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Abstract: We propose a new simplified crop irrigation model and study the optimal control which consists in maximizing the biomass production at harvesting time, under a constraint on the total amount of water used. Under water scarcity, we show that a strategy with a singular arc can be better than a simple bang-bang control as commonly used. The gain is illustrated on numerical simulations. This result is a promising first step towards the application of control theory to the problem of optimal irrigation scheduling.

Keywords: Crop irrigation, water management, optimal control, state constraint.

1. INTRODUCTION

Irrigation scheduling is the process of defining, at any time of a crop growing season, the amount of water delivered using an irrigation system. It is a major crop management issue in the context of increasing water scarcity. Dynamic crop growth models can help defining optimal strategies depending on various criteria (Allen et al., 1998). Several methodological approaches have been used so far to tackle the question of model-based irrigation scheduling:

(1) Numerical comparison of a set of predefined irrigation scenarios (which represent relatively small discrete sets of possible solutions) using complex simulation models, see (Saseendran et al., 2008; Vico and Porporato, 2013; Cheviron et al., 2016) among many others.
(2) Numerical optimization of a parameterized scheduling problem (e.g. optimal 2-events scheduling with water budget) using complex simulation models (Wen et al., 2017; Li et al., 2018)
(3) Numerical optimization based on optimal control theory applied to simplified models (Li et al., 2011; Ramanathan et al., 2013).

Most existing approaches lack analytical insight on the theoretical properties of optimal solutions. This is probably due to the complexity of the global optimization problem when dealing with detailed simulation models. However as showed in the work of (Shani et al., 2004), a deeper analysis on optimal solutions might lead to better practical guidelines providing that the model accurately represents crop response to water stress. In this work, in the spirit (Shani et al., 2004), we introduce a theoretical model based on the simplification of some existing crop models in order to study optimal irrigation scheduling. We derive a first promising analytical result on this model and illustrate numerically its implication on the shape of optimal solutions.

2. THE MODEL

We consider a simplified dynamical model of crop irrigation, inspired from Pelak et al. (2017), where $S(t)$ and $B(t)$ stand respectively for the relative soil humidity (a number between 0 and 1) and the crop biomass at time t belonging to an interval $[0, T]$ representing the crop growth season:

$$\dot{S} = k_1(-\varphi(t)K_S(S) - (1-\varphi(t))K_R(S) + k_2 u(t)) \quad (1)$$

$$\dot{B} = k_3\varphi(t)K_S(S) \quad (2)$$

with the initial condition (at the sowing date 0)

$$S(0) = 1 \quad (3)$$

$$B(0) = 0 \quad (4)$$

and T being the harvesting date. The control variable $u(t) = F(t)/F_{max} \in [0, 1]$ is the ratio of the input water flow rate $F(t)$ at time t over the maximal flow F_{max} that the irrigation device allows.

On an agronomic point of view, (1) represents the variation of a vertically averaged soil moisture as influenced by three fluxes: crop transpiration, crop evaporation, and crop irrigation. Unlike Pelak et al. (2017) and Shani et al. (2004), we use here the simplified hypothesis made in (Bertrand et al., 2018): transpiration and evaporation can be partitioned using a variable $\varphi(t)$ representing the crop radiation use efficiency and independent of water stress. Both transpiration and evaporation fluxes are regulated by soil moisture as in (Pelak et al., 2017) using two functions K_S and K_R (see Assumption 1 and Fig. 1 below). Equation

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(2) determines the amount of biomass produced per time unit. It is simply related to the transpiration flux using the water use efficiency principle (Steduto et al., 2009; Pelak et al., 2017). Note also that the proposed model does not consider rainfall inputs and might be better associated to greenhouse grown crops.

Assumption 1. $K_S(\cdot)$ and $K_R(\cdot)$ are continuous piece-wise linear functions from $[0,1]$ to $[0,1]$:

$$K_S(S) = \begin{cases} 0 & S \in [0, S_w] \\ \dfrac{S - S_w}{S^\star - S_w} & S \in [S_w, S^\star] \\ 1 & S \in [S^\star, 1] \end{cases}$$

$$K_R(S) = \begin{cases} 0 & S \in [0, S_h] \\ \dfrac{S - S_h}{1 - S_h} & S \in [S_h, 1] \end{cases}$$

where the parameters S_w, $S^\star$ and S_h are such that

$$0 < S_h < S_w < S^\star < 1$$

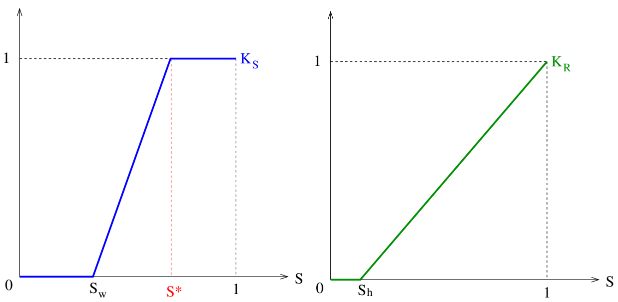

Fig. 1. Graphs of the functions K_S and K_R

Assumption 2. $\varphi(\cdot)$ is a L^1 increasing function from $[0,T]$ to $[0,1]$ with $\varphi(0) = 0$ and $\varphi(T) = 1$.

Assumption 3. k_1, k_2, k_3 are positive parameters with

$$k_2 > 1$$

One can easily check that under these assumptions, any solution $S(\cdot)$ of (1) with initial condition (3) verifies $S(t) > S_h$ for any $t \geq 0$. The condition $k_2 > 1$ is a *controllability* assumption, in the sense that it allows the variable S to stay equal to 1 with the constant control $u = 1/k_2$. However, the controlled system is naturally subject to the constraint

$$S(t) \leq 1, \quad t \in [0,T] \tag{5}$$

The set of admissible controls $u(\cdot)$ are measurable functions taking value in $[0,1]$, and such that the solution of (1),(3) verifies the constraint (5). To each such control function, we associate the total water delivered on the time interval $[0,T]$, and the biomass production at the harvesting date T, given respectively by

$$Q[u(\cdot)] := F_{max} \int_0^T u(t)\, dt, \quad B_T[u(\cdot)] := B(T)$$

3. THE CONTROL PROBLEM

Our objective in the present work is to study admissible strategies $u(\cdot)$ maximizing the biomass production $B_T[u(\cdot)]$ under a constraint on the total water quantity

$$Q[u(\cdot)] \leq \bar{Q} \tag{6}$$

For convenience, we shall denote, for any $t_0 \in [0,T]$ and $S_0 \in [0,1]$, $S_{t_0,S_0,0}(\cdot)$, resp. $S_{t_0,S_0,1}(\cdot)$, for the solution of the differential equation (1) with $S(t_0) = S_0$ and the constant control $u = 0$, resp. $u = 1$. The following definition will be useful in the following.

Definition 4. Let $\underline{S}(\cdot) := S_{0,1,0}(\cdot)$ and

$$\underline{t} := \sup\{t \in [0,T] \text{ s.t. } \underline{S}(t) > S^\star\}$$

Define also the number

$$B_T^\star := k_3 \int_0^T \varphi(t)\, dt$$

Straightforwardly, on has the first result.

Lemma 5.

(i) The inequality $B_T[u(\cdot)] \leq B_T^\star$ is fulfilled for any admissible control $u(\cdot)$.
(ii) If $\underline{t} = T$, then any admissible control $u(\cdot)$ gives $B_T[(\cdot)] = B_T^\star$.

For non trivial cases for which $\underline{t} < T$, let us consider the following *singular* control.

Definition 6. Let

$$\tilde{u}_{S^\star}(t) := \frac{\varphi(t) + (1 - \varphi(t))K_R(S^\star)}{k_2}, \quad t \in [\underline{t}, T] \tag{7}$$

and define

$$Q^\star := F_{max} \int_{\underline{t}}^T u_{S^\star}(t)\, dt$$

Notice that under Assumption 3, this control is admissible as one has

$$\tilde{u}_{S^\star}(t) < 1, \quad t \in [\underline{t}, T] \tag{8}$$

One can easily check that the following statement is satisfied.

Lemma 7. Assume $\underline{t} < T$.

(i) For any $\bar{Q} \geq Q^\star$, the control
$$\tilde{u}(t) = \begin{cases} 0 & t \in [0, \underline{t}) \\ \tilde{u}_{S^\star}(t) & t \in [\underline{t}, T] \end{cases} \tag{9}$$
is admissible with $Q[\tilde{u}(\cdot)] = Q^\star$, and gives $B_T[\tilde{u}(\cdot)] = B_T^\star$.
(ii) For any $\bar{Q} < Q^\star$ and admissible control $u(\cdot)$ satisfying the constraint (6), one has $B_T[u(\cdot)] < B_T^\star$.

Consequently, when $\underline{t} = T$ or $\bar{Q} \geq Q^\star$, a simple optimal admissible control is known, giving the maximal biomass production $B_T^\star$. In the remaining of the paper, we shall consider the complementary cases, that is the following hypothesis.

Hypothesis 8. $\underline{t} < T$ and $\bar{Q} < Q^\star$.

This hypothesis corresponds to situations of water scarcity, for which there is not any enough water available for the time horizon $[0,T]$ to maintain the soil humidity constantly above or equal to the level $S^\star$ which provides the maximal production $B_T^\star$ at the harvesting time.

4. A COMPARISON RESULT

The main result given in this section is playing an important role in the satisfaction of the state constraint

(5) while maximizing B_T under the integral constraint (6). We introduce below the MRAP (for *Most Rapid Approach Path*) to $S = S^\star$ controls. Such kind of controls have already been considered in several optimal control problems in the plane, characterizing their optimality (e.g. Miele (1962); Hermes and Lasalle (1969); Hartl and G. Feichtinger (1987) or related to the so-called "turnpike" property (see e.g. Rapaport and Cartigny (2004); Trélat and Zuazua (2015); Faulwasser et al. (2017)). Here, we use it in a different way. We do not pretend that these controls are necessarily optimal (and indeed they are not), but they respect the state constraint (5) and can locally improve the cost, providing then a comparison result given in Proposition 11 below.

We begin by some definitions.

Definition 9. For $(t_0, S_0) \in [0, T) \times (S^\star, 1]$, we define

$$t^+(t_0, S_0) = \begin{cases} T & \text{if } S_{t_0,S_0,0}(t) > S^\star,\ t \in [t_0, T] \\ \inf\{t > t_0\ ;\ S_{t_0,S_0,0}(t) = S^\star\} & \text{otherwise} \end{cases}$$

For any $(t_0, S_0) \in (0, T] \times (S^\star, 1]$, we define

$$t^-(t_0, S_0) = \begin{cases} 0 & \text{if } S_{t_0,S_0,1}(t) > S^\star,\ t \in [0, t_0] \\ \sup\{t < t_0\ ;\ S_{t_0,S_0,1}(t) = S^\star\} & \text{otherwise} \end{cases}$$

Definition 10. For any $(t_1, S_1) \in [0, T) \times [S^\star, 1]$ and $(t_2, S_2) \in (t_1, T] \times [S^\star, 1]$ such that S_2 is attainable from (t_1, S_1) at time t_2 with an admissible control, we associate the MRAP control $\tilde{u}(\cdot)$ on the time interval $[t_1, t_2]$:

i) If $t_-(t_2, S_2) \geq t_+(t_1, S_1)$:

$$\tilde{u}(t) := \begin{cases} 0 & \text{if } t \in [t_1, t^+(t_1, S_1)) \\ \tilde{u}_{S^\star}(t) & t \in [t_+(t_1, S_1), t^-(t_2, S_2)] \\ 1 & \text{if } t \in (t_-(t_2, S_2), t_2] \end{cases} \tag{10}$$

ii) If $t^-(t_2, S_2) < t^+(t_1, S_1)$:

$$\tilde{u}(t) := \begin{cases} 0 & \text{if } t \in [t_1, \bar{t}(t_1, S_1, t_2, S_2)) \\ 1 & \text{if } t \in (\bar{t}(t_1, S_1, t_2, S_2), t_2] \end{cases}$$

where $\bar{t}(t_1, S_1, t_2, S_2)$ is the unique $\bar{t} \in [t_1, t_2]$ such that $S_{t_1,S_1,0}(\bar{t}) = S_{t_2,S_2,1}(\bar{t}) > S^\star$ (one can easily verify that the function $I(t) := S_{t_1,S_1,0}(t) - S_{t_2,S_2,1}(t)$ is decreasing on $[t_1, t_2]$ with $I(t_1) \geq 0$ and $I(t_2) \leq 0$, which gives the existence and uniqueness of $\bar{t}(t_1, S_1, t_2, S_2)$).

Proposition 11. Let $S(\cdot)$ be a solution of (1) on $[t_1, t_2]$ (with $0 \leq t_1 < t_2 \leq T$) for an admissible control $u(\cdot)$ such that $S(t) \geq S^\star$ for any $t \in [t_1, t_2]$. Denote $S_1 = S(t_1)$ and $S_2 = S(t_2)$. Then, the solution $\tilde{S}(\cdot)$ of (1) on $[t_1, t_2]$ with $\tilde{S}(t_1) = S_1$ and the MRAP control $\tilde{u}(\cdot)$ (see Definition 10) satisfies the following properties:

$$\tilde{S}(t_2) = S_2 \tag{11}$$

$$\tilde{S}(t) \leq S(t), \quad t \in [t_1, t_2] \tag{12}$$

$$\int_{t_1}^{t_2} \tilde{u}(t)\, dt \leq \int_{t_1}^{t_2} u(t)\, dt \tag{13}$$

Moreover, the last inequality is strict when $S(\cdot)$ and $\tilde{S}(\cdot)$ are not identical.

This proposition leads to our main result.

Proposition 12. Assume that Hypothesis 8 is satisfied. One has the following properties.

(i) $u(t) = 0$ for $t \in [0, \underline{t}]$ is optimal.
(ii) Any optimal solution verifies $S(t) \leq S^\star$ for any $t \in [\underline{t}, T]$.
(iii) An optimal solution verifies $Q[u(\cdot)] = \bar{Q}$.

The proofs of Propositions 11 and 12 are given in appendix.

5. OPTIMALITY OF ONE SHOT CONTROLS

One Shot (OS) controls represent a class of widely used irrigation strategies, typically when drip irrigation is not available. They consists in delivering water at maximum flow rate during a single irrigation period. The OS control $u^{OS}_{t_S}(\cdot)$ is therefore defined as a bang-bang control, parameterized by the triggering time $t_S \in [0, T]$.

Definition 13.

$$u^{OS}_{t_S}(t) := \begin{cases} 0 & \text{if } t < t_S \text{ or } t > \min(t_S + \bar{Q}/F_{max}, T)) \\ 1 & \text{if } t \in [t_S, \min(t_S + \bar{Q}/F_{max}, T)) \end{cases}$$

A triggering time t_S is said to be admissible when the control $u^{OS}_{t_S}(\cdot)$ is admissible i.e. such that the constraint (5) is satisfied.

However, Proposition 12 shows that a time t_S below $\underline{t}$ or above $T - \bar{Q}/F_{max}$ (the later being the very last time to allow all the water quantity $\bar{Q}$ to be delivered by the date T) cannot be optimal. Moreover, some values of $t_S \in [\underline{t}, T - Q/F_{max}]$ could conduct the humidity rate S above the value $S^\star$, which cannot be optimal neither. Therefore, we introduce instead the *Saturated One Shot* (SOS) feedback control

$$(t, S, V) \mapsto \psi^{SOS}_{t_S}(t, S, V)$$

which is a bang-singular-bang or bang-bang strategy, also parameterized by a triggering time $t_S \in [\underline{t}, T - Q/F_{max}]$, defined as follows.

Definition 14.

$$\psi^{SOS}_{t_S}(t, S, V) := \begin{cases} 0 & \text{if } t < t_S \text{ or } V = \bar{Q} \\ \tilde{u}_{S^\star}(t) & \text{if } t \geq t_S,\ S = S^\star \text{ and } V < \bar{Q} \\ 1 & \text{if } t \geq t_S,\ S < S^\star \text{ and } V < \bar{Q} \end{cases}$$

where $\tilde{u}_{S^\star}(\cdot)$ is given in (7) and $V(t)$ denotes the volume of water consumed up to time t, that is

$$V(t) = F_{max} \int_0^t u(\tau)\, d\tau$$

Note that this control is necessarily admissible.

In the next section, we have performed numerical comparisons between the OS ans SOS strategies for their respective best t_S (i.e. maximizing B_T).

6. NUMERICAL SIMULATIONS AND DISCUSSION

We present in Fig. 2 the simulations performed with irrigation strategies OS and SOS and with inputs data given in Table 1. For illustrative purposes only, we have

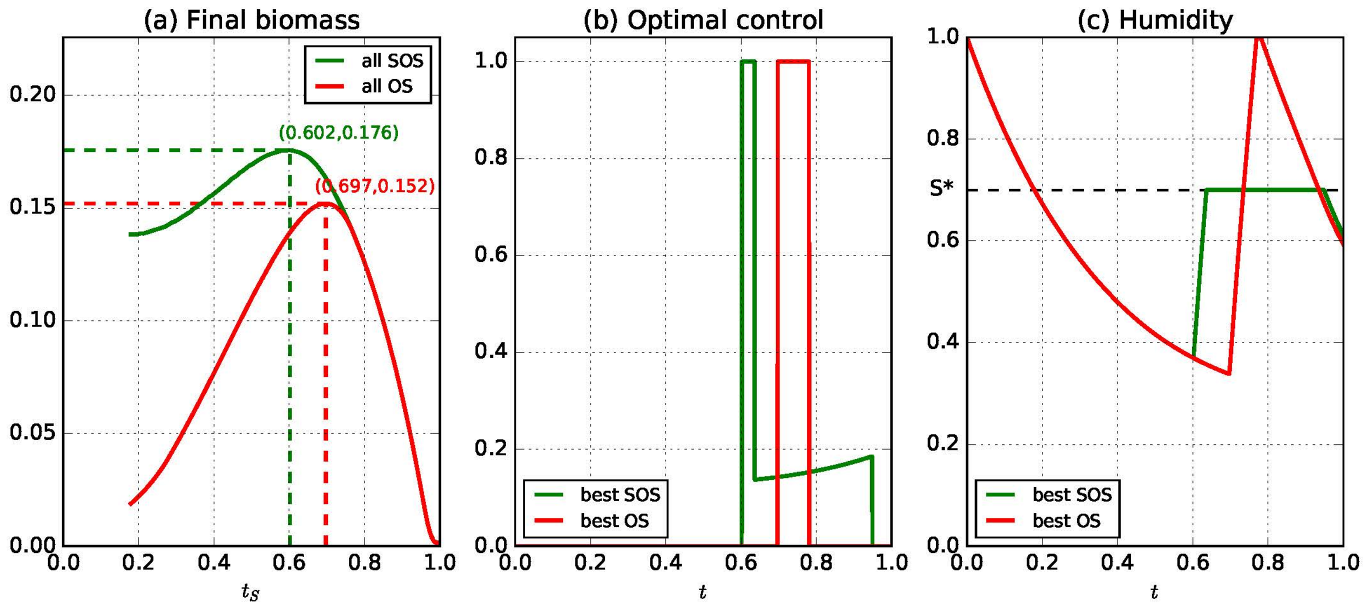

Fig. 2. Comparison of OS and *SOS* controls strategies on one typical example. Model parameters used are given in Table 1.

Table 1. Normalized parameters used for the simulations

T	k_1	k_2	k_3	$S^\star$	S_w	S_h	F_{max}	$\bar{Q}$	α
1	2.1	5	1	0.7	0.4	0.2	1.2	0.1	4

considered dimensionless parameters (by normalizing the units) and function φ in the family of $t \mapsto (t/T)^\alpha$ ($\alpha > 0$).

The optimal OS strategy was obtained for $t_S = 0.697$ and produced a biomass $B(T) = 0.152$. The corresponding humidity dynamics is plotted in Fig. 2c. It can be seen that some value of S are above $S^\star$. It can be therefore concluded from the application of of Proposition 12 that an OS strategy cannot be optimal. This is further illustrated by applying the SOS strategies for the same inputs data. We find that the best SOS strategy gives a final biomass $B(T) = 0.176$ which is 15% higher than what gives the best OS strategy. The associated control is a bang-singular-bang (see Fig. 2b).

Under water scarcity, the OS strategy is empirically used by practitioners, even though there is no model to help deciding the best triggering time. To our knowledge, the SOS strategy and more particularly its potentially late triggering time parameter t_s, is new and has not been yet tested on the field. Notice that the SOS strategy requires more knowledge or online measurements than the OS control for its real application (as the expression of the singular control (7) needs the function $\varphi(\cdot)$ and the values $S^\star$, k_2 and $K_R(S^\star)$). Moreover it change gradually the input flow rate during the singular arc phase. This is why it can be considered as a more *sophisticated* strategy.

7. CONCLUSION

We have introduced a simple crop irrigation model in order to study optimal irrigation scheduling using a mathematical analysis. We have shown, using a comparison tool, that the state constraint of this model is never activated for the optimal control problem solutions. Moreover we have shown that, under water scarcity, an optimal trajectory has to reach as fast as possible the domain for which the relative humidity is below or equal to the threshold of maximal crop transpiration, and then do not leave this domain until the harvesting time. However, due to water scarcity, it has to be below the threshold at some stage. We have then compared two control strategies: the one-shot (OS), commonly used in practice and a more sophisticated one, the saturated one-shot (SOS), that could exhibit a singular arc. We have shown numerically the superiority of this last strategy. We have not been yet able to prove that no more than one "saturated shot" is optimal. We conjecture that the SOS strategy is indeed an optimal control for this model. This would be a promising result since SOS irrigation schemes are not so intuitive controls and because they can be also tested on more detailed simulation models. Moreover, other criteria such as minimizing the water consumption for a desired biomass production, or maximizing the productivity could be also of interest. This shall be the matter of a future work.

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Appendix A. PROOF OF PROPOSITION 11

By construction, the solution $\tilde{S}(\cdot)$ verifies $\tilde{S}(t_1) = S(t_1)$ and $S(t_2) = S(t_2)$. Thus, property (11) is verified.

From standard comparison results of scalar differential equation with right hand sides that are Lipschitz continuous w.r.t. the state variable (see e.g. Walter (1998)), one has for any solution $S(\cdot)$ of (1) with $S(t_0) = S_0$ and any admissible control function $u(\cdot)$, the following frame.

$$S_{t_0,S_0,0}(t) \leq S(t) \leq S_{t_0,S_0,1}(t), \quad t \in [t_0, T] \tag{A.1}$$

Therefore, property (12) is verified.

Consider then the function $\delta(t) := S(t) - \tilde{S}(t)$. From expression (1), one can write

$$\begin{aligned} d\delta = -k_1 \left(F(t, S(t)) - F(t, \tilde{S}(t)) \right) dt \\ +k_1 k_2 \left(u(t) - \tilde{u}(t) \right) dt \end{aligned} \tag{A.2}$$

where we posit

$$F(t, S) = \varphi(t) K_S(S) + (1 - \varphi(t)) K_R(S)$$

Integrating (A.2) between $t = t_1$ and $t = t_2$, one obtains

$$\begin{aligned} \delta(t_2) - \delta(t_1) = -k_1 \int_{t_1}^{t_2} \left(F(t, S(t)) - F(t, \tilde{S}(t)) \right) dt \\ +k_1 k_2 \left(\int_{t_1}^{t_2} u(t)\, dt - \int_{t_1}^{t_2} \tilde{u}(t)\, dt \right) \end{aligned}$$

As F is non-decreasing w.r.t. S and $S(t) \geq \tilde{S}(t)$ for $t \in [t_1, t_2]$, one obtains

$$\int_{t_1}^{t_2} u(t)\, dt - \int_{t_1}^{t_2} \tilde{u}(t)\, dt \geq \frac{\delta(t_2) - \delta(t_1)}{k_1 k_2} = 0$$

which proves property (13).

Appendix B. PROOF OF PROPOSITION 12

Let $\tilde{u}(\cdot)$ be the MRAP control for $(t_1, S_1) = (0, 1)$ and $(t_2, S_2) = (T, S^\star)$ (see Definition 10).

Consider any $S(\cdot)$ solution of (1),(3) for an admissible control $u(\cdot)$ satisfying the constraint (6). Notice first that the set

$$E := \{t \in [0, T] \text{ s.t. } S(t) < S^\star\}$$

is non-empty, otherwise one would have $B(T) = B_T^\star$, which is excluded by Lemma 7.ii. Let $t^\star := \inf E < T$. By continuity of $S(\cdot)$, one has necessarily $S(t^\star) = S^\star$ and by Proposition 11 one has

$$\int_0^{t^\star} \tilde{u}(t)\, dt \leq \int_0^{t^\star} u(t)\, dt \tag{B.1}$$

Notice that one has $\tilde{u}(t) = \tilde{u}_{S^\star}(t)$ for $t \in [t^\star, T]$. From Hypothesis 8, the inequality

$$Q[u(\cdot)] = F_{max} \int_0^T u(t)\, dt < Q^\star = F_{max} \int_0^T \tilde{u}(t)\, dt \tag{B.2}$$

is fulfilled. Consequently, (B.1) and (B.2) give the inequality

$$\int_{t^\star}^{T} u(t)\,dt < \int_{t^\star}^{T} \tilde{u}_{S^\star}(t)\,dt$$

where $\tilde{u}_{S^\star}(t) < 1$ for $t \in [t^\star, T]$ (cf property (8)). Therefore, the set

$$E_1 := \{t \in [t^\star, T] \text{ s.t. } u(t) < 1\}$$

is necessarily of non-null measure. Moreover, the set $E \cap E_1$ is also of non-null measure (otherwise one would have $u(t) = 1$ for a.e. $t \in E$ that would imply that $S(\cdot)$ is increasing on E, which contradicts $S(t^\star) = S^\star$).

If $t^\star > \underline{t}$, inequality (B.1) is strict (by Proposition 11), and one can consider a control $v(\cdot)$ such that

$$\begin{array}{ll} v(t) = \tilde{u}(t), & t \in [0, t^\star], \\ v(t) = u(t), & t \in [t^\star, T] \setminus (E \cap E_1), \\ v(t) \in [u(t), 1], & t \in E \cap E_1 \end{array}$$

with

$$0 < \int_{E \cap E_1} \big(v(t) - u(t)\big)\,dt \leq \int_0^{t^\star} \big(u(t) - \tilde{u}(t)\big)\,dt$$

Then, one has

$$Q[v(\cdot)] \leq Q[u(\cdot)] \leq \bar{Q}$$

which guarantees that $v(\cdot)$ satisfies the constraint (6). Its associated solution $S_v(\cdot)$, $B_v(\cdot)$ satisfies then $S_v(t) \geq S(t)$ for any $t \in [0, T]$ with

$$\int_{E \cap E_1} S_v(t)\,dt > \int_{E \cap E_1} S(t)\,dt$$

As $S(t) < S^\star$ for $t \in E \cap E_1$, one obtains under Assumption 1 the inequality

$$\int_{E \cap E_1} \varphi(t) K_S(S_v(t))\,dt > \int_{E \cap E_1} \varphi(t) K_S(S(t))\,dt \quad \text{(B.3)}$$

which yields

$$\begin{aligned} B_v(T) &= k_3 \int_0^T \varphi(t) K_S(S_v(t))\,dt \\ &> k_3 \int_0^T \varphi(t) K_S(S(t)) = B(T) \end{aligned} \quad \text{(B.4)}$$

We conclude that an optimal solution has to verify $t^\star = \underline{t}$, that is such that

$$S(t) = \underline{S}(t), \quad t \in [0, \underline{t}]$$

or equivalently that $u(t) = 0$ for $t \in [0, \underline{t}]$ is optimal.

Consider now a solution $S(\cdot)$, $B(\cdot)$ with an admissible control $u(\cdot)$ that is null on $[0, \underline{t}]$ and satisfies the constraint (6), with the set

$$F := \{t \in [\underline{t}, T] \text{ s.t. } S(t) > S^\star\}$$

non empty. From Proposition 11, one has

$$\int_F \tilde{u}(t)\,dt < \int_F u(t)\,dt$$

Let us consider an admissible control $v(\cdot)$ such that

$$\begin{array}{ll} v(t) = \tilde{u}(t), & t \in F, \\ v(t) = u(t), & t \in [0, T] \setminus (F \cup (E \cap E_1)), \\ v(t) \in [u(t), 1], & t \in E \cap E_1 \end{array}$$

with

$$0 < \int_{E \cap E_1} \big(v(t) - u(t)\big)\,dt \leq \int_F \big(u(t) - \tilde{u}(t)\big)\,dt$$

Its solution $S_v(\cdot)$, $B_v(\cdot)$ satisfies $S_v(t) = S^\star$ for $t \in F$ and $S_v(t) \geq S^\star$ for $t \in [0, T] \setminus F$ with

$$\int_{E \cap E_1} S_v(t)\,dt > \int_{E \cap E_1} S(t)\,dt$$

As before, we obtain inequalities (B.3), (B.4), and conclude that an optimal solution has to verify $F = \emptyset$, that is such that $S(t) \leq S^\star$ for $t \in [\underline{t}, T]$.

Finally, consider an admissible control $u(\cdot)$ that is null on $[0, \underline{t}]$ with $S(t) \leq S^\star$ for $t \in [\underline{t}, T]$ and $Q[u(\cdot)] < \bar{Q}$. As previously, one can consider another admissible control $v(\cdot)$ such that:

$$\begin{array}{ll} v(t) = u(t), & t \in [0, T] \setminus (E \cap E_1), \\ v(t) \in [u(t), 1], & t \in E \cap E_1 \end{array}$$

with

$$0 < F_{max} \int_{E \cap E_1} \big(v(t) - u(t)\big)\,dt \leq \bar{Q} - Q[u(\cdot)]$$

Its solution $S_v(\cdot)$, $B_v(\cdot)$ satisfies $S_v(t) \geq S(t)$ for $t \in [0, T]$ with

$$\int_{E \cap E_1} S_v(t)\,dt > \int_{E \cap E_1} S(t)\,dt$$

One obtains again inequality (B.4), which shows that the control $u(\cdot)$ cannot be optimal. Therefore, an optimal control $u(\cdot)$ has to satisfy $Q[u(\cdot)] = \bar{Q}$.