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ON THE CHOW RING OF CERTAIN HYPERSURFACES IN A GRASSMANNIAN

ROBERT LATERVEER

ABSTRACT. This note is about Plücker hyperplane sections $X$ of the Grassmannian $\text{Gr}(3, V_{10})$. Inspired by the analogy with cubic fourfolds, we prove that the only non–trivial Chow group of $X$ is generated by Grassmannians of type $\text{Gr}(3, W_6)$ contained in $X$. We also prove that a certain subring of the Chow ring of $X$ (containing all intersections of positive–codimensional subvarieties) injects into cohomology.

1. INTRODUCTION

Let $\mathcal{L}$ be the Plücker polarization on the complex Grassmannian $\text{Gr}(3, V_{10})$, and let

$$X \in |\mathcal{L}|$$

be a smooth hypersurface in the linear system of $\mathcal{L}$. The Hodge diamond of the 20–dimensional variety $X$ is

\[
\begin{array}{cccccccccccc}
1 & & & & & & & & & & & \\
2 & & & & & & & & & & & \\
3 & & & & & & & & & & & \\
* & & & & & & & & & & & \\
\vdots & & & & & & & & & & & \\
* & & & & & & & & & & & \\
0 & \ldots & \ldots & 0 & 1 & 30 & 1 & 0 & \ldots & \ldots & 0 \\
* & & & & & & & & & & & \\
\vdots & & & & & & & & & & & \\
* & & & & & & & & & & & \\
3 & & & & & & & & & & & \\
2 & & & & & & & & & & & \\
1 & & & & & & & & & & & \\
\end{array}
\]

(where $*$ indicates some unspecified number, and all empty entries are 0). This looks much like the Hodge diamond of a cubic fourfold. To further this analogy, Debarre and Voisin [?] have constructed, for a general such hypersurface $X$, a hyperkähler fourfold $Y$ that is associated (via an Abel–Jacobi isomorphism) to $X$. Just as in the famous Beauville–Donagi construction starting from a cubic fourfold [?], the hyperkähler fourfolds $Y$ form a 20–dimensional family, deformation equivalent to the Hilbert square of a $K3$ surface.

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In this note we are interested in the Chow ring $A^i(X)_\mathbb{Q}$ of the hypersurface $X$. Using her celebrated method of spread of algebraic cycles in families, Voisin [?, Theorem 2.4] has already proven a form of the Bloch conjecture for $X$: one has vanishing

$$A^i_{\text{hom}}(X)_\mathbb{Q} = 0 \quad \forall \ i \neq 11$$

(where $A^i_{\text{hom}}(X)_\mathbb{Q}$ is defined as the kernel of the cycle class map to singular cohomology). This is the analogue of the well–known fact that the only non–trivial Chow group of a cubic fourfold is the Chow group of 1–cycles.

We complete Voisin’s result, by describing the only non–trivial Chow group of $X$:

**Theorem (=theorem ??).** Let $L$ be the Plücker polarization on $\text{Gr}(3, V_{10})$. Let $X \in |L|$ be a smooth hypersurface for which the associated hyperkähler fourfold $Y$ is smooth. Then $A^1_{\text{hom}}(X)_\mathbb{Q}$ is generated by Grassmannians $\text{Gr}(3, W_6) \subset X$ (where $W_6 \subset V_{10}$ is a six–dimensional vector space).

This is reminiscent of the famous result about the Chow ring of a $K3$ surface [?]. It is also an analogue of the fact that for a cubic fourfold $V \subset \mathbb{P}^5(\mathbb{C})$, the Chow group $A^2(V)_\mathbb{Q}$ is generated by lines [?]. Theorem ?? is readily proven using the spread method of [?]; as such, theorem ?? could naturally have been included in [?].

The second result of this note concerns the ring structure of the Chow ring of $X$, given by the intersection product:

**Theorem (=theorem ??).** Let $L$ be the Plücker polarization on $\text{Gr}(3, V_{10})$, and let $X \in |L|$ be a smooth hypersurface. Let $R^{11}(X) \subset A^{11}(X)_\mathbb{Q}$ be the subgroup containing intersections of two cycles of positive codimension, the Chern class $c_{11}(T_X)$ and the image of the restriction map $A^{11}(\text{Gr}(3, V_{10}))_\mathbb{Q} \to A^{11}(X)_\mathbb{Q}$. The cycle class map induces an injection

$$R^{11}(X) \hookrightarrow H^{22}(X, \mathbb{Q}).$$

This is reminiscent of the famous result about the Chow ring of a $K3$ surface [?]. It is also an analogue of the fact that for a cubic fourfold $V$, the subgroup $A^2(V)_\mathbb{Q} \cdot A^1(V)_\mathbb{Q} \subset A^3(V)_\mathbb{Q}$ is one–dimensional. Theorem ?? suggests that the hypersurfaces $X$ might have a multiplicative Chow–Künneth decomposition, in the sense of Shen–Vial [?]. This seems difficult to establish, however (cf. remark ??).

**Conventions.** In this note, the word variety will refer to a reduced irreducible scheme of finite type over $\mathbb{C}$. For a smooth variety $X$, we will denote by $A^j(X)$ the Chow group of codimension $j$ cycles on $X$ with $\mathbb{Q}$–coefficients.

The notation $A^j_{\text{hom}}(X)$ will be used to indicate the subgroups of homologically trivial cycles.

For a morphism between smooth varieties $f : X \to Y$, we will write $\Gamma_f \in A^*(X \times Y)$ for the graph of $f$, and $\Gamma_f^t \in A^*(Y \times X)$ for the transpose correspondence.

We will write $H^*(X) = H^*(X, \mathbb{Q})$ for singular cohomology with $\mathbb{Q}$–coefficients.

2. Generators for $A^{11}$

**Theorem 2.1.** Let $L$ be the Plücker polarization on $\text{Gr}(3, V_{10})$. Let $X \in |L|$ be a smooth hypersurface for which there is an associated smooth hyperkähler fourfold $Y$. Then $A^{11}_{\text{hom}}(X)$ is generated by the classes of Grassmannians $\text{Gr}(3, W_6) \subset X$ (where $W_6 \subset V_{10}$ is a six–dimensional vector space).
Proof. As mentioned in the introduction, Voisin [?, Theorem 2.4] has proven that
\[ A^i_{\text{hom}}(X) = 0 \quad \forall i > 11. \]
Using the Bloch–Srinivas “decomposition of the diagonal” method [?] (cf. also [?, Chapter 3]), this readily implies that actually
\[ A^i_{\text{hom}}(X) = 0 \quad \forall i \neq 11, \]
and so
\[ \text{Niveau}(A^*(X)) \leq 2, \]
in the language of [?]. That is, the 20–dimensional variety \( X \) motivically looks like a surface, and so in particular the Hodge conjecture is true for \( X \) [?, Proposition 2.4].

Let
\[ X \to B \]
denote the universal family of smooth hypersurfaces in the linear system \(|L|\). The base \( B \) is the Zariski open in \( \mathbb{P}(\wedge^3 V_{10}^*) \) parametrizing 3–forms \( \sigma \) such that the corresponding hyperplane section
\[ X_\sigma \subset \text{Gr}(3, V_{10}) \subset \mathbb{P}(\wedge^3 V_{10}) \]
is smooth.

Let \( B' \subset B \) be the Zariski open such that the fibre \( X_\sigma \) has an associated hyperkähler fourfold \( Y_\sigma \), in the sense of [?]. That is, \( B' \) parametrizes 3–forms \( \sigma \) such that both \( X_\sigma \) and
\[ Y_\sigma := \{ W_6 \in \text{Gr}(6, V_{10}) \text{ such that } \sigma|_{W_6} = 0 \} \subset \text{Gr}(6, V_{10}) \]
are smooth of the expected dimension.

We rely on the spread result of Voisin’s, in the following form:

**Theorem 2.2** (Voisin [?]). Let \( \Gamma \in A^{20}(X \times_B X) \) be a relative correspondence with the property that
\[ (\Gamma|_{X_\sigma \times X_\sigma})_* H^{11,9}(X_\sigma) = 0 \quad \text{for very general } \sigma \in B. \]
Then
\[ (\Gamma|_{X_\sigma \times X_\sigma})_* A^{11}_{\text{hom}}(X_\sigma) = 0 \quad \text{for all } \sigma \in B. \]

(For basics on the formalism of relative correspondences, cf. [?, Section 8.1].) Since theorem ?? is not stated precisely in this form in [?], we briefly indicate the proof:

**Proof.** (of theorem ??) The assumption on \( \Gamma \) (plus the shape of the Hodge diamond of \( X_\sigma \), and the truth of the Hodge conjecture for \( X_\sigma \)) implies that for the very general \( \sigma \in B \) there exist 10–dimensional subvarieties \( V^i_\sigma, W^i_\sigma \) such that
\[ \Gamma|_{X_\sigma \times X_\sigma} = \sum_{i=1}^s V^i_\sigma \times W^i_\sigma \quad \text{in } H^{40}(X_\sigma \times X_\sigma). \]
By Noether–Lefschetz, the subvarieties \( V^i_\sigma, W^i_\sigma \) are obtained by restriction from subvarieties of \( \text{Gr}(3, V_{10}) \), hence they exist universally. (Instead of evoking Noether–Lefschetz, one could also
apply Voisin’s Hilbert scheme argument [?, Proposition 3.7] to obtain that the \( V^i, W^i \) exist universally. That is, there exist 10–codimensional subvarieties \( \mathcal{V}^i, \mathcal{W}^i \subset \mathcal{X} \), and a cycle \( \delta \) supported on \( \bigcup \mathcal{V}^i \times_B \mathcal{W}^i \), such that

\[
(\Gamma - \delta)|_{X_\sigma \times X_\sigma} = 0 \quad \text{in } H^{40}(X_\sigma \times X_\sigma), \quad \text{for very general } \sigma \in B.
\]

We define

\[
R := \Gamma - \delta \in A^{20}(\mathcal{X} \times_B \mathcal{X}).
\]

For brevity, let us now write \( M := \text{Gr}(3, V_{10}) \). Since \( M \) has trivial Chow groups, we are in the set–up of \[?\]. As in loc. cit., we consider the blow–up \( \widetilde{\mathcal{X}} \) of \( \mathcal{X} \) along the relative diagonal \( \Delta_{\mathcal{X}} \). There is an open inclusion \( \mathcal{X} \times_B \mathcal{X} \subset I \). Hence, given \( \sigma \in A^* \mathcal{X} \), there exists a (non–canonical) cycle \( \bar{R} \in A^n(I) \) such that

\[
\bar{R}|_{\mathcal{X} \times_B \mathcal{X}} = f^* (R) \quad \text{in } A^n(\mathcal{X} \times_B \mathcal{X}).
\]

Hence, we have

\[
\bar{R}|_{X_\sigma \times X_\sigma} = (f^*(R))|_{X_\sigma \times X_\sigma} = (f_\sigma)^*(R)|_{X_\sigma \times X_\sigma} = 0 \quad \text{in } H^{40}(X_\sigma \times X_\sigma),
\]

for \( \sigma \in B \) very general, by assumption on \( R \). (Here, as one might guess, the notation

\[
f_\sigma : X_\sigma \times X_\sigma \to X_\sigma \times X_\sigma
\]

indicates the blow–up along the diagonal \( \Delta_{X_\sigma} \).)

We now apply [?, Proposition 1.6] to the cycle \( \bar{R} \). The result is that there exists a cycle \( \gamma \in A^{20}(\Delta_{\mathcal{X}} \times_B \mathcal{X}) \) such that there is a rational equivalence

\[
R|_{X_\sigma \times X_\sigma} = (f_\sigma)_*(\bar{R}|_{X_\sigma \times X_\sigma}) = \gamma|_{X_\sigma \times X_\sigma} \quad \text{in } A^{20}(X_\sigma \times X_\sigma) \quad \forall \sigma \in B.
\]

But the restriction of \( \gamma \) acts as zero on \( A^{11}_{\hom}(X_\sigma) \) (indeed, the action of \( \gamma|_{X_\sigma \times X_\sigma} \) on \( A^{11}_{\hom}(X_\sigma) \) factors over \( A^{12}_{\hom}(M) = 0 \), and so

\[
(R|_{X_\sigma \times X_\sigma})_* = 0 : A^{11}_{\hom}(X_\sigma) \to A^{11}_{\hom}(X_\sigma) \quad \forall \sigma \in B.
\]

For any given \( \sigma \in B \), one can construct the subvarieties \( \mathcal{V}^i, \mathcal{W}^i \subset \mathcal{X} \) in the above argument in such a way that they are in general position with respect to the fibre \( X_\sigma \). This implies that the restriction

\[
\delta|_{X_\sigma \times X_\sigma} \in A^{20}(X_\sigma \times X_\sigma)
\]
is a completely decomposed cycle, i.e. a cycle supported on a union of subvarieties $V_j^\sigma \times W_j^\sigma \subset X_\sigma \times X_\sigma$ with $\text{codim}(V_j^\sigma) + \text{codim}(W_j^\sigma) = 20$. But completely decomposed cycles do not act on $A^*_\text{hom}(\) [?]) and so

\[
(\Gamma|_{X_\sigma \times X_\sigma})_* = ((R + \delta)|_{X_\sigma \times X_\sigma})_* = 0:\ A^{11}_\text{hom}(X_\sigma) \to A^{11}_\text{hom}(X_\sigma) \forall \sigma \in B.
\]

This ends the proof of theorem ??.

Let us now pick up the thread of the proof of theorem ?? As in [?, Section 2], for any 3–form $\sigma \in B'$ let

\[
G_\sigma := \left\{ (W_3, W_6) \in \text{Gr}(3, V_{10}) \times \text{Gr}(6, V_{10}) \mid W_3 \subset W_6, \sigma|_{W_6} = 0 \right\}
\]

denote the incidence variety, with projections

\[
G_\sigma \overset{p_\sigma}{\longrightarrow} X_\sigma \\
\downarrow_{q_\sigma} \\
Y_\sigma.
\]

The fibres of $q_\sigma$ are 9–dimensional Grassmannians $\text{Gr}(3, W_6)$.

Let $Y \to B'$ denote the universal family of Debarre–Voisin fourfolds (i.e., $Y \subset \text{Gr}(6, V_{10}) \times B'$ is the subvariety of pairs $(W_6, \sigma)$ such that $\sigma|_{W_6} = 0$), and let $G \to B'$ be the relative version of $G_\sigma$, with projections

\[
G \overset{p}{\longrightarrow} X \\
\downarrow q \\
Y.
\]

We will also rely on the following Abel–Jacobi type result:

**Lemma 2.3.** Let $\sigma \in B'$ be very general. Then there is an isomorphism

\[(q_\sigma)_*(p_\sigma)^* : H^{20}(X_\sigma, \mathbb{Q})_{\text{van}} \cong H^2(Y_\sigma, \mathbb{Q})_{\text{van}}.\]

The inverse isomorphism is given by

\[
H^2(Y_\sigma, \mathbb{Q})_{\text{van}} \xrightarrow{\mu g^2} H^6(Y_\sigma, \mathbb{Q})_{\text{van}} \xrightarrow{(p_\sigma)_*(q_\sigma)^*} H^{20}(X_\sigma, \mathbb{Q})_{\text{van}}.
\]

(Here $\mu \in \mathbb{Q}$ is a non–zero number, $g \in A^1(Y_\sigma$ is the Plücker polarization, and the vanishing cohomology $H^{20}(X_\sigma, \mathbb{Q})_{\text{van}}$ and $H^*(Y_\sigma, \mathbb{Q})_{\text{van}}$ is defined with respect to the inclusion of $X_\sigma$ and $Y_\sigma$ in $\text{Gr}(3, V_{10})$ resp. in $\text{Gr}(6, V_{10})$.)

**Proof.** The first part (i.e. the fact that $(q_\sigma)_*(p_\sigma)^*$ is an isomorphism on the vanishing cohomology) is [?, Theorem 2.2 and Corollary 2.7]. For the second part, we observe that the dual map (with respect to cup product)

\[(p_\sigma)_*(q_\sigma)^* : H^6(Y_\sigma, \mathbb{Q})_{\text{van}} \to H^{20}(X_\sigma, \mathbb{Q})_{\text{van}}
\]

is also an isomorphism. In particular, using hard Lefschetz, this means that the composition

\[
H^2(Y_\sigma, \mathbb{Q})_{\text{van}} \xrightarrow{g^2} H^6(Y_\sigma, \mathbb{Q})_{\text{van}} \xrightarrow{(p_\sigma)_*(q_\sigma)^*} H^{20}(X_\sigma, \mathbb{Q})_{\text{van}} \xrightarrow{(q_\sigma)_*(p_\sigma)^*} H^2(Y_\sigma, \mathbb{Q})_{\text{van}}
\]
is non–zero. Hence, the assignment

\[ < \alpha, \beta >_{\Gamma} := < \alpha, (q_\sigma)_*(p_\sigma)^* (q_\sigma)^* (g^2 \cdot \beta) >_{Y_\sigma} \]

defines a polarization on \( H^2(Y_\sigma, \mathbb{Q})_{\text{van}} \). Here, \(< \alpha, \beta >_{Y_\sigma} \) is the Beauville–Bogomolov form. However, as explained in [\ref{2}], for very general \( \sigma \) the Hodge structure on \( H^2(Y_\sigma, \mathbb{Q})_{\text{van}} \) is simple, and admits a unique polarization up to a coefficient. That is, there exists a non–zero number \( \mu \in \mathbb{Q} \) such that

\[ < \alpha, \beta >_{\Gamma} = \mu < \alpha, \beta >_{Y_\sigma} . \]

The Beauville–Bogomolov form being non–degenerate, this proves that

\[ (q_\sigma)_*(p_\sigma)^* (q_\sigma)^* (g^2 \cdot \beta) = \mu \beta \quad \forall \beta \in H^2(Y_\sigma, \mathbb{Q})_{\text{van}} . \]

Reasoning likewise starting from \( H^{20}(X_\sigma, \mathbb{Q})_{\text{van}} \), we find that the other composition is also the identity. \( \square \)

Let us define the relative correspondence

\[ \Gamma := \mu \Delta_X - \Gamma_p \circ \iota \Gamma_q \circ \Gamma_g \circ \iota \Gamma_p \in A^{20}(\mathcal{X} \times_{B'} \mathcal{X}) , \]

where \( \Gamma_{g^2} \in A^{6}(\mathcal{Y} \times_{B'} \mathcal{Y}) \) is the correspondence acting fibrewise as intersection with two Plücker hyperplanes. Lemma ?? implies that

\[ (\Gamma|_{X_\sigma \times X_\sigma})_* H^{20}(X_\sigma, \mathbb{Q})_{\text{van}} = 0 \quad \text{for very general } \sigma \in B' . \]

That is, the relative correspondence \( \Gamma \) satisfies the assumption of theorem ??, Thanks to theorem ??, we thus conclude that

\[ (\Gamma|_{X_\sigma \times X_\sigma})_* A^{11}_{\text{hom}}(X_\sigma) = 0 \quad \forall \sigma \in B . \]

Unraveling the definition of \( \Gamma \), this means in particular that there is a surjection

\[ (p_\sigma)_*(q_\sigma)^* : A^{4}_{\text{hom}}(Y_\sigma) \twoheadrightarrow A^{11}_{\text{hom}}(X_\sigma) \quad \forall \sigma \in B' . \]

As we have seen, for any point \( y \in Y_\sigma \) the fibre \((q_\sigma)^{-1}(y)\) is a 9–dimensional Grassmannian \( \text{Gr}(3, W_6) \) such that the 3–form \( \sigma \) vanishes on \( W_6 \). Such a Grassmannian is contained in the hypersurface \( X_\sigma \), and so

\[ (p_\sigma)_*(q_\sigma)^*(y) = \text{Gr}(3, W_6) \quad \text{in } A^{11}(X_\sigma) \quad \forall y \in Y_\sigma . \]

The theorem is proven. \( \square \)

**Remark 2.4.** The above argument actually shows that

\[
A^{11}_{\text{hom}}(X_\sigma) \xrightarrow{(q_\sigma)_*(p_\sigma)^*} A^4_{\text{hom}}(Y_\sigma) \xrightarrow{g^2} A^4_{\text{hom}}(Y_\sigma) \xrightarrow{(p_\sigma)_*(q_\sigma)^*} A^{11}_{\text{hom}}(X_\sigma)
\]

is a non–zero multiple of the identity, for any \( \sigma \in B' \). This is very much reminiscent of cubic fourfolds and their Fano varieties of lines [\ref{3}], [\ref{4}]. Inspired by this analogy, it is tempting to ask the following: can one somehow prove that

\[
\text{Im}(A^{11}(X_\sigma) \twoheadrightarrow A^4(Y_\sigma))
\]

is the same as the subgroup of 0–cycles supported on a uniruled divisor?
3. AN INJECTIVITY RESULT

Theorem 3.1. Let $\mathcal{L}$ be the Plücker polarization on $\text{Gr}(3, V_{10})$, and let $X \in |\mathcal{L}|$ be a smooth hypersurface. Let $R^{11}(X) \subset A^{11}(X)_G$ be the subgroup containing intersections of two cycles of positive codimension, the Chern class $c_{11}(T_X)$ and the image of the restriction map $A^{11}(\text{Gr}(3, V_{10})) \to A^{11}(X)$. The cycle class map induces an injection
\[ R^{11}(X) \hookrightarrow H^{22}(X, \mathbb{Q}) . \]

In order to prove theorem ??, we first establish a “generalized Franchetta conjecture” type of statement (for more on the generalized Franchetta conjecture, cf. [?], [?], [?]):

Theorem 3.2. Let $X \to B$ denote the universal family of Plücker hyperplanes in $\text{Gr}(3, V_{10})$ (as in section ??). Let $\Psi \in A^{11}(\tilde{X})$ be such that
\[ \Psi|_{\tilde{X}} = 0 \quad \text{in } H^{22}(X, \mathbb{Q}) \quad \forall \sigma \in B . \]

Then
\[ \Psi|_{\tilde{X}} = 0 \quad \text{in } A^{11}(X, \mathbb{Q}) \quad \forall \sigma \in B . \]

Proof. This is a two–step argument:

Claim 3.3. There is equality
\[ \text{Im} \left( A^{11}(\tilde{X}) \to A^{11}(X, \mathbb{Q}) \right) = \text{Im} \left( A^{11}(\text{Gr}(3, V_{10})) \to A^{11}(X, \mathbb{Q}) \right) \quad \forall \sigma \in B . \]

Claim 3.4. Restriction of the cycle class map induces an injection
\[ \text{Im} \left( A^{11}(\text{Gr}(3, V_{10})) \to A^{11}(X, \mathbb{Q}) \right) \hookrightarrow H^{22}(X, \mathbb{Q}) \quad \forall \sigma \in B . \]

Clearly, the combination of these two claims proves theorem ??.

Let us prove claim ??.

Claim 3.3. There is equality
\[ \text{Im} \left( A^{11}(\tilde{X}) \to A^{11}(X, \mathbb{Q}) \right) = \text{Im} \left( A^{11}(\text{Gr}(3, V_{10})) \to A^{11}(X, \mathbb{Q}) \right) \quad \forall \sigma \in B . \]

Claim 3.4. Restriction of the cycle class map induces an injection
\[ \text{Im} \left( A^{11}(\text{Gr}(3, V_{10})) \to A^{11}(X, \mathbb{Q}) \right) \hookrightarrow H^{22}(X, \mathbb{Q}) \quad \forall \sigma \in B . \]

Clearly, the combination of these two claims proves theorem ??.

To prove claim ??, let $\bar{B} := \mathbb{P}H^0(\text{Gr}(3, V_{10}), \mathcal{L})$ and let
\[ \tilde{X} \xrightarrow{\pi} \text{Gr}(3, V_{10}) \]
\[ \downarrow \phi \]
\[ \bar{B} \]
denote the universal hyperplane (including the singular hyperplanes). The morphism $\pi$ is a projective bundle, and so any $\Psi \in A^{11}(\tilde{X})$ can be written
\[ \Psi = \sum \pi^*(a_\ell) \cdot \phi^*(h^\ell) \quad \text{in } A^{11}(\tilde{X}) , \]
where $a_\ell \in A^{11-\ell}(\text{Gr}(3, V_{10}))$ and $h := c_1(\mathcal{O}_{\bar{B}}(1)) \in A^1(\bar{B})$. For any $\sigma \in B$, the restriction of $\phi^*(h)$ to the fibre $X_\sigma$ vanishes, and so
\[ \Psi|_{X_\sigma} = a_0|_{X_\sigma} \quad \text{in } A^{11}(X, \mathbb{Q}) , \]
which establishes claim ??.

Let us prove claim ??.

For any given $\sigma \in B$, let $\iota_\sigma : X_\sigma \to \text{Gr}(3, V_{10})$ denote the inclusion morphism. We know that
\[ \iota_\sigma^* : A^j(\text{Gr}(3, V_{10})) \to A^{j+1}(\text{Gr}(3, V_{10})) \]
equals multiplication by the ample class $c_1(\mathcal{L}) \in A^1(\text{Gr}(3,V_{10}))$. Now let

$$b \in A^{11}(\text{Gr}(3,V_{10}))$$

be such that the restriction $\iota^*(b) \in A^{11}(X_\sigma)$ is homologically trivial. Then we have that also

$$b \cdot c_1(\mathcal{L}) = \iota_* \iota^*(b) = 0 \quad \text{in } H^{24}(\text{Gr}(3,V_{10})) = A^{12}(\text{Gr}(3,V_{10})).$$

To conclude that $b = 0$, it suffices to show that

$$\cdot c_1(\mathcal{L}) : A^{11}(\text{Gr}(3,V_{10})) \to A^{12}(\text{Gr}(3,V_{10}))$$

is injective (and hence, by hard Lefschetz, an isomorphism). By hard Lefschetz, this is equivalent to showing that

$$\cdot c_1(\mathcal{L}) : A^9(\text{Gr}(3,V_{10})) \to A^{10}(\text{Gr}(3,V_{10}))$$

is surjective (hence an isomorphism).

According to [?, Theorem 5.26], the Chow ring of the Grassmannian is of the form

$$A^*(\text{Gr}(3,V_{10})) = \mathbb{Q}[c_1,c_2,c_3]/I,$$

where $c_j \in A^j(\text{Gr}(3,V_{10}))$ are Chern classes of the universal subbundle, and $I$ is a certain complete intersection ideal generated by 3 relations in degree 8, 9, 10. Writing out the relations in $I$, we find that

$$A^{10}(\text{Gr}(3,V_{10})) = \mathbb{Q}[c_1^{10}, c_1^5c_2, c_1^6c_2^2, c_1^4c_2^3, c_1^7c_3, c_1c_2^2c_3, c_1^3c_2c_3, c_1c_2^2c_3, c_1^4c_2^2, c_1^2c_2^2c_3^2]$$

is 10–dimensional (the classes $c_1^2c_2^3$, $c_2^5$ are eliminated thanks to the relation in degree 8 containing $c_2^5$; the class $c_1c_2^3$ is eliminated thanks to the relation in degree 9; the class $c_2^3c_3^2$ is eliminated thanks to the relation in degree 10). We observe that the inclusion

$$c_1 \cdot A^9(\text{Gr}(3,V_{10})) \subset A^{10}(\text{Gr}(3,V_{10}))$$

is an equality. This proves claim ??.

It remains to prove theorem ??:

**Proof.** (of theorem ??) Clearly, the Chern class is universally defined: for any $\sigma \in B$, we have

$$c_{11}(T_{X_\sigma}) = c_{11}(T_{X/B})|_{X_\sigma}.$$

Also, the image

$$\text{Im}(A^{11}(\text{Gr}(3,V_{10})) \to A^{11}(X_\sigma))$$

consists of universally defined cycles. (For a given $a \in A^{11}(\text{Gr}(3,V_{10}))$, the relative cycle

$$(a \times B)|_X \in A^{11}(\mathcal{X})$$

do the job.)

Likewise, for any $j < 10$ the fact that $A^j_{\text{hom}}(X_\sigma) = 0$, combined with weak Lefschetz in cohomology, implies that

$$A^j(X_\sigma) = \text{Im}(A^j(\text{Gr}(3,V_{10})) \to A^j(X_\sigma)),$$

and so $A^j(X_\sigma)$ consists of universally defined cycles for $j < 10$. In particular, all intersections

$$A^j(X_\sigma) \cdot A^{11-j}(X_\sigma) \subset A^{11}(X_\sigma), \quad 1 < j < 10$$

...
consist of universally defined cycles.

It remains to make sense of intersections

\[ A^{10}(X_\sigma) \cdot A^1(X_\sigma) \subset A^{11}(X_\sigma). \]

To this end, we note that \( A^1(X_\sigma) \) is 1–dimensional, generated by the restriction \( g \) of the Plücker line bundle \( \mathcal{L} \). Let \( \iota : X_\sigma \to \text{Gr}(3, V_{10}) \) denote the inclusion. The normal bundle formula implies that

\[ a \cdot g = \iota^* \iota_*(a) \quad \text{in} \quad A^{11}(X_\sigma) \quad \forall \ a \in A^{10}(X_\sigma). \]

It follows that

\[ A^{10}(X_\sigma) \cdot A^1(X_\sigma) \subset \text{Im}(A^{11}(\text{Gr}(3, V_{10})) \xrightarrow{\iota^*} A^{11}(X_\sigma)) \]

also consists of universally defined cycles.

In conclusion, we have shown that \( R^{11}(X_\sigma) \) consists of universally defined cycles, and so theorem ?? is a corollary of theorem ??.

\begin{flushright}
\Box
\end{flushright}

**Remark 3.5.** Theorem ?? is an indication that perhaps the hypersurfaces \( X \subset \text{Gr}(3, V_{10}) \) have a *multiplicative Chow–Künneth decomposition*, in the sense of [?, Chapter 8]. Unfortunately, establishing this seems difficult; one would need something like theorem ?? for

\[ A^{40}(\mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X}). \]

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**References**


[16] C. Voisin, Bloch’s conjecture for Catanese and Barlow surfaces, J. Differential Geometry 97 (2014), 149—175,

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