set, which is not convex anymore, in order to repair its convexity by making the convex hull and deflating it by removing some digital points.

2   Digital convexity and combinatorics on words

In this section, we introduce the basic notions needed in the following sections. Precisely, we recall basic notions in combinatorics on words. Then we define digital convexity and two particular families in combinatorics on words, that are Christoffel and Lyndon words [1, 16].

2.1   Basic notions of Combinatorics

An **alphabet** A is a finite set of symbols such that its elements are called **letters**. A **word** w over an alphabet A is a finite sequence of letters over A. In another way, a word w is obtained by *concatenating* letters of A, we write $w \in A^*$, where A^* represents the set of all the words formed by A, with the **empty word** ϵ being the identity element for the concatenation. We denote $w^n \in A^*$ the concatenation of the word w, n times, such that: $w^n = w.w.w \cdots w$ where w is repeated n times with the convention of $w^0 = \epsilon$. A word w is said **primitive** if it is not the power of a nonempty word.

Let $w \in A^*$. The **length** of w is the number of letters of w denoted by $|w|$. Note that $|\epsilon| = 0$. For all $\ell \in A$, $|w|_\ell$ denotes the number of occurrences of the letter ℓ in the word w, so that: $|w| = \sum_{\ell \in A} |w|_\ell$.

Two words w and w' are ***conjugate*** of order k, with $k \neq 0$, denoted by $w \equiv_k w'$, if and only if there exist $u, v \in A^*$ such that $|u| = k$, $w = u.v$ and $w' = v.u$. When the exact value of k is not relevant, we simply write $w \equiv w'$. We can set several order relations over A^*. Here, we choose the total **lexicographic order**, or simply the *dictionary order* that is a total order and denoted by $<$.

Definition 1. *Let $w_1, w_2 \in A^*$, we say that:*

$$w_1 < w_2 \quad if \begin{cases} w_2 = w_1.w', \ where \ w' \in A^*; or \\ w_1 = u.a.v_1 \ and \ w_2 = u.b.v_2 \quad where \ u, v_1, v_2 \in A^* \ and \ a < b \in A. \end{cases}$$

2.2   Polyomino and boundary words

Let us consider the lattice $\mathbb{Z}^2$, which is the set of all the vectors having integer components. The canonical basis of the Euclidean vector space $\mathbb{R}^2$ is $\{e_1, e_2\}$. We call by a finite discrete set any finite subset of $\mathbb{Z}^2$. We define a path in $\mathbb{Z}^2$ from point X to point Y a sequence of points $(p_i)_{1 \leq i \leq n}$ with $p_i \in \mathbb{Z}^2$ where $p_1 = X$ and $p_n = Y$. A path in $\mathbb{Z}^2$ is 4-connected if for any $1 \leq i \leq n - 1$, $p_{i+1} - p_i \in \{\pm e_1, \pm e_2\}$. A *polyomino P* is a simply 4-connected finite set of digital points, without hole, that can be visualized as set of a unit squares. The Freeman chain code allows us to represent any path over $\mathbb{Z}^2$ as a word over an alphabet of four letters $A = \{0, 1, 2, 3\}$ [13] such that:

- 0 denotes a right horizontal step;
- 2 denotes a left horizontal step;
- 1 denotes an upside vertical step;
- 3 denotes a downward vertical step;

The boundary word of a polyomino P, denoted by $Bd(P)$, is the word obtained by coding its perimeter using the alphabet A. For each polyomino, we can have an equivalent class of words that are the conjugates to each other. We choose the representative of these words to be the one that starts from the left lowermost side of the polyomino.

In the last decades, many researchers [3] have studied the digital convexity and worked on finding the link between it and combinatorics on words. In the following two sections, we introduce two families of words, that are essential for this work and we introduce some results.

2.3   Christoffel words and digital line segments

The first family of words is *Christoffel words*. From the geometric point of view, we can consider any Christoffel word as the discretization of a line segment of rational slope [1, 2]. In this case, we restrict our work to the binary alphabet $A = \{0, 1\}$, and start by defining a Christoffel path which is a sequence of unitary steps joining two points of integer lattice. Let a and b be two coprime numbers, the *lower* Christoffel path of slope a/b is the path joining the origin $O(0,0)$ to the point (b, a) and respecting the following characteristics:

1. it is the nearest path below the line segment joining these two points;
2. there is no point of $\mathbb{Z}^2$ between the path and line segment.

Analogously, the *upper* Christoffel path is the path that lies above the line segment. By convention, the Christoffel path is exactly the *lower* Christoffel path. We remark that the *upper* Christoffel path of slope a/b is the image of the *lower* Christoffel path obtained by a rotation of angle $180°$ (see Figure 1 for example).

Using the binary alphabet $A = \{0, 1\}$, and by assigning to each horizontal (resp. vertical) step the letter 0 (resp. 1), we get the *Christoffel word* w of slope $\frac{a}{b}$, denoted by $C(\frac{a}{b})$, where the fraction $\frac{a}{b}$ is exactly $\frac{|w|_1}{|w|_0}$ (see Figure 2). We define the morphism $\rho : A^* \longrightarrow \mathbb{Q} \cup \{\infty\}$ by:

$$\rho(\epsilon) = 1 \text{ and } \rho(w) = \frac{|w|_1}{|w|_0}, \ \forall \ w \neq \epsilon \in A^*;$$

where $\frac{1}{0} = \infty$. This morphism determines the slope for each given word in A^*. In particular:

1. for $a = 0$, we get the Christoffel word of slope 0, $C(\frac{0}{1}) = 0$.
2. for $b = 0$, we have the Christoffel word of slope ∞, $C(\frac{1}{0}) = 1$.

Christoffel words have several properties and characterizations [1, 8, 16]. Here, we only mention a few of them. Let $\widetilde{w} = w_n \cdots w_2.w_1$ be the **reversal** of the word $w = w_1 w_2 \ldots w_n$; we say that w is a **palindrome** if $\widetilde{w} = w$.

Property 1. The Christoffel word of slope $\frac{a}{b} \neq 0, \infty$ can be written as: $C\left(\frac{a}{b}\right) = 0p1$, where p is a palindrome.
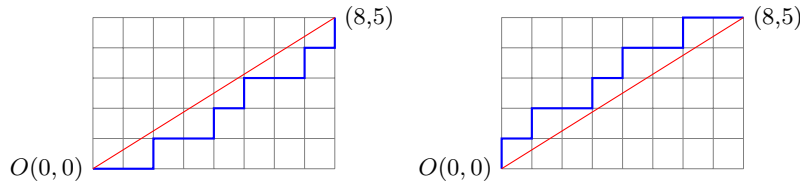
Fig. 1. The lower and upper Christoffel paths of slope 5/8 respectively.
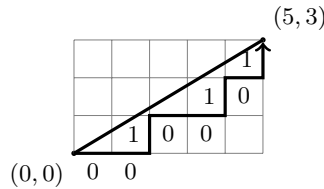
Fig. 2. The Christoffel path from (0,0) to (5,3) and the Christoffel word $C(\frac{3}{5}) = 00100101$.

In addition, any Christoffel word can be factorized into two Christoffel words, using the standard factorization [2] defined as follows.

Property 2. Let w be any Christoffel word of slope $\frac{a}{b}$, w can be written in a unique way as $w = (w_1, w_2)$, where both w_1 and w_2 are Christoffel words and of minimal length equal to 1.

Note that the slopes of the new Christoffel words, obtained by the standard factorization, bound the initial slope.

Property 3. Let $w = (w_1, w_2)$, such that $\rho(w_1) = \frac{a}{b}$ and $\rho(w_2) = \frac{c}{d}$. We have:

1. $\frac{a}{b} < \rho(w) < \frac{c}{d}$,
2. $\det(\rho(w_1), \rho(w_2)) = \det \begin{pmatrix} |w_1|_0 & |w_2|_0 \\ |w_1|_1 & |w_2|_1 \end{pmatrix} = cb - ad = 1.$

Using this factorization, we are able to define two functions ϕ_0 and ϕ_1 from $A^* \times A^*$ into itself as follows: $\phi_0(w_1, w_2) = (w_1, w_1 w_2)$; $\phi_1(w_1, w_2) = (w_1 w_2, w_2)$.

Property 4. Any Christoffel word is constructed by an iteration of these two functions ϕ_0 *and* ϕ_1 on $(0, 1)$.

Consequently, the standard factorization of any Christoffel word is unique, i.e, there exists a unique point of the Christoffel path, where we can do this factorization. This point is exactly the closest point of the Christoffel path with respect to the line segment, that divides the initial Christoffel word into two concatenated Christoffel words. For Figure 2, we note that the standard factorization of $C(\frac{3}{5}) = (001, 00101)$.

2.4   Lyndon factorization

The second family of words that is useful for our purpose is the family of *Lyndon words* [17].

Definition 2. *A word $w \in A^*$ is a* Lyndon *word if for all $u, v \in A^* \setminus \{\epsilon\}$ such that $w = u.v$, we have $w < v.u$.*

In other words, we can say that w is a Lyndon word if w is the smallest word among all its conjugates. Based on this definition, we can deduce one of the properties of this family of words. In fact, *Lyndon* words are primitive words; otherwise, we can have an equality between at least one of the conjugates of w. Therefore, if w is a Lyndon word, $w.w$ is not Lyndon. For example, let $w = 0110$, and X be the set of all the possible conjugates of w. The smallest element of X in lexicographic order is $w' = 0011$. Therefore, w is not a Lyndon word. A factorization was introduced in 1958 by Chen, Fox and Lyndon [7].

Definition 3. *Every non-empty word w admits a unique factorization as a lexicographically decreasing sequence of Lyndon words: $w = l_1^{n_1} l_2^{n_2} \cdots l_k^{n_k}$, s.t $\rho(l_1) > \rho(l_2) > \cdots > \rho(l_k)$ where $n_i \geq 1$ and l_i are Lyndon words.*

This factorization is called **Lyndon factorization**. In 1980 Duval proved that this factorization can be computed in a linear time [12]. For example, the Lyndon factorization of $w = 100101100101010$ is $(1)(001011)(0010101)(0)$.

2.5   Digital convexity by combinatorics

Now, we introduce the link between these two families of words and digital convexity for polyominos.

Definition 4. *A polyomino P is said digitally convex if the convex hull of P, $conv(P)$, satisfies $conv(P) \cap \mathbb{Z}^2 \subset P$.*
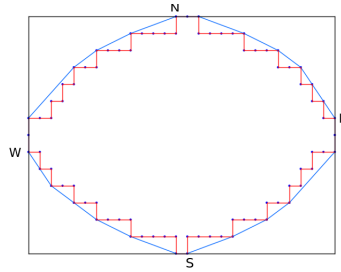
Each digital convex polyomino P can be bounded by a rectangle box. We denote the points of intersection of each side of the bounding box with P by (W)est, (N)orth, (E)ast and (S)outh, starting from the leftmost side and in a clockwise order as shown in Figure 3.

The boundary word of P, $Bd(P)$, can be divided into 4 sub-words, w_1, w_2, w_3 and w_4, where each word codes respectively the WN, NE, ES and SW side of P. We note that w_1 starts with the letter 1, while w_2, w_3 and w_4 start with 0, 3 and 2 respectively. Each word uses only two letters among the four of the alphabet, and hence, we deal on each side with a binary word.

In this section, we consider the WN-word of the boundary word of the digital convex set, and introduce the result of Brleck et al. [3] about digital convexity: a characterization for this path using the two previous families of words.

Property 5. [3] A word w is WN-convex if and only if its unique Lyndon factorization $w_1^{n_1} w_2^{n_2} \ldots w_k^{n_k}$ is such that all w_i are primitive Christoffel words.

See Fig. 4 for an example of Lyndon factorization: $(1)^1(011)^1(01)^2(0001)^1(0)^1$ where 1, 011, 01, 0001 and 0 are all Christoffel words. The Christoffel words are arranged in a weakly decreasing order of slopes: $\frac{1}{0} > \frac{2}{1} > \frac{1}{1} > \frac{1}{3} > \frac{0}{1}$. With this result, the WN convex path is decomposed into line segments with decreasing slopes using a linear algorithm [3]. As mentioned before, we know that line segments are represented by Christoffel words, and the decreasing lexicographic order that appears in the Lyndon factorization can explain the condition of decreasing order of slopes.

Fig. 3. A digital convex polyomino bounded in a rectangle, where W, N, E and S are the first points of intersection with the left, upper, right and lower sides of the bounding box.
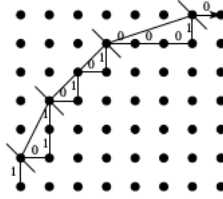
Fig. 4. Example of a WN-convex word 101101010001, formed of Christoffel words using the Lyndon factorization.

Due to this, we have the following property:

Property 6. [3] Each vertex of the convex hull of any digital convex set C is exactly at the end of each factor of the Lyndon factorization of its boundary word $Bd(C)$.

Such a vertex is exactly at the factor of either 10, 03, 32 or 21, depending on which side of C it is located, either in WN, NE, ES and SW respectively.

3   Contraction of digitally convex set

Deflating a digital convex set in a geometric point of view can be obtained by removing, step by step, at most one integer point. In the digital sense, we can see that this deformation is investigated by removing one pixel at each step. In [11], the authors introduced the *split operator*, that determines the unique accepted position, for each line segment, to add a digital point and conserving the convexity. In this paper, we work on removing digital points under the same condition. Since the vertices, as mentioned before, are digitally represented by the factor 10 in the WN path, then removing a point means switching the factor 10 (resp. 03, 32 and 21) to 01, (resp. 30, 23 and 12).

3.1   Removable points and contraction

The points that we are allowed to remove on the boundary are exactly the vertices of the convex hull. In fact, we might face several possibilities depending on the position of the points and the convex shape. In this section, we will mention all the possible situations that we can encounter and the different algorithms that we can use in order to remove a single point from the digital convex set at each step. Given a digitally convex set C, on the boundary word $Bd(C)$, we can have the vertices of the convex hull of C, also called corners. These vertices can be grouped in two categories. The first ones, called the internal vertices, that belong to one of the boundary sub-words. The second ones are external or critical vertices that are at the intersection between two consecutive sub-words. Let w_i

and w_{i+1} be two consecutive sub-words, the corner is exactly the last letter of w_i and the first letter of w_{i+1}., We define such contraction operator on Christoffel words. Note that Christoffel words on WN, NE, ES and SW sides are binary words over the binary alphabets $\{0,1\}$, $\{0,3\}$, $\{2,3\}$ and $\{1,2\}$ respectively. Christoffel words on each side are defined as follows: $x = 0h1$, $y = 3k0$, $z = 2l3$ and $t = 1m2$ where h, k, l and m are palindroms.

3.2   Switch operator

In this section, we consider the four boundary binary sub-words w_1, w_2, w_3 and w_4 that belong to $A^* = \{0,1,2,3\}^*$. On these sub-words, we set an operation at a certain position k, from A^* into A^* such that it switches the two consecutive letters of this word at position k.

Definition 5. *Let $w = l_1 l_2 \ldots l_k l_{k+1} \ldots l_n$ be a word in $\{0,1,2,3\}^*$,*

$$\mathrm{switch}_k(w) = l_1 l_2 \ldots l_{k+1} l_k \ldots l_n,$$

where $02 = 20 = 13 = 31 = \epsilon$.

Given a boundary word $Bd(C)$, thanks to Property 6, we can apply this switch operator to $Bd(C)$ at the last letter of each Christoffel word obtained by the Lyndon factorization of $Bd(C)$. In this way, we remove a corner of the form 10, 03, 32 or 21. We have two cases, depending on the position of a corner. The first case is when the corner belongs to one of the four paths, either WN, NE, ES or SW. In this case, the operation is directly applied at position k of the Christoffel word w, where $k = |w|$, as seen in Fig. 5. In the second case, we deal with the corners that are at the intersection position between w_1, w_2, w_3 and w_4. In fact, only at these positions, it might occurs some particular situations. We might obtain, after a switch operation, one of the factors 02, 20, 13, and 31 that will be replaced by ϵ by definition. But in this case, a re-arrangement for the four boundary words has to be considered; see an example in Fig. 6.

To keep the set 4-connected, we must avoid the existence of the factor $1^k 0^l 3^k 2^l$ and its conjugates on the boundary word. Therefore, if after a certain
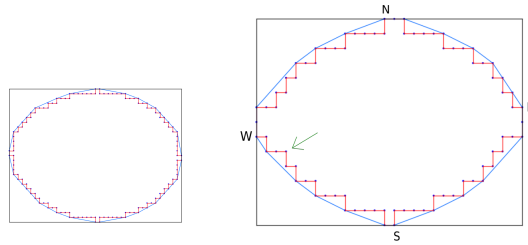
Fig. 5. The fourth boundary word of C is: $w_4 = (2)(12222)(1222)(122)^2(12)^2$. The switch operator is applied on the fifth factor, at position 2. We obtain the new boundary word $w_4' = (2)(12222)(1222)(122)(12122)(12)$.
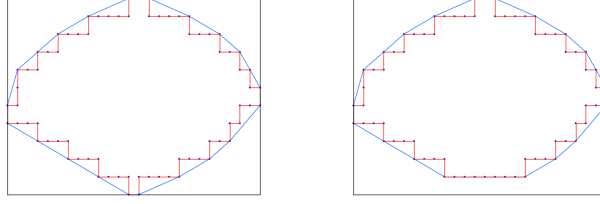
Fig. 6. Re-arrangement of the boundary words, w_4 and w_3, after removing the factor 31 from the boundary word $Bd(C)$ of the digital convex set C.

operation, one of these four factors appear in the boundary word, the switch operation must be canceled.

By applying the switch operator locally all around the boundary word and sub-words, the possible words that can be obtained are of the following form:

Theorem 1. *Let u and v be two consecutive Christoffel words such that $|u| = k$, and $\rho(u) > \rho(v)$. By applying the switch operator one of these cases appears:*

1. $\text{switch}_k(uv) = w^\ell$ *where $\rho(w) < \rho(u)$ and $\ell \geq 1$;*
2. $\text{switch}_k(uv) = m'_1 \dots m'_i$ *where $\rho(m'_1) < \rho(u)$ and $\rho(m'_1) > \dots \rho(m'_i)$.*

In order to give the proof, we must recall Pick's theorem [19] that gives the area of a triangle using its interior and boundary integer points:
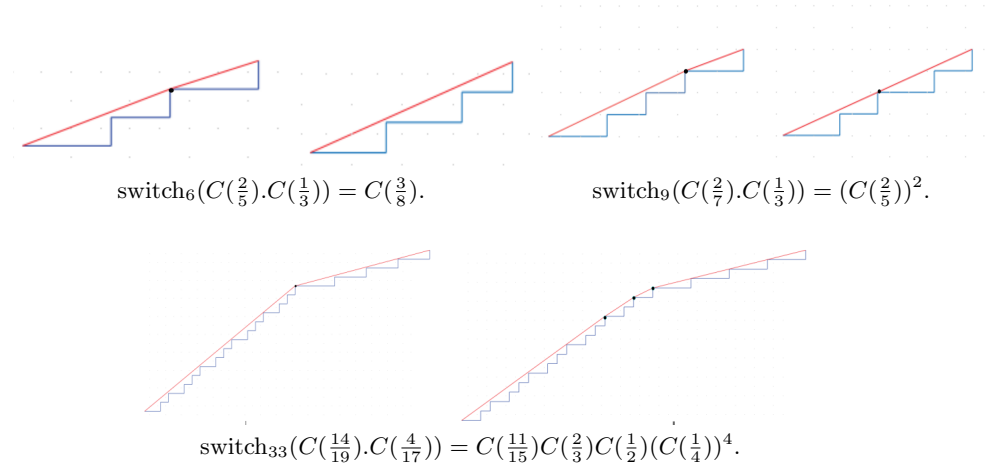
Theorem 2. *[18] The area of a given polygon in Euclidean plane, whose vertices have integer coordinates, is equal to $A = i + \frac{b}{2} - 1$ where i is the number of interior integer points and b is the number of boundary integer points of the polygon.*

After this theorem, we can pass to a sketch of the proof of Theorem 1.

Proof. From Properties 2 and 3, we write $m_1 = u.v$ and $m_2 = x.y$ with $\rho(u) < \rho(m_1) < \rho(v)$ and $\rho(x) < \rho(m_2) < \rho(y)$. Recall that the area of a triangle ABC is equal to $\frac{1}{2} \det(\overrightarrow{AB}, \overrightarrow{AC}) = \frac{1}{2} \det \begin{pmatrix} x_{\overrightarrow{AB}} & x_{\overrightarrow{AC}} \\ y_{\overrightarrow{AB}} & y_{\overrightarrow{AC}} \end{pmatrix} = i + \frac{b}{2} - 1$. Hence, we can conclude that: $i = \frac{\det(\overrightarrow{AB}, \overrightarrow{AC}) - b + 2}{2}$. With a simple vectorial calculation we can conclude that if $\det(\rho(m_1), \rho(m_2)) = 1$, we get $i = 0$. Hence we get the first case for $\ell = 1$ since in this case we get $b = 3$, that are exactly the points A, B and C. We can note that this result is the inverse of the split operator defined in [11]. Similarly, and by noticing that each Christoffel word is obtained by applying several morphisms of ϕ_0 and ϕ_1 defined in Property 4, we can prove the other cases.

Some examples are illustrated in Fig. 7.

The number of possible new segments is related to the solutions of the a variant of the *Knapsack problem* in integer linear programming: given a triangle

$$\text{switch}_6(C(\tfrac{2}{5}).C(\tfrac{1}{3})) = C(\tfrac{3}{8}).$$ $$\text{switch}_9(C(\tfrac{2}{7}).C(\tfrac{1}{3})) = (C(\tfrac{2}{5}))^2.$$

$$\text{switch}_{33}(C(\tfrac{14}{19}).C(\tfrac{4}{17})) = C(\tfrac{11}{15})C(\tfrac{2}{3})C(\tfrac{1}{2})(C(\tfrac{1}{4}))^4.$$

Fig. 7. Examples of three different cases of switch operator.

T with rational vertices in a $[0, N] \times [0, N]$ domain ($N \in \mathbb{Z}$), the number of vertices in the convex hull of $T \cap \mathbb{Z}^2$ is in $O(\log N)$ [14]. This result has been used to efficiently compute the convex hull of interior grid points of rational convex polygon [6]. In other words, the number of $m'_i s$ and the value of k is bounded by $O(\log N)$.

4   Choosing the removal order

As mentioned above, we can find all points that can be removed in order to deflate a digital convex set C while preserving its convexity. These points are exactly the vertices of the convex hull of C. Here arises the natural question: is there any specific order for choosing which digital point to remove at each step? If so, is there a preferred order? In fact, we investigate and test three ordering techniques to deflate C.

Let V be a removable point, and V_p and V_n be its previous and next vertices of the convex hull of C. After removing V from C, let us assume that the set of new vertices $\mathbf{W}$ is obtained, instead of V.

4.1   Ordering by determinant obtained from two consecutive line segments

For each removable point V, we calculate

$$\triangle(V) = \frac{1}{2}\det(\overrightarrow{V_pV}, \overrightarrow{VV_n}),$$

which is the (signed) area of triangle V_pVV_n. Then, all removable points are listed in increasing order with respect to this area measure, and the point with min-

imum value is chosen for each elementary contraction step. The list is updated locally whenever a point is removed.

4.2   Ordering by area change

For each removable point V, we calculate

$$\triangle_c(V) = \triangle(V) - \frac{1}{2} \sum_{W \in \mathbf{W}} \det(\overrightarrow{V_pW}, \overrightarrow{WW_n})$$

which is the area difference between the convex hulls of C and $C \setminus \{V\}$. Similarly to the previous case, all removable points are listed in increasing order with respect to this area change, and the point with minimum value is chosen for each elementary contraction step. Once a point is removed, the list is updated locally.

4.3   Ordering by angle-sum change

For each removable point V, we calculate the difference between the line-segment angle at V and the sum of such angles at all $W \in \mathbf{W}$, such that

$$\theta_c(V) = \theta(V) - \sum_{W \in \mathbf{W}} \theta(W)$$

where

$$\theta(V) = \cos^{-1}\left(\frac{\overrightarrow{V_pV} \cdot \overrightarrow{VV_n}}{\|\overrightarrow{V_pV}\|\|\overrightarrow{VV_n}\|}\right).$$

Similarly to the previous cases, all removable points are listed in increasing order with respect to this angle measure, and the point with minimum value is chosen from the list for each elementary contraction step. Once a point is removed, the list is updated locally.

5   Experimental results

Deforming digitally convex sets with digital convexity preservation is crucial, and has different applications in digital image processing. One of them is rotating digitally convex sets without loss of their convexity, which is not trivial at all.

Figure 8 shows the experimental results of applying our methods to a rotated digitized ellipse, which was originally digitally convex before the rotation but is not anymore (see Fig. 8 (b)). In order to repair the digital convexity, we made the convex hull and digitized it (see Fig. 8 (c)). As the difference between the rotated digitized ellipse and the digitized convex hull are 45 pixels, which are colored in gray in Fig. 8 (b), we decided to remove 45 pixels while preserving the digital convexity by using our method with the three different strategies. Note that, in Section 4, the point to be removed at each iteration has to be with
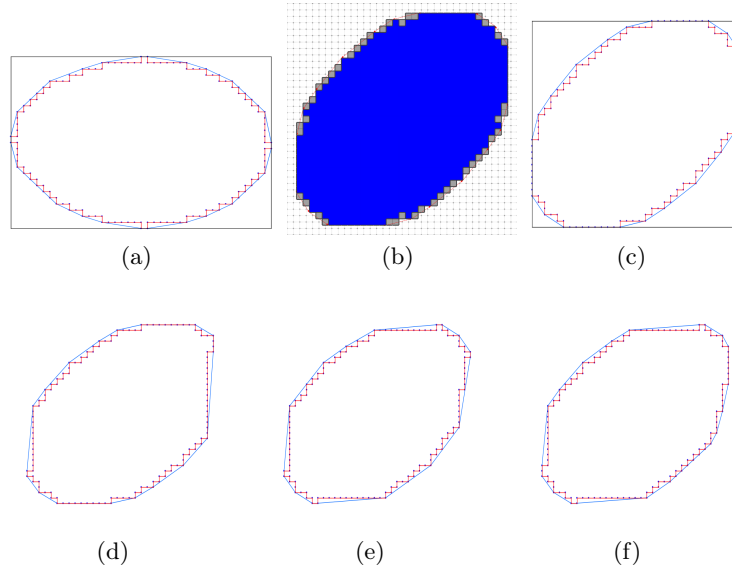
Fig. 8. (a) A digitized ellipse, which is digitally convex; (b) its rotated image with angle $\frac{\pi}{4}$ (in blue), which is not anymore digitally convex; (c) the convex hull of the rotated shape of (b) where the difference from the rotated image is visualized with 45 gray pixels in (b); the contraction results for the digitized convex hull of the rotated digitized ellipse of (b) after removing 45 points with the three ordering strategies: by (d) determinant, (e) area change, and (f) angle-sum change.
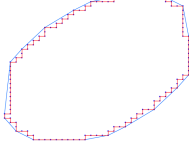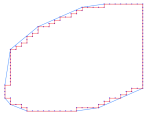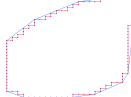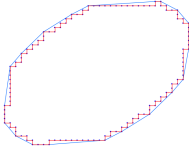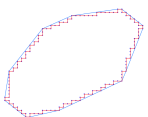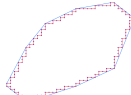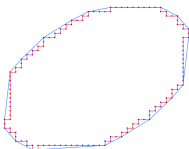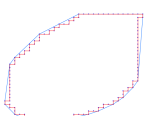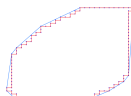
a minimal value for the three strategies. We might encounter the case where several points can give us the same minimal value. Our algorithm chooses in a random way a removal point among all the points having the minimal value at each iteration. Using this algorithm, the results after 45 iterations can be seen in Fig. 8 (d,e,f), respectively. Table 1 also shows the results of the three different strategies after 22, 150 and 200 iterations.

While all of them preserve digital convexity, the deformation tendencies are slightly different. The first removal-point ordering strategy based on area tends to extend vertical and horizontal line segments while it keeps curved parts as well. On the other hand, it is seen that the second (resp. third) strategy based on area (resp. angle) change tend to straighten curved parts.

6   Conclusion

We presented the method for contraction of digitally convex sets by removing digital points one by one while preserving their digital convexity, with a help of combinatorics on words. Such tools enable us to detect a set of removable points and to define such convexity-preserving contraction of a digital set as an operation of re-writing its boundary word. In order to choose one of removable

Table 1. The contraction results for the digitized convex shape illustrated in Fig. 8 (c) after removing 22, 150 and 200 points with the three ordering strategies: determinant, area change, and angle-sum change.

points for each contraction step, we present three geometrical strategies, which are related to vertex angle and area changes of the convex hull of a digitally convex set. We showed experimental results of applying the methods to repair some non-convex digital sets, which are obtained by a rotation of convex digital sets. Our further work will consist of deforming non-convex sets while preserving their convex and concave parts. Another idea can be to write the algorithm needed to add a point at each step instead of removing.

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