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Amadeu Almeida Coco
ICD-LOSI, Université de Technologie de Troyes
DCC-UFMG, Universidade Federal de Minas Gerais

Elyn L. Solano-Charris
ICD-LOSI, Université de Technologie de Troyes
AMLI-EICEA, Universidad de La Sabana

Andréa Cynthia Santos
ICD-LOSI, Université de Technologie de Troyes

Christian Prins
ICD-LOSI, Université de Technologie de Troyes

Thiago Noronha
DCC-UFMG, Universidade Federal de Minas Gerais

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Amadeu Almeida Coco[1,2], Elyn L. Solano-Charris[1,3]
Andréa Cynthia Santos[1], Christian Prins[1], Thiago Noronha[2]

[1] ICD-LOSI, UMR CNRS 6281, Université de Technologie de Troyes
12, rue Marie Curie, CS 42060, 10004, Troyes CEDEX, France.
{andrea.duhamel},{elyn.solano_charris},{christian.prins}@utt.fr

[2] DCC-UFMG, Universidade Federal de Minas Gerais
Avenida Antônio Carlos 6627, Belo Horizonte, MG, Brazil.
{amadeuac},{tfn}@dcc.ufmg.br

[3] AMLI-EICEA, Universidad de La Sabana
Autopista norte de Bogotá D.C., Chía (Cundinamarca), Colombia.
erlyn.solano@unisabana.edu.co

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Abstract

Uncertain parameters appear in many optimization problems raised by real-world applications. To handle such problems, several approaches to model uncertainty are available, such as stochastic programming and robust optimization. This study is focused on robust optimization, in particular, the criteria to select and determine a robust solution. We provide an overview on robust optimization criteria and introduce two new classifications criteria for measuring the robustness of both scenarios and solutions. They can be used independently or coupled with classical robust optimization criteria and could work as a complementary tool for intensification in local searches.

1 Introduction

Uncertain parameters appear in many optimization problems raised by real-world applications. In general, uncertain data arise due to incomplete information and undifferentiated information [37]. Several approaches are available to model them in the context of optimization problems, in particular stochastic programming where uncertain data are modeled as random variables [56, 61]. However, such approach is limited to the cases where uncertainties have a stochastic nature (which is not always the case) and when it is possible to identify the probability distribution, which can be difficult to obtain, especially for large-scale problems [9]. Many studies deal with stochastic programming approaches and the pioneer models were proposed by Beale [8] and Dantzig [19].

Robust optimization has been applied in general as a way to self-protection against undesirable impacts due to vague approximations, incomplete, imprecise, or ambiguous data. Readers are referred to [53] for an interesting survey on different uses of robustness in the field of Operations Research. This study is dedicated to robust optimization problems for which uncertain data are defined as a set of possible values, usually called as scenarios. In this context, the literature covers a large number of applications [43, 54, 57, 66], such as scheduling [25, 28], facility location [3, 7, 26, 41], inventory [11, 17], finance [15, 22, 27, 18, 55], queuing networks, stochastic systems and game theory [13, 32, 35, 51], machine learning and statistics [10, 63, 62], and energy systems [14, 52].

The robust optimization approach considered here is then, an alternative to stochastic programming where the optimization is done over a set of scenarios. A scenario denotes the way to structure the uncertain data, where two common ways of modeling are *interval* or *discrete data*. Interesting definitions for a *robust model* and a *robust solution* are introduced by [36, 46]. Roughly, a *robust solution* is related to the robust optimization criteria. For instance, whenever a *minmax* objective function is addressed, a robust solution is the one which minimizes the worst case. For the sake of clarity, let us consider the classical Robust Shortest Path (RSP) problem as an example to show the use of interval and discrete data. The RSP problem is defined on a digraph $G = (V, A)$, where V denotes the vertex-set while A is the set of arcs. An origin vertex $t \in V$ and a destination vertex $d \in V$ are given, but the costs associated with each arc $(i, j) \in A$ are uncertain. Thus, such uncertain data can be modeled as an interval (RSP-I) $[l_{ij}, u_{ij}]$, with $l_{ij}, u_{ij} \in \mathbb{R}$, where $u_{ij} \geq l_{ij} \geq 0$, or else by a set of discrete scenarios (RSP-D) $S = \{1, 2, ..q\}$, where every scenario $k \in S$ specifies a cost value c_{ij}^k for each arc. Without loss of generality, using the interval data, a scenario k is an assignment of arc costs $c_{ij}^k \in [l_{ij}, u_{ij}]$, for all $(i, j) \in A$.

Three main components are related in the application of robust optimization methods. These components are: (i) the way uncertain data are modeled (e.g. with the use of scenarios), (ii) the selection of an appropriate robust optimization criterion such as *minmax*, *minmax regret*, *minmax relative regret*, *α-robustness*, *bw-robustness*, *pw-robustness*, referred here as Robust Optimization Criteria (ROC), and (iii) the choice of a mathematical model and methods to generate robust solutions [36].

In this study, different ROC and their main applications are reviewed. Then, we introduce two Robust Classifications Criteria (RCC) that can be applied for measuring the robustness of both scenarios and solutions. The alternative RCC proposed here can be used independently of other criteria, as an additional support to classical ROC, or still as a tool for intensification in local search strategies. It is important to highlight that the proposed RCC are not optimization criteria, instead they are a mesure of quality based on ranking (classification) of solutions over all the scenarios considered. The remain of this work is organized as follows: a state-of-the-art on ROC is presented in Section 2, followed by a detailed description of the proposed RCC in Section 3. Finally, concluding remarks are given in Section 4.

2 State of the art on ROC

The robust optimization approach has been typically modeled with a set of scenarios defined with discrete or interval values [5], as previously mentioned. In either case, decisions can be made according to one of the following ROC: *minmax, minmax regret, minmax relative regret*, *lexicographical criteria*, *α-robustness*, *bw-robustness* and *pw-robustness*, etc. These are the most common ROC to deal with robust optimization problems, and their use mainly depend on the application and the optimization goals. Below we review such criteria and provide some practice cases for them.

The *minmax* family criteria, *i.e*, the absolute *minmax*, the *minmax regret* and the *minmax relative regret* [53], are usually refereed as conservative criteria since they aim at minimizing the possible losses whenever the worst case scenario occurs [36].

The *minmax* criterion has been initially developed for game theory by Von Neumann [48] for a two-player zero-sum game. The *minmax* dual problem relies on maximizing the minimum gain, named *maxmin*. Both the *minmax* and the *maxmin* are used whenever the worst case scenario can imply a major damage. Extending the work of Von Neumann [48], Wald proposed a non-probabilistic decision-making model based on the worst-case of a decision [59]. According to the author's definition, the decisions are ranked based on their worst-case outcomes. This strategy discards the worst case scenario among the possible decisions in an optimization process. Actually, the *minmax* criterion is one of the most studied ROC and has been successfully applied to different problems [12, 33, 36], such as: competitive situations, punctual risky decisions, robust shortest path [31, 45, 65], robust minimum spanning tree [34, 44, 64], among others.

Another possible ROC is the *lexicographic minmax* introduced by Dresher [20], which extends the work of Von Neumann known as the nucleolus of a matrix game in game theory. The idea consists in selecting a subset of optimal strategies, based on the optimal solution of *minmax*, which take advantage of the opponent player's mistakes. In the *lexicographic minmax* not only the worst case is minimized, as in the classical *minmax.* It also considers the second worst case, the third worst, etc. Therefore, the *lexicographic minmax* refines the concept of solution in the *minmax* approach, since it selects a unique set of outcomes, but not necessarily a unique solution (if there is more than one solution, all solutions selected have the same distribution).

Extending the work [20], the *lexicographic minmax* has been studied by [49] to improve the *minmax* when the uncertain data are modeled with discrete scenarios. The author in [49] also showed that the *lexicographic minmax* complies with both "Pareto-Optimality Principle" and the "Principle of the Transfers", whereas the standard *minmax* may violate both principles. In accordance with the "Pareto-Optimality Principle" all the objective functions are treated on the same way without preferences and/or specific assumptions, and the Principle of the Transfers is commonly recognized as the essential axiom for equity measure [39]. Finally, the author showed that the solutions obtained by the *lexicographic minmax* are better than those obtained by the *minmax* applying this criterion to location problems [49]. The *lexicographic minmax* has been successfully applied to allocation problems [38, 60], location problems [47, 50], and sensor node placement [2], as well as it has been modeled and integrated to mathematical programming [1, 29]

The α-*robustness* has been proposed by Kalaï [30] to be less conservative than the *minmax*. It extends the *lexicographic minmax* and is applied whenever the uncertain data are modeled using discrete scenarios [49]. Both criteria, the *lexicographic minmax* and the α-*robustness*, rank solutions considering the worst scenarios. Concerning the α-*robustness*, a vector of solutions, called "ideal solutions" are defined. An ideal solution is the best solution obtained for each scenario, and it is used to compute deviations. Given a solution θ, an ideal solution θ^* and a set of scenarios S, the cost of θ, also called distance, is the scenario $k \in S$ where the difference between the cost of θ and the cost of θ^* in scenario k is maximum. For example, considering a problem with two scenarios, if the ideal solution values for each scenario $k \in S$ are given by $cost(\theta^*, k) = \{10, 8\}$ and the cost of a solution θ for each scenario $k \in S$ is $cost(\theta, k) = \{15, 10\}$, then the maximum deviation between $cost(\theta, k)$ and $cost(\theta^*, k)$ is equal to 5. Recently, this criterion is used to treat uncertain attribute evaluations in the outranking methods [21] which build a list of preferences following a set of predetermined alternatives.

An alternative ROC to the *minmax* using discrete scenarios is presented by Bertsimas and Sim [16]. The authors propose a robust integer programming model that allows to control the conservatism degree of a solution by using probabilistic bounds and violation constraints. Let n be the number of binary variables for a discrete optimization problem with uncertain data associated with the cost coefficients. Then, we solve the robust counterpart by solving $n + 1$ instances of the original problem. Besides, the authors have proposed an algorithm for robust network flows using their model. This criterion has been mostly used for solving facility location problems [6], robust prize-collecting Steiner tree problems [4], robust knapsack problem [42] and the robust network loading problem with dynamic routing [40].

Roy (2010) [53] argues the *minmax* approach fails to translate the robustness since it focuses only on minimizing the worst case. Thus, the robust optimization is defined in his work as the ability to prevent undesirable impacts when uncertain data are handled. The author proposed a new ROC as an alternative for the *minmax*, named *bw-robustness*. A non-negative value b establishes a threshold that cannot be exceeded on most of the scenarios, while the constant w defines the value that cannot be exceeded by any scenario represented as follows, $w \leq max_\theta min_k(v_k(\theta))$, where k is a scenario, θ is a feasible solution and $v_k(\theta)$ is the cost of the feasible solution in a scenario. The *bw-robustness* is used only when the scenarios correspond to discrete values and the number of scenarios is very large. Like for the *minmax* family criteria, the *bw-robustness* can be addressed as *bw-absolute robustness*, *bw-absolute deviation* and relative regret *bw-relative deviation*. The *bw-robustness* has been successfully applied to the robust shortest path problem [24] and to the robust military mission planning problem [58] which consists of allocating resources and scheduling tasks during a military mission in order to protect the city borders.

The *pw-robustness* criterion is an extension of the *bw-robustness* criterion [23]. Thus, a solution is said to be robust whenever it ensures a w value for all scenarios, and if it reaches a value b in a p percentage of scenarios.

3 Robust Classification Criteria

We suggest here two RCC for measuring the solutions quality, as well as the impact of the scenarios over the solutions, named *robust absolute rank* and *robust mean rank*. In terms of solutions quality, the general idea of the RCC is to provide an evaluation of solutions based on their evaluation over all the scenarios. They do not look only for avoiding the worst case, neither for computing the worst case, the second one, etc., like in the α-*robustness*, the *bw-robustness* and the *pw-robustness*. The RCC focus on the number of times a solution obtain the best results in its evaluation over a given set of scenarios. Thus, in terms of evaluating a solution, it means that a solution is a good one if it is good in a higher number of scenarios.

In terms of scenarios, the proposed criteria provide an evaluation considering the impact of the solutions found for a given scenario, by identifying the scenarios which produce a higher number of better solutions.

The proposed RCC are less conservative than the *minmax* family and the *lexicographic* family of criteria, and they can be applied to evaluate independently the impact of applying an individual solution, as well as deciding about the scenarios which has produced the best solutions. The RCC can be used as main evaluation criteria, as well as a decision support tool for existing ROC. An example for the latter is whenever one looks for avoiding the worst case, and after that to classify the remaining solutions, *i.e.* a two stage evaluation. These criteria provide complementary and useful information for intensification in local search for a given scenario. An example where the intensification can be done is whenever one deals with a robust version of a NP-hard problem which implies to handle a NP-hard problem per scenario.

Let us consider a number of feasible solutions. Then, a way to evaluate the solutions over all scenarios can be done by a classification, as well as to know which scenarios have a higher impact on the solutions. For each known feasible solution, the *robust absolute rank* denotes the number of times for which the considered solution is the best over all known solutions for a specific scenario. Whenever a solution is the best for a specific scenario, it is assigned one point. At the end, the number of times a solution has a minimal value, is kept. If for a given scenario, two or more solutions draw for best solution, they all count as best and one point is added to its *robust absolute rank*. Considering a solution, the higher is its *robust absolute rank*, the better the solution is. Considering a set of scenarios S, the *robust absolute rank* can also be applied for evaluating S by taking the sum of all *robust absolute rank* for each known solution of S.

The *robust mean rank* is computed for every solution considering each scenario. Let r be the position of a solution in a classification over a set of solutions. Then, for a given solution and considering a set of S scenarios, the solution that returns the best result for a specific scenario k receives one point, the one that follows two points, and the one that returns the worst result receives r^{th} points. Whenever several solutions obtain the best result, they are all assigned one point. Whenever two solutions have the second best value, they are both assigned two points. For every solution and every scenario, the corresponding *robust mean rank* is equal to the mean number of points assigned to each solution. The smaller the *robust mean rank*, the better the corresponding solution is. Considering a scenario k, the *robust mean rank* can also be applied for evaluating S by taking the sum of the *robust mean rank* for each feasible solution of S.

An example for the RSP-D is provided in Figure 1. Tables 1 and 2 summarize the corresponding values for some ROC and the proposed RCC. In Table 1, each line corresponds to a possible solution for the graph depicted in Figure 1. The next columns stand for their cost considering respectively, the first, the second and the third scenario. Then, the remaining columns contain respectively the values for the following criteria: *minmax*, *minmax regret*, and *robust absolute rank*. Table 2 presents the classification for each solution in each scenario according to the *robust mean rank*. The three columns of classification indicate the points assigned for each solution in each scenario. The last column indicates the *robust mean rank* per solution.

For the solution $t-0-2-3-d$, the *robust absolute rank* is set to a value of 2 because this solution is the best for two scenarios out of three scenarios. It can be noticed that the solution $t-0-3-d$ stands for the best one using the mean, the *minmax* and the *minmax regret* criteria. On the contrary, the best solutions using the *robust absolute rank* and the *robust mean rank* are $t-0-2-3-d$ and $t-1-2-4-d$ because their cost evaluation is the best for a higher number of scenarios. In fact, for some applications, the robustness may enclose this idea: "robust solutions are those which are good and could be applied in most of the scenarios considered". Obviously, whenever a decision maker may avoid the worst case, these criteria cannot be applied alone. But, they can be coupled with a criteria from the *minmax* family considering the complementary information that can be obtained obtain with the proposed RCC (quality of solutions and the impact they have in the scenarios), opening opportunities for sensitivity analysis.

Figure 1: An example of graph with uncertain data represented by a discrete set of values.

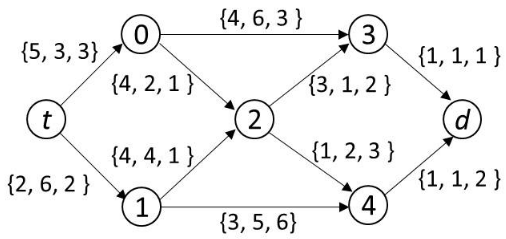

Table 1: Evaluating solutions for the robust shortest path problem.

Solutions	Scenarios costs C^1	C^2	C^3	Evaluation Mean	*minmax*	*minmax regret*	*robust absolute rank*
$t-0-3-d$	10	10	7	**9.00**	**10**	**4**	1
$t-0-2-3-d$	13	7	7	**9.00**	13	7	**2**
$t-0-2-4-d$	11	8	9	9.67	11	5	1
$t-1-4-d$	6	12	10	9.33	12	5	1
$t-1-2-3-d$	10	12	6	9.33	12	5	1
$t-1-2-4-d$	8	13	8	9.67	13	6	**2**
robust absolute rank per scenario	**2**	**2**	**4**				

Table 2: Applying the *robust mean rank*.

Solutions	Classification $k=1$	$k=2$	$k=3$	*robust mean rank* per solution
$t-0-3-d$	2	2	1	1.67
$t-0-2-3-d$	2	1	1	**1.33**
$t-0-2-4-d$	3	1	2	2.00
$t-1-4-d$	1	3	2	2.00
$t-1-2-3-d$	1	3	2	2.00
$t-1-2-4-d$	1	2	1	**1.33**
robust mean rank per scenario	1.83	2.00	**1.33**	

4 Concluding remarks

In spite of its simplicity, the proposed RCC enclose the idea of better solutions in a higher number of scenarios, as well as, they provide an easy way to identify the scenarios which has produced a higher number of better solutions. Contrary to the *minmax* family criteria and the *lexicographic* ROC, the proposed criteria provide a way to quantify a robust solution in a global perspective as the impact of the scenarios over the solutions. The RCC can be used as independent or be coupled with a ROC as a complementary information for the decision-making. A possible direction is to apply these RCC for intensification in local searches.

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