3D COLORED MESH STRUCTURE-PRESERVING FILTERING WITH ADAPTIVE P-LAPLACIAN ON DIRECTED GRAPHS

S. Bougleux, O. Lézoray *

Normandie Univ, UNICAEN, ENSICAEN
CNRS, GREYC, 14000 Caen, France

A. Nouri

Vision Lab. Isen Brest, L@bISEN
29228 Brest Cedex, France

ABSTRACT

Editing of 3D colored meshes represents a fundamental component of nowadays computer vision and computer graphics applications. In this paper, we propose a framework based on the p-laplacian on directed graphs for structure-preserving filtering. This relies on a novel objective function composed of a fitting term, a smoothness term with a spatially-variant pTV norm, and a structure-preserving term. The last two terms can be related to formulations of the p-Laplacian on directed graphs. This enables to impose different forms of processing onto different graph areas for better smoothing quality.

Index Terms— 3D colored meshes, graph signal, structure preserving filtering, p-Laplacian.

1. INTRODUCTION

Structure-preserving filtering is one of the most important processing tasks dedicated to image editing and computational photography. During the last decade, many structure-preserving smoothing filters have been proposed so far [1, 2, 3, 4]. To enable the editing of an image, they decompose it into a base layer containing the rough structures with preserved edges and several layers of increasing level of details. From this hierarchical representation, image editing tasks such as smoothing, abstraction, and sharpening can be performed [5]. At the same time, the advent of low cost 3D sensors has enabled the large development of 3D scanning. Using either a real 3D scanner or a set of images obtained by moving around an object, one can easily obtain 3D colored point clouds or meshes where each point or vertex is described by its 3D position and RGB color. However, the visual quality of the acquired data is not always of sufficient especially when a further 3D printing is planned. In addition, if the 3D data has to be used in virtual reality applications, it often has to be post-processed to be visually simplified. With such 3D processing tasks to be performed, there is interest into the development, as this has been done for images, of structure-preserving filters that operate on 3D colored data. Very few works have considered this kind of problem. Some structure-preserving filters have been extended to 3D meshes but they only consider the editing of the 3D points' coordinates to sharpen the mesh [6, 7]. Some recent works have considered the extension for 3D colored meshes by a 3D extension of unsharp masking [8] or with the use of morphological filters to obtain a hierarchical decomposition of the 3D color information [9, 10] .

In this paper we propose a structure-preserving filter based on an objective function composed of three terms (Sec. 3): a data-fitting term, a smoothness term and a structure preserving term. To have a more adaptive filtering, the smoothness term relies on a spatially-variant p-total variation norm (pTV). This extends our previous works [11] that considered only the two first terms with a fixed pTV norm. The smoothness term can be related to formulations of the p-Laplacian on directed graphs (Sec. 2). Finally the structure-preserving term enables to preserve the most important structures. This enables to impose different forms of regularization onto different 3D areas for better smoothing quality, as illustrated and discussed in Sec. 4.

2. P-LAPLACIANS ON DIRECTED GRAPHS

Graphs provide a powerful and common structure for representing and processing structured and unstructured data, like images, colored 3D point clouds or meshes. The data are assumed to be composed of elements living in $\mathbb{R}^{n_c}$, *e.g.* 3D points or RGB colors. A data of n_v elements is represented by a real matrix $\mathrm{F} = [f_{i,c}]_{i \in N_v, c \in N_c}$, with $N_v = \{1, \ldots, n_v\}$ and $N_c = \{1, \ldots, n_c\}$, so that $f_{i,c}$ encodes the c-th component of the i-th element. Alternatively, the i-th element can be represented by a vector $\mathrm{f}_i = [f_{i,c}]_{c \in N_c}$. The elements are connected by edges to form a graph used as a domain for processing the data. Since connections do not usually have the same strength, a weight is associated to each edge, and the graph can be fully represented by its weighted vertex-vertex adjacency matrix $\mathrm{W} = [w_{i,j}]_{i \in N_v, j \in N_v}$, with $w_{i,j} \in (0, +\infty)$ if the elements i and j are connected by an edge from i to j, or $w_{i,j} = 0$ if they are not. Graphs are here assumed to

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be without self-loops ($w_{i,i} = 0$) and directed ($w_{i,j}$ and $w_{j,i}$ can be different). This is motivated by the construction of graphs based on k-nearest neighbors, that are known to produce asymmetric connections in general, *i.e.* an edge from i to j does not necessarily imply an edge from j to i. Such graphs are commonly symmetrized, *i.e.* W is replaced by $\frac{1}{2}(\mathrm{W} + \mathrm{W}^T)$, before processing the data. While this is mathematically correct for most methods based on the unnormalized Laplacian $\mathrm{L} = \mathrm{diag}((\mathrm{W}+\mathrm{W}^T)1_{n_v})-(\mathrm{W}+\mathrm{W}^T)$ [12, 13, 14], this is not the case for most second-order operators, especially non-linear operators like curvatures and p-Laplacians [15, 11]. These operators are the main ingredients of filtering techniques based on variational principles, but few of them have been considered for directed graphs. We have recently proposed several formulations of the p-Laplacian [11]. In this paper we describe the unnormalized case used by the filtering process presented in the next section.

Given a data F, its gradient over an weighted graph W can be defined as $\nabla^{\mathrm{W}}\mathrm{F} = (\partial_j^{\mathrm{W}}\mathrm{F})_{j\in N_v}$, with $\partial_j^{\mathrm{W}}\mathrm{F} = [\sqrt{w_{i,j}}(f_{j,c} - f_{i,c})]_{i\in N_v, c\in N_c}$ the directional differences according to vertex j. The gradient at a vertex i is thus given by $\nabla_i^{\mathrm{W}}\mathrm{F} = [\sqrt{w_{i,j}}(f_{j,c} - f_{i,c})]_{j\in N_v, c\in N_c}$. Its magnitude

$$|\nabla_i^{\mathrm{W}}\mathrm{F}| = \left(\sum_{j=1}^{n_v} w_{i,j} \sum_{c=1}^{n_c} (f_{j,c} - f_{i,c})^2\right)^{1/2}$$
$$= \left(\sum_{j=1}^{n_v} w_{i,j} \|\mathrm{f}_j - \mathrm{f}_i\|^2\right)^{1/2} = \left(\sum_{c=1}^{n_c} |\nabla_i^{\mathrm{W}}\mathrm{F}_c|^2\right)^{1/2},$$

with $\|\mathrm{f}\| = \sqrt{\sum_{c=1}^{n_c} f_c^2}$ and $\mathrm{F}_c \in \mathbb{R}^{n_v}$ the restriction of F to the c-th component, provides a basic tool for measuring the variations of F at each vertex. The regularity of F over the graph can then be measured by its p-total variation (pTV) defined as the L_1 norm of a power of the gradient magnitude:

$$\|\mathrm{F}\|_{p\mathrm{TV}} = \| \, |\nabla^{\mathrm{W}}\mathrm{F}|^p \|_1 = \sum_{i=1}^{n_v} |\nabla_i^{\mathrm{W}}\mathrm{F}|^p \qquad (1)$$

where $p \in [1, +\infty)$ controls the degree of regularity, and $|\nabla^{\mathrm{W}}\mathrm{F}| = [\, |\nabla_i^{\mathrm{W}}\mathrm{F}| \,]_{i\in N_v}$. As for undirected graphs [16, 17], it is easy to show that $\nabla\|\mathrm{F}\|_{p\mathrm{TV}} = p\,\mathrm{L}_{p,\mathrm{F}}\mathrm{F}$ holds, with

$$\mathrm{L}_{p,\mathrm{F}} = \mathrm{diag}((\mathrm{W}_{p,\mathrm{F}} + \mathrm{W}_{p,\mathrm{F}}^T)1_{n_v}) - (\mathrm{W}_{p,\mathrm{F}} + \mathrm{W}_{p,\mathrm{F}}^T) \qquad (2)$$

and $\mathrm{W}_{p,\mathrm{F}} = \mathrm{diag}(|\nabla^{\mathrm{W}}\mathrm{F}|^{p-2})\mathrm{W}$. In other terms, we have:

$$[\mathrm{L}_{p,\mathrm{F}}\mathrm{F}]_{i,c} = \sum_{j=1}^{n_v} \left(\frac{w_{i,j}}{|\nabla_i^{\mathrm{W}}\mathrm{F}|^{2-p}} + \frac{w_{j,i}}{|\nabla_j^{\mathrm{W}}\mathrm{F}|^{2-p}}\right)(f_{i,c} - f_{j,c})$$

This defines the p-Laplacian of F on W. The Laplacian L is retrieved for $p=2$, and for $p=1$, $\mathrm{L}_{1,\mathrm{F}}\mathrm{F}$ defines the weighted (mean) curvature of F. The p-Laplacian is non-linear, and $\mathrm{L}_{p,\mathrm{F}}$ is symmetric, positive semi-definite, and can be viewed as a data-dependent Laplacian. It can be rewritten as $\mathrm{L}_{p,\mathrm{F}} = \mathrm{diag}(\overline{\mathrm{W}}_{p,\mathrm{F}}1_{n_v}) - \overline{\mathrm{W}}_{p,\mathrm{F}}$ with $\overline{\mathrm{W}}_{p,\mathrm{F}} = \mathrm{W}_{p,\mathrm{F}} + \mathrm{W}_{p,\mathrm{F}}^T$, equivalent to a data-dependent Laplacian on an undirected graph.

The regularity measured by the pTV norm and the p-Laplacian depends on the global parameter p. As proposed in the following section, more flexibility can be obtained by adapting this parameter locally at each vertex.

3. ADAPTIVE P-LAPLACIAN STRUCTURE-PRESERVING FILTERING

We present a structure-preserving smoothing filter based on an adaptive p-Laplacian and an adaptive preservation of gradient magnitudes. This is inspired by a recent work in image processing [18] which considers other measures of local variations and a different formulation of their local adaptation.

3.1. Energy formulation

Given a data $\mathrm{F}^0 \in \mathbb{R}^{n_v \times n_c}$, the proposed method consists to find a smoother version that minimizes an objective function $E(\mathrm{F}, \mathrm{F}^0) = \lambda_d E_d(\mathrm{F}, \mathrm{F}^0) + \lambda_s E_s(\mathrm{F}, \mathrm{F}^0) + \lambda_r E_r(\mathrm{F})$, with E_d a data term, E_r a regularity term, E_s a structure-preserving term, and $\lambda_d, \lambda_s, \lambda_e \in [0, +\infty)$ constant balancing weights.

The data term E_d aims at providing a solution close to F^0. This is measured according to the mean square error:

$$E_d(\mathrm{F}, \mathrm{F}^0) = \|\mathrm{F} - \mathrm{F}^0\|^2 \qquad (3)$$

with $\|\mathrm{F}\| = \sqrt{\mathrm{tr}(\mathrm{F}^T\mathrm{F})}$. The local differences $(f_{i,c} - f_{i,c}^0)$ should be null, or low, at vertices representing parts of the data that must be preserved. This is guided by the structure-preserving term E_s. It measures a weighted mean square error between the gradient magnitude of F and the one of F^0 on the same weighted graph $\mathrm{S} = [s_{i,j}]_{i,j\in N_v}$:

$$E_s(\mathrm{F}, \mathrm{F}^0) = \frac{1}{2n_s} \sum_{i=1}^{n_v} \alpha_i \left(|\nabla_i^{\mathrm{S}}\mathrm{F}|^2 - |\nabla_i^{\mathrm{S}}\mathrm{F}^0|^2\right)^2 \qquad (4)$$

with $\alpha = [\alpha_i]_{i\in N_v}$, $\alpha_i \in [0, 1]$ the degree of structure preservation, and $n_s = n_c \sum_{i=1}^{n_v} \alpha_i$ a normalization term. The term α behaves as a mask that enforces more or less the preservation of the gradient magnitude of F^0. To ensure that structures in F^0 are preserved , it should be defined from an indicator or a detector of saliency, as detailed in Sec. 3.3.

The local differences involved in the approximation terms E_d and E_s should be higher at vertices representing unimportant details. This is obtained by reducing the variations of F^0 at these vertices according to an adaptive pTV function:

$$E_r(\mathrm{F}) = \sum_{i=1}^{n_v} \frac{1}{p_i} |\nabla_i^{\mathrm{W}}\mathrm{F}|^{p_i} \qquad (5)$$

where $p_i \in [1, 2]$ fixes the degree of regularity at vertex i and ensures convexity of (5). Intuitively, it should depend on F^0

and be inversely proportional to α_i, *i.e.* large at vertices where the data needs to be smoothed and low where the data must be preserved. Note that the graphs W and S are different. In practice, it is more coherent and computationally effective that they are structurally equivalent, *i.e.* $s_{i,j} = 0 \Leftrightarrow w_{i,j} = 0$.

Since the terms E_d, E_s and E_r are convex functions of F, the objective function E is also convex, and several methods can be used to compute a solution. In this paper we derive a simple filter related to weighted mean filtering.

3.2. Filtering process

To optimize the objective function E w.r.t. F, we consider the gradient $\nabla E(\mathrm{F}) = [\partial E(\mathrm{F})/\partial f_{i,c}]_{i\in N_v, c\in N_c}$ and try to solve the system of nonlinear partial difference equations $\nabla E(\mathrm{F}) = 0$ with

$$\nabla E(\mathrm{F}) = 2(\mathrm{F} - \mathrm{F}^0) + \lambda_r \mathrm{L}_{r,\mathrm{F}}\mathrm{F} + \tfrac{2}{n_s}\lambda_s \mathrm{L}_{s,\mathrm{F}}\mathrm{F} \qquad (6)$$

where $\mathrm{L}_{r,\mathrm{F}}$ and $\mathrm{L}_{s,\mathrm{F}}$ are defined below.

Indeed, it is straightforward that $\nabla E_d(\mathrm{F}) = 2(\mathrm{F} - \mathrm{F}^0)$, and the gradient of the regularity term E_r is given by $\nabla E_r(\mathrm{F}) = \mathrm{L}_{r,\mathrm{F}}\mathrm{F}$, with

$$[\mathrm{L}_{r,\mathrm{F}}\mathrm{F}]_{i,c} = \sum_{j=1}^{n_v} \left(\frac{w_{i,j}}{|\nabla_i^{\mathrm{W}}\mathrm{F}|^{2-p_i}} + \frac{w_{j,i}}{|\nabla_j^{\mathrm{W}}\mathrm{F}|^{2-p_j}} \right) (f_{i,c} - f_{j,c})$$

Contrary to the p-Laplacian described in Sec. 2, $\mathrm{L}_{r,\mathrm{F}}$ uses here a regularity degree $\mathrm{p} = [p_i]_{i\in N_v}$ adapted to each vertex.

Similarly, the gradient of the structure preserving term E_s is given by $\nabla E_s(\mathrm{F}) = \frac{2}{n_s}\mathrm{L}_{s,\mathrm{F}}\mathrm{F}$ with

$$[\mathrm{L}_{s,\mathrm{F}}\mathrm{F}]_{i,c} = \sum_{j=1}^{n_v} (a_i s_{i,j} + a_j s_{j,i}) (f_{i,c} - f_{j,c}) \qquad (7)$$

with $a_i = \alpha_i(|\nabla_i^{\mathrm{S}}\mathrm{F}|^2 - |\nabla_i^{\mathrm{S}}\mathrm{F}^0|^2)$.

We propose to use a linearized Gauss-Jacobi iterative method to find a solution to the system of non-linear equations. Let t be an iteration step, and $\mathrm{F}^{(t)}$ be the solution at step t. Starting with $\mathrm{F}^{(0)} = \mathrm{F}^0$, and $\mathrm{g}^0 = |\nabla^{\mathrm{S}}\mathrm{F}^0|^2$, the method iterates the following steps:

$$\begin{aligned}
\forall i, \quad & h_i \leftarrow |\nabla_i^{\mathrm{W}}\mathrm{F}^{(t)}|^{p_i-2} \\
\forall i, \quad & g_i \leftarrow \alpha_i \left(|\nabla_i^{\mathrm{S}}\mathrm{F}^{(t)}|^2 - g_i^0 \right) \\
\forall i, \forall j, \quad & \overline{w}_{i,j} \leftarrow \tfrac{1}{2}(h_i w_{i,j} + h_j w_{j,i}) \\
\forall i, \forall j, \quad & \overline{s}_{i,j} \leftarrow \tfrac{1}{n_s}(g_i s_{i,j} + g_j s_{j,i}) \\
\forall i, \forall c, \quad & f_{i,c}^{(t+1)} \leftarrow \frac{\lambda_d f_{i,c}^0 + \sum_{j=1}^{n_v} (\lambda_r \overline{w}_{i,j} + \lambda_s \overline{s}_{i,j}) f_{j,c}^{(t)}}{\lambda_d + \sum_{j=1}^{n_v} \lambda_r \overline{w}_{i,j} + \lambda_s \overline{s}_{i,j}}
\end{aligned}$$

until convergence or a given number of steps is reached. An iteration computes $\mathrm{F}^{(t+1)}$, at each vertex, as a weighted mean of $\mathrm{F}^{(t)}$ in the neighborhood of the vertex in the graph $\overline{\mathrm{W}} + \overline{\mathrm{S}}$. Contrary to the family of filters based on pTV on undirected [16, 17] and directed graphs [11], the proposed filter adapts the regularity parameter p locally and enforces the gradients to be preserved according to a structure-preserving map.

3.3. Parameters for colored 3D point clouds

The proposed structure-preserving filter depends on several parameters: the graphs W and S, the vector of local regularity degrees p, and the vector of local structure-preserving strengths α. They are determined by the nature of the data and the desired filtering effect. We describe them for smoothing colored 3D point clouds. Given a set $X = \{x_i\}_{i=1,\ldots,n_v}$ of n_v points $x_i \in \mathbb{R}^3$, and the RGB colors $\mathrm{F}^0 \in \mathbb{R}^{n_v \times 3}$ associated to these points ($n_c = 3$), we assume a connected graph S^0 has already been constructed for connecting the points. When X samples the surface of a 3D object, S^0 should be the graph induced by the edges of a mesh having X as vertices. But any graph can be used as long as it is connected and represents the geometric structure of the data.

The graphs S and W are constructed by connecting each point $x_i \in X$ to its k nearest points, within an α-hop $\mathcal{N}_i^\alpha$ in S^0, according to the dissimilarity measure $d(x_i, x_j) = d_{\mathrm{EMD}}(H(\Phi_i^\tau), H(\Phi_j^\tau))$, where $\Phi^\tau = (\Phi_i^\tau)_{i=1,\ldots,n_v}$ associates to each point x_i a feature $\Phi_i^\tau = (\mathrm{f}_j^0)_{j\in\mathcal{N}_i^\tau\cup\{i\}}$, *i.e.* the colors of the points x_j around x_i within a τ-hop $\mathcal{N}_i^\tau$ in S^0. Since S^0 is not assumed to be regular, features do not have the same size, and a simple L_2 norm cannot be used to compare them. The Earth Mover Distance d_{EMD} [19] between the histograms H of the features in the $L^*a^*b^*$ color space is more appropriate. The retained connections are then directly used to construct the graph S. This graph is unweighted, *i.e.* $s_{i,j} = 1$ if x_j is among the k nearest neighbors of x_i, or 0 otherwise. The graph W has the structure of S but it is weighted according to the distances between the features defined above by a parameter-less similarity:

$$w_{i,j} = 1 - \frac{d_{\mathrm{EMD}}(H(\Phi_i^\tau), H(\Phi_j^\tau))}{\max\limits_{\substack{l=1,\ldots,n_v \\ s_{i,l}=1}} d_{\mathrm{EMD}}(H(\Phi_i^\tau), H(\Phi_l^\tau))} \qquad (8)$$

if $s_{i,j} = 1$, and 0 else. Remark that S and W are directed.

The two other parameters p and α are defined from a common saliency map $\mathrm{m} = [m_i]_{i=1,\ldots,n_v}$ which indicates, for each point x_i, the local degree of structure in F^0. This is measured by the normalized sum of distances within a ρ-hop in S^0: $m_i = \frac{1}{|\mathcal{N}_i^\rho|}\sum_{j\in\mathcal{N}_i^\rho} d_{\mathrm{EMD}}(H(\Phi_i^{\rho-1}), H(\Phi_j^{\rho-1}))$. The parameters are then defined by :

$$p_i = 1 + \frac{1}{1 + m_i^2}, \qquad \alpha_i = \frac{m_i - \min_j m_j}{\delta_i(\max_j m_j - \min_j m_j)} \qquad (9)$$

with δ_i the number of edges starting at i in S (outgoing degree). This enables to normalize the gradients in the objective function E_s. As discussed in the previous section, p and α defined by Eq. 9 are antagonists, one for smoothing the data and the other for preserving its main structures.

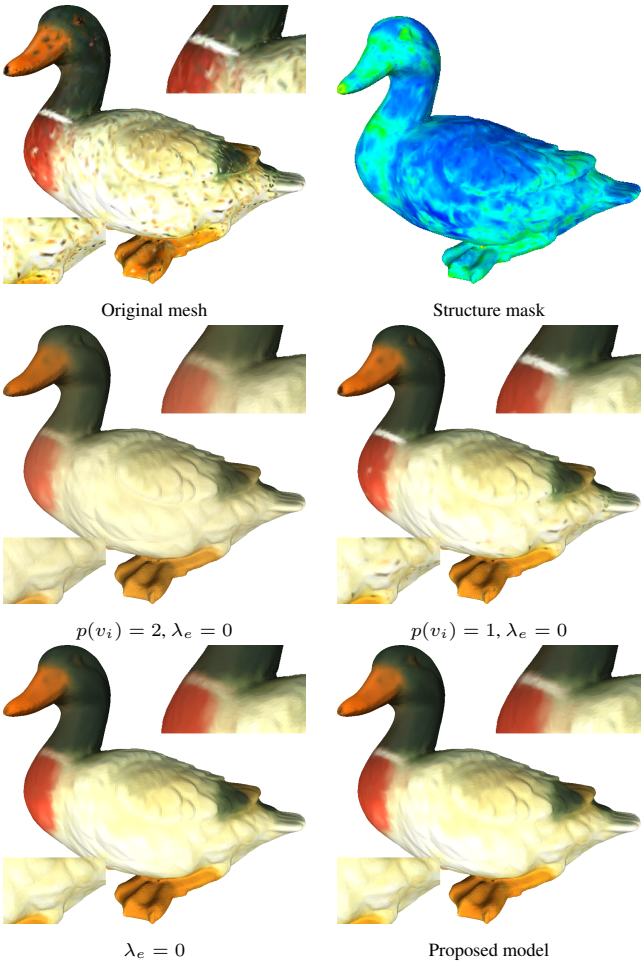


Fig. 1. Illustration of the proposed model.

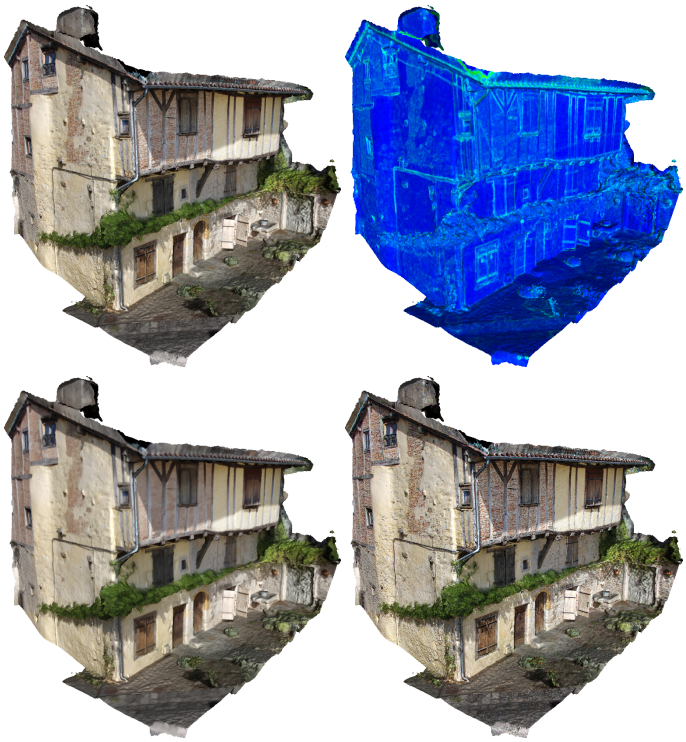


Fig. 2. First row: original mesh, structure mask. Second row: filtering and sharpening results.

4. EXPERIMENTAL RESULTS AND CONCLUSION

To illustrate the proposed feature-preserving filter, we consider colored 3D point clouds acquired by digitizing an object by means of a scanner. A mesh obtained by reconstruction techniques is also provided. We use this mesh to define the initial graph S^0. For all the experiments, the parameters for constructing the graphs S and W, and the saliency map m, are fixed to $\alpha = 5$, $\tau = 1$ and $\rho = 2$ for the sizes of the hops, and the number of neighbors is set to $k = 10$. The balancing terms in the objective function are set to $\lambda_d = 10^{-3}$, $\lambda_r = 1$, and $\lambda_s = 0.25$. The number of iterations is fixed to 25.

Figure 1 shows results on a low-resolution mesh (a scan of a stuffed duck with 19247 vertices and 38490 faces) to illustrate the benefit of the proposed approach (each image shows a caption of the mesh and two cropped and zoomed areas). First row shows the original mesh and its structure mask M (shown with a heat map LUT). Second row shows classical results obtained with fixed values of p and no structure-preserving term. With $p = 2$ (the Laplacian) a strong smoothing effect is obtained, whereas with $p = 1$ (the Graph Total Variation), the edges are better preserved. However some small structures are also kept. Better filtering results are obtained with a spatially varying norm and the unwanted small details are removed. However, some parts such as the duck's collar are now more blurry. This problem is then corrected by using the whole model we propose that incorporates an additional structure-preserving term. The final result is inline with our objective: to eliminate unimportant fine-scale details while maintaining the primary structures.

Figure 2 shows an exemple of 3D colored mesh editing for sharpening. The mesh is a high-resolution scan (780977 vertices and 1557701 faces) of a medieval house in the city of Bellac, France. As it can be seen, in the filtering result, the small fine details have been suppressed while preserving the strongest structures. This can be used as the base layer of a sharpness enhancement procedure [20]. The difference between the original mesh and its filtering is computed, boosted by a factor 0.8 and added back to the filtered version. As it can be seen, the sharpened result has enhanced the local contrast without artifact magnification or detail loss. Note that the enhancements are better visible at high resolution. This shows the benefit of our approach for 3D editing tasks. Further works will consider other minimization schemes as well as other editing tasks such as inpainting and abstraction.

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