ASYMPTOTICS OF SIGNED ENTIERE SERIES

Luc Abergel

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Abstract:
This article focuses on the effects of several pertubations that would be applied to the exponential for large negative values. It's about identifying which ones are important and what are their effects.

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Introduction

The purpose of this work is to study some perturbations in the series $e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$.

So we want to give an asymptotic at $+\infty$ of $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} u(n)$ for explicit sequences made using expressions such as $(n+a)^k$ ou $\ln^k(n+a)$.

The basic difficulty lies in the fact that an error on a term u_n gives a perturbation of the sum whose order of magnitude is x^n. The case of the e^{-x} series shows that such an error then hides the asymptotic of the sum .
We therefore do not expect a priori to obtain any meaningful results for this question. We can even think that in fact such series have erratic behavior in $+\infty$.
It turns out that this is not the case for the sequenses envisaged, and that on the contrary, there are very regular results, with even a simple and regular dependence on the parameters (a and k in the examples cited). We will see that if the serie u_n is not defined for all n, choosing to look at the sum from a certain rank or for the n values for which the terms u_n are defined does not change anything about asymptotics. Another surprising result, we will also see that there is an order structure on successive perturbations that could be applied to the e^{-x} series. For example, we can give an infinite number of terms for a perturbation such as $(n+a)^k$ and that in the case of 2 perturbations $(n+a_1)^{k_1}$ and $(n+a_2)^{k_2}$, only one of them plays a role on an asymptotic of the sum, the other contributing only through a constant by which the effect of the major peeturbation is multiplied.

The working method is to give an integral representation of u_n then the sum $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} u(n)$. It is not so obvious to do, for example by the fact that we could not treat sequences that would not tend towards

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0 by this method *stricto sensu*.

We'll start with a case that's very simple to obtain and motivates the work that follows, then we'll present definitions that allow for the full development of certain sequences, then we'll deal with concrete examples, and we will end with the study of the case of superposition of perturbations which is the essence of this article, as well as examples illustrating the effectiveness of this approach.

Notations : We will note indifferently $\Gamma(a) = (a-1)!$ for $a \notin -\mathbb{N}$ and $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt$.

1 Case of a rational fraction.

We're going to study here $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} R(n)$ where R is a rational fraction. We will note here that if u is polynmial, then $F_u(x) = O(\rho^x)$
(this being done by treating the case of the polynomial $X(X-1)\cdots(X-k+1)$).

1.1 A first case.

We're studying the case $u(x) = \frac{1}{x+a}$ with $\Re(a) > 0$.
On the one hand, we write $\int_x^{\infty} t^{a-1} e^{-t}\ dt = O(x^{a-1}e^{-x})$,
on the other hand $\int_0^x t^{a-1}e^{-t}\ dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+a}}{n!(n+a)}$.
This shows the formula :

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!(n+a)} = x^{-a}\Gamma(a) + O(x^{a-1}e^{-x}).$$

The error term is well in $O(\rho^x)$.

1.2 Generalization to the case $a \notin -\mathbb{N}$.

Proposition : The case $u(x) = \frac{1}{(x+a)}$ with $a \notin -\mathbb{N}$.
We have the following asymptotic :

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!(n+a)} \underset{\infty}{\sim} \frac{\Gamma(a)}{x^a} \text{ si } a \notin -\mathbb{N}.$$

Proof :
As example, we will only deal with the case of $-1 < \Re(a) \leq 0$.
Let $a = -1 + b$ with $\Re(b) > 0$.
We write $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!(n+a)} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!(n-1+b)} = \frac{1}{-1+b} + \sum_{n=0}^{\infty} \frac{-(-1)^n x^{n+1}}{(n+1)!(n+b)}$.
Then $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!(n+a)} = \frac{1}{b-1} - x\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} \frac{1}{(n+1)(n+b)} = \frac{1}{b-1} - x\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} \left[\frac{1}{n+1} - \frac{1}{n+b}\right] \frac{1}{b-1}$.
Let $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!(n+a)} = \frac{1}{b-1} - \frac{x}{b-1}\left[\frac{\Gamma(1)}{x} - x^{-b}\Gamma(b) + O(\rho^x)\right]$, and we deduce :

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!(n+a)} = x^{-b+1}\frac{\Gamma(b)}{b-1} + O(\rho^x).$$

This is the formula for 1.1 but in the more general case $\Re(a) > -1$.
The case $a \notin -\mathbb{N}$ is also written in the same way. ■

1.3 Case of a multiple pole.

Case $u(x) = \frac{1}{(x+a)^k}$ with $a \notin -\mathbb{N}$ and $p \in \mathbb{N}$.
We have the following asymptotic :

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!(n+a)^k} \underset{\infty}{\sim} \frac{\Gamma(a)}{x^a} \frac{\ln^{k-1}(x)}{(k-1)!} \text{ si } a \notin -\mathbb{N}.$$

Proof :

Again, we will only deal with one case, the $k = 2$ case, the working method becoming generalized without any difficulty for the other values of k.
We star with $\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+a-1}}{n!(n+a)} = \frac{\Gamma(a)}{x} + O(\rho^x)$ that we integrate on $[0, x]$. ■

1.4 Case of a rational fraction.

The case of a rational fraction is now available.
Note the case where u is polynomial :
We're dealing with the case $u_k(x) = x(x-1)...(x-k+1)$ that gives $F_{u_k}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(n-k)!}$.
And so $F_{u_k}(x) = (-x)^k e^{-x} = O(\rho^x)$.

The case of a rational fraction.
If we give $u(x) = \frac{P(x)}{\prod_{1 \leq i \leq p} (x+a_i)^{p_i}}$ with P polynomial, with integer exponents p_i strictly positive, if we have $\Re(a_1) < \Re(a_i)$ pour tout $i \neq 1$, and if we finally notice $p = p_1$, $a = a_1$ so

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} u(n) \underset{\infty}{\sim} A \frac{\Gamma(a)}{x^a} \frac{\ln^{p-1}(x)}{(p-1)!}.$$

with $A = \frac{P(-a)}{\prod_{1 \leq i \leq p} (a_i - a)^{p_i}}$.

Proof :
It's enough to decompose into simple elements and apply the previous results. ■

Generalization in the case of a rational fraction that may have negative integer poles.

Again by decomposition into simple elements, it is enough to treat the case of a monome.
Le cas $\sum_{n=p+1}^{\infty} \frac{(-1)^n x^n}{n!(n-p)^k}$.
We have the asymptotic

$$\sum_{n=p+1}^{\infty} \frac{(-1)^n x^n}{n!(n-p)^k} \underset{\infty}{\sim} (-1)^{k+1} \frac{x^p}{p!} \frac{\ln^k(x)}{k!}.$$

Proof :

We therefore want to study, for example, the case of $\sum_{n=p+1}^{\infty} \frac{(-1)^n x^n}{n!(n-p)^k} = (-x)^{p+1} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!(n+1)^{k+1}(n+2)\cdots(n+p+1)}$
According to the case of a rational factorized fraction, we obtain an equivalent by looking at the part relating to the pole $\frac{1}{(n+1)^{k+1}}$:
$\frac{1}{(n+1)^{k+1}(n+2)\cdots(n+p+1)} = \frac{1}{p!} \frac{1}{(n+1)^{k+1}} + \cdots$.
The equivalent sought is therefore $(-x)^{p+1} \frac{1}{p!} \frac{\ln^k(x)}{k!} \frac{\Gamma(1)}{x}$.
So $\sum_{n=p+1}^{\infty} \frac{(-1)^n x^n}{n!(n-p)^k} \underset{\infty}{\sim} (-1)^{p+1} \frac{x^p}{p!} \frac{\ln^k(x)}{k!}$.
It should be noted, given the equivalent obtained, that the result is the same for $\sum_{n \neq p} \frac{(-1)^n x^n}{n!(n-p)^k}$. ■

An example where several poles have even real parts. Let's deal with the case $F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!(n^2+1)}$.

We write $\frac{1}{n^2+1} = \frac{i}{2}\left[\frac{1}{x+i} - \frac{1}{x-i}\right]$. This gives us $F(x) = \frac{i}{2}\left[\Gamma(i)x^{-i} - \Gamma(-i)x^{i}\right] + O(\rho^x)$, or, by writing $\Gamma(i) = \rho e^{i\theta}$:

$$F(x) \underset{+\infty}{=} \rho \sin(\ln(x) - \theta) + O(r^x) \text{ avec } r < 1.$$

2 Integral form and applications.

Notations and working context :

- We consider functions defined from a certain rank n_0 and we will talk about sequences by looking at the values of such functions over fairly large integers.
- For a given sequence u we note $F_u(x) = \sum_{n\geq n_0}^{\infty} \frac{(-1)^n x^n}{n!} u(n)$.

For a given sequence u :

- If we can write $u(x) = \int_0^1 t^x \pi(t)\, dt$ for a function π and for any $x \geq 0$, then we'll say that the u is density sequence. It will be said to be associated with π that we will note π_u.
- If we can write $u(x) = \int_0^1 \left(t^x - \sum_{i=0}^{p-1} \frac{(x\ln(t))^i}{i!}\right) \pi(t)\, dt$ for an integer p, then we will say that u is deduced from a density sequence.

In practice we will assume π monotonous around 0.

The working context will be that of density sequences or suites that are deduced from them. We will naturally see these last ones appear by operations on density sequences.

It should be noted that the case of a defined sequence, for example for $x \geq 1$, is not that of a density sequence, but only of a sequence that is deduced from it, as will be seen later.

In the case of a density sequence, $F_u(x) = \int_0^1 e^{-tx}\pi_u(t)\, dt$, it is then the study of this integral that gives an asymptotic of F_u in $+ + \infty$. So we'll look for an asymptotic at $O(\rho^x)$ with $\rho < 1$.

Therefore, sometimes we will write'=' for two terms that differ by a $O(\rho^x)$ (always with $\rho < 1$).

Finally, let's note that the values of $u(x)$ for x large essentially involve the behavior of π in 1, while those of F_u in $+\infty$ that of π in 0.

3 First properties.

- **Derivation :**

  If u is a density sequence associated to π_u, then u' also is, associated to $\pi_{u'} = \ln(t)\pi_u(t)$.

- **Primitivation :**

  If $U(x) = \int_0^x u(t)\, dt$, then $U(t) = \int_0^1 (1 - t^x)\frac{\pi_u(t)}{\ln(1/t)}\, dt$. This sequence is therefore deduced from a density sequence.

- **Truncation :**

  The truncated sequence $Tu(0) = 0$ et $Tu(n) = u(n)$ for $n \geq 1$ verifies
  $F_{Tu}(x) = \displaystyle\int_0^1 (e^{-xt} - 1)\pi_u(t)\ dt.$
  If π is integrable in 0 (so that $u(0)$ is defined), noting Π_u the primitive of π_u which is vanishes in 1,

  $$F_{Tu}(x) = x \int_0^1 e^{-xt}\Pi_u(t)\ dt.$$

  We will see just after that that it means that this sequence is deduced from a density sequence.

  This is immediate after an integration by parts.

- **Shift :**

  We note S^α the operator who has a sequence u associates the sequence $x \to u(x+\alpha)$ where α is real. If u is a density sequence, then $S^\alpha(u)$ is associated to $t^\alpha \pi_u(t)$.
  And so $F_{S^\alpha(u)}(x) = \displaystyle\int_0^\infty e^{-xt} t^\alpha \pi_u(t)\ dt.$

Proof :

We write $S^\alpha(u)(x) = \displaystyle\int_0^1 t^x t^\alpha \pi_u(t)\ dt.$

This then gives $F_{S^\alpha(u)}(x) = \displaystyle\int_0^\infty e^{-xt} t^\alpha \pi_u(t)\ dt.$ ■

Remark :

For example, if we consider the sequence $u(x) = \frac{1}{\ln(x+2)}$ defined for $x \geq 0$, the sequence $S^{-1}(u)$ is only defined from the rank 1. It is therefore not a priori a density sequence, but it is deduced by truncation from the u sequence from the rank 1.
It is therefore necessary to consider $F_{TS^{-1}}(u)(x) = x \displaystyle\int_0^1 e^{-xt}\Pi_u(t)\ dt$ with $\Pi_u(t) = -\displaystyle\int_t^1 \frac{\pi_u(s)}{s}\ ds.$

Cases of sequences that are deduced from density sequences.

We also want to study sequences that do not tend towards 0 in $+\infty$. We will be able to show them as sequences deduced from density sequences.

given a density of π, we're interested in the sequence $u(x) = \displaystyle\int_0^1 \left(t^x - \sum_{i=0}^{p-1} \frac{(x\ln(t))^i}{i!} \right) \pi(t)\ dt$ and we want to study F_u in $+\infty$.
This will require some definitions.

We recall the notation $\Delta(P) = P(X+1) - P(X)$ for a polynomial.

- For a function π that can be integrated on $]0.1[$, we define the operator Φ by

  $$\Phi(\pi)(t) = \frac{1}{t}\int_0^t \pi(s)\ ds.$$

- For a positive integer p, we note $Q_{\alpha,p}$ the polynomial defined by $S_{\alpha,p}(X) = \displaystyle\sum_{i=0}^{p} b_{\alpha,i}\frac{(-X)^i}{i!}$
  with $b_{\alpha,i} = \Delta^i((X+\alpha)^p)(0)$. In the case of $\alpha = 0$, we will simply note $b_i = b_{\alpha,i}$ and $S_p = S_{0,p}$.

Let us note a remark that will be used later :

Note on coefficients $b_{\alpha,i}$:
$\sum_{i=0}^{p}(-1)^i b_{\alpha,i}\binom{\alpha+i}{i} = (-1)^p$.

Proof :

This comes from $\binom{-\alpha-1}{i} = (-1)^i\binom{\alpha+i}{i}$ and from $(X+\alpha)^p = \sum_{i=0}^{p} b_{\alpha,i}\binom{X}{i}$
which is applied in $X = -\alpha-1$. ■

Let be π an integrable function on $]0,1[$.
If $u(x) = \int_0^1 \left(t^x - \sum_{i=0}^{p-1}\frac{(x\ln(t))^i}{i!}\right)\pi(t)\ dt$, then :

- $u(x) = (-x)^p \int_0^1 t^x \Phi^p(\pi)(t)\ dt$.
- $F_u(x) = (-1)^p \int_0^1 e^{-xt} S_p(xt)\Phi^p(\pi)(t)\ dt$.

Proof :

First point :
We integrate by parts $\int_0^1 \left(t^x - \sum_{i=0}^{p-1}\frac{(x\ln(t))^i}{i!}\right)\pi(t)\ dt$.
We then obtain
$\left[\left(t^x - \sum_{i=0}^{p-1}\frac{(x\ln(t))^i}{i!}\right)t\Phi(\pi)(t)\right]_0^1 - \int_0^1 x\left(t^{x-1} - \sum_{i=1}^{p-1}\frac{(x\ln(t))^{i-1}}{t(i-1)!}\right)t\Phi(\pi)(t)\ dt$,
then $(-x)\int_0^1 \left(t^x - \sum_{i=0}^{p-2}\frac{(x\ln(t))^i}{i!}\right)\Phi(\pi)(t)\ dt$,
and we conclude by induction.

Second point :
Note $N_i(X) = \frac{X(X-1)\cdots(X-i+1)}{i!} = \binom{X}{i}$.
We write $X^p = \sum_{i=0}^{p} b_i N_i(X)$ avec $b_i = \Delta^i(X^p)(0)$.
So we have $u(x) = (-1)^p \sum_{i=0}^{p} b_i N_i(x)\int_0^1 t^x\Phi^p(\pi)(t)\ dt$.
For $v_i(x) = N_i(x)\int_0^1 t^x\Phi^p(\pi)(t)\ dt$, we have
$F_{v_i}(x) = \sum_{n\geq 0}\frac{(-1)^n x^n}{n!}N_i(n)\int_0^1 t^n\Phi^p(\pi)(t)\ dt = \sum_{n\geq i}\frac{(-1)^n x^n}{i!(n-i)!}\int_0^1 t^n\Phi^p(\pi)(t)\ dt$, and then
$F_{v_i}(x) = \int_0^1 \sum_{n\geq i}\frac{(-1)^n (xt)^n}{i!(n-i)!}\Phi^p(\pi)(t)\ dt = \int_0^1 e^{-xt}\frac{(-xt)^i}{i!}\Phi^p(\pi)(t)\ dt$.
So $F_u(x) = (-1)^p\int_0^1 e^{-xt}\sum_{i=0}^{p} b_i\frac{(-xt)^i}{i!}\Phi^p(\pi)(t)\ dt = (-1)^p\int_0^1 e^{-xt}S_p(xt)\Phi^p(\pi)(t)\ dt$. ■

Application to an integral representation of $S^\alpha u$ and of $F_{S^\alpha u}$.
For such a sequence u, per shift we have :

- $S^\alpha(u)(x) = (-1)^p(x+\alpha)^p \int_0^1 t^{x+\alpha}\Phi^p(\pi)(t)\ dt$.

- $F_{S^\alpha u}(x) = (-1)^p \int_0^1 e^{-xt} S_{\alpha,p}(xt) t^\alpha \Phi^p(\pi)(t)\ dt.$

Proof :

First point is obvious.

For the second, we must write $(X+\alpha)^p = \sum_{i=0}^{p} a_{\alpha,i} \binom{X}{i}$ with $a_{\alpha,i} = \Delta^p((X+\alpha)^p)(0)$.

Thus we obtain the result by the relation $S_{\alpha,p}(X) = \sum_{i=0}^{p} a_{\alpha,i} \frac{(-X)^i}{i!}$. ■

4 Some technical results.

Let us mention here some asymptotics that will be used, without dwelling on the demonstrations.

Si $t^\alpha \pi(t)$ is integrable on $]0,1]$ et si $\rho < 1$, so $\rho^x = o\left(F_u(x)\right)$.
Furthermore, if $\pi_1 \underset{0}{\sim} \pi_2$ in 0, then $F_{u_1} \underset{+\infty}{\sim} F_{u_2}$.

For the first point :
$\int_0^1 e^{-xt}\pi(t)\ dt = \int_0^1 t^{-\alpha} e^{-xt} t^\alpha \pi(t)\ dt \geq a^{-\alpha} e^{-xa} \int_a^1 t^\alpha \pi(t)\ dt$ for $a > 0$,
this leading term ρ^x if we choose a small enough. ■

For the second :
If $a(t) = o(\pi(t))$ in 0, let's check that $\int_0^1 e^{-xt} a(t)\ dt = o\left(\int_0^1 e^{-xt}\pi(t)\ dt\right)$.
We write $|a(t)| \leq \varepsilon \pi(t)$ over $]0,\alpha]$ and so $\int_0^1 e^{-xt} a(t)\ dt \leq \varepsilon \int_0^1 e^{-xt}\pi(t)\ dt + e^{-\alpha x} \int_\alpha^1 a(t)\ dt.$
The increase in the first point then makes it possible to conclude. ■

And finally for $a > 0$ (voir [2]):

$$* \int_0^1 e^{-xt} t^{a-1} \ln^b(1/t)\ dt \underset{\infty}{\sim} \Gamma(a) \frac{\ln^b(x)}{x^a}$$
$$= \frac{1}{x^a} \sum_{i=0}^{p} \binom{i}{b} \ln^{b-i}(x)(-1)^i \Gamma^{(i)}(a) + o\left(\frac{\ln^{b-p}}{x^a}\right)$$

$$** \int_0^{1/e} e^{-xt} t^{a-1} \ln^b(1/t) \ln^c(\ln(1/t))\ dt \underset{\infty}{\sim} \Gamma(a) \frac{\ln^b(x)\ln^c(\ln(x))}{x^a}$$
$$= \frac{1}{x^a} \sum_{0 \leq i+k \leq p \text{ and } j \leq q} \binom{i}{b} \ln^{b-i-k}(x) \ln^{c-j}(\ln(x))(-1)^i \Gamma^{(i+k)}(a) + o\left(\frac{\ln^{b-p}\ln^{c-q}(\ln(x))}{x^a}\right)$$

5 Practical calculations.

5.1 Case $u(x) = \frac{1}{(x+a)^k}$ with a and k positive.

For $u(x) = \frac{1}{x+1}$ we have $\pi_u(t) = 1_{[0,1]}$. We notice π or even π_u this function.
By calculating the derivative, we have $\frac{(-1)^k k!}{(x+1)^{k+1}} = \int_0^1 t^x \ln^k(t)\ dt$.
That is, for $u_k(x) = \frac{1}{(x+1)^k}$, $\pi_{u_k}(t) = \frac{(-1)^{k-1}\ln^{k-1}(t)}{(k-1)!}$ on the interval $]0,1]$.
The case of a positive integer exponent k was treated as such.

This is widespread in : if $u_k(x) = \frac{1}{(x+1)^k}$ with $k \in \mathbb{R}^+$, then $\pi_{u_k}(t) = \frac{\ln^{k-1}(1/t)}{(k-1)!} 1_{]0,1]}$.

This relation is easily verified by puting $t = e^{-u}$ in $\dfrac{1}{(k-1)!}\displaystyle\int_0^1 t^x \ln^{k-1}(1/t)\ dt$,
remembering the rating $(k-1)! = \Gamma(k)$.
This results in the following formula $F_{u_k}(x) = \dfrac{1}{(k-1)!}\displaystyle\int_0^1 e^{-xt}\ln^{k-1}(t)\ dt$.

We will now present here the case $v_k(x) = \frac{1}{(x+a)^k}$ with $a > 0$.

We just saw that if $u_k(x) = \frac{1}{(x+1)^k}$, then $\pi_{u_k}(t) = \frac{\ln^{k-1}(1/t)}{(k-1)!}1_{]0,1]}$.
By the shift operator we obtain $v_k = S^{a-1}(u_k)$.
So $\pi_{v_k}(t) = t^{a-1}\frac{\ln^{k-1}(1/t)}{(k-1)!}1_{]0,1]}$,
and then $F_{v_k}(x) = \dfrac{1}{(k-1)!}\displaystyle\int_0^1 e^{-xt}t^{a-1}\ln^{k-1}(1/t)\ dt$.

For real positive a and k, we have the asymptotic

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!(n+a)^k} = x^{-a}\sum_{i=0}^{p}\binom{k-1}{i}\ln^{k-1-i}(x)(-1)^i\Gamma^{(i)}(a) + o\left(\frac{\ln^{k-p}(x)}{x^a}\right).$$

And in particular

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!(n+a)^k} \underset{\infty}{\sim} \frac{\Gamma(a)}{x^a}\frac{\ln^{k-1}(x)}{(k-1)!}.$$

This is directly deduced from the integral expression and the $*$ relation. ■

Important remark :

It should be noted that everything that has been mentioned is true for an exponent $k > -1$ since the functions that appear to be integrable on]0.1[.

In conclusion :

Pour $u(x) = \frac{1}{(x+a)^k}$ with $k > -1$ and $a > 0$, we have

$$\pi_u(t) = t^{a-1}\frac{\ln^{k-1}(1/t)}{(k-1)!}$$

$$F_u(x) \underset{\infty}{\sim} \frac{\Gamma(a)}{x^a}\frac{\ln^{k-1}(x)}{(k-1)!}$$

5.2 Case x^{-k} with k positive.

We will present here the case $u(x) = x^{-k}$ et $u(0) = 0$.
So we're looking at $F_u(x) = \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n!n^k}$.
Here the shift and truncation operators will intervene.

We know that for $v(x) = (x+1)^{-k}$ we have

- $v(x) = \displaystyle\int_0^1 t^x \frac{\ln^{k-1}(1/t)}{(k-1)!}\ dt$.
- $F_v(x) = \displaystyle\int_0^1 e^{-xt}\frac{\ln^{k-1}(1/t)}{(k-1)!}\ dt$.

By shifting and then truncation (see page 5), we have $F_u(x) = x\displaystyle\int_0^1 e^{-xt}\Pi(t)\ dt$ with $\Pi(t) = -\displaystyle\int_t^1 \frac{\ln^{k-1}(1/s)}{s(k-1)!}\ ds$.
By truncation, we have thus shown $F_u(x) = -x\displaystyle\int_0^1 e^{-xt}\frac{\ln^k(1/t)}{k!}\ dt$.

For k real positive, we have the asymptotic

$$\sum_{n=1}^{+\infty} \frac{(-1)^n x^n}{n! n^k} = -\frac{1}{k!} \sum_{i=0}^{p} \binom{k}{i} \ln^{k-i}(x)(-1)^i \Gamma^{(i)}(1) + O\left(\ln^{k-p}(x)\right).$$

And in particular

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n! n^k} \underset{\infty}{\sim} -\frac{\ln^k(x)}{k!}.$$

This is directly deduced from the integral expression and the relation $*$ ■

In conclusion :

For $u(x) = \frac{1}{n^k}$ with k real positive and $x > 0$, we have $$F_u(x) = -x \int_0^1 e^{-xt} \frac{\ln^k(1/t)}{k!}\ dt$$ $$F_u(x) \underset{\infty}{\sim} -\frac{\ln^k(x)}{k!}$$

5.3 Case of a negative exponent.

We will study the case $u(x) = (x+a)^k$ with $k > 0$.

We will start with the case $u(x) = (x+1)^k$ then conclude with the shift operator.

Integral representation of $u(x) = (x+1)^{p-k}$:

If $0 < k < 1$ and if p is a positive integer, then

$$(x+1)^{p-k} = \int_0^1 \left(t^x - \sum_{i=0}^{p-1} \frac{(x \ln(t))^i}{i!} \right) \frac{\ln^{k-p-1}(1/t)}{(k-p-1)!}\ dt + P_k(x).$$

where P_k is polynomial.

Proof :

It should be noted first that $t \to \left(t^x - \sum_{i=0}^{p-1} \frac{(x \ln(t))^i}{i!} \right) \frac{\ln^{k-p-1}(1/t)}{(k-p-1)!}$ is integrable over $]0,1[$.

Let us note $u_p(x) = (x+1)^{p-k}$ and $v_p(x) = \int_0^1 \left(t^x - \sum_{i=0}^{p-1} \frac{(x \ln(t))^i}{i!} \right) \frac{\ln^{k-p-1}(1/t)}{(k-p-1)!}\ dt + Q_k(x)$ with Q_k a polynomial whose derivative is P_k.
A derivation calculation left to the reader shows that $u_p^{'}(x) = (p-k)u_{p-1}(x)$ et $v_p^{'}(x) = (p-k)v_{p-1}(x) + P_k(x)$.
The case p =0 corresponds to $u(x) = \frac{1}{(x+a)^{-k}}$ with $-k > -1$ which has been treated in section 5.1 (under the title important remark).
It is then sufficient to adjust $P_k(0)$ to state that $u_k = v_k$.
We can therefore conclude by induction that the Proposition is verified for any $p \in \mathbb{N}$. ■

Remark :

We have already reported that for a polynomial u function, $F_u(x) = O(\rho^x)$.
The appearance of the polynomials P_k is therefore of no importance to the asymptotics sought, even if clearly we could calculate these polynomials.
The general case $u_k(x) = (x+a)^k$.

For $k \notin \mathbb{N}^*$ we have

$$F_{u_k}(x) \underset{\infty}{\sim} \frac{\Gamma(a)}{x^a} \frac{\ln^{-k-1}(x)}{(-k-1)!}.$$

Proof :

We'll actually take $0 < k < 1$ and process the case $u(x) = (x+a)^{k-p}$.
Let's pose $\pi_k(t) = \frac{\ln^{k-p-1}(1/t)}{(k-p-1)!}$.
In the case $a = 1$ as u is deduced from a density sequence, we know that

$$F_u(x) = (-1)^p \int_0^1 e^{-xt} S_p(-xt)\Phi^p(\pi_k)(t)\ dt.$$

So by the shift operator (see page 4)

$$F_u(x) = (-1)^p \int_0^1 e^{-xt} S_{a-1,p}(-xt)t^{a-1}\Phi^p(\pi_k)(t)\ dt.$$

We clearly have the implication : If $\pi(t) \underset{0}{\sim} \frac{\ln^\alpha(1/t)}{\alpha!}$ then $\Phi(\pi)(t) \underset{0}{\sim} \frac{\ln^\alpha(1/t)}{\alpha!}$.
So here, $\Phi^p(\pi_k)(t) \underset{0}{\sim} \frac{\ln^{k-p-1}(1/t)}{(k-p-1)!}$.
Let's write $S_{a-1,p}(X) = \sum_{i=0}^{p} b_{a-1,i}\frac{(-X)^i}{i!}$.
As $\int_0^1 e^{-xt}(xt)^i t^{a-1}\Phi^p(\pi_k)(t)\ dt \underset{\infty}{\sim} x^i \frac{\Gamma(a+i)}{x^{a+i}}\frac{\ln^{k-p-1}(x)}{(k-p-1)!}$,
we thus obtain $F_u(x) = (-1)^p \sum_{i=0}^{p}(-1)^i b_{a-1,i}\frac{\Gamma(a+i)}{i!x^a}\frac{\ln^{k-p-1}(x)}{(k-p-1)!} + o\left(\frac{\ln^{k-p-1}(x)}{x^a}\right)$ at $+\infty$.
We write $\frac{\Gamma(a+i)}{i!} = \frac{(a+i-1)\cdots a}{i!}\Gamma(a) = \binom{i+a-1}{i}\Gamma(a)$.
The relation $\sum_{i=0}^{p}(-1)^i b_{a-1,i}\binom{i+a-1}{i} = (-1)^p$ quoted and established in page 5 then makes it possible to conclude. ■

In conclusion :

$$u(x) = (x+a)^k \textbf{ with } k > 0,$$

$$u(x) = \int_0^1 \left(t^x - \sum_{i=0}^{p-1}\frac{(x\ln(t))^i}{i!}\right)\frac{\ln^{k-p-1}(1/t)}{(k-p-1)!}\ dt + Q_k(x) \textbf{ with } Q_k \textbf{ polynomial}$$

$$\textbf{For } k \notin \mathbb{N}^* \textbf{ we have } F_u(x) \underset{\infty}{\sim} \frac{\Gamma(a)}{x^a}\frac{\ln^{-k-1}(x)}{(-k-1)!}$$

5.4 Case of a negative pole a.

Asymptotic of $F_u(x) = \sum_{n \geq n_0}\frac{(-1)^n x^n}{n!}(n+a)^k$ avec $k > 0$.

Let $n_0 - 1 < a \leq n_0$ **and let** $u(x) = \frac{1}{(x-a)^k}$
If $n_0 - 1 < a < n_0$ **then** $F_u(x) \underset{\infty}{\sim} (-1)^{n_0} x^a \Gamma(-a)\frac{\ln^{-k-1}(x)}{(-k-1)!}$
If $a = n_0$ **then**

$$F_u(x) \underset{\infty}{\sim} (-1)^{n_0+1}\frac{x^{n_0}}{n_0!}\frac{\ln^{-k}(x)}{(-k)!}$$

Proof :

We put $a = n_0 - \varepsilon$ with $0 \leq \varepsilon < 1$.

If we note $n = m + n_0$, we rewrite $F_u(x)$ as $F_u(x) = (-x)^{n_0} \sum_{m \geq 0} \frac{(-1)^m x^m}{m!(m+1)\cdots(m+n_0)(m+\varepsilon)^k}$. If k is an integer, by the case of a rational fraction treated at the beginning, we have

$F_u(x) \underset{\infty}{\sim} (-x)^{n_0} K F_v(x)$ where $v(x) = \frac{1}{(x+\varepsilon)^k}$ and $K = [(1-\varepsilon)(2-\varepsilon))\cdots(n_0-\varepsilon)]^{-1}$.
For the other values of k, we will find this result by the principle of no superposition of perturbations in the example paragraph.

- Cas $n_0 - 1 < a < n_0$:
  As $F_v(x) \underset{\infty}{\sim} \frac{\Gamma(\varepsilon)}{x^\varepsilon} \frac{\ln^{-k-1}(x)}{(-k-1)!}$ and by the relation $\Gamma(\varepsilon) = (\varepsilon - 1)\cdots(\varepsilon - n_0)\Gamma(\varepsilon - n_0)$,
  we thus obtain the relation $F_u(x) \underset{\infty}{\sim} (-1)^{n_0} x^a \Gamma(-a) \frac{\ln^{-k-1}(x)}{(-k-1)!}$.
- Case $a = n_0$:
  In this case $v(x) = \frac{1}{x^k}$ and $F_v(x) \underset{\infty}{\sim} -\frac{\ln^{-k}(x)}{(-k)!}$.
  So $F_u(x) \underset{\infty}{\sim} (-1)^{n_0+1} \frac{x^{n_0}}{n_0!} \frac{\ln^{-k}(x)}{(-k)!}$. ■

It should be noted that, as in the case of rational fractions, we have the same equivalent if we study $F_u(x) = \sum_{n \neq n_0} \frac{(-1)^n x^n}{n!} u(n)$.

5.5 Conclusion.

We consider $u(x) = \frac{1}{(x+a)^k}$ with x and k real
We define $n_0 = 0$ if $a > 0$ and $n_0 - 1 < a \leq n_0$ if $a \leq 0$

$$F_u(x) \underset{\infty}{\sim} (-1)^{n_0} \frac{\Gamma(a)}{x^a} \frac{\ln^{k-1}(x)}{(k-1)!} \textbf{ si } a \notin -\mathbb{N}$$

$$F_u(x) = O(\rho^x) \textbf{ si } a \in -\mathbb{N}$$

6 Logarithmic cases.

6.1 Cas $\ln(x+a)$.

We will present here the case $u(x) = \ln(x+a)$.

Case $u(x) = \ln(x+a)$.

If $u(x) = \ln(x+a)$, then

$$F_u(x) \underset{\infty}{\sim} -\frac{\Gamma(a)}{x^a \ln(x)}$$

Proof :

Let $u(x) = \ln(x+1)$.
Clearly, by primitization, $u(x) = \int_0^1 \frac{t^x - 1}{\ln(t)}\, dt$.
Let $Li(t) = \int_0^t \frac{ds}{\ln(s)}$.
Because u is deduced from a density sequence, we then have $F_u(x) = -\int_0^1 e^{-xt} S_1(xt) \Phi(Li)(t)\, dt$.
We have the asymptotic $Li(t) \underset{0}{\sim} \frac{t}{\ln(t)}$,

and then $\Phi(Li)(t) \sim \frac{1}{\ln(t)}$,
so, $\displaystyle\int_0^1 e^{-xt} Li(t)\, dt \underset{\infty}{\sim} x \int_0^1 e^{-xt} \frac{t}{\ln(1/t)}\, dt$ (Remember $S_1(X) = -X$).
Thus, by the relation $*$, $F_u(x) \underset{\infty}{\sim} -\frac{1}{x\ln(x)}$.

Let us treat the case $a \neq 1$ in the same way :
For $v(x) = \ln(x+a)$, by shifting u by S^{a-1}, $F_v(x) = -\displaystyle\int_0^1 e^{-tx} t^{a-1} S_{1,a-1}(xt)\Phi(Li)(t)\, dt$,
and then $F_v(x) = -\displaystyle\int_0^1 e^{-xt}(a-1-xt)Li(t)t^{a-1}\, dt$,
and finally, after calculations left to the reader, $F_v(x) \underset{\infty}{\sim} -\frac{\Gamma(a)}{x^a \ln(x)}$. ■

In conclusion :

$$u(x) = \ln(x+a)$$

$$F_u(x) = -\int_0^1 e^{-xt}(a-1-xt)Li(t)t^{a-1}\, dt$$

$$F_u(x) \underset{\infty}{\sim} -\frac{\Gamma(a)}{x^a \ln(x)}$$

6.2 Case $\ln^k(x+a)$ with k integer.

Case $u(x) = \ln^k(x+a)$ with k positive integer.

If $v_k(x) = \ln^k(x+a)$ for a positive integer k, then

$$F_{v_k}(x) \underset{\infty}{\sim} (-1)^k k \frac{a!}{x^a} \frac{\ln^{k-1}(\ln(x))}{\ln(x)}.$$

Proof :

We're going to proceed in 3 steps.

1) Study of $I_k(x) = \displaystyle\int_0^1 (1-t^x)\frac{\ln^k(\ln(1/t))}{\ln(1/t)}\, dt$.
In this paragraph we will note $c_k = \Gamma^{(k)}(1) = \displaystyle\int_0^\infty \ln^k(t)e^{-t}\, dt$.
By changing the variable $e^{-t} = u$ we get $\Gamma^{(k)}(1) = \displaystyle\int_0^1 \ln^k(\ln(1/u))\, du$.
We clearly have $I'_k(x) = \displaystyle\int_0^1 t^x \ln^k(\ln(1/t))\, dt$.
We write $I'_k(x) = \displaystyle\int_0^x t^{x-1} t \ln^k(\ln(1/t))\, dt$ that we integrate by parts :

$$I'_k(x) = \underbrace{\left[\frac{(t^x-1)}{x} t \ln^k(\ln(1/t))\right]_0^1}_{0} - \int_0^1 \frac{(t^x-1)}{x} \underbrace{\left(\ln^k(\ln(1/t)) - kt\ln^{k-1}(\ln(1/t))\frac{-1/t}{\ln(1/t)}\right)}_{\ln^k(\ln(1/t)) + k\frac{\ln^{k-1}(\ln(1/t))}{\ln(1/t)}} dt.$$

Thus $xI'_k(x) = -I'_k(x) + c_k - kI_{k-1}(x)$.
We then obtain the relation $(x+1)I'_k(x) = c_k - kI_{k-1}(x)$.
We then define P_k by $I_k(x) = P_k(\ln(x+1))$.
The recurrence relation is written as $P'_k = c_k - kP_{k-1}$ with $P_k(0) = 0$ and also $P_0 = X$ since $I_0(x) = \ln(x+1)$.

Let's check by induction the relation

$$(k+1)P_k(X) = \int_0^{\infty} e^{-t}\left[\ln^{k+1}(t) - (\ln(t)-X)^{k+1}\right]\ dt :$$

Note $J_k(X)$ the integral $\dfrac{1}{k+1}\displaystyle\int_0^{\infty} e^{-t}\left[\ln^{k+1}(t) - (\ln(t)-X)^{k+1}\right]\ dt.$
For $k=0$ we clearly have $J_0(X) = X$.
As $J_k(0) = 0$, to check the relation to the k rank by induction, just check that J_k satisfies the relation $J_k'(X) = c_k - kJ_{k-1}(X)$:

$J_k'(X) = \displaystyle\int_0^{\infty} e^{-t}(\ln(t)-X)^k\ dt = -kJ_{k-1}(X) + k\int_0^{\infty} e^{-t}\ln^k(t)\ dt = -kJ_{k-1} + c_{k-1}.$
Thus

$$\int_0^1 (1-t^x)\frac{\ln^k(\ln(1/t))}{\ln(1/t)}\ dt = \frac{1}{k+1}\int_0^{\infty} e^{-t}\left[\ln^{k+1}(t) - (\ln(t)-\ln(x+1))^{k+1}\right]\ dt = P_k(\ln(x+1)).$$

We will remember that P_k a polynomial of degree $k+1$ verifying $P_k(0) = 0$ and leading coefficient $\frac{(-1)^k}{k+1}$.

2) Calculation of the density associated with $\ln^k(x+a)$.

Let's write $X^{k+1} = a_{0,k}P_0(X) + a_{1,k}P_1(X) + ... + a_{k-1,k}P_k(X)$, and pose $Q_k(X) = a_{0,k} + a_{1,k}X ... + a_{k-1,k}X^k$.
By looking at the leading coefficient we have $a_{k-1,k} = (-1)^k(k+1)$.
By taking $X = \ln(x+1)$ we have $\ln^{k+1}(x) = \displaystyle\int_0^1 (1-t^x)\sum_{i=0}^{k-1} a_{i,k}\frac{\ln^i(\ln(1/t))}{\ln(1/t)}.$
So we have

$$\ln^{k+1}(x+1) = \int_0^1 (1-t^x)\frac{Q_k(\ln(\ln(1/t)))}{\ln(1/t)}\ dt.$$

Thus, for $u_k(x) = \ln^k(x+1)$, we have $\pi_{u_k}(t) = \frac{Q_{k-1}(\ln(\ln(1/t)))}{\ln(1/t)}$, that we'll write π_k in the following.
In particular $\pi_{u_k}(t) \underset{0}{\sim} (-1)^{k-1}k\frac{\ln^{k-1}(\ln(1/t))}{\ln(1/t)}$.

3) Equivalent of F_{v_k} at $+\infty$ with $v_k(x) = \ln^k(x+a)$.

We so have $u_k(x) = -\displaystyle\int (t^x - 1)\frac{Q_{k-1}(\ln(\ln(1/t)))}{\ln(1/t)}\ dt,$
and then, for $v_k(x) = \ln^k(x+a)$:

$$F_{v_k}(x) = \int_0^1 e^{-x}S_{1,a-1}(xt)t^{a-1}\Phi(\pi_k)\ dt$$

As $\Phi(\pi_k)(t) \sim \pi_k(t) \sim (-1)^{k-1}k\frac{\ln^{k-1}(\ln(1/t))}{\ln(1/t)}$,

so we have $F_{v_k}(x) \sim \displaystyle\int_0^1 e^{-xt}(a-1-xt)t^{a-1}(-1)^{k-1}k\frac{\ln^{k-1}(\ln(1/t))}{\ln(1/t)}\ dt,$
which gives as equivalent
$(-1)^{k-1}k(a-1)\displaystyle\int_0^1 e^{-xt}t^{a-1}\frac{\ln^{k-1}(\ln(1/t))}{\ln(1/t)}\ dt + (-1)^k kx\int_0^1 e^{-xt}t^a\frac{\ln^{k-1}(\ln(1/t))}{\ln(1/t)}\ dt$
(provided that the main parts are not simplified)

or, after using the relation $**$:

$F_{v_k}(x) \sim \frac{(-1)^{k-1}k\ln^{k-1}(\ln(x))}{x^a\ln(x)}\left((a-1)\Gamma(a) - \Gamma(a+1)\right),$
which shows that $F_u(x) \underset{\infty}{\sim} (-1)^k kx\frac{\Gamma(a)}{x^a}\frac{\ln^{k-1}(\ln(x))}{\ln(x)}$. ■

In conclusion :

$$u(x) = \ln^k(x+a)$$

$$\pi_u(t) = t^{a-1} \frac{Q_{k-1}(\ln(\ln(1/t)))}{\ln(1/t)}$$

$$\pi_u(t) \underset{0}{\sim} (-1)^{k-1} k t^{a-1} \frac{\ln^{k-1}(\ln(1/t))}{\ln(1/t)}$$

$$F_u(x) \underset{\infty}{\sim} (-1)^k k x \frac{\Gamma(a)}{x^a} \frac{\ln^{k-1}(\ln(x))}{\ln(x)}$$

7 Principle of superposition of perturbations.

7.1 Preliminaries.

Product of two density sequences.

Proposition :
Let u_i two density sequences π_i.
The sequence $u_1 u_2$ admits $\pi = \pi_1 \star \pi_2$ as density with $\pi(z) = \displaystyle\int_z^1 \pi_1(s)\pi_2(z/s)\ \frac{ds}{s}$.

Proof :

$u_1(x)u_2(x) = \displaystyle\int_0^1 \int_0^1 t^x s^x \pi_1(t)\pi_2(s)\ dsdt.$
By puting $u = st$ we obtain $\displaystyle\int_0^1 \int_0^t u^x \frac{1}{t}\pi_1(t)\pi_2(u/t)\ dt\ du = \int_0^1 \int_u^1 \frac{1}{t}\pi_1(t)\pi_2(u/t)\ dt\ du = \int_0^1 z^x \pi(z)\ dz$,
with $\pi(z) = \displaystyle\int_z^1 \pi_1(s)\pi_2(z/s)\ \frac{ds}{s}$. ■

Properties of the $\star$ operator :

1 $\star$ is assoiative and commutative.

2 Under conditions of derivability, $(\pi_1 \star \pi_2)'(z) = -\frac{1}{z}\pi_1(z)\pi_2(1) + \displaystyle\int_z^1 \frac{1}{s^2}\pi_1(s)\pi_2'(z/s)\ ds$.

3 Equivalent formulas :
$\pi_1 \star \pi_2(z) = \displaystyle\int_z^{\sqrt{z}} \frac{1}{s}\left(\pi_1(s)\pi_2(z/s) + \pi_1(z/s)\pi_2(s)\right)\ ds$**.**
$z\left(\pi_1 \star \pi_2\right)'(z) = -\pi_1(\sqrt{z})\pi_2(\sqrt{z}) + \displaystyle\int_z^{\sqrt{z}} \left(\pi_1(z/s)\pi_2'(s) + \pi_1'(s)\pi_2(z/s)\right)\ ds$**.**

Proof :

The first point is left to the reader.

The second is obtained by considering $G(x,y) = \displaystyle\int_x^1 \pi_1(s)\pi_2(y/s)\ \frac{ds}{s}$.

For the third one :

We consider $\displaystyle\int_{\sqrt{z}}^1 \frac{1}{s^2}\pi_1(s)\pi_2'(z/s)\ ds$
where the variable change $u = z/s$ is performed, which gives
$\displaystyle\int_z^{\sqrt{z}} \pi_1(z/s)\pi_2'(s)\ ds$ which is integrated by parts and gives the result.

Some examples :

- For $\pi_1(t) = t^a$ et $\pi_2(t) = t^b$:
We have $\pi_1 \star \pi_2(z) = \frac{z^b - z^a}{b-a}$ if $a \neq b$, $z^a \ln(1/z)$ if $a = b$.

- For $\pi_1(t) = \ln^a(1/t)$ et $\pi_2(t) = \ln^b(1/t)$, we have $\pi_1 \star \pi_2(z) = B(a+1, b+1)\ln^{a+b+1}(1/z)$, B designating the Euler function.

Indeed: The definition gives $\displaystyle\int_z^1 \ln^a(t/z)\ln^b(1/t)\ \frac{dt}{t} \overset{u=\ln(1/t)}{=} \int_0^{\ln(1/z)} (\ln(1/z)-u)^a u^b\ du$,
the change of variable $u = v\ln(1/z)$ giving the result.

- For $\pi_1(t) = t^a$ et $\pi_2(t) = t^b\ln^c(1/t)$, we have $\pi_1 \star \pi_2(z) = \displaystyle\int_z^1 t^{b-a-1}\ln^c(1/t)\ dt \underset{+\infty}{\sim} z^{b-a-1}\ln^c(1/z)$.

- For $\pi(t) = t^{a-1}$ on a $\underbrace{\pi \star \cdots \star \pi}_{k \text{ fois}}(t) = t^{a-1}\frac{\ln^{k-1}(1/t)}{(k-1)!}$ (by an easy induction),
which makes it possible to find the case $u(x) = \frac{1}{(x+a)^k}$.

7.2 Power part of a function.

Definition :
If f is a given function over $]0,1]$ and under the condition of existence, we will talk about the power part of f, noted $a(f) = \lim_{t\to 0}\frac{tf'(t)}{f(t)}$.

Remarks :

If f admits a power part a, then $F(x) = \displaystyle\int_0^x f(t)\ dt$ admits a power part $a+1$.

If f and g admit a power part then $a(fg) = a(f) + a(g)$.

If f admits a power part then we have $c_1 t^{a(f)+\varepsilon} \leq f(t) \leq c_2 t^{a(f)-\varepsilon}$ over $]0,1]$.

Proof:

- $a(f) = a$ is equivalent to $a(g) = 0$ for $g(t) = t^{-a}f(t)$.

- If $a(f) = 0$, then $f(t) = e^{\alpha(t)}$ where $\alpha'(t) = o(1/t)$.
If $F(x) = \displaystyle\int_0^x f(t)\ dt$, then

$$\int_0^x e^{\alpha(t)}\ dt = \left[te^{\alpha(t)}\right]_0^x - \underbrace{\int_0^x t\alpha'(t)e^{\alpha(t)}\ dt}_{=o(F(x))}.$$

Thus $\frac{xF'(x)}{F(x)} \to 1$.

The other two points are immediate.

7.3 Problem of overlapping perturbations.

Theorem :

We give u_i with density π_i $i = 1, 2$, so monotonous around 0.

1 Principle of non-superposition of perturbations :

If $\pi_2(t) = O\left(t^b\right)$ with $b > a(\pi_1)$
and if π_1' admits a power part, then

$$\pi_1 \star \pi_2 \underset{0}{\sim} C(\pi_1, \pi_2)\pi_1$$

and therefore

$$F_{u_1.u_2} \underset{\infty}{\sim} CF_{u_1}$$

where

$$C(\pi_1, \pi_2) = \int_0^1 t^{-a(\pi_1)-1}\pi_2(t)\ dt$$

2 Principle of superposition of perturbations :

If π_1 and π_2 admit a power part with $a(\pi_1) = a(\pi_2)$
then

$$a(\pi_1) = a(\pi_2) = a(\pi_1 \star \pi_2)$$

Let's mention two very important applications before giving the proof :

If $a(\pi_1) < a(\pi_2), \cdots, a(\pi_k)$ with π_1' admitting a power part, then
for $2 \leq i \leq k$, we have $\pi_i(t) = O(t^b)$ for one $b > a(\pi_1)$,
so $\pi_2 \star \cdots \star \pi_k(t) = O(t^b)$,
and thus

$$F_{u_1 \cdots u_k} \underset{+\infty}{\sim} C(u_1, u_2 \star \cdots \star \pi_k)F_{u_1}$$

If $a(\pi_1) = a(\pi_2) < a(\pi_3), \cdots, a(\pi_k)$, meanwhile

$$F_{u_1 \cdots u_k} \underset{+\infty}{\sim} C(u_1 \star u_2, \pi_3 \star \cdots \star \pi_k)F_{u_1 u_2}$$

Proof :

First case :
Let $a = a(\pi_1) > b$, $\pi_1(t) = t^a\pi(t)$.
We thus know that $a(\pi) = 0$ and $t^{-a}\pi_2(t) = O(t^\alpha)$ with $\alpha > 0$.
put $f(t) = t^{-a-1}\pi_2(t)$ whitch is integrable over $]0, 1]$.
$\pi_1 \star \pi_2(z) = = z^a \displaystyle\int_z^1 f(s)\pi(z/s)\ ds$.
It is therefore sufficient to show $\displaystyle\int_z^1 f(s)\pi(z/s)\ ds \underset{0}{\sim} \pi(z) \int_0^1 f(s)\ ds$.
To do this, we integrate by parts by puting $F(z) = \displaystyle\int_0^z f(s)\ ds$:
$\displaystyle\int_z^1 f(s)\pi(z/s)\ ds = F(1)\pi(z) - F(z)\pi(1) + \int_z^1 \frac{z}{s^2}F(s)\pi'(z/s)\ ds$.
$a(f) > -1$ thus $a(F) > 0$ and so we have an inequality such as $F(z) \leq cz^\varepsilon$,
which shows that $F(z)\pi(1) = o(\pi(z))$.

By $z/s = u$, the last integral is equal to $\displaystyle\int_z^1 F(z/s)\pi(s)\ ds$,
and we have to show that it's negligible in front of $pi(z)$.
$a(f) > -1$ so $a(F) > 0$ and we have an inequality such as $F(t) \leq Ct^\alpha$.
The last integral is therefore increased by $\displaystyle\int_z^1 \left(\frac{z}{s}\right)^\alpha \pi'(s)\ ds = H(z)$.

$$H(z) = z^\alpha \int_z^1 s^{-\alpha+1} s\pi'(s)\ ds = \frac{z^\alpha}{\alpha}\left[-s^{-\alpha}s\pi'(s)\right]_z^1 + \frac{z^\alpha}{\alpha}\int_z^1 s^{-\alpha}(s\pi'(s))'\ ds$$
$$= O(z^\alpha) + \frac{1}{\alpha}\underbrace{z\pi'(z)}_{o(\pi(z))} + \frac{z^\alpha}{\alpha}\int_z^1 s^{-\alpha}(s\pi'(s))'\ ds.$$

But as $a(s\pi'(s)) = 0$, $(s\pi'(s))' = o(\pi'(s))$ there are two cases.
- If $\frac{\pi'(s)}{s^\alpha}$ is integrable over $]0,1]$, then $\displaystyle\frac{z^\alpha}{\alpha}\int_z^1 s^{-\alpha}(s\pi'(s))'\ ds = O(z^\alpha)$.
- Else, $\displaystyle\frac{z^\alpha}{\alpha}\int_z^1 s^{-\alpha}(s\pi'(s))'\ ds = o\left(z^\alpha \int_z^1 \frac{\pi'(s)}{s^\alpha}\ ds\right) = o(H(z))$.
Finally, the relation $a(\pi) = 0$ provides $z^\alpha = o(\pi(z))$. ■

Enfin la relation $a(\pi) = 0$ fournit $z^\alpha = o(\pi(z))$. ■

Second case :

Let's start with a lemma :

Lemma :

If π_1, π_2 are locally positive and monotonous around 0, from part to zero power, then $\displaystyle\pi_1(\sqrt{z})\pi_2(\sqrt{z}) = o\left(\int_z^{\sqrt{z}} \pi_1(s)\pi_2(z/s)\ \frac{ds}{s}\right)$**.**

Proof of the lemma :

Let's note $\pi(z)$ for the integral. First of all, let's notice the equality:
$$\int_z^{\sqrt{z}} \pi_1(s)\pi_2(z/s)\ \frac{ds}{s} = \int_{\sqrt{z}}^1 \pi_1(z/s)\pi_2(s)\ \frac{ds}{s}$$

Case π_1 decreasing and π_2 increasing :

$$\int_z^{\sqrt{z}} \frac{1}{s}\pi_1(s)\pi_2(z/s)\ ds \geq \pi_1(\sqrt{z})\int_z^{\sqrt{z}} \pi_2(z/s)\ \frac{ds}{s} = \pi_1(\sqrt{z})\int_{\sqrt{z}}^1 \pi_2(s)\ \frac{ds}{s} \geq \pi_1(\sqrt{z})\pi_2(\sqrt{z})\ln(1/\sqrt{z}).$$

Cas π_1 increasing et π_2 decreasing :

$$\int_{\sqrt{z}}^1 \pi_1(z/s)\pi_2(s)\ \frac{ds}{s} \geq \pi_2(\sqrt{z})\int_{\sqrt{z}}^1 \pi_1(z/s)\ \frac{ds}{s} = \pi_2(\sqrt{z})\int_z^{\sqrt{z}} \pi_1(s)\ \frac{ds}{s} \geq \pi_1(\sqrt{z})\pi_2(\sqrt{z})\ln(1/\sqrt{z}).$$

Cas π_1 et π_2 decreasing :

If $\pi_2(0) = 0$, by monotony and positivity we would have the trivial case $\pi_2 = 0$,
so $\displaystyle\int_0^1 \frac{\pi_2(s)}{s}\ ds$ diverge.
But because $a(\pi_2) = 0$, we have $\displaystyle\pi_2(1/2) - \pi_2(z) = \int_z^{1/2} \pi_2'(s)\ ds = o\left(\int_z^1 \frac{\pi_2(s)}{s}\ ds\right)$
and $\displaystyle\pi_2(z) = o\left(\int_z^1 \frac{\pi_2(s)}{s}\ ds\right)$.
Thus $\displaystyle\pi_2(\sqrt{z}) = o\left(\int_{\sqrt{z}}^1 \frac{\pi_2(s)}{s}\ ds\right)$.
The inequality $\displaystyle\pi(z) = \int_z^{\sqrt{z}} \frac{1}{s}\pi_1(s)\pi_2(z/s)\ ds \geq \pi_1(\sqrt{z})\int_{\sqrt{z}}^1 \frac{\pi_2(s)}{s}\ ds$ allows us to conclude.

Cas π_1 et π_2 increasing :

$$\pi_2(\sqrt{z}) - \pi_2(z) = \int_z^{\sqrt{z}} \pi_2'(s)\ ds = o\left(\int_z^{\sqrt{z}} \frac{\pi_2(s)}{s}\ ds\right).$$

Futhermore, $\displaystyle\int_z^{\sqrt{z}} \frac{\pi_2(s)}{s}\ ds \geq \pi_2(z)\,[\ln(t)]_z^{\sqrt{z}} = \pi_2(z)\ln(1/z)$,

so $\pi_2(z) = o\left(\displaystyle\int_z^{\sqrt{z}} \frac{\pi_2(s)}{s}\ ds\right)$, puis $\pi_2(\sqrt{z}) = o\left(\displaystyle\int_z^{\sqrt{z}} \frac{\pi_2(s)}{s}\ ds\right)$.

Finally $\pi(z) \geq \displaystyle\int_{\sqrt{z}}^1 \pi_1(s)\pi_2(z/s)\ \frac{ds}{s} \geq \pi_1(\sqrt{z})\int_{\sqrt{z}}^1 \frac{\pi_2(z/s)}{s}\ ds = \pi_1(\sqrt{z})\int_z^{\sqrt{z}} \frac{\pi_2(s)}{s}\ ds$, hence the result. ■

Remark :
A proof of this lemma not using monotony would be welcome.

Proof of the second case :

If $a(\pi_1) = a(\pi_2) = a$:
By puting $\pi_i(s) = t^a\pi_i^*(s)$, and by $\pi_1 \star \pi_2(z) = z^a\pi_1^* \star \pi_2^*(z)$, we'll come back in case $a = 0$.
We will therefore assume $a(\pi_i) = 0$ and note $\pi = \pi_1 \star \pi_2$.
We have
$z\pi'(z) = -\pi_1(\sqrt{z})\pi_2(\sqrt{z}) + \displaystyle\int_z^{\sqrt{z}} (\pi_1(z/s)\pi_2'(s) + \pi_1'(s)\pi_2(z/s))\ ds$
The relations $\pi_i'(z) = o(\pi_i(z)/z)$ as well as the nature of the integration interval then show that the integral is negligeable in front of $pi(z)$,
and so it remains to show $\pi_1(\sqrt{z})\pi_2(\sqrt{z}) = o(\pi(z))$, which is ensured by the lemma. ■

7.4 Exemples.

Fisrt example

As promised, we will give an equivalent of $\displaystyle\sum_{m\geq 0} \frac{(-1)^m x^m}{(m+n_0)!(m+\varepsilon)}$ si $0 \leq \varepsilon < 1$
thanks to the principle of non-superposition of perturbations.

We write $\left[(m+1)\cdots(m+n_0)(m+\varepsilon)^k\right]^{-1} = \displaystyle\frac{1}{(m+\varepsilon)^k}\sum_{i=1}^{n_0} \frac{a_i}{m+i}$.

For $v(x) = \frac{1}{(x+\varepsilon)^k}$ we have $\pi_v(t) = t^{\varepsilon-1}\frac{\ln^{k-1}(1/t)}{(k-1)!}$ and for $u_i(x) = \frac{1}{x+i}$ we have $\pi_i(t) = t^{i-1}$.

As $a(\pi_v) = \varepsilon - 1 < a(\pi_i) = i - 1$, by non-superposition, we have
$\pi_v \star \pi_i \sim C_i\pi_v$ avec $C_i = \displaystyle\int_0^1 t^{-(\varepsilon-1)-1}t^{i-1}\ dt = \frac{1}{i-\varepsilon}$.
Thus for $u(x) = \left[(x+1)\cdots(x+n_0)(x+\varepsilon)^k\right]^{-1}$, we have
$\pi_u \sim \pi_v \displaystyle\sum_{i=1}^{n_0} \frac{a_i}{i-\varepsilon} = \frac{1}{(1-\varepsilon)\cdots(n_0-\varepsilon)}$,
whitch gives $F_u(x) \underset{\infty}{\sim} (-x)^{n_0}KF_v(x)$ ou $v(x) = \frac{1}{(x+\varepsilon)^k}$ and $K = [(1-\varepsilon)(2-\varepsilon))\cdots(n_0-\varepsilon)]^{-1}$ as wished.

Second example

We're going to focus on $w_n = \frac{\ln(n+a)}{(n+b)^k}$.
We want an equivalent at $+\infty$ of

$$\boxed{F_w(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n x^n \ln(n+a)}{(n+b)^k}}$$

It should be noted that this is not the product of two density sequences.

We therefore pose $w(x) = \frac{\ln(x+a)}{(x+b)^k}$.
We know that $\ln(x+a) = x\displaystyle\int_0^1 t^x t^{a-1} Li(t)\ dt$.
So we're going to pose $u(x) = \displaystyle\int_0^1 t^x t^{a-1} Li(t)\ dt$, and $v(x) = \frac{1}{(x+b)^k} = \displaystyle\int_0^1 t^x t^{b-1} \frac{\ln^{k-1}(1/t)}{(k-1)!}\ dt$,
let $\pi_u(t) = t^{a-1} Li(t) \sim \frac{t^a}{\ln(t)}$ and $\pi_v(t) = t^{b-1}\frac{\ln^{k-1}(1/t)}{(k-1)!}$, and finally $\pi = \pi_1 \star \pi_2$.
We know that $F_w(x) = \displaystyle\int_0^1 e^{-xt} S_1(xt)\Phi(\pi)(t)\ dt$ with $S_1(X) = -X$,

$a(\pi_u) = a$ and $a(\pi_v) = b-1$.

There are therefore three cases.

First case $a < b-1$:

$\pi_{u.v} \sim C(\pi_u, \pi_v)\pi_u$ with $C = \displaystyle\int_0^1 t^{-a-1} t^{b-1} \frac{\ln^{k-1}(1/t)}{(k-1)!}\ dt$,
$\Phi(\pi)(t) \sim C\frac{t^a}{(a+1)\ln(t)}$.
Thus $F_w(x) \sim -C\displaystyle\int_0^1 e^{-xt} xt \frac{t^a}{(a+1)\ln(t)}\ dt$,
or $F_w(x) \sim -\frac{C\Gamma(a+2)x}{(a+1)x^{a+2}\ln(x)}$,
or even

$$\boxed{F_w(x) \underset{+\infty}{\sim} -\frac{C\Gamma(a+1)}{x^{a+1}\ln(x)}}$$

Second case $a > b-1$:

This time, $C = C(\pi_v, \pi_u) = \displaystyle\int_0^1 t^{-(b-1)-1} ta - 1Li(t)\ dt$,
and $\pi_{u.v} \sim C\pi_v$, $\Phi(\pi_v)(t) \sim bt^{b-1}\frac{\ln^{k-1}(1/t)}{(k-1)!}$
Thus $F_w(x) \sim -Cbx\displaystyle\int_0^1 e^{-xt} t^b \frac{\ln^{k-1}(1/t)}{(k-1)!}\ dt$

and finally

$$\boxed{F_w(x) \sim -\frac{Cb\Gamma(b+1)\ln^{k-1}(x)}{(k-1)!x^b}}$$

Third case $a = b-1$:

$\pi(z) = \displaystyle\int_z^1 t^{a-1} Li(t)(z/t)^{b-1} \frac{\ln^{k-1}(t/z)}{(k-1)!}\frac{1}{t}\ dt$.
This gives $\pi(z) \sim z^a \dfrac{\ln^{k-1}(1/z)}{(k-1)!}\displaystyle\int_z^1 \frac{Li(t)}{t^2}\ dt \sim -z^a\frac{\ln^{k-1}(1/z)}{(k-1)!}\ln(\ln(1/z))$
(we will not justify here the first equivalent),
puis $\phi(\pi) \sim \frac{1}{a+1}\pi$.

In this way $F_w(x) \sim x\displaystyle\int_0^1 e^{-xt} t \frac{t^a}{a+1}\frac{\ln^{k-1}(1/t)}{(k-1)!}\ln(\ln(1/t))\ dt$,
and at last

$$\boxed{F_w(x) \underset{+\infty}{\sim} \frac{\Gamma(a+1)\ln^{k-1}(x)\ln(\ln(x))}{(a+1)x^a(k-1)!}}$$

Third example

We're going to treat the example $u(x) = \frac{1}{(x^\alpha - a^\alpha)^k}$ with $0 < a < 1$, $\alpha > 0$ and $k \in \mathbb{N}$.

First method with the principle of non-superposition of perturbations :

Put $u(x) = \frac{1}{(x^\alpha - a^\alpha)^k}$, $v(x) = \frac{1}{(x-a)^k}$ and $w(x) = \left(\frac{x-a}{x^\alpha - a^\alpha}\right)$.
We assume that v, and therefore w, are deduced from density sequences admitting a power part.

By truncation, $F_v(x) = x \int_0^1 e^{-xt} \pi_V(t)\, dt$ avec $\pi_V(t) = -\int_t^1 \pi_v(s)\, ds$,
so $\pi_v = -\pi_V'$, and the same is true for w.

We have the relations $a(\pi_V) = a(\pi_v) + 1$, $a(\Phi^2(\pi_V)) = a(\pi_V)$.
In addition, by practical calculations, $\pi_V(t) = t^{-a-1} \frac{\ln^{k-1}(1/t)}{(k-1)!}$.

Since W is defined for some $x < a$, by holomorphy, $f(t) = t^{a-\varepsilon}\pi_W(t)$ is can be integrated in 0.

Having a power part, $tf'(t)$ is also
and by an integration by parts, it is easy to see that
$f(t) = O(1/t)$, whitch gives $\pi_W(t) = O(t^{-1-a+\varepsilon})$.

So $a(\pi_W) > a(\pi_V)$ and by non-superposition, $\pi_V \pi_W \sim C\pi_V$.
Thus $u(x) \sim Cx^2 \int_0^1 t^x \pi_V(t)\, dt$, and then $F_u(x) \sim \int_0^1 e^{-xt} S_2(xt)\Phi^2(\pi_V)(t)\, dt$.
$S_2(X) = X^2 + X$ and $F_u(x) \sim \int_0^1 e^{-xt}(xt + x^2t^2)t^{-a-1} \ln^{k-1}(1/t)\, dt$.
This gives $F_u(x) \sim C' \ln^{k-1}(x)(x\frac{\Gamma(-a+1)}{x^{-a+1}} + x^2 \frac{\Gamma(-a+2)}{x^{-a+2}})$
(because main parts don't simplifie).
After simplification we find

$$\boxed{F_u(x) \underset{+\infty}{\sim} Kx^a \ln^{k-1}(x)}$$

Second method, by direct work :

It's more delicate.

If we give $0 < a < 1$ et $\alpha > 0$ and if we put $u(x) = \frac{1}{(x^\alpha - a^\alpha)^k}$, then

$$\boxed{F_u(x) \underset{\infty}{\sim} \alpha^{-k} a^{k(1-\alpha)} \Gamma(-a) x^a \frac{\ln^{k-1}(x)}{(k-1)!}}$$

Proof : We write $u(x) = \sum_{i=0}^{\infty} \binom{-k}{i} \frac{(-1)^i a^{\alpha i}}{x^{\alpha(k+i)}}$.
The case $v(x) = x^{-p}$ gives $F_v(x) = -x \int_0^1 e^{-xt} \frac{\ln^k(1/t)}{k!}\, dt$.
By this formula, we obtain here
$F_u(x) = -x \int_0^1 e^{-xt} \sum_{i \geq 0} \binom{-k}{i} (-1)^i \frac{a^{\alpha i} \ln^{\alpha(k+i)}(1/t)}{(\alpha(k+i))!}\, dt = -x \int_0^1 \ln^{\alpha k}(1/t)\varphi(t) e^{-xt}\, dt$
with $\varphi(t) = \sum_{i=0}^{\infty} \binom{-k}{i} \frac{(-1)^i (a \ln(1/t))^{\alpha i}}{(\alpha(k+i))!}$.
Let's note $c_i = \binom{-k}{i}(-1)^i \frac{(\alpha i)!}{(\alpha(k+i))!} = \binom{k+i-1}{i} \frac{(\alpha i)!}{(\alpha(k+i))!}$,
so that $\varphi(t) = \sum_{i=0}^{\infty} c_i \frac{(a \ln(1/t))^{\alpha i}}{(\alpha i)!}$.
As $\frac{(x+a)!}{x!} \underset{x \to \infty}{\sim} x^a$, so we have here $c_i = \frac{(k+i-1)!(\alpha i)!}{i!(k-1)!(\alpha i + \alpha k)!} \underset{\infty}{\sim} \frac{i^{k-1}}{(k-1)!(\alpha i)^{\alpha k}}$.

We note $b=(1-\alpha)k-1$ and $K=\frac{\alpha^{-\alpha k}}{(k-1)!}$, so that $c_i \underset{\infty}{\sim} K i^b$.

Because of the equivalent for c_i, we have $\varphi(t) \underset{0}{\sim} K \sum_{i\geq 0} i^b \frac{(a\ln(1/t))^{\alpha i}}{(\alpha i)!}$.

But we have $\sum_{i\geq 0} i^b \frac{(a\ln(1/t)^{\alpha i}}{(\alpha i)!} \underset{0}{\sim} \frac{(a\ln(1/t))^b}{\alpha^{b+1} t^a}$ (see [2] averaging page 7).

And so $\varphi(t) \underset{\infty}{\sim} \frac{K}{\alpha}\alpha^{-b}(a\ln(1/t))^b t^{-a} = \frac{\alpha^{-k}}{(k-1)!}(a\ln(1/t))^{(1-\alpha)k-1}t^{-a}$ after simplification.

so $F_u \underset{\infty}{\sim} -x\frac{\alpha^{-k}}{(k-1)!}\int_0^1 e^{-xt}\ln^{\alpha k}(1/t)a^{(1-\alpha)k-1}\ln^{(1-\alpha)k-1}(1/t)\frac{1}{t^a}\,dt$.

Here again, after simplification, we have $F_u(x) \underset{\infty}{\sim} -x\frac{\alpha^{-k}}{(k-1)!}a^{(1-\alpha)k-1}\int_0^1 e^{-xt}\ln^{k-1}(1/t)\frac{1}{t^a}\,dt$,

the, by the relation $*$, $F_u(x) \underset{\infty}{\sim} \alpha^{-k}a^{k(1-\alpha)}\Gamma(-a)x^a\frac{\ln^{k-1}(x)}{(k-1)!}$. ■

8 Generalizations.

There are two possible generalizations.

If u satisfies $u(x)=\sum_{i=1}^{\infty}\frac{c(i)}{(x+1)^i}$ alors $\pi_u(t)=\sum_{i=1}^{\infty}c(i)\frac{(-1)^{i-1}\ln^{i-1}(t)}{(i-1)!}$.
For example, if c is a regular (and therefore positive) sequence, that means $\frac{c'}{c}(x) \underset{\infty}{=} o(\frac{1}{\sqrt{x}})$, then we have the asymptotic $\pi_u(t) \underset{0}{\sim} \frac{c(\ln(1/t))}{t}$ (see [1]).

Example :

Let $u(x)=-\ln(1-\frac{1}{x+a})=\sum_{k\geq 1}\frac{1}{k(x+a)^k}$ with $a>1$,

$\pi_u(t)t^{a-1}\sum_{k\geq 1}\frac{\ln^{k-1}(1/t)}{k(k-1)!}=\frac{t^{a-1}}{\ln(1/t)}(\frac{1}{t}-1)\sim\frac{t^{a-2}}{\ln(1/t)}$,

and $F_u(x)\sim\int_0^1 e^{-xt}\frac{t^{a-2}}{\ln(1/t)}\,dt$, then

$$F_u(x) \underset{+\infty}{\sim} \frac{\Gamma(a-1)}{x^{a-1}\ln(x)}$$

.

If u satisfies $u(x)=\sum_{i=1}^{\infty}\frac{(-1)^{i-1}c(i)}{(x+1)^i}$ alors $\pi_u(t)=\sum_{i=1}^{\infty}c(i)\frac{\ln^{i-1}(t)}{(i-1)!}$.
This requires an asymptotic of $\sum_{i=1}^{\infty}c(i)\frac{\ln^{i-1}(t)}{(i-1)!}$ in 0 to get one of F_u. This can be done, again in the case where $c(k)$ is a positive sequence, with the techniques proposed in this article.

First example :

If $\pi_u(t)=\frac{t^{a-1}}{\ln(t)}\sum_{k\geq 1}\frac{(-1)^{k-1}\ln^k(1/t)}{k^\alpha k!}$.

We use $\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n!n^\alpha} \underset{\infty}{\sim} -\frac{\ln^\alpha(x)}{\alpha!}$,

and so $\pi_u(t)\sim\frac{t^{a-1}\ln^\alpha(\ln(1/t))}{\alpha!\ln(1/t)}$, and finally

$$F_u(x) \underset{+\infty}{\sim} \frac{\Gamma(a)\ln^{\alpha-1}(x)}{\alpha! x^a}$$

Second example :

We'll shamefully add up equivalents (but we'll say where the crimes take place), by studying $u(x) = \frac{1}{x+a}$, which will validate the calculations on the logarithms.

We write :
$u(x) = \sum_{k\geq 0} \frac{(-1)^k}{k!} \underbrace{\ln^k(x+a)}_{u_k}$.
We know that $\pi_{u_k}(t) \sim (-1)^{k-1} k t^{a-1} \frac{\ln^{k-1}(\ln(1/t))}{ln(1/t)}$.
So we're going to assume that we have (it's here !) :
$\pi_u(t) \sim \sum_{k\geq 0} \frac{(-1)^k}{k!}(-1)^{k-1} k t^{a-1} \frac{\ln^{k-1}(\ln(1/t))}{ln(1/t)}$,
whitch gives $\pi_u(t) \sim -\frac{t^{a-1}}{\ln(1/t)} \sum_{k\geq 1} \frac{\ln^{k-1}(\ln(1/t))}{\ln(1/t)} = t^{a-1}$.
Furthermore, $F_{u_k}(x) \sim (-1)^k k \frac{\Gamma(a)\ln^{k-1}(\ln(x))}{x^a \ln(x)}$.
We'll also assume (it's also here !):
$F_u(x) \sim \sum_{k\geq 1} (-1)^k k \frac{\Gamma(a)\ln^{k-1}(\ln(x))}{x^a \ln(x)}$,
whitch gives $F_u(x) \sim \sum_{k\geq 0} \frac{\Gamma(a)\ln^{k-1}(\ln(x))}{(k-1)! x^a \ln(x))}$.
We find well $F_u(x) \underset{+\infty}{\sim} \frac{\Gamma(a)}{x^a}$.

Finally, it would remain to treat the case $\ln^k(x+a)$ with k and a real.

Références :

[1] https://hal.archives-ouvertes.fr/hal-00747720v4/document

[2] https://hal.archives-ouvertes.fr/hal-00644237/document

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