Numerical Algorithm for the Topology of Singular Plane Curves

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Abstract

We are interested in computing the topology of plane singular curves. For this, the singular points must be isolated. Numerical methods for isolating singular points are efficient but not certified in general. We are interested in developing certified numerical algorithms for isolating the singularities. In order to do so, we restrict our attention to the special case of plane curves that are projections of smooth curves in higher dimensions. In this setting, we show that the singularities can be encoded by a regular square system whose isolation can be certified by numerical methods. This type of curves appears naturally in robotics applications and scientific visualization.

1 Introduction

Computing the topology of a curve C (i.e., a set of dimension one that is the zero locus of smooth maps) means computing a piecewise-linear graph that can be deformed continuously toward that curve. If C is smooth (i.e., the tangent space exists and is of dimension one at every point of C), there are several certified numerical methods for computing the topology. One can mention for example the global subdivision [8, 12] and certified continuation approaches [9]. On the other hand, if the curve is singular (i.e., not smooth), computing the topology is more complicated. We need, first, to isolate its singularities, second, to compute the topology in a neighborhood of those singularities and third to compute the topology of the smooth remaining part of that curve (see Figure 1). The main challenge is to isolate the singular points of the curve.

State-of-art methods to isolate the singular points of a curve are symbolic in general e.g., based in resultant and sub-resultant theory [2], Gröbner basis or rational univariate representations [13, 3]. However, symbolic approaches suffer from inefficiency while numerical methods fail to be certified for singular curves. Our goal is to develop efficient numerical methods that avoid losing the correctness that symbolic ones offer.

We show that this could be achieved for the specific class of plane curves that are projections of smooth curves in higher dimension. This is achieved by exhibiting a regular and square system that characterizes, under some generic assumptions, the singular points of the plane projection of a generic curve (see Theorem 4.2). This system, being square and regular, satisfies the conditions to apply certified numerical isolation methods [10, Chapter 8].

2 Assumptions

We first recall the definitions of a node and an ordinary cusp (see Figure 2).

▶ **Definition 2.1.** Let C be a curve in $\mathbb{R}^2$ and $p \in C$. We call p an ordinary cusp (resp. a node) if there exists an open subset V of $\mathbb{R}^2$, a smooth diffeomorphism $\varphi : V \to V$ such that V contains p and $\varphi(V \cap C) = V \cap C'$, where C' is the zero set of the map $x_1^2 - x_2^3$ (resp. $x_1^2 - x_2^2$), for some local coordinates system (x_1, x_2) at p.

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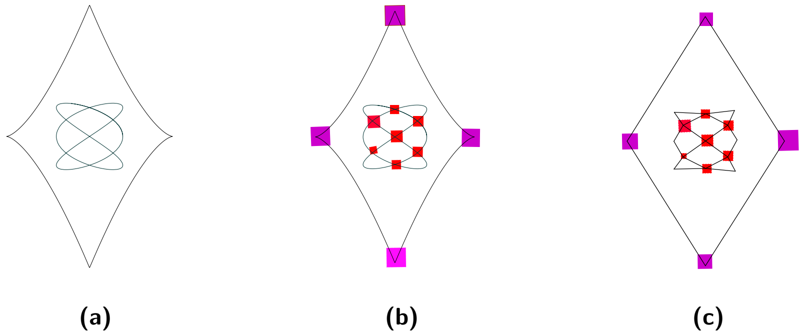

Figure 1 (a): A given curve. (b) Isolating and classifying the singular points. (c) A graph homeomorphic to the given curve.

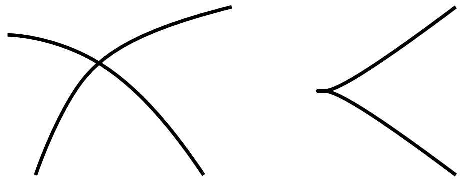

Figure 2 A node and an ordinary cusp.

Let $n \geq 3$ be an integer and U be a bounded open set in $\mathbb{R}^n$. Let $C^\infty(U, \mathbb{R}^{n-1})$ be the space of smooth maps (i.e., the set of maps that are differentiable infinitely many times) from U to $\mathbb{R}^{n-1}$ equipped with the weak topology defined in [4, §3.9.2] or [5, p.34]. Consider the map $P = (P_1, \dots, P_{n-1}) \in C^\infty(U, \mathbb{R}^{n-1})$. We denote by C_P the solution set of the system $P_1(x) = \cdots = P_{n-1}(x) = 0$, with $x = (x_1, \dots, x_n) \in U$. Also, consider the projection π from C_P to $\mathbb{R}^2$ that sends x to (x_1, x_2). Let J_P be the Jacobian matrix of size $(n-1) \times n$ of the system $P_1(x) = \cdots = P_{n-1}(x) = 0$. We assume that:

Assumption 1. For all $q \in C_P$, $\operatorname{rank}(J_P(q)) = n - 1$, which implies that C_P is a smooth curve.[1]

Assumption 2. The set of $q \in C_P$ that have tangent line orthogonal to $\mathbb{R}^2$ (the plane defined by the first two coordinates of $\mathbb{R}^n$), is finite.

Assumption 3. π is injective except at a finite number of points.

Assumption 4. The preimage of any point of $\mathbb{R}^2$ under π is at most two points (counted with multiplicity).

Assumption 5. Every singular point of $\pi(C_P)$ is either an ordinary cusp or a node.

▶ **Definition 2.2.** We denote by $\mathbb{U} \subset C^\infty(U, \mathbb{R}^{n-1})$ the set of maps for which all the above assumptions are satisfied.

We prove in Theorem 4.1 that our above assumptions are generic. Genericity in topology is similar to the notion of almost everywhere in measure theory, see Section 3.2.6. in [4] for more details. More formally:

[1] Note that the converse is not true as the vertical (double) line defined by $x_1^2 = x_2 = 0$ in $\mathbb{R}^3$ is smooth but the rank of its Jacobian is never full.

▶ **Definition 2.3.** [4, Definition 3.2.5] A subset of $C^\infty(U, \mathbb{R}^{n-1})$ is called residual if it contains the intersection of a countable family of dense open subsets in $C^\infty(U, \mathbb{R}^{n-1})$. A property is generic if it holds for a residual subset.

▶ Remark. By [4, Proposition 3.9.3], since $C^\infty(U, \mathbb{R}^{n-1})$ is a Baire space, every residual subset of $C^\infty(U, \mathbb{R}^{n-1})$ is dense.

Given a map $P \in \mathbb{U}$, our goal is to isolate the singular points of $\pi(C_P)$, that is to find a set of boxes (i.e., Cartesian products of 2 intervals) in $\mathbb{R}^2$ such that each box contains exactly one singular point of $\pi(C_P)$. This is the first step towards computing the topology of the curve $\pi(C_P)$.

3 Regular sytems

We recall the definition of regularity and illustrate it in Figure 3.

▶ **Definition 3.1.** A regular system is such that its Jacobian matrix is full rank at its solutions.

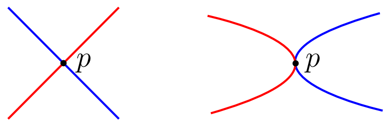

Figure 3 $p = (0, 0)$ is a regular solution of the system $\{x_1 + x_2 = x_1 - x_2 = 0\}$ (left) and it is not a regular solution of the system $\{x_1 + x_2^2 = x_1 - x_2^2 = 0\}$(right).

In order to isolate singular points of $\pi(C_P)$ using certified numerical methods [10, Chapter 8], we first need to encode them as the solutions of a regular zero-dimensional system of equations. To define such a system in Proposition 4.1, we introduce some notation.

▶ **Definition 3.2.** Let y, r be two sets of $n-2$ real variables and x_1, x_2, t be real variables. For a smooth map $f : U \to \mathbb{R}$, with $U \subseteq \mathbb{R}^n$, we define the maps:

$$S \cdot f(x_1, x_2, y, r, t) = \begin{cases} \frac{1}{2}[f(x_1, x_2, y + r\sqrt{t}) + f(x_1, x_2, y - r\sqrt{t})] & \text{for } t \neq 0 \\ f(x_1, x_2, y), & \text{for } t = 0. \end{cases}$$

$$D \cdot f(x_1, x_2, y, r, t) = \begin{cases} \frac{1}{2\sqrt{t}}[f(x_1, x_2, y + r\sqrt{t}) - f(x_1, x_2, y - r\sqrt{t})] & \text{for } t \neq 0 \\ \nabla f \cdot (0, 0, r), & \text{for } t = 0. \end{cases}$$

4 Contributions

▶ **Theorem 4.1.** *Assumptions 1 − 5 in Section 2 are generic, that is,* $\mathbb{U}$ *is residual in* $C^\infty(U, \mathbb{R}^{n-1})$.

We omit the proof Theorem 4.1 which is based on Thom's Transversality Theorem [4, Theorem 3.9.4].

▶ **Theorem 4.2.** *For* $P \in \mathbb{U}$ *there exists a regular zero-dimensional system* Ball(P) *(defined in Proposition 4.1) of* $2n-1$ *variables such that every real solution of* Ball(P) *projects bijectively to a singular point in* $\pi(C_P)$*, where the projection here is the one that sends every point in* $\mathbb{R}^{2n-1}$ *to its first two coordinates.*

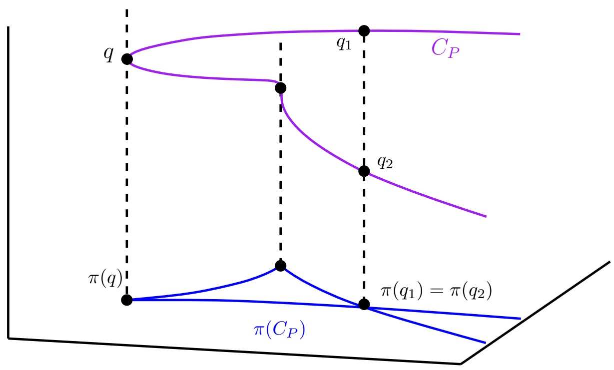

Figure 4 Illustration of Proposition 4.1 in the case $n = 3$.

More precisely, Proposition 4.1 explicits this bijection and Proposition 4.2 proves the regularity.

▶ Remark. Analogously to [7], we can check, using a semi-algorithm, whether a given map $P \in C^\infty(U, \mathbb{R}^{n-1})$ satisfies Assumptions $1 - 4$. This semi-algorithm stops if and only if Assumptions $1 - 4$ are satisfied. The reformulation of Assumption 5 as the regularity of system $\text{Ball}(P)$ in Proposition 4.2 also enables to check it via a semi-algorithm.

▶ **Proposition 4.1.** *Consider a map $P = (P_1, \ldots, P_{n-1}) \in C^\infty(U, \mathbb{R}^{n-1})$ that satisfies Assumptions $1 - 5$. Let $\text{Ball}(P)$ be the system:*

$$\begin{cases} S \cdot P_1 = \cdots = S \cdot P_{n-1} = 0 \\ D \cdot P_1 = \cdots = D \cdot P_{n-1} = 0 \\ r_1^2 + \cdots + r_{n-2}^2 = 1. \end{cases} \tag{1}$$

Denote by $M_P \subseteq \mathbb{R}^{2n-1}$ the set of real solutions of $\text{Ball}(P)$. Let $X = (x_1, x_2, y, r, t) \in \mathbb{R}^{2n-1}$, with $r_1^2 + \cdots + r_{n-2}^2 = 1$ and consider $q_1 = (x_1, x_2, y + r\sqrt{t})$ and $q_2 = (x_1, x_2, y - r\sqrt{t})$ in $\mathbb{R}^n$. Then, X is in M_P if and only if one of the following cases holds:

- *$q_1 \neq q_2$, $q_1, q_2 \in C_P$ and $\pi(q_1)$ is a node.*
- *$q_1 = q_2$, C_P contains q_1, $(0, 0, r) \in T_{q_1} C_P$ and $\pi(q_1)$ is an ordinary cusp.*

▶ **Proposition 4.2.** *Assume that $P \in C^\infty(U, \mathbb{R}^{n-1})$ satisfies Assumptions $1 - 4$, then $\text{Ball}(P)$ is regular at its solution if and only if Assumption 5 is satisfied.*

Proof. (Sketch) Let $X = (x_1, x_2, y, r, t)$ be a solution of $\text{Ball}(P)$. We consider two cases depending on t and prove that X is a regular solution of $\text{Ball}(P)$ if and only if (x_1, x_2) is either a node (when $t \neq 0$) or an ordinary cusp (when $t = 0$).

Case $t \neq 0$. Let $q_1 = (x_1, x_2, y + r\sqrt{t})$ and $q_2 = (x_1, x_2, y - r\sqrt{t})$ and $J_{\text{Ball}(P)}$ be Jacobian matrix of $\text{Ball}(P)$. By linear operations on $J_{\text{Ball}(P)}$, we get that the latter is of full rank if and only if the matrix $M = \begin{pmatrix} N_P(q_1) & 0 & M_P(q_1) \\ N_P(q_2) & M_P(q_2) & 0 \end{pmatrix}$ is full rank, where $M_P(q_1), M_P(q_2)$ are the $((n-1) \times (n-2))$-submatrices that are obtained respectively by removing the first two columns from $J_P(q_1), J_P(q_2)$ and $N_P(q_1), N_P(q_2)$ are the $((n-1) \times 2)$-submatrices formed by the first two columns of $J_P(q_1), J_P(q_2)$ respectively.

Now, $\text{Ball}(P)$ is not regular at X is equivalent to $\det(M) = 0$, which is equivalent to the fact that there exist $\alpha \in \mathbb{R}^2 \setminus \{0\}$ and $\beta, \gamma \in \mathbb{R}^{n-2}$ such that (α, β, γ) is in the kernel of M.

Under our assumption, the last statement is equivalent to say that (α, β) and (α, γ) are in $T_{q_1}C_P$ and $T_{q_2}C_P$ respectively and none of them is trivial. Equivalently, $\pi(q_1)$ is not a node. This concludes the proof in the case where $t \neq 0$ because we proved that X is not a regular solution of Ball(P) if and only if $\pi(q_1)$ is not a node. Hence, $\pi(q_1)$ is a node if and only if X is a regular solution of Ball(P).

Case $t = 0$. By Proposition 4.1, q_1 is in C_P. Using implicit function theorem [4, Corollary 2.7.2] and Hadamard‘s Lemma [4, Proposition 4.2.3] we can prove that, in a neighborhood of q in U, there exists a local coordinates system $(z_1, \dots, z_n)$ such that a neighborhood of $\pi(q_1)$ in $\pi(C_p)$ is the zero set of the equation $z_1^2 - z_2^{2k+1} = 0$, for some integer $k \geq 1$. By computing explicitly the Jacobian of Ball(P) in this new coordinate system, we can see that Ball(P) is regular at X if and only if $k = 1$. Thus, in the above local coordinate system, $\pi(C_p)$ has equation $z_1^2 - z_2^3 = 0$ and thus $\pi(q_1)$ is an ordinary cusp by Definition 2.1. Hence, X is a regular solution of Ball(P) if and only if $\pi(q_1)$ is an ordinary cusp of $\pi(C_p)$. ◀

5 Algorithmic aspects

Propositions 4.1 and 4.2 pave the way to the following algorithm:

The input of the algorithm is an integer $n \geq 3$, a bounded open subset U in $\mathbb{R}^n$ and $P \in \mathbb{U}$. The output is a set of boxes S each of which lives in $\mathbb{R}^2$ and contains exactly one singular point of $\pi(C_P)$. The idea of the algorithm is, first, to compute the system Ball(P). Second, to isolate the solutions of Ball(P) using a certified numerical solver (see for example [11]), computing a set of boxes S_0 in $\mathbb{R}^{2n-1}$ (the Cartesian product of $2n-1$ intervals) each of which contains exactly one solution of Ball(P). We can shrink every box of S_0 in such a way that their projections to $\mathbb{R}^2$ are pairwise disjoint. Finally, we project each box of S_0 and we add the projection to S, where the projection here is the one that sends every point in $\mathbb{R}^{2n-1}$ to its first two coordinates. Thus, every projected box in $\mathbb{R}^2$ contains exactly one singular point of $\pi(C_P)$.

The bottleneck of this method is the resolution of Ball(P) in $\mathbb{R}^{2n-1}$. For the certified numerical solver we can use homotopy methods [1] or subdivision methods [11]. Moreover, if we use subdivision methods, we might try to use the structure of the Ball system to reduce the dimension, as done in [6] for $n = 3$.

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