A shakedown analysis in hardening plasticity

Quoc Son Nguyen

On shakedown analysis in hardening plasticity

Quoc-Son Nguyen*

Laboratoire de Mecanique des Solides, CNRS-umr7649, Ecole Polytechnique, 91128 Palaiseau, Paris, France

Abstract

The extension of classical shakedown theorems for hardening plasticity is interesting from both theoretical and practical aspects of the theory of plasticity. This problem has been much discussed in the literature. In particular, the model of generalized standard materials gives a convenient framework to derive appropriate results for common models of plasticity with strain-hardening. This paper gives a comprehensive presentation of the subject, in particular, on general results which can be obtained in this framework. The extension of the static shakedown theorem to hardening plasticity is presented at first. It leads by min–max duality to the definition of dual static and kinematic safety coefficients in hardening plasticity. Dual static and kinematic approaches are discussed for common models of isotropic hardening of limited or unlimited kinematic hardening. The kinematic approach also suggests for these models the introduction of a relaxed kinematic coefficient following a method due to Koiter. Some models for soils such as the Cam-clay model are discussed in the same spirit for applications in geomechanics. In particular, new appropriate results concerning the variational expressions of the dual kinematic coefficients are obtained.

Keywords: A. Plasticity; Strain hardening; B. Shakedown; C. Duality; Variational principles

1. Introduction

The elastic shakedown phenomenon is related to the long-term behaviour of a solid under variable loads and expresses the fact that the mechanical response of solids becomes purely elastic if the load amplitude is small enough or if the hardening effect is strong enough, especially in cyclic plasticity, immaterial of the initial state of the evolution. The possibility of shakedown is interesting in the analysis of quasi-static or dynamic response of elastic–p1astic solids under cyclic loads. Indeed, in

* Tel.: +33-1-69333375; fax: +33-1-69333026.

E-mail address: son@lms.polytechnique.fr (Q.-S. Nguyen).

cyclic plasticity, an uncontrolled progressive or alternating plastic deformation is often the origin of undesirable effects for the resistance of a solid. For example, the existence of shakedown will prevent the fatigue phenomenon under plastic strains, which results in failure under a small number of cycles, in contrast with the fatigue under elastic strain with much higher number of cycles (cf. Dang Van and Papadopoulos, 1999). For this reason, shakedown conditions are discussed in a large number of papers, for different applications in the design of structures. Classical shakedown theorems in a quasi-static deformation takes its definitive form from the pioneering works of Bleich (1932), Melan (1936) and Koiter (1960). Its generalization to dynamics has been discussed (cf. Corradi and Maier, 1974). Further extensions to hardening plasticity, non-standard plasticity, to visco-plasticity or to damage mechanics and poroplasticity can be found in a large number of references (e.g. Maier, 1972; Konig, 1987; Polizzotto et al., 1991; Debordes, 1976; Weichert and Maier, 2000; Maier, 2001; Bodovillé and de Saxcé, 2001). In particular, the reader can refer to Martin (1975), Polizzotto (1982), Corigliano et al. (1995) for a rather complete presentation of the theory and historical survey, to Maier (2001), Ponter and Chen (2001) and Hachemi and Weichert, 1998 for new directions on the related subjects.

The objective of this paper is to give a presentation of shakedown theorems in hardening plasticity, available for common models of strain hardening. Since the extension of shakedown theory into hardening plasticity has been much discussed (e.g. Maier, 1972; Mandel, 1976; Nguyen, 1976; Polizzotto et al., 1991; Pycko and Mroz, 1992; Corigliano et al., 1995; Fuschi, 1999 etc.) a complete survey on general results for hardening plasticity in the spirit of Koiter's discussion (Koiter, 1960) is certainly useful. However, because of the complexity and the diversity of hardening laws, it is clear that such a task is difficult and it will be easier to give only a less ambitious presentation on general theorems which can be derived within some description. In this spirit, our attention is focussed on the framework of generalized standard models of plasticity. This framework is a straightforward extension of perfect plasticity, with the same ingredients of convexity and normality and has been shown to be large enough to cover most common models of hardening plasticity (Halphen and Nguyen, 1975; Nguyen, 2000). As in perfect plasticity (Koiter, 1960; Debordes, 1976; Pycko and Mroz, 1992), the method of min–max duality can be followed within this framework. The starting point is a static shakedown theorem, given previously in Nguyen (1976); Polizzotto et al. (1991). This theorem leads to the definition of the safety coefficient with respect to shakedown and, by a min–max duality, to dual expressions of the safety coefficient obtained respectively from static and kinematic approaches. This method is then applied to the particular cases of strain hardening materials for which hardening parameters are the plastic strain or equivalent plastic strain.

This discussion also suggests the introduction of a relaxed kinematic safety coefficient, following a method due to Koiter, in the so-called Koiter's second shakedown theorem (Koiter, 1960). Some of the obtained results have been announced in Nguyen and Pham (2001). Min–max duality is also considered for common models of geomechanics such as the Cam-clay model. Finally, a simple example is given in the last section to illustrate the kinematic approach in limited kinematic hardening. Our principal goal is to obtain appropriate theorems for common models of limited isotropic or kinematic hardening or for pressure-dependent geomaterials.

2. Static shakedown theorems

The quasi-static evolution of an elastic plastic solid under variable loads on the interval $[0, +\infty[$ is considered here. For dynamic-conditions, it is well known that additional terms due to the inertial forces can be taken into account and the same conclusion remains valid as it has been shown in the literature. Let $(u(t), \sigma(t), \varepsilon^p(t))$ be the elastic–plastic response of the solid starting from a given initial state on the interval $t \geqslant 0$. By definition, this response will shake down if the existence of the limit

$$\lim_{t\to\infty} \varepsilon^p(t) \tag{1}$$

is ensured. It can be noted that this property also ensures that $\lim_{t\to\infty} u(t) - u_{\mathrm{el}}(t)$ and $\lim_{t\to\infty}\sigma(t) - \sigma_{\mathrm{el}}(t)$ exist, under certain additional assumptions (e.g. Debordes and Nayroles, 1976), where $\sigma_{\mathrm{el}}(t), u_{\mathrm{el}}(t)$ denote the fictitious response of the same solid, assumed to be purely elastic, to the same loading. The proof of this statement is clear for discrete systems while for continua, some difficulties remain concerning the choice of a relevent functional space in the case of perfect plasticity (Debordes, 1976).

2.1. Perfect plasticity

The classical static theorem of shakedown in perfect plasticity, also known as Melan's theorem, states that:

Static shakedown theorem. *If there exists a self-stress field $s^*(x)$, a safety coefficient $m > 1$ and a time τ such that the stress field $m(s^* + \sigma_{\mathrm{el}}(t))$ satisfies everywhere and for all $t \geqslant \tau$ the plastic criterion*

$$f(m(s^*(x) + \sigma_{\mathrm{el}}(x,t))) \leqslant 0 \quad \forall x,\ \forall t > \tau$$

then there is shakedown, immaterial of the initial conditions.

The proof of Melan's theorem can be obtained in two steps. In the first step, it is shown that under the assumptions of the theorem, the dissipated energy W^{d} is necessarily bounded. In the second step, this property ensures the existence of $\lim_{t\to\infty}\varepsilon^p(t)$.

To prove the first step, it is useful to note that for all plastically admissible stress fields $\tilde{\sigma}$, the following inequality holds:

$$\int_\Omega (\sigma - \tilde{\sigma}) : \dot{\varepsilon}^p \, \mathrm{d}\Omega \geqslant 0.$$

By taking $\tilde{\sigma} = m(s^* + \sigma_{\mathrm{el}}(t))$ which is plastically admissible by assumption, it follows that

$$\int_\Omega (\sigma - \sigma^*):\ \dot{\varepsilon}^p \geqslant \frac{m-1}{m} \int_\Omega d_{\mathrm{in}}\, \mathrm{d}\Omega$$

with $\sigma^* = s^* + \sigma_{\mathrm{el}}$, where $d_{\mathrm{in}} = \sigma:\ \dot{\varepsilon}^p$. Since $\sigma - \sigma^*$ is a self-stress field and since $\dot{u} - \dot{u}_{\mathrm{el}} = 0$ on S_u, one obtains

$$0 = \int_\Omega (\sigma - \sigma^*):\ (\dot{\varepsilon} - \dot{\varepsilon}_{\mathrm{el}})\, \mathrm{d}\Omega = \int_\Omega (\sigma - \sigma^*):\ \dot{\varepsilon}^p + \int_\Omega (\sigma - \sigma^*):\ L^{-1}:\ (\dot{\sigma} - \dot{\sigma}^*)\, \mathrm{d}\Omega.$$

It follows that

$$-\int_\Omega (\sigma - \sigma^*):\ L^{-1}:\ (\dot{\sigma} - \dot{\sigma}^*)\, \mathrm{d}\Omega \geqslant \frac{m-1}{m} \int_\Omega d_{\mathrm{in}}\, \mathrm{d}\Omega,$$

which leads, after a time integration on the interval $[\tau, t]$, to

$$I(\tau) - I(t) \geqslant \frac{m-1}{m} W^{\mathrm{d}} \tag{2}$$

with

$$I(t) = \int_\Omega \frac{1}{2} (\sigma - \sigma^*):\ L^{-1}:\ (\sigma - \sigma^*)\, \mathrm{d}\Omega.$$

The dissipated energy $W^{\mathrm{d}}(t) = \int_\tau^t \int_\Omega \sigma:\ \dot{\varepsilon}^p\, \mathrm{d}\Omega\, \mathrm{d}\theta$ thus remains bounded for all initial conditions.

The second step consists of proving that $\varepsilon^p(t)$ tends to a limit. The fact that the dissipated energy remains bounded already ensures the existence of a limit of $\varepsilon^p(t)$ for any appropriate functional space which is complete with respect to the norm associated with the dissipation, since there exists a constant $c > 0$ such that $\sigma:\ \dot{\varepsilon}^p \geqslant c\|\dot{\varepsilon}^p\|$. This inequality follows from the fact that the origin of stress is strictly inside the elastic domain.

This convergence ensures immediately the existence of the limits $\lim_{t\to\infty} u(t) - u_{\mathrm{el}}(t)$ and $\lim_{t\to\infty} \sigma(t) - \sigma_{\mathrm{el}}(t)$ for discrete systems. A mathematical difficulty, however, remains for continuous media and concerns an appropriate choice of the norm. A method due to Nayroles (1977) can be applied concerning the stress field when the plastic criterion is symmetric with respect to the origin of stress. It consists of proving that the sequence $s_i = \sigma(t_i) - \sigma_{\mathrm{el}}(t_i)$ is a Cauchy sequence with the energy norm

$$\|s\|_{\mathrm{e}}^2 = \int_\Omega s:\ L^{-1}:\ s\, \mathrm{d}\Omega$$

and thus converges. Indeed, for all $t_2 > t_1 > t_0$, we have

$$\frac{1}{2}\|s_1 - s_2\|_{\mathrm{e}}^2 = \int_{t_1}^{t_2} \int_\Omega \dot{s}:\ L^{-1}:\ (s - s_1)\, \mathrm{d}\Omega\, \mathrm{d}t = -\int_{t_1}^{t_2} \int_\Omega \dot{\varepsilon}^p:\ (s - s_1)\, \mathrm{d}\Omega\, \mathrm{d}t.$$

The last integrand can also be written as

$$\begin{aligned} -\dot{\varepsilon}^p:\ (s - s_1) &= -\dot{\varepsilon}^p:\ ((\sigma - \sigma^*) - (\sigma_1 - \sigma_1^*)) \\ &= -\dot{\varepsilon}^p:\ (\sigma - \sigma^*) + \sigma_1:\ \dot{\varepsilon}^p - \dot{\varepsilon}^p:\ \sigma_1^*. \end{aligned}$$

In this expression, the first term is non-positive and the second is bounded by $d_{\mathrm{in}}=\sigma\colon \dot{\varepsilon}^p$. If the plastic criterion is symmetric with respect to the origin of stress, such as in the case of Mises or of Tresca's criterion, the stress $-\sigma_1^*$ is also plastically admissible, thus the third term is also bounded by d_{in}. It follows that

$$\Delta = \|s_1 - s_2\|_{\mathrm{e}}^2 \leqslant 4(W^{\mathrm{d}}(t_2) - W^{\mathrm{d}}(t_1)).$$

Since the dissipated energy is bounded and cannot decrease, s_i is a Cauchy sequence and so this sequence converges. It follows that $\sigma(t) - \sigma_{\mathrm{el}}(t)$ tends to a limit when $t \to \infty$.

In the presentation of the shakedown theorem in perfect plasticity, it can be assumed equivalently that there exists a particular plastic strain field ε^{p*} generating a self-stress s^* such that the stress field $m\sigma^*$ is plastically admissible. It should be also emphasized that two principal ingredients are involved in the proof of Melan's theorem. Firstly, the property of contraction of the distance separating two solutions starting from two initial states holds in perfect plasticity. Secondly, the possibility to have a constant solution for the plastic response exists by assumption.

2.2. *Hardening plasticity*

This discussion is limited to the case of generalized standard models of hardening plasticity. A generalized standard material is defined by the following conditions:

- State variables are (ε, α), which represent, respectively, the strain tensor and a set of internal parameters. Internal parameters α include the plastic strain and other hardening variables. There exists an energy potential $W(\varepsilon, \alpha)$ which leads to associated forces

$$\sigma = W_{,\varepsilon}, \quad A = -W_{,\alpha} \tag{3}$$

  and to a dissipation per unit volume d_{in}

$$d_{\mathrm{in}} = A\dot{\alpha}. \tag{4}$$

- The force A must be plastically admissible, this means that physically admissible forces A^* must remain inside a convex domain C, called the elastic domain and defined by the plastic criterion $f(A^*) \leqslant 0$.
- Normality law is satisfied for α:

$$\dot{\alpha} = \mu \frac{\partial f}{\partial A}, \quad \mu \geqslant 0, \ \ \mu f = 0. \tag{5}$$

  Thus, the following maximum dissipation principle is satisfied:

$$A^* \dot{\alpha} \leqslant A\dot{\alpha} = D(\dot{\alpha}) = \max_{A^* \in C} A^* \dot{\alpha} \tag{6}$$

in the spirit of Hill's maximum principle in perfect plasticity. Finally, as in perfect plasticity, the notions of convexity, normality, energy and generalized forces are four principal ingredients of the generalized standard models. The dissipation potential $D(\dot{\alpha})$ is convex, positively homogeneous of degree 1. This function is state independent, i.e. independent of the present value of state variables (ε, α), if the plastic criterion is state independent.

The following result has been obtained under some additional assumptions.

Proposition 1 (static shakedown theorem in hardening plasticity). *It is assumed that the plastic criterion is state-independent and that*

$$W = W_1(\varepsilon, \alpha) + W_2(\alpha), \tag{7}$$

where W_1 is quadratic and non-negative with respect to (ε, α) and $W_2(\alpha)$ is an arbitrary convex and differentiable function. Then there is shakedown whatever the initial state if there exists a time τ, a constant internal parameter field α^* and a safety coefficient $m > 1$ such that the force field $mA^*(t)$ is plastically admissible for all $t > \tau$, where $A^* = -W_{,\alpha}(\varepsilon^*, \alpha^*)$ denotes the force defined from the associated response $u^*(t)$.

The associated response u^* denotes the response of a corresponding elastic solid admitting the linear elastic relationship

$$\sigma = W_{,\varepsilon} = W_{1,\varepsilon} = L\colon \varepsilon + Y\alpha = L\colon (\varepsilon - \varepsilon^{p*}) \quad \text{with } \varepsilon^{p*} = -L^{-1}\colon Y\alpha^*.$$

under the same loading (prescribed forces and displacements) and under the inherent plastic strain ε^{p*}. In particular, the associated stress is $\sigma^*(t) = \sigma_{\mathrm{el}}(t) + s^*$ where s^* denotes the self-stress due to the inherent plastic strain ε^{p*}.

The proof of this proposition is relatively straightforward. Indeed, let $(u(t), \alpha(t))$ be a solution of the evolution problem starting from a given initial condition. Since under the assumptions introduced, mA^* is plastically admissible for all $t \geqslant \tau$, the following inequality holds:

$$\int_\Omega (A - A^*)\dot{\alpha}\, \mathrm{d}\Omega \geqslant \frac{m-1}{m} \int_\Omega A\dot{\alpha}\, \mathrm{d}\Omega.$$

From the fact that

$$\begin{aligned}&(\sigma - \sigma^*)\colon (\dot{\varepsilon} - \dot{\varepsilon}^*) - (A - A^*)(\dot{\alpha} - \dot{\alpha}^*)\\ &\quad = \frac{\mathrm{d}}{\mathrm{d}t}\{W_1(\varepsilon - \varepsilon^*, \alpha - \alpha^*) + W_2(\alpha) - W_2(\alpha^*) - W_{2,\alpha}(\alpha^*)(\alpha - \alpha^*)\}\end{aligned}$$

since $\dot{\alpha}^* = 0$, and that

$$\int_\Omega (\sigma - \sigma^*)\colon (\dot{\varepsilon} - \dot{\varepsilon}^*)\, \mathrm{d}\Omega = 0,$$

the following equation results:

$$\begin{aligned}&-\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \{W_1(\varepsilon - \varepsilon^*, \alpha - \alpha^*) + W_2(\alpha) - W_2(\alpha^*) - W_{2,\alpha}(\alpha^*)(\alpha - \alpha^*)\}\, \mathrm{d}\Omega\\ &\quad \geqslant \frac{m-1}{m} \int_\Omega d_{\mathrm{in}}\, \mathrm{d}\Omega.\end{aligned}$$

Thus estimate (2) is again obtained with

$$I(t) = \int_{\Omega} \{W_1(\varepsilon(t) - \varepsilon^*(t), \alpha(t) - \alpha^*)$$
$$+W_2(\alpha(t)) - W_2(\alpha^*) - W_{2,\alpha}(\alpha^*)(\alpha(t) - \alpha^*)\}\, \mathrm{d}\Omega.$$

Since $I(t)$ is non-negative as a sum of two non-negative terms, the dissipated energy $W^{\mathrm{d}}(t)$ is bounded by $I(\tau)$ for all $t > \tau$. Repeating the same arguments, the existence of the limit

$$\lim_{t\to\infty} \alpha(t) - \alpha^*$$

is ensured again from this estimate on the dissipated energy. In the same spirit as Melan's theorem, Proposition 2 gives a sufficient condition ensuring the shakedown of a solid subjected to a given loading path.

If the plastic criterion is symmetric with respect to the origin in the force space, Nayroles' method is still valid and leads then to the convergence of the stress field $\sigma(t)$ in the elastic energy norm. Indeed, the following expression holds:

$$\sigma\colon \dot{\varepsilon}^p = A\dot{\alpha} + \alpha(X - Y^{\mathrm{T}}L^{-1}Y)\dot{\alpha} + W_{2,\alpha}\dot{\alpha}$$

with the notation $W_1 = 1/2(\varepsilon\colon L\colon \varepsilon + \varepsilon\colon Y\alpha + \alpha X\alpha)$. Thus

$$-\dot{\varepsilon}^p\colon (\sigma - \sigma^*) + \sigma_1\colon \dot{\varepsilon}^p - \sigma_1^*\dot{\varepsilon}^p = -\dot{\alpha}\colon (A - A^*) + A_1\dot{\alpha} - A_1^*\dot{\alpha}$$
$$-W_{2,\alpha}\dot{\alpha} - W_{2,\alpha 1}\dot{\alpha} - (\alpha - \alpha_1)(X - Y^{\mathrm{T}}L^{-1}Y)\dot{\alpha}$$

and the same argument leads to

$$\Delta \leqslant 4(W^{\mathrm{d}}(t_2) - W^{\mathrm{d}}(t_1)) - (W_2(\alpha_2) - W_2(\alpha_1) - W_{2,\alpha 1})(\alpha_2 - \alpha_1))$$
$$-\frac{1}{2}(\alpha_2 - \alpha_1)(X - Y^{\mathrm{T}}L^{-1}Y)(\alpha_2 - \alpha_1) \leqslant 4(W^{\mathrm{d}}(t_2) - W^{\mathrm{d}}(t_1))$$

since the two last terms of the middle expression are non-positive after the assumptions of Proposition 2.

3. Min–max duality and dual safety coefficients

In the spirit of the min–max duality method, developed by Debordes and Nayroles (1976) or Pycko and Mroz (1992) for shakedown theorems in perfect plasticity, the definition of a safety coefficient with respect to shakedown is now introduced and dual static and kinematic approaches are considered.

3.1. Static approach and safety coefficient

The definition of a safety coefficient with respect to shakedown can be given from the previous results. For a given loading history on the time interval $[0 + \infty[$,

Proposition 2. *Let m_s be the safety coefficient defined by the maximum*

$$m_s(\tau) = \max_{\alpha^*} m \tag{8}$$

among all constant fields α^* such that

$$\begin{aligned} &\forall t \geqslant \tau, \quad u^*(t) CA, \quad \sigma^*(t) SA, \quad mA^* PA, \\ &\sigma^* = W_{,\varepsilon}(\varepsilon(u^*), \alpha^*), \quad A^* = -W_{,\alpha}(\varepsilon(u^*), \alpha^*), \end{aligned} \tag{9}$$

where CA, SA, PA, respectively, stand for kinematically, statically and plastically admissible fields. Then there is shakedown if there exists $\tau \geqslant 0$ such that $m_s(\tau) > 1$ while no conclusion is available if $m_s(\tau) < 1$ for all $\tau \geqslant 0$.

Indeed, if $m_s > 1$ then there exists a constant field α^* in the spirit of Proposition 1 and the conclusion holds. In perfect plasticity, this statement reduces to Melan's theorem. In kinematic hardening for example, the linear Zieger–Prager model is often adopted. It is defined by state variables $\varepsilon, \varepsilon^p$ with $\varepsilon^p_{kk} = 0$. The energy potential $W = \frac{1}{2}(\varepsilon - \varepsilon^p)\colon L\colon (\varepsilon - \varepsilon^p) + \frac{1}{2}\varepsilon^p\colon h\colon \varepsilon^p$ leads to associated force $A = -W_{,\varepsilon^p} = \sigma' - h\colon \varepsilon^p$. The plastic criterion (Mises) is $\|A\| - k \leqslant 0$ and the dissipation potential is $D(\dot{\varepsilon}^p) = k\|\dot{\varepsilon}^p\|$ with the notation $\|A\| = \sqrt{A_{ij}A_{ij}}$. From Proposition 2, Proposition 3 follows.

Proposition 3. *In linear kinematic hardening, the static safety coefficient is defined as the maximum*

$$m_s(\tau) = \max_{\varepsilon^{p*}} m \tag{10}$$

among constant plastic strain fields ε^{p*} satisfying $\varepsilon^{p*}_{kk} = 0$ such that

$$\|m(s^{*\prime}(\varepsilon^{p*}) + \sigma'_{\mathrm{el}} - h\colon \varepsilon^{p*})\| \leqslant k \tag{11}$$

for all $x \in \Omega$ and $t \geqslant \tau$.

It is well known in linear elasticity that the self-stress associated with a given initial strain $s(\varepsilon^{\mathrm{I}})$ is obtained from the resolution of a linear elastic problem with initial strain. In particular, the self-stress associated with a given field of initial strains ε^{I} is the solution of the minimization problem of total complementary energy of the elastic solid

$$\min_{s\ \text{self-stress}} \|s + L\colon \varepsilon^{\mathrm{I}}\|^2,$$

where $\|s\|$ denotes the elastic energy norm

$$\|s\|^2 = \int_\Omega s\colon L^{-1}\colon s \, \mathrm{d}\Omega.$$

Thus, the associated self-stress can be schematically written in the form

$$s(\varepsilon^{\mathrm{I}}) = -Z(\varepsilon^{\mathrm{I}}) \tag{12}$$

where the linear and symmetric operator Z is a projection in the sense of the energy norm. In particular, this linear operator is non-negative since

$$\langle Z(\varepsilon^{\mathrm{I}})|\varepsilon^{\mathrm{I}}\rangle = \int_{\Omega} -s(\varepsilon^{\mathrm{I}})\colon \varepsilon^{\mathrm{I}}\,\mathrm{d}\Omega = \|s(\varepsilon^{\mathrm{I}})\|^2 \geqslant 0.$$

Let us assume that there exists a state of stress deviator Σ^* such that

$$\|\sigma'_{\mathrm{el}}(t) - \Sigma^*\| < k' < k \quad \forall t \geqslant \tau. \tag{13}$$

Then the plastic criterion $\|A^*\| < k$ is always satisfied if a field ε^{p*} is such that

$$(Z + hI)\varepsilon^{p*} = \Sigma^* \tag{14}$$

could be found. Since this equation admits a solution for all Σ^* from the fact that the linear, symmetric operator $Z + hI$ is positive-definite when $h > 0$. Thus, a particular field ε^{p*} satisfying the static theorem can then be obtained and there is shakedown. Finally, in linear kinematic hardening, the shakedown problem leads to the discussion of conditions (13) which can be easily solved as it has been shown in the literature (cf. Gittus and Zarka, 1986 for example). It is well known that there is shakedown if the amplitude of the elastic stress does not exceed the diameter of the yield surface.

3.2. *Min–max duality*

Eqs. (8) or (10) are convex optimization problems. The dual approach consists of considering dual problems obtained by relaxing some constraints. For this, the initial problem (8) is first written as the search of maximum of m in the set of constant fields α^* and time-dependent fields $\tilde{A}(t)$ and of coefficients m such that

$$\begin{aligned}
&\forall t \geqslant \tau, \quad u^*(t)\mathrm{CA}, \quad \sigma^*(t)\mathrm{SA}, \quad m\tilde{A}(t)\mathrm{PA},\\
&\sigma^* = W_{,\varepsilon}(\varepsilon(u^*),\alpha^*), \quad A^* = -W_{,\alpha}(\varepsilon(u^*),\alpha^*), \quad \tilde{A}(t) = A^*(t).
\end{aligned} \tag{15}$$

The last constraint is relaxed by the introduction of Lagrange multipliers $\beta(t)$ associated with the constraint $m(\tilde{A}(t) - A^*(t)) = 0$ and the Lagrangian

$$\boldsymbol{\Lambda}(\beta, m, \alpha^*, \tilde{A}) = m + \int_{\tau}^{\infty} \int_{\Omega} m(\tilde{A} - A^*)\beta\,\mathrm{d}\Omega\,\mathrm{d}t \tag{16}$$

The saddle-point problem

$$\max_{\alpha^*,\tilde{A},m}\ \min_{\beta} \boldsymbol{\Lambda} \tag{17}$$

in the set of arbitrary fields $\beta(t)$ and α^*, $\tilde{A}(t), m$ such that

$$\begin{aligned}
&\forall t \geqslant \tau, \quad u^*(t)\mathrm{CA}, \quad \sigma^*(t)\mathrm{SA}, \quad m\tilde{A}(t)\mathrm{PA},\\
&\sigma^* = W_{,\varepsilon}(\varepsilon(u^*),\alpha^*), \quad A^* = -W_{,\alpha}(\varepsilon(u^*),\alpha^*),
\end{aligned} \tag{18}$$

leads to the initial problem since the result of the minimization with respect to β then gives $\min_\beta \Lambda = m$ if $\tilde{A}(t) = A^*(t)$ for all $t \geqslant \tau$ and $\min_\beta \Lambda = -\infty$ otherwise.

Proposition 4. *The dual problem*

$$m_k = \min_{\beta} \max_{\alpha^*, \tilde{A}, m} \Lambda \tag{19}$$

in the set of arbitrary fields $\beta(t)$ and $\alpha^*, \tilde{A}(t)$ such that

$$\begin{aligned} &\forall t \geqslant \tau, \quad u^*(t) CA, \quad \sigma^*(t) SA, \quad m\tilde{A}(t) PA, \\ &\sigma^* = W_{,\varepsilon}(\varepsilon(u^*), \alpha^*), \quad A^* = -W_{,\alpha}(\varepsilon(u^*), \alpha^*), \end{aligned} \tag{20}$$

defines the dual approach to compute the safety coefficient. It is clear that

$$m_s(\tau) \leqslant m_k(\tau), \tag{21}$$

according to general results of saddle-point duality.

In particular, if $m_k < 1$, then it is not possible to find a self-stress satisfying the condition of the previous propositions.

4. Dual kinematic approach

In the following sections, the dual kinematic approach will be considered for strain hardening models admitting as internal parameters α the plastic strains or the equivalent plastic strain in view of common models in hardening plasticity.

4.1. Perfect plasticity

In this case, Eq. (10) leads to the strain rate history $\beta = d^p$ and to the Lagrangian

$$\Lambda = m + \int_\tau^\infty \int_\Omega m(\tilde{\sigma} - \sigma^*) \colon d^p \, \mathrm{d}\Omega \, \mathrm{d}t, \tag{22}$$

where $m\tilde{\sigma}$ must be plastically admissible and $\sigma^* = s^* + \sigma_{\mathrm{el}}$. The operation $\max_{s^*} \Lambda$ leads to the calculation of

$$\max_{s^*} \int_\tau^\infty \int_\Omega -s^* \colon d^p \, \mathrm{d}\Omega \, \mathrm{d}t \tag{23}$$

and of

$$\max_{m\tilde{\sigma}\,\mathrm{PA}} \int_\tau^\infty \int_\Omega m\tilde{\sigma} \colon d^p \, \mathrm{d}\Omega \, \mathrm{d}t. \tag{24}$$

The last problem is trivial:

$$\max_{m\tilde{\sigma}\ \mathrm{PA}} \int_\tau^\infty \int_\Omega m\tilde{\sigma}\colon d^p \,\mathrm{d}\Omega \,\mathrm{d}t = \int_\tau^\infty \int_\Omega D(d^p) \,\mathrm{d}\Omega \,\mathrm{d}t$$

with $D(d^p) = \max_{m\tilde{\sigma}\ \mathrm{PA}} m\tilde{\sigma}\colon d^p$. The first problem leads to

$$\max_{s^*\ \text{self-stress}} \int_\Omega -s^*\colon \int_\tau^\infty d^p \mathrm{d}t \,\mathrm{d}\Omega$$

$$= 0 \quad \text{if } E^p = \int_\tau^\infty d^p \,\mathrm{d}t \text{ is a compatible field,}$$

$$+\infty \quad \text{if} \quad E^p \text{ is not a compatible field.}$$

A compatible field means that there exists a displacement field u^p with $u^p = 0$ on S_u such that $E^p = (\nabla u^p)$. It follows that

$$\max_{\varepsilon^{p*},\tilde{\sigma}} \Lambda = m\left(1 - \int_0^\infty \int_\Omega \sigma_{\mathrm{el}}\colon d^p \,\mathrm{d}\Omega \,\mathrm{d}t\right) + \int_\tau^\infty \int_\Omega D(d^p) \,\mathrm{d}\Omega \,\mathrm{d}t.$$

when d^p is admissible and $E^p = \int_\tau^\infty d^p \,\mathrm{d}t$ is compatible. Admissible rates d^p must be considered in order to ensure a finite value of $D(d^p)$. For example, if Mises criterion is satisfied, the plastic rate is admissible if $d^p_{kk} = 0$. It is concluded after the maximization with respect to m that the following result holds (cf. Koiter, 1960; Debordes, 1976; Pycko and Mroz, 1992):

In perfect plasticity, the dual kinematic approach leads to a coefficient $m_k \geqslant m_s$, defined as the minimum

$$m_k = \min_{d^p} \int_\tau^\infty \int_\Omega D(d^p) \,\mathrm{d}\Omega \,\mathrm{d}t \tag{25}$$

among the plastic rates d^p satisfying

$$d^p \text{ admissible,} \quad E^p = \int_\tau^\infty d^p \,\mathrm{d}t \text{ compatible,} \quad \int_\tau^\infty \int_\Omega \sigma_{\mathrm{el}}\colon d^p \,\mathrm{d}\Omega \,\mathrm{d}t = 1. \tag{26}$$

It is useful to note that $m_k \geqslant 1$ if the following inequality holds:

$$\int_\tau^\infty \int_\Omega D(d^p) \,\mathrm{d}\Omega \,\mathrm{d}t \geqslant \int_\tau^\infty \int_\Omega \sigma_{\mathrm{el}}\colon d^p \,\mathrm{d}\Omega \,\mathrm{d}t \tag{27}$$

for all plastic strain rate d^p admissible and satisfying the condition $E^p = \int_\tau^\infty d^p \,\mathrm{d}t$ compatible. This result is classically known as Koiter's shakedown theorem (Koiter, 1960).

As usually established in min–max duality, equality $m_s = m_k$ is generically satisfied. In particular, this equality holds if the plastic domain is bounded in the stress space as it has been shown by Debordes and Nayroles (1976). It is expected that this equality always holds although a rigorous proof is lacking for continuum solids.

4.2. *Isotropic hardening*

The isotropic hardening model with Mises criterion is defined by state variables $\varepsilon, \varepsilon^p, e^{-p}$ with energy $W = \frac{1}{2}(\varepsilon - \varepsilon^p)\colon L\colon (\varepsilon - \varepsilon^p) + V(e^{-p})$ and plastic criterion

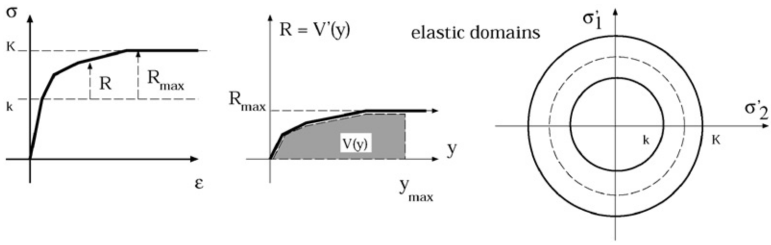

Fig. 1. A model for isotropic hardening.

$\|\sigma'\| - R(e^{-p}) - k \leqslant 0$ where $R(y) = V'(y)$. Since $V(y)$ is a convex function, its derivative $R(y)$ is a non-decreasing function. The associated force is $A = (\sigma', -R)$ and the dissipation is $d_{\text{in}} = \sigma' \colon \dot{\varepsilon}^p - R(e^{-p})\|\dot{\varepsilon}^p\| = D = k\|\dot{\varepsilon}^p\|$.

If $R(y)$ is bounded and attains its maximum $R(y) = R_{\max}$ for $y \geqslant y_{\max}$ as shown in Fig. 1, the static coefficient m_s will be bounded. Let $K = k + R_{\max}$, the calculation can be done in the same spirit and leads to

$$\max_{\varepsilon^{p*}, e^{-p*}} \Lambda = m\left(1 - \int_\tau^\infty \int_\Omega \sigma_{\text{el}} \colon d^p \, \mathrm{d}\Omega \, \mathrm{d}t\right) + \int_\tau^\infty \int_\Omega (k + R_{\max})\|d^p\| \mathrm{d}\Omega \, \mathrm{d}t$$

if $E^p = \int_\tau^\infty d^p \, \mathrm{d}t$ is a compatible field, while the result is infinite otherwise. It is concluded that

Proposition 5. *In isotropic hardening with Mises criterion, the dual kinematic approach leads to a coefficient $m_k \geqslant m_s$, defined as the minimum*

$$m_k = \min_{d^p} \int_\tau^\infty \int_\Omega K\|d^p\| \, \mathrm{d}\Omega \, \mathrm{d}t \tag{28}$$

among the plastic rates d^p satisfying

$$d^p_{kk} = 0, \quad E^p = \int_\tau^\infty d^p \, \mathrm{d}t \ \textit{compatible}, \quad \int_\tau^\infty \int_\Omega \sigma_{\text{el}} \colon d^p \mathrm{d}\Omega \, \mathrm{d}t = 1. \tag{29}$$

This result is quite natural in the sense that the behaviour of the solid is the same as in perfect plasticity with yield stress K. As in perfect plasticity, this discussion is given here as a simple illustration of the method.

4.3. Linear kinematic hardening

If Zieger–Prager's model is considered, it is not difficult to establish that

$$\max_{\varepsilon^{p*}} - \int_\Omega (s^* - h \colon \varepsilon^{p*}) \colon \int_\tau^\infty d^p \, \mathrm{d}t \, \mathrm{d}\Omega = \begin{cases} 0 & \text{if } E^p = \int_\tau^\infty d^p \, \mathrm{d}t = 0, \\ +\infty & \text{if } E^p \neq 0. \end{cases}$$

The same method then leads to Shakedown theorem (cf. for example to Koiter (1960), Pham (2001))

Proposition 6. *In linear kinematic hardening with Mises criterion, the dual kinematic approach leads to a coefficient $m_k \geqslant m_s$, defined as the minimum*

$$m_k = \min_{d^p} \int_\tau^\infty \int_\Omega D(d^p)\, \mathrm{d}\Omega\, \mathrm{d}t \tag{30}$$

among the plastic rates d^p satisfying

$$d^p_{kk} = 0, \quad E^p = \int_\tau^\infty d^p\, \mathrm{d}t = 0, \quad \int_\tau^\infty \int_\Omega \sigma_{\mathrm{el}}\colon d^p\, \mathrm{d}\Omega\, \mathrm{d}t = 1. \tag{31}$$

Thus, closed cycles of plastic rates must be considered instead of compatible plastic cycles as in perfect plasticity or in isotropic hardening.

It is clear from this proposition that the result does not depend on the hardening tensor h, the only restriction is its positiveness. The fact that closed cycles must be considered leads to the consideration of the amplitude of the elastic stress and to the trivial result that there is shakedown if this amplitude is smaller than the diameter of the yield surface.

Combined isotropic and linear kinematic hardening models can also be discussed in the same spirit. Again, closed cycles of plastic rates must be considered. This result, to the knowledge of the author, has not been given in the literature and deserves to be underlined although shakedown analysis in linear kinematic hardening is a trivial problem.

4.4. Limited kinematic hardening

The model of nonlinear and limited kinematic hardening of Fig. 2 is considered. For this model, the state variables are $\varepsilon, \varepsilon^p$ with $\varepsilon^p_{kk} = 0$. The energy potential is $W = \frac{1}{2}(\varepsilon - \varepsilon^p)\colon L\colon (\varepsilon - \varepsilon^p) + V(\|\varepsilon^p\|)$ where $R(y) = V'(y)$ is the monotone function previously introduced in Fig. 1. Thus the generalized force is $A = -W_{,\varepsilon^p} = \sigma' - C$ with $C = R(\|\varepsilon^p\|)\varepsilon^p/\|\varepsilon^p\|$. With Mises criterion $\|A\| - k \leqslant 0$, the elastic domain is a sphere of radius k and with center C in the force space. This center C remains near the origin

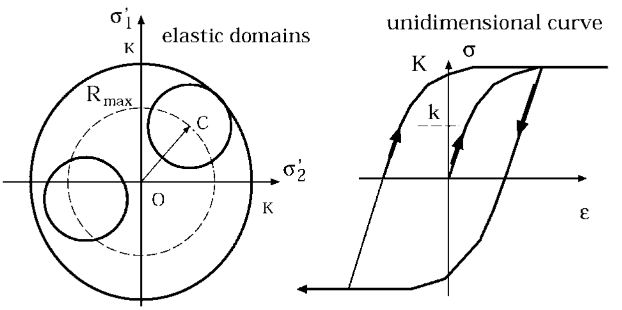

Fig. 2. A model for limited kinematic hardening.

since $\|C\| \leqslant R_{\max}$. It is not difficult to establish that

$$\max_{\varepsilon^{p*}} - \int_{\Omega} \left(s^* - R(\|\varepsilon^{p*}\|) \frac{\varepsilon^{p*}}{\|\varepsilon^{p*}\|} \right) : \int_{\tau}^{\infty} d^p \, \mathrm{d}t \, \mathrm{d}\Omega$$

$$= \int_{\Omega} R_{\max} \|E^p\| \quad \text{if } E^p \text{ is compatible,}$$

$$+\infty \quad \text{if } E^p \text{ is not compatible.}$$

The same method then leads to

Proposition 7. *For this limited kinematic hardening model, the dual kinematic approach leads to a coefficient* $m_k \geqslant m_s$, *defined as the minimum*

$$m_k = \min_{d^p} \int_{\Omega} R_{\max} \|E^p\| \, \mathrm{d}\Omega + \int_{\tau}^{\infty} \int_{\Omega} k \|d^p\| \, \mathrm{d}\Omega \, \mathrm{d}t \tag{32}$$

among plastic rates d^p *satisfying*

$$d^p_{kk} = 0, \quad E^p = \int_{\tau}^{\infty} d^p \, \mathrm{d}t \; \textit{compatible}, \quad \int_{\tau}^{\infty} \int_{\Omega} \sigma_{\mathrm{el}} : d^p \, \mathrm{d}\Omega \, \mathrm{d}t = 1. \tag{33}$$

Again, compatible plastic cycles must be considered as in perfect plasticity or in isotropic hardening.

In particular, if m^0_k, m^{lkh}_k and m^{nkh}_k denote, respectively, the kinematic safety coefficients in perfect plasticity, in linear kinematic hardening and in nonlinear kinematic hardening with the same yield stress k, it follows from their definition that

$$m^0_k(\tau) \leqslant m^{nkh}_k(\tau) \leqslant m^{lkh}_k(\tau) \quad \text{for all } \tau. \tag{34}$$

The case of limited hardening (Proposition 7) appears as the penalization of the unlimited case (Proposition 6), $R_{\max}$ is the penalty parameter associated with the constraint $\|E^p\| = 0$.

4.5. A relaxed kinematic coefficient

If incompressible plastic strains and Mises yield criterion are assumed, the plastic rates d^p to be considered in the dual kinematic approach must satisfy both compatibility and admissibility conditions:

$$d^p_{kk} = 0 \quad \text{and} \quad E^p = \int_{\tau}^{\infty} d^p \, \mathrm{d}t \quad \text{compatible.}$$

A relaxation of the first condition has been introduced by (Koiter, 1960) in order to replace in perfect plasticity the computation of m_k by a smaller coefficient defined as the minimum. A translation following the p-axis can also be introduced in order to include the origin of stresses inside the elastic domain

$$m_\ell = \min_{d^p} \int_{\tau}^{\infty} \int_{\Omega} D(d^{p'}) \, \mathrm{d}\Omega \, \mathrm{d}t \tag{35}$$

among the rates β satisfying

$$E^p = \int_\tau^\infty d^p \, \mathrm{d}t \text{ compatible}, \qquad \int_\tau^\infty \int_\Omega \sigma_{\mathrm{el}} \colon d^{p'} \, \mathrm{d}\Omega \, \mathrm{d}t = 1 \tag{36}$$

where d^p is a symmetric second order tensor and $d^{p'}$ its deviatoric part. From this definition and from the expression of m_k in Proposition 5, it is clear that m_ℓ, denoted as the relaxed kinematic safety coefficient, must satisfy

$$m_\ell \leqslant m_k. \tag{37}$$

Moreover, the following result has been established and known as Koiter's second shakedown theorem.

Koiter's second shakedown theorem. *If $m_\ell > 1$, then the plastic dissipated energy is necessarily bounded for any elastic–plastic evolution of a solid submitted to a given loading path from any initial condition. Thus, there is shakedown.*

This theorem can be easily understood under the assumption of the generic equality $m_s = m_k$. Indeed, it follows that $m_s \geqslant m_\ell > 1$, thus there is shakedown from the static approach. However, a direct proof has been given by (Koiter, 1960) without this assumption.

The same theorem also holds for the limited kinematic hardening model.

Proposition 8 (relaxed kinematic coefficient). *In the case of the previous limited kinematic hardening model, the dissipated energy is necessarily bounded and there is shakedown if there exists $\tau \geqslant 0$ such that the relaxed safety coefficient $m_\ell \leqslant m_k$, defined by*

$$m_\ell = \min_{d^p} \int_\Omega R_{\max} \|E^{p'}\| \, \mathrm{d}\Omega + \int_\tau^\infty \int_\Omega k \|d^{p'}\| \, \mathrm{d}\Omega \, \mathrm{d}t \tag{38}$$

among the symmetric rates $d^p(x,t)$ such that

$$E^p = \int_\tau^\infty d^p \, \mathrm{d}t \quad \textit{compatible} \quad \int_\tau^\infty \int_\Omega \sigma_{\mathrm{el}} \colon d^{p'} \, \mathrm{d}\Omega \, \mathrm{d}t = 1, \tag{39}$$

satisfies $m_\ell \geqslant 1$.

A direct proof is given here without the assumption $m_s = m_k$ following Koiter's method. Indeed, the response of a hardening solid obeying this model under a given loading path is considered starting from any initial condition. A special compatible rate history d^p is now constructed on $[\tau, +\infty[$ from the plastic strain rate $\dot{\varepsilon}^p(t)$, $t \in [\tau, T]$ for a chosen time $T > \tau$. With the notation $\varepsilon = \varepsilon_{\mathrm{el}} + \varepsilon_r + \varepsilon^p$ where $\varepsilon_r = L^{-1} s$, the following rate is considered:

$$d^p(t) = \dot{\varepsilon}^p(t) \quad \text{if } \tau \leqslant t \leqslant T,$$

$$d^p(t) = \tfrac{1}{T} \Delta\varepsilon_r(T) \quad \text{if } 2T > t > T,$$

$$d^p(t) = 0 \quad \text{if } t \geqslant 2T$$

with $\Delta\varepsilon(t) = \varepsilon(t) - \varepsilon(\tau)$. It is clear that

$$E^p = \int_\tau^\infty d^p(t)\,\mathrm{d}t = \Delta\varepsilon^p(T) + \Delta\varepsilon_r(T) = \Delta\varepsilon(T) - \Delta\varepsilon_{\mathrm{el}}(T)$$

is a compatible but not necessarily admissible field. We are interested in the expression of the quantity $\int_\tau^\infty \int_\Omega \sigma_{\mathrm{el}}\colon d^{p'}\,\mathrm{d}\Omega\,\mathrm{d}t$. Since for all $t \in [0, T]$

$$\int_\Omega (\sigma\colon \dot\varepsilon^p + s\colon \dot\varepsilon_r - \sigma_{\mathrm{el}}\colon \dot\varepsilon^p) = \int_\Omega (\sigma - \sigma_{\mathrm{el}})(\dot\varepsilon - \dot\varepsilon_{\mathrm{el}})\,\mathrm{d}\Omega = 0,$$

$$\int_\Omega \sigma_{\mathrm{el}}\colon d^{p'}\,\mathrm{d}\Omega = \int_\Omega (\sigma\colon \dot\varepsilon^p + s\colon \dot\varepsilon_r)\,\mathrm{d}\Omega,$$

it follows that

$$\int_\tau^{2T} \int_\Omega \sigma_{\mathrm{el}}\colon d^{p'}\,\mathrm{d}\Omega\,\mathrm{d}t = \int_\tau^T \int_\Omega \sigma\colon \dot\varepsilon^p + s\colon L^{-1}\colon \dot s\,\mathrm{d}\Omega\,\mathrm{d}t$$
$$+\frac{1}{T}\int_T^{2T} \mathrm{d}t \int_\Omega \sigma_{\mathrm{el}}\colon \Delta\varepsilon_r'(T)\,\mathrm{d}\Omega.$$

This rate history satisfies necessarily after the definition of m_ℓ

$$\int_\tau^{+\infty} \int_\Omega \sigma_{\mathrm{el}}\colon d^{p'}\,\mathrm{d}\Omega\,\mathrm{d}t \leqslant \frac{1}{m_\ell}\int_\tau^{+\infty} \int_\Omega k\|d^{p'}\|\,\mathrm{d}\Omega\,\mathrm{d}t + \frac{1}{m_\ell}\int_\Omega R_{\max}\|E^{p'}\|\,\mathrm{d}\Omega.$$

It follows that

$$\begin{aligned}
\frac{m_\ell - 1}{m_\ell}\int_\tau^T \int_\Omega A\colon \dot\varepsilon^p\,\mathrm{d}\Omega\,\mathrm{d}t \leqslant \int_\Omega &\{W^{\mathrm{c}}(s(\tau)) - W^{\mathrm{c}}(s(T)) + V(\|\varepsilon^p(\tau)\|) \\
&- V(\|\varepsilon^p(T)\|) + \frac{1}{m_\ell}(k + R_{\max})\|\Delta\varepsilon_r'(T)\| \\
&+ \frac{1}{m_\ell}R_{\max}\|\varepsilon^p(\tau)\| + \frac{1}{m_\ell}R_{\max}\|\varepsilon^p(T)\|\Big\}\,\mathrm{d}\Omega \\
&- \frac{1}{T}\int_\Omega \int_T^{2T} \sigma_{\mathrm{el}}\colon \Delta\varepsilon_r'(T)\,\mathrm{d}t\,\mathrm{d}\Omega,
\end{aligned} \tag{40}$$

where $A\colon \dot\varepsilon^p = k\|\dot\varepsilon^p\|$ and $W^{\mathrm{c}}(s) = \frac{1}{2}s\colon L^{-1}\colon s$. The next step of the proof is to check that the second member of this inequality remains bounded. On the one hand, after the introduced assumption on the function $V(y)$ as shown in Fig. 1, the estimate

$$V(\|\varepsilon^p\|) \geqslant R_{\max}(\|\varepsilon^p\| - y_{\max})$$

holds and gives

$$\frac{1}{m_\ell}R_{\max}\|\varepsilon^p(T)\| - V(\|\varepsilon^p(T)\| \leqslant y_{\max}R_{\max} + \frac{1 - m_\ell}{m_\ell}R_{\max}\|\varepsilon^p(T)\| \leqslant y_{\max}R_{\max}.$$

On the other hand, ε'_r remains bounded from the fact that

$$\|A\| = \|s' + \sigma'_{\text{el}} - C\| \leqslant k \quad \Rightarrow \quad \|s'\| \leqslant k + R_{\max} + \|\sigma'_{\text{el}}\|.$$

It is thus concluded that the second member of Eq. (40) is bounded for all T and Proposition 8 holds.

5. Pressure-dependent models of geomaterials

In particular, the min–max duality method can be discussed for some common models of soil mechanics.

5.1. Cam-clay model

This model, proposed by the Cambridge school (Burland and Roscoe, 1968; Schofield and Wroth, 1968), is very popular in soil mechanics since it describes correctly the plastic dilatation or contraction observed experimentally.

A generalized standard version of Cam-clay model is considered here. If p denotes the mean pressure, $p = \sigma_{kk}/3$ and $q = \|\sigma'\|$, the plastic criterion is given by the yield function

$$((p+R)^2 + bq^2)^{1/2} - R \leqslant 0,$$

where $R \geqslant 0$ is a critical pressure denoted as the consolidation pressure and $b > 0$ is a material constant.

The elastic domain is thus a family of growing ellipses as shown in Fig. 3. This family is homothetical with respect to the origin and there is isotropic hardening. Hardening effects are described by the variation of the consolidation pressure R. For example, the following relation holds:

$$R = R_0 \exp(y),$$

where y is the consolidation parameter, a non-decreasing variable with the plastic deformation to ensure that the consolidation pressure is non-decreasing. It can be defined

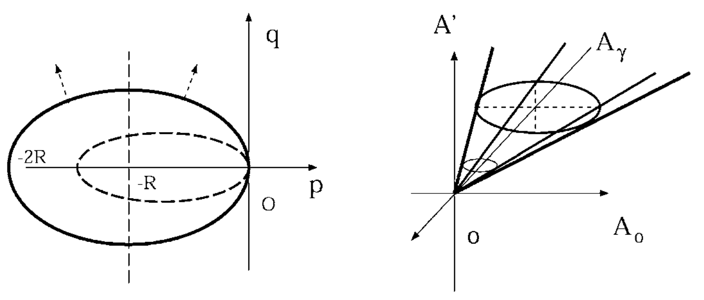

Fig. 3. Cam-clay model of geomaterials: the elastic domain is limited by a family of ellipses in the stress space ($p \times q$) and represented by a cone in the force space ($A_0 \times A' \times A_\gamma$).

by the difference between an equivalent plastic strain γ and the plastic dilatation ε_{kk}^{p}

$$y = \gamma - \varepsilon_{kk}^{p}.$$

This choice will be shown to be appropriate. It also ensures that the consolidation is possible in compression and not in tension. This is a generalized standard model with internal parameters ε^{p}, γ and energy

$$W = \tfrac{1}{2}(\varepsilon - \varepsilon^{p})\colon L\colon (\varepsilon - \varepsilon^{p}) + V(y) \quad \text{with } V'(y) = R \quad \text{thus } V(y) = R_0 \exp(y).$$

The generalized forces associated with the plastic dilatation and plastic deviator are $A_0 = p + R$ and $A' = \sigma'$, respectively. The generalized force associated with γ is $A_\gamma = -R$. The elastic domain is state-independent since it is defined by the convex cone

$$\sqrt{A_0^2 + bA'\colon A'} + A_\gamma \leqslant 0.$$

Normality law gives

$$\dot{\varepsilon}_{kk}^{p} = \mu \frac{A_0}{\sqrt{A_0^2 + bA'\colon A'}} = \mu \frac{(p+R)}{R},$$

$$\dot{\varepsilon}^{p'} = \mu b \frac{A'}{\sqrt{A_0^2 + bA'\colon A'}} = \mu b \frac{\sigma'}{R},$$

$$\dot{\gamma} = \mu \geqslant 0.$$

It follows that the plastic volume rate is dilative if $p > -R$ and contractive if $p > -R$ and that

$$\dot{\gamma} = \left\{ |\dot{\varepsilon}_{kk}^{p}|^2 + \frac{1}{b} \|\dot{\varepsilon}^{p'}\|^2 \right\}^{1/2}.$$

This relation gives the physical interpretation of γ as a measure of equivalent plastic strain. In particular, the consolidation parameter and the consolidation pressure are non-decreasing since $\dot{y} \geqslant 0$.

The consolidation pressure R is not limited in the previous description since $R(y)$ varies from 0 to $+\infty$ when y varies from $-\infty$ to $+\infty$. Its expression can be in fact replaced and defined from any suitable monotone increasing curve such that $\lim_{y\to-\infty} R(y) = 0$ and $\lim_{y\to+\infty} R(y) = R_{\max} < +\infty$. In this case, a model of limited consolidation is obtained.

For such a model, Eq. (16) is now considered with $\tilde{A}_\gamma = A_\gamma^*$. It is clear that

$$\max_{\varepsilon^{p*}} - \int_\Omega (s^* + R^* I)\colon \int_\tau^\infty d^p \,\mathrm{d}t \,\mathrm{d}\Omega = \begin{cases} -\displaystyle\int_\Omega R^* E_{kk}^{p} \,\mathrm{d}\Omega & \text{if } E^p \text{ is compatible,} \\ +\infty & \text{if not.} \end{cases}$$

Finally, R^* must maximise the quantity

$$\int_\tau^\infty \int_\Omega R^* \left\{ (d_{kk}^{p})^2 + \frac{1}{b} \|d^{p'}\|^2 \right\}^{1/2} \mathrm{d}\Omega \,\mathrm{d}t \,\mathrm{d}\Omega \,\mathrm{d}t - \int_\Omega R^* E_{kk}^{p} \,\mathrm{d}\Omega \geqslant 0.$$

Thus the following result holds from min–max saddle method.

Proposition 9. *For a Cam-clay model with limited consolidation, the kinematic approach leads to a coefficient $m_k \geqslant m_s$, defined as the minimum*

$$m_k = \min_{d^p} \underset{\max}{R} \left\{ \int_\tau^\infty \int_\Omega \left\{ (d^p_{kk})^2 + \frac{1}{b} \|d^{p'}\|^2 \right\}^{1/2} \mathrm{d}\Omega \, \mathrm{d}t \, \mathrm{d}\Omega \, \mathrm{d}t - \int_\Omega E^p_{kk} \, \mathrm{d}\Omega \right\}$$

$$= \int_\tau^\infty \int_\Omega D_m(d^p) \, \mathrm{d}\Omega \, \mathrm{d}t \quad \textit{with } D_m(d^p) = \max_{\sigma \in C_{\max}} \sigma \colon d^p \tag{41}$$

among the plastic rates d^p satisfying

$$E^p = \int_\tau^\infty d^p \, \mathrm{d}t \ \textit{compatible}, \qquad \int_\tau^\infty \int_\Omega \sigma_{\mathrm{el}} \colon d^p \, \mathrm{d}\Omega \, \mathrm{d}t = 1. \tag{42}$$

As expected, the result is the same as an elastic perfectly-plastic material with a elastic domain $C_{\max}$ associated with the extreme ellipse.

5.2. Kinematic hardening

In the spirit of the previous Cam-clay model, a particular model of kinematic hardening can also be introduced. Here, the consolidation parameter can increase or decrease and is taken simply as

$$y = -\varepsilon^p_{kk}$$

with the plastic criterion

$$((p+R)^2 + bq^2)^{1/2} - k \leqslant 0,$$

where $k > 0$ is a constant and $R(y) \geqslant 0$ is a given non-decreasing function satisfying

$$\lim_{y \to -\infty} R(y) = R_{\min} \geqslant 0 \quad \text{and} \quad \lim_{y \to +\infty} R(y) = R_{\max} > 0.$$

The elastic domain is now a family of ellipses of the same size, centered on axis $p \leqslant 0$ of the $(p \times q)$-plane. This is a generalized standard model of kinematic hardening, in the same spirit as Ziegler–Prager's model, with internal parameter ε^p. The translation of the center of ellipse in the interval $[-R_{\max}, -R_{\min}]$ is due to the variation of the plastic dilatation.

When $R_{\max} < +\infty$, there is limited kinematic hardening. In this case, the choice of R^* must maximize $-R^* E^p_{kk}$ thus $R^* = R_{\max}$ if $E^p_{kk} < 0$ and $R^* = R_{\min}$ if $E^p_{kk} > 0$. Finally, the following proposition holds.

Proposition 10. *For the Cam-clay model of limited kinematic hardening, the kinematic approach leads to a coefficient $m_k \geqslant m_s$, defined as the minimum*

$$m_k = \min_{d^p} \left\{ \begin{array}{l} \displaystyle\int_\tau^\infty \int_\Omega k \left\{ (d^p_{kk})^2 + \frac{1}{b} \|d^{p'}\|^2 \right\}^{1/2} \mathrm{d}\Omega \, \mathrm{d}t \\ \displaystyle + \int_\Omega (R_{\max} \langle E^p_{kk} \rangle^- - R_{\min} \langle E^p_{kk} \rangle^+) \, \mathrm{d}\Omega \end{array} \right\} \tag{43}$$

among the plastic rates d^p satisfying

$$E^p = \int_\tau^\infty d^p \, \mathrm{d}t \text{ compatible}, \qquad \int_\tau^\infty \int_\Omega \sigma_{\mathrm{el}} \colon d^p \, \mathrm{d}\Omega \, \mathrm{d}t = 1. \tag{44}$$

In this expression, the notations $\langle E_{kk}^p \rangle^-$ and $\langle E_{kk}^p \rangle^+$ mean, respectively, the negative part and the positive part of E_{kk}^p, for example,

$$\langle E_{kk}^p \rangle^+ = \begin{cases} E_{kk}^p & \text{if } E_{kk}^p > 0, \\ 0 & \text{if } E_{kk}^p \leqslant 0. \end{cases}$$

When $R_{\max} = +\infty$, there is unlimited kinematic hardening and the following proposition holds.

Proposition 11. *For the Cam-clay model of unlimited kinematic hardening, the kinematic approach leads to a coefficient $m_k \geqslant m_s$, defined as the minimum*

$$m_k = \min_{d^p} \int_\tau^\infty \int_\Omega k \left\{ (d_{kk}^p)^2 + \frac{1}{b} \| d^{p'} \|^2 \right\}^{1/2} \mathrm{d}\Omega \, \mathrm{d}t - \int_\Omega R_{\min} E_{kk}^p \, \mathrm{d}\Omega \tag{45}$$

among the plastic rates d^p satisfying

$$E^p = \int_\tau^\infty d^p \, \mathrm{d}t \text{ compatible}, \quad E_{kk}^p \geqslant 0, \qquad \int_\tau^\infty \int_\Omega \sigma_{\mathrm{el}} \colon d^p \, \mathrm{d}\Omega \, \mathrm{d}t = 1. \tag{46}$$

This result is easily understood in the spirit of Propositions 6 and 7 since the back-stress is a pressure in this particular case. Again, the limited case (Proposition 10) appears as a penalization of the unlimited case (Proposition 11), $R_{\max}$ is a penalty parameter associated with the constraint $E_{kk}^p \geqslant 0$. In particular, if $R_{\min} = k$, the expression to be minimized (45) is exactly the dissipated energy associated with the yield surface defined by the ellipse of center $-k$.

6. Shakedown conditions for a domain of load values

In the classical presentation of shakedown theorems, the load history in conveniently presented by the elastic response $\sigma_{\mathrm{el}}(t)$ for $t \in [0, +\infty[$. It is well known that instead of a load history, a set of possible values $\sigma_{\mathrm{el}}(\lambda)$ depending on n load parameters $\lambda = (\lambda^1, \ldots, \lambda^n)$ for $\lambda \in S$ can be introduced in the case of cyclic loads. The conditions of safety with respect to shakedown for such a domain of loads can be easily written by adapting the previous theorems. Such a condition means that there is shakedown for any load history defined by a curve $\lambda = \lambda(t)$ taking values in S for all $t \geqslant 0$.

Let $u_{\mathrm{el}}(\lambda)$, $\sigma_{\mathrm{el}}(\lambda)$ be the elastic response in displacement and stress under the load λ. The associated response u^*, σ^* is given by $u^* = u_{\mathrm{el}}(\lambda) + u_p^*$ and $\sigma^* = \sigma_{\mathrm{el}}(\lambda) + s^*$. Static shakedown theorem Proposition 1 can be written in this case under the form of Proposition $1'$:

Proposition 1′. *For generalized standard materials obeying assumptions* (7), *there is shakedown under any load path* $\lambda(t)\in S$ *whatever the initial state if there exists an internal parameter field* α^* *and a safety coefficient* $m>1$ *such that the force field* $mA^*(\lambda)$ *is plastically admissible for all* $\lambda\in S$, *where* $A^*(\lambda)=-W_{,\alpha}(\varepsilon^*(\lambda),\alpha^*)$.

The particular case of a domain S such that the associated convex domain S_c defined from S by convexification is a bounded polyhedral domain with corners Λ_r, $r=1,q$, can be considered for practical situations. Since $A^*(\lambda)=-Y^{\mathrm{T}}\varepsilon_{\mathrm{el}}(\lambda)-X\alpha^*-W_{2,\alpha}(\alpha^*)$, the dependence of A^* on $\varepsilon_{\mathrm{el}}$ is linear and leads to a simpler statement.

Proposition 1″. *For generalized standard materials obeying assumptions* (7), *there is shakedown under any load path* $\lambda(t)\in S_c$ *whatever the initial state if there exists an internal parameter field* α^* *and a safety coefficient* $m>1$ *such that the force field* $mA^*(\Lambda_r)$ *is plastically admissible for all* $r=1,q$, *where* $A^*(\lambda)=-W_{,\alpha}(\varepsilon^*(\lambda),\alpha^*)$.

In the same spirit, for a limited kinematic hardening model, Proposition 7 leads to the following statement

Proposition 7′. *For the limited kinematic hardening model, the dual kinematic approach leads to the computation of a coefficient* m_k, *defined as the minimum*

$$m_k=\min_{d^p} R_{\max}\int_\Omega\left\|\sum_{r=1}^{q} d_r^p\right\|\,\mathrm{d}\Omega+\sum_{r=1}^{q} k\int_\Omega\|d_r^p\|\,\mathrm{d}\Omega \tag{47}$$

among the plastic strains d_r^p, $r=1,q$ *satisfying the constraints*

$$\mathrm{Tr}(d_r^p)=0,\quad E^p=\sum_{r=1}^{q} d_r^p\ \textit{compatible},\quad \sum_{r=1}^{q}\int_\Omega \sigma_{\mathrm{el}}(\Lambda_r):\,d_r^p\,\mathrm{d}\Omega=1. \tag{48}$$

7. Illustrative example

The simple case of a symmetric three-bar system is considered here as an illustrative example. The system is composed of three elastic–plastic bars (2,1,2) connected by rigid bars as shown in Fig. 4. It is submitted to a central force F which is maintained constant while the temperature elevation $T-T_0$ of the central bar is varied and takes its values in the interval $[-\Delta T,+\Delta T]$. The shakedown condition with respect to thermo-mechanical cyclic loading inside this domain of values is discussed.

Taking account of the symmetry, static and kinematic equations are reduced to

$$\sigma_1+2\sigma_2=f,$$

$$\varepsilon=\varepsilon_1^p+\varepsilon^{\mathrm{T}}+\sigma_1=\varepsilon_2^p+\sigma_2,$$

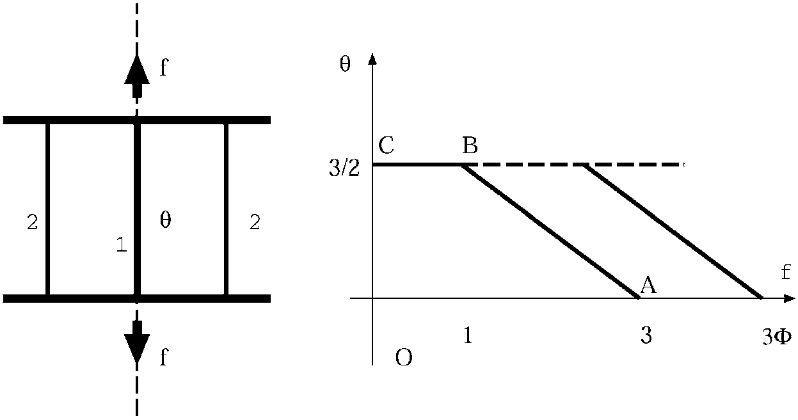

Fig. 4. A symmetric three-bar system and shakedown domains in perfect plasticity, limited kinematic hardening and linear kinematic hardening.

in dimensionless variables $f = F/k$, $\sigma_i = F_i/k$, $\varepsilon = E\tilde{\varepsilon}/k$, $\varepsilon^{\mathrm{T}} = E\beta(T - T_0)/k$. The elastic response associated with the extreme loads $\varepsilon^{\mathrm{T}} = \pm\theta$ is

$$\sigma_1^{\mathrm{el}\pm} = \frac{f \mp 2\theta}{3}, \quad \sigma_2^{\mathrm{el}\pm} = \frac{f \pm \theta}{3}.$$

For this system, a self-stress state respecting the symmetry has the form $(\rho, -2\rho, \rho)$ and a compatible state of strain respecting the symmetry the form $(\varepsilon, \varepsilon, \varepsilon)$.

In perfect plasticity, the shakedown is ensured if there exists a self-stress state $(\rho, -2\rho, \rho)$ such that the plastic criterion is satisfied for the extreme loads, i.e. such that

$$\left| -2\rho + \frac{f \mp 2\theta}{3} \right| \leqslant 1, \quad \left| \rho + \frac{f \pm \theta}{3} \right| \leqslant 1.$$

It is well known that the resulting shakedown domain is a convex domain OABC in Fig. 4, completed by symmetry with respect to axes f and θ.

The kinematic approach consists of finding the safety coefficient m_k of the loads $f, \pm\theta$. For this, plastic increments (d_1^+, d_1^-) and (d_2^+, d_2^-) in bars 1 and 2, associated with the extreme loads, are introduced. The kinematic approach gives m_k as the minimum of the dissipated work

$$m_k = \min_d 2|d_2^+| + 2|d_2^-| + |d_1^+| + |d_1^-| \tag{49}$$

among compatible increments i.e.

$$d_2^+ + d_2^- = d_1^p + d_1^- = E^p$$

satisfying

$$2\sigma_2^{\mathrm{el}+} d_2^+ + 2\sigma_2^{\mathrm{el}-} d_2^- + \sigma_1^{\mathrm{el}+} d_1^+ + \sigma_1^{\mathrm{el}-} d_1^- = 1.$$

This condition gives

$$\tfrac{4}{3}\theta(d_2^+ - d_1^+) + E^p f = 1.$$

Shakedown is ensured if $m_k > 1$.

In linear kinematic hardening, the shakedown domain can be found easily. There is shakedown if the amplitude of the elastic response is small enough to satisfy the plastic criterion. Thus, there is shakedown if $|\theta| < 3/2$ as shown in Fig. 4 and no shakedown if $|\theta| \geqslant 3/2$. In the last cases, the responses are elastic–plastic and periodic for periodic loadings. The kinematic approach in kinematic hardening leads to the search of the minimum of the same expression (49) under the last condition among closed plastic increments

$$d_2^+ + d_2^- = d_1^p + d_1^- = 0.$$

It follows that

$$m_k = \min_d 4|d_2^+| + 2|d_1^+| \quad \text{such that } \tfrac{4}{3}\theta(d_2^+ - d_1^+) = 1$$

and thus, $m_k = 3/2\theta$. Thus, $m_k > 1$ if $\theta < \frac{3}{2}$ which is the result obtained by the static approach as shown in Fig. 4.

In limited kinematic hardening, from the static shakedown theorem (Proposition 1), there is shakedown if a plastic strain state $(\varepsilon_1^{p*}, \varepsilon_2^{p*})$ can be found such that the associated self-stresses $(\rho_2^*, \rho_1^*, \rho_2^*)$, defined by

$$\rho_1^* + 2\rho_2^* = 0,$$

$$\varepsilon^* = \varepsilon_2^{p*} + \rho_2^* = \varepsilon_1^{p*} + \rho_1^*$$

lead to plastically admissible stress states. This means that the inequalities

$$\left|\frac{2}{3}(\varepsilon_2^{p*} - \varepsilon_1^{p*}) + \frac{f \pm \theta}{3} - r(\varepsilon_1^{p*})\right| \leqslant 1, \quad \left|\frac{1}{3}(\varepsilon_1^{p*} - \varepsilon_2^{p*}) + \frac{f \pm 2\theta}{3} - r(\varepsilon_2^{p*})\right| \leqslant 1,$$

must be satisfied, where $r(\varepsilon^p) = (1/k)R(\varepsilon^p)$. The kinematic approach consists of solving the problem

$$m_k = \min_{d, E^p} m(d, E^p) \quad \text{with } m(d, E^p) = 3r_{\max}|E^p| + 2|d_2^+| + 2|d_2^-| + |d_1^+| + |d_1^-|$$

among increments $(d_2^\pm, d_1^\pm, d_2^\pm)$ and amplitudes E^p such that

$$E^p = d_2^+ + d_2^- = d_1^+ + d_1^- \quad \text{(compatibility)},$$

$$2\sigma_2^{\text{el}+} d_2^+ + 2\sigma_2^{\text{el}-} d_2^- + \sigma_1^{\text{el}+} d_1^+ \sigma_1^{\text{el}-} d_1^- = 1.$$

To compute m_k, it is then necessary to consider two possibilities:

- If $E^p = 0$, it is already shown that $\min_d m(d, 0) = 3/2\theta$.
- If $E^p \neq 0$, since $\min_d m(d, E^p) = (3r_{\max} + 3)|E^p|$ is attained when d_2^+ and d_1^+ are arbitrary in the interval $[0, E^p]$ if $E^p \geqslant 0$ or in the interval $[E^p, 0]$ if $E^p \leqslant 0$, it follows that $\min_{E^p \neq 0} m(d, E^p) > 1$ if $3r_{\max} + 3 > |f| + \frac{4}{3}|\theta|$. The shakedown domain is thus given by the conditions $|\theta| < \frac{3}{2}$ and $3r_{\max} + 3 > |f| + \frac{4}{3}|\theta|$ as shown in Fig. 4 where $\Phi = r_{\max} + 1$.

8. Conclusion

In this paper, new results concerning the expression of kinematic safety coefficients, available for common models of limited isotropic and kinematic hardening, are given. The expressions obtained from the min–max duality such as (19), (30), (32), (38), (41) and (45) provide a useful complement to the static approach. Propositions 7–11 are new compared to the existing results of the literature in shakedown analysis since they deal principally with the kinematic approach. They are particularly simple and can be easily exploited to approximate by upper bounds the theoretical values in hardening plasticity as it is usually done for shakedown and limit analyses in perfect plasticity. In particular, the amplitude of the plastic rate path, denoted as E^p in this discussion, figures explicitly in the expression of the kinematic safety coefficients.

For the models considered here, it should be emphasized that the material always offers its maximum allowable resistance given by the saturation limit of the yield surface. It is true that this remarkable capability holds only for appropriate flow laws of the plastic strain and internal parameters, as it has been emphasized in the literature (cf. for example, Fuschi, 1999). However, this conclusion could possibly remain available for a class of hardening materials larger than the generalized standard models. It may be then interesting to discuss the least restrictive assumption on constitutive equations ensuring this result and the validity of the static and kinematic approaches. For non-associated flow laws, it is already known that no general variational principle or dual theorems could be derived for shakedown analysis or for limit analysis.

References

Bleich, H., 1932. Uber die bemessung statisch unbestimmter stahltragwerke unter beruschsichtigung der elastisch-plastischen verhaltens des baustoffes. J. Bauingenieur, Berlin 19, 261–269.

Bodovillé, G., de Saxcé, G., 2001. Plasticity with nonlinear kinematic hardening: modelling and shakedown analysis by the bipotential approach. Eur. J. Mech. A 20, 99–112.

Burland, I., Roscoe, K., 1968. On the generalized stress/strain behaviour of wet clay. In: J. Heyman, F.A. Leckie (Eds.), Engineering Plasticity. Cambridge University Press, Cambridge.

Corigliano, A., Maier, G., Pycko, S., 1995. Dynamic shakedown analysis and bounds for elasto–plastic structures with non-associated internal variable constitutive laws. Int. J. Solids Struct. 32, 3145–3166.

Corradi, L., Maier, G., 1974. Dynamic non-shakedown theorem for elastic perfectly-plastic continua. J. Mech. Phys. Solids 22, 401–413.

Dang Van, K., Papadopoulos, I., 1999. High-Cycle Metal Fatigue From Theory to Application. CISM, Springer, Wien.

Debordes, O., 1976. Dualité des théorèmes statique at cinématique sur la théorie de l'adaptation des milieux continus élasto-plastiques. C. R. Acad. Sci. 282, 535–537.

Debordes, O., Nayroles, B., 1976. Sur la théorie et le calcul à l'adaptation des structures élasto-plastiques. J. Mécanique 20, 1–54.

Fuschi, P., 1999. Structural shakedown for elastic–plastic materials with hardening saturation surface. Int. J. Solids Struct. 36, 219–240.

Gittus, J., Zarka, J., 1986. Modelling Small Deformations of Polycrystals. Elsevier, New York.

Hachemi, A., Weichert, D., 1998. Numerical shakedown analysis of damaged structures. Comput. Methods Appl. Mech. Eng. 160, 57–70.

Halphen, B., Nguyen, Q., 1975. Sur les matériaux standard généralisés. J. Mecanique 14, 1–37.

Koiter, W., 1960. General problems for elastic–plastic solids. In: Sneddon, Hill (Eds.), Progress in Solid Mechanics, Vol. 4. North-Holland, Amsterdam, pp. 165–221.

Konig, J., 1987. Shakedown of Elastic–Plastic Structures. Elsevier, Amsterdam.

Maier, G., 1972. A shake-down matrix theory allowing for workhardening and second order geometric effects. In: A. Sawzuk (Ed.), Symposium on Foundations of Plasticity, Noordhoff International Publishing, Leyden, Warsaw, pp. 417–433.

Maier, G., 2001. On some issues in shakedown analysis. J. Appl. Mech. 68, 799–808.

Mandel, J., 1976. Adaptation d'une structure plastique écrouissable. Mech. Res. Comm. 3, 251–256.

Martin, J., 1975. Plasticity: Fundamentals and General Results. MIT Press, Cambridge, MA.

Melan, E., 1936. Theorie Statisch unbestimmter systeme aus ideal-plastischen baustoff. Sitz. Berl. Ák. Wiss. 145, 195–218.

Nayroles, B., 1977. Tendances récentes et perspectives à moyen terme en élastoplasticité asymptotique des constructions. In: Proceedings, Congrès Français de Mécanique, Grenoble, France.

Nguyen, Q., 1976. Extension des théorèmes d'adaptation et d'unicité en écrouissage non linéaire. C. R. Acad. Sci. 282, 755–758.

Nguyen, Q., 2000. Stability and Nonlinear Solid Mechanics. Wiley, Chichester.

Nguyen, Q., Pham, D., 2001. On shakedown theorems in hardening plasticity. C. R. Acad. Sci. 329, 307–314.

Pham, D., 2001. Shakedown kinematic theorem for elastic–plastic bodies. Int. J. Plast. 17, 773–780.

Polizzotto, C., 1982. A unified treatment of shakedown theory and related bounding techniques. Solid Mech. Arch. 7, 19–75.

Polizzotto, C., Borino, G., Cademi, S., Fuschi, P., 1991. Shakedown problems for mechanical models with internal variables. Eur. J. Mech. A 10, 621–639.

Ponter, A., Chen, H., 2001. A minimum theorem for cyclic load in excess of shakedown with application to the evaluation of a ratchet limit. J. Mech. Phys. Solids 20, 539–554.

Pycko, S., Mroz, Z., 1992. Alternative approach to shakedown as a solution of min–max problem. Acta Mech. 93, 205–222.

Schofield, A., Wroth, C., 1968. Critical State Soil Mechanics. McGraw-Hill, London.

Weichert, D., Maier, G., 2000. Inelastic Analysis of Structures Under Variable Repeated Loads. Kluwer, Dordrecht.