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New Detection Thresholds and Stop Rules for CUSUM Online Detection

Nassim Sahki, Anne Gégout-Petit, Sophie Wantz-Mézières

03 September 2019
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</table>
Change-point

(a) Changepoint in the mean

(b) Changepoint in the variance

(c) Changepoint in the mean and variance

(d) Slope changepoint
Context of analysis

Offline context:
- All data are received and processed in one go;
- The primary aim is accurate detection of changes;
- Inference about all change-points simultaneously.

Online context:
- Data arrives either as single data-points or in batches;
- Data must be processed quickly before new data arrives;
- The aim is the quickest detection of a change after it has occurred;
- Inference about most recent change only.
Online change-point detection
Hypothesis test

Let $\{X_i\}_{i=1,\ldots,n}$ a series of observations sequentially observed.

$X_n$ is the last observed point in the dataset.

Statistically, the problem of change-point detection is to sequentially test for each new observation $x_n$, the hypotheses:

$$
\begin{align*}
H_{0,n} : v > n & \quad X_i \sim f_0(\cdot) \quad \forall i = 1, \ldots, n \\
H_{1,n} : \exists \, v \leq n, & \quad X_i \sim f_0(\cdot) \quad \forall i = 1, \ldots, (v-1) \\
& \quad X_i \sim f_1(\cdot) \quad \forall i = v, \ldots, n
\end{align*}
$$

(1)

Where ("distribution pre-change") $f_0 \neq f_1$ ("distribution post-change")

The "instantaneous" Log Likelihood Ratio (LLR) is defined by:

$$L_i = \log(\Lambda_i) = \log \left( \frac{f_1(x_i)}{f_0(x_i)} \right), \quad i \geq 1$$
Hypothesis test

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\]
Recursive detection statistics

- The Cumulative Sum ”CUSUM” statistics is written recursively [Page(1954)]:

\[
W_n = \max\{0, W_{n-1} + L_n\}, \quad n \geq 1, \quad W_0 = 0
\] (2)

When the two distributions \(f_0\) and \(f_1\) are unknown;

⇒ [Tartakovsky, A. G. and all (2006)] suggests replacing the log likelihood ratio \(L_n\) through a score function \(S_n = S_n(X_1, \ldots, X_n)\).
Recursive detection statistics

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\[ \Rightarrow \] [Tartakovsky, A. G. and all (2006)] suggests replacing the log likelihood ratio \( L_n \) through a score function \( S_n = S_n(X_1, \ldots, X_n) \).
The score $S_n$ is defined for a mean and variance change-point detection by:

$$S_n(\delta, q) = C_1 \cdot Y_n + C_2 \cdot Y_n^2 - C_3$$

(3)

$Y_n = (X_n - \mu_0)/\sigma_0$ : the centered and standardized data under $H_0$.

$C_1 = \delta \cdot q^2$, $C_2 = \frac{1-q^2}{2}$, $C_3 = \frac{\delta^2 \cdot q^2}{2} - \log(q)$

$\delta = (\mu_1 - \mu_0)/\sigma_0$  \hspace{1cm} q = \sigma_0/\sigma_1$

- $\delta$ : minimum level of change in the mean that is required to be detected.

No changepoint detection on the mean :

$\mu_1 = \mu_0 \Rightarrow \delta = 0$, therefore $C_1 = 0$.

- $q$ : minimum level of change in the variance that is required to be detected.

No changepoint detection on the variance :

$\sigma_1^2 = \sigma_0^2 \Rightarrow q = 1$, therefore $C_2 = 0$. 
The score function can only be used to the knowledge of the parameters mean and variance of pre-change data $\mu_0, \sigma_0^2$.

- Use a portion of observed data on the normal state without change-point
  $\Rightarrow$ **Estimate** $\mu_0$ and $\sigma_0^2$.

- Depending on the objective (mean and/or variance) and level of change that we want to detect: $\delta = (\mu_1 - \mu_0)/\sigma_0$, $q = \sigma_0/\sigma_1$
  $\Rightarrow$ **Fixed** $\mu_1$ and $\sigma_1^2$. 
Note

The score function can only be used to the knowledge of the parameters **mean and variance of pre-change data** $\mu_0, \sigma_0^2$.

- Use a portion of observed data on the normal state without change-point
  - $\Rightarrow$ **Estimate** $\mu_0$ and $\sigma_0^2$.

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  - $\Rightarrow$ **Fixed** $\mu_1$ and $\sigma_1^2$. 
The statistics is calculated recursively:

\[ W_n = \max\{0, W_{n-1} + S_n\}, \quad n \geq 1, \quad W_0 = 0 \]

Online detection is based on a **Stopping Rule**:

\[ T_h = \min\{n \geq 1 : W_n \geq h\}, \quad h \geq 0 : \text{threshold.} \]

When \( W \) exceeds the threshold \( h \):

\[ \Rightarrow \text{The procedure triggers an alarm (Stopping Time) to signal that a change-point has occurred.} \]
Denote $T$ a stopping time, such as:

$$T = \min\{i \geq 1 : W_i \geq h\}$$

* $T \geq v$: detection with a delay $(T - v)$.
* $T < v$: false alarm.
* $T = +\infty$: non detection.
Detection parameters

Let $P_0[.]$, $E_0[.]$: respectively the probability and the expectation before the change-point $v$.
Let $P_1[.]$, $E_1[.]$: respectively the probability and the expectation after the change-point $v$.

### Parameters evaluated under $P_0$.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Formula</th>
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<tbody>
<tr>
<td>Mean Time Between False Alarm (MTBFA)</td>
<td>$MTBFA = E_0[T]$</td>
</tr>
<tr>
<td>Instantaneous False Alarm Rate (IFAR)</td>
<td>$\alpha = \frac{1}{E_0[T]}$</td>
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### Parameter evaluated under $P_1$.

<table>
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<tr>
<td>Average Detection Delay (ADD)</td>
<td>$ADD = E_1[T]$</td>
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Nassim SAHKI
New Detection Thresholds and Stop Rules for CUSUM Online Detection
The conventional detection threshold used in the literature is based on the Wald's inequality [Egea-Roca et al (2017)].

⇒ This threshold is constant. It is given after fixing \( \alpha \) "the tolerated \( IFAR \), by:

\[
h_\alpha \leq - \ln(\alpha)
\]  

(7)


1 New detection thresholds constructed by an empirical method;
2 New stopping rules by modifying the classical rule.
CUSUM statistics under pre-change regime

- Simulate a series $X_n$ of $n = 200$ observations of Gaussian distribution ($\mu_0 = 0$ et $\sigma_0^2 = 1$);
- Compute W- statistics according to different levels of $\delta$, ($q = 1$).

The behavior (variability) of the W-statistics depends on the level of $\delta$;
⇒ Build thresholds according to $\delta$. 

(A): $\delta = 0.5$

(B): $\delta = 1$

(C): $\delta = 2$
CUSUM statistics under pre-change regime

- Simulate a series $X_n$ of $n = 200$ observations of Gaussian distribution ($\mu_0 = 0$ et $\sigma_0^2 = 1$);
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$\Rightarrow$ Build thresholds according to $\delta$. 

(A): $\delta = 0.5$  
(B): $\delta = 1$  
(C): $\delta = 2$
Empirical method: perform simulations of the statistics under $\mathbb{P}_0$ and build the detection threshold according to empirical quantile of law of the statistics under pre-change regime.

Construction steps:

1. Under $\mathbb{P}_0$: simulate $B$ series of $n$ observations

   $\{X_{ij}^j\}_{i=1,..,n; j=1,..,B}$. 
Empirical constant threshold

**Empirical method**: perform simulations of the statistics under $\mathbb{P}_0$ and build the detection threshold according to empirical quantile of law of the statistics under pre-change regime.

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**Construction steps**:

1. Under $P_0$: simulate $B$ series of $n$ observations
   \[ \{ X_{ij} \}_{i=1}^n, j=1,..,B. \]
2. Choice of the objective of detection $(\delta, q)$ and compute $w_j^i(\delta, q)$;
3. Choice of instantaneous false alarm rate $\alpha$ tolerated;

\[ \delta = 1, \quad q = 1, \quad \alpha = 0.01 \]
\[ W(1, 1) \]
Empirical constant threshold

**Empirical method**: perform simulations of the statistics under $\mathbb{P}_0$ and build the detection threshold according to empirical quantile of law of the statistics under pre-change regime.

**Construction steps**:

1. **Under $\mathbb{P}_0$**: simulate $B$ series of $n$ observations 
   \[ \{X_i^j\}_{i=1,\ldots,n; j=1,\ldots,B} \]

2. Choice of the objective of detection ($\delta, q$) and compute $w_i^j(\delta, q)$;

3. **Choice of instantaneous false alarm rate $\alpha$ tolerated**;

4. For each series $\{x_i^j\}_{1 \leq i \leq n}$, compute the **maximum of statistics**:
   \[ m^j(\delta, q) = \max_{1 \leq i \leq n} w_i(\delta, q). \]
**Empirical method**: perform simulations of the statistics under $P_0$ and build the detection threshold according to empirical quantile of law of the statistics under pre-change regime.

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   \]

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4. For each series $\{ x^j_i \}_{1 \leq i \leq n}$, compute the maximum of statistics:
   \[
   m^j(\delta, q) = \max_{1 \leq i \leq n} w_i(\delta, q).
   \]

5. The constant threshold would be the empirical quantile of order $(1-\alpha n)$:
   \[
   h(\delta, q) = q_{1-\alpha n} \left[ \left( m^j(\delta, q) \right)_{1 \leq j \leq B} \right].
   \]
Empirical constant threshold

\[ q = 1, \quad \alpha = 0.02 \]

![Graph](image)

\[ h(\delta, q) \]

\[ \delta \]

<table>
<thead>
<tr>
<th>0.5</th>
<th>1</th>
<th>2</th>
</tr>
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</table>

Constant Wald

Constant empir
Empirical instantaneous threshold

Construction steps:

1. Under $P_0$: simulate $B$ series of $n$ observations

$$\{X^j_i\}_{i=1,..,n; j=1,..,B}.$$
Empirical instantaneous threshold

**Construction steps:**

1. Under $\mathbb{P}_0$: simulate $B$ series of $n$ observations
   \[ \{X_i^j\}_{i=1,..,n; j=1,..,B}. \]
2. Choice of the objective of detection ($\delta, q$) and compute $w_i^j(\delta, q)$;
3. Choice of instantaneous false alarm rate $\alpha$ tolerated;

\[ \delta = 1, \; q = 1, \; \alpha = 0.01 \]
Empirical instantaneous threshold

Construction steps:

1. Under \( P_0 \): simulate \( B \) series of \( n \) observations \( \{X_{ij}\}_{i=1,\ldots,n;\ j=1,\ldots,B} \).

2. Choice of the objective of detection \((\delta, q)\) and compute \( w_j^i(\delta, q) \).

3. Choice of instantaneous false alarm rate \( \alpha \) tolerated;
Empirical instantaneous threshold

Construction steps:

1. Under $P_0$: simulate $B$ series of $n$ observations
   \[ \{ X_i^j \}_{i=1}^n \quad j=1,..,B. \]

2. Choice of the objective of detection $(\delta, q)$ and compute $w_i^j(\delta, q)$;

3. Choice of instantaneous false alarm rate $\alpha$ tolerated;

4. The instantaneous threshold would be the empirical quantile of order $(1-\alpha)$:
   \[
   h_t(\delta, q) = \mathbf{q}_{(1-\alpha)} \left[ (w_t^j(\delta, q))_{1 \leq j \leq B} \right],
   \]
   \[ t = 1, .., n \]
Empirical instantaneous threshold

$q = 1$, $\alpha = 0.02$

- Constant Wald
- Empirical constant
- Empirical instantaneous
Propose a **dynamic instantaneous threshold (data-driven)**: \( h_t - Z_{N_t}(\delta, q) \)

- Use the built instantaneous threshold and adapt it to the behavior of the statistic;
- Moving the threshold whenever statistics returns to its initial value (zero).

Where \( N_t = \sum_{i=1}^{t} \mathbf{1}_{\{W_i = 0\}} \), and \( Z_{N_t} = \inf\{i \geq Z_{N_{t-1}}; W_i = 0\} \) (renewal process)

**Graphs:**

(A): \( h_t(0.5, 1) \) is not dynamic

(B): \( h_t - Z_{N_t}(0.5, 1) \) is dynamic
Propose a **dynamic instantaneous threshold (data-driven)**: $h_t - Z_{N_t}(\delta, q)$

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Where $N_t = \sum_{i=1}^{t} 1\{W_i = 0\}$, and $Z_{N_t} = \inf\{i \geq Z_{N_{t-1}}; W_i = 0\}$ (renewal process)
Dynamic instantaneous threshold

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Simulation results
Objective
- Evaluate the different detection thresholds.

Data simulation
- Choice of real pre-change regime $P^R_0: \mu^R_0, \sigma^R_0$ (supposed known);
- Choice of real post-change regime $P^R_1: \mu^R_1, \sigma^R_1$ (and fixed $\delta^R, q^R$).

Objective of detection
- Choice the type and level of the expected change ($\delta, q$).

Estimation of $MTBFA, \alpha$ and $ADD$
- Knowing that we simulated series limited to $n = 100$ observations each, we used an empirical estimate taking into account the censoring (survival analysis).
Simulation

- **Objective**
  - Evaluate the different detection thresholds.

- **Data simulation**
  - Choice of real pre-change regime $\mathbb{P}_0^R : \mu_0^R, \sigma_0^R$ (supposed known);
  - Choice of real post-change regime $\mathbb{P}_1^R : \mu_1^R, \sigma_1^R$ (and fixed $\delta^R, q^R$).

- **Objective of detection**
  - Choice the type and level of the expected change ($\delta, q$).

- **Estimation of** $MTBFA, \alpha$ and $ADD$
  - Knowing that we simulated series limited to $n = 100$ observations each, we used an empirical estimate taking into account the censoring (survival analysis).
Objective
- Evaluate the different detection thresholds.

Data simulation
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- Choice of real post-change regime \( \mathbb{P}_1^R : \mu_1^R, \sigma_1^R \) (and fixed \( \delta^R, q^R \)).

Objective of detection
- Choice the type and level of the expected change (\( \delta, q \)).

Estimation of \( MTBFA, \alpha \) and \( ADD \)
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Objective
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- Knowing that we simulated series limited to $n = 100$ observations each, we used an empirical estimate taking into account the censoring (survival analysis).
Results under $\mathbb{P}_0$ : *MTBFA* and *IFAR*

**Table** – $B = 100000$, $n = 100$, $\alpha = 0.02$

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<tr>
<th>Threshold</th>
<th>$\hat{\delta}$</th>
<th>MTBFA</th>
<th>$\hat{\alpha}$</th>
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<tr>
<td><strong>Wald</strong></td>
<td>0.5</td>
<td>779</td>
<td>0.001</td>
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</tr>
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<td></td>
<td>1</td>
<td>318</td>
<td>0.003</td>
<td>27125</td>
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<td></td>
<td>2</td>
<td>239</td>
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The more the objective $\delta$ is large, the more we have false alarms.
Results under $P_0$: MTBFA and IFAR

\textbf{Table} – $B = 100000$, $n = 100$, $\alpha = 0.02$

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<td>37</td>
<td>0.027</td>
<td>93740</td>
</tr>
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Similar $\hat{\alpha}$ whatever is $\delta$; $\hat{\alpha}$ slightly exceeds the tolerated $\alpha$. 
### Results under $\mathbb{P}_0 : MTBFA$ and $IFAR$

**Table** — $B = 100000, n = 100, \alpha = 0.02$

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<td>$h_t(2)$</td>
<td>2</td>
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- The same behavior of Wald’s threshold results, with higher levels of FA but always respecting the tolerated $\alpha$. 

Nassim SAHKI

New Detection Thresholds and Stop Rules for CUSUM Online Detection
Results under $\mathbb{P}_0 : MTBFA$ and $IFAR$

**Table** – $B = 100000$, $n = 100$, $\alpha = 0.02$

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<td>$h_t(0.5)$</td>
<td>291</td>
<td>0.003</td>
<td>27953</td>
</tr>
<tr>
<td></td>
<td>$h_t(1)$</td>
<td>147</td>
<td>0.007</td>
<td>48564</td>
</tr>
<tr>
<td></td>
<td>$h_t(2)$</td>
<td>73</td>
<td>0.014</td>
<td>74391</td>
</tr>
<tr>
<td>Inst. Empir</td>
<td>$h_t(0.5)$</td>
<td>75</td>
<td>0.013</td>
<td>73466</td>
</tr>
<tr>
<td></td>
<td>$h_t(1)$</td>
<td>65</td>
<td>0.015</td>
<td>78544</td>
</tr>
<tr>
<td></td>
<td>$h_t(2)$</td>
<td>58</td>
<td>0.017</td>
<td>81940</td>
</tr>
</tbody>
</table>

- More homogeneous results; $\hat{\alpha}$ is close to the tolerated one but never exceeds it.
### Results under $\mathbb{P}_1 : \text{ADD}$

**Table** – $B = 100000$, $n = 100$, $v = 50$, $\alpha = 0.02$

<table>
<thead>
<tr>
<th>Threshold</th>
<th>$\delta$</th>
<th>$\delta^R = 1$</th>
<th>$\delta^R = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Wald</strong></td>
<td>3.91</td>
<td>$\text{ADD}$ 9 $\text{Mdn}$ 9 No-detect</td>
<td>$\text{ADD}$ 4 $\text{Mdn}$ 4 No-detect</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>9.45 9 0</td>
<td>4.31 4 0</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>7.44 6 2</td>
<td>2.94 3 0</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>9.37 7 212</td>
<td>2.58 2 0</td>
</tr>
</tbody>
</table>

- Detection is not so fast when $\delta^R > \delta$;
- The change-point is quickly detected when $\delta^R$ is large, whatever is $\delta$. 
### Results under $\mathbb{P}_1: ADD$

**Table** \( B = 100000, n = 100, v = 50, \alpha = 0.02 \)

<table>
<thead>
<tr>
<th>Threshold</th>
<th>$\delta$</th>
<th>$\widehat{ADD}$</th>
<th>$\widehat{Mdn}$</th>
<th>No-detect</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Wald</strong></td>
<td>0.5</td>
<td>9.45</td>
<td>9</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>7.44</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>9.37</td>
<td>7</td>
<td>212</td>
</tr>
<tr>
<td><strong>Const.</strong></td>
<td>1.42</td>
<td>3</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td><strong>Empir</strong></td>
<td>1.94</td>
<td>3.62</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>2.04</td>
<td>4.54</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

$\widehat{ADD}$ is considerably better; Change-point is quickly detected as long as $\delta^R \geq \delta$. 

Nassim SAHKI

*New Detection Thresholds and Stop Rules for CUSUM Online Detection*
### Results under $\mathbb{P}_1 : ADD$

**Table** — $B = 100000, n = 100, v = 50, \alpha = 0.02$

<table>
<thead>
<tr>
<th>Threshold</th>
<th>$\delta$</th>
<th>$\delta^R = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\overline{ADD}$</td>
</tr>
<tr>
<td>Wald</td>
<td>0.5</td>
<td>9.45</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>7.44</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>9.37</td>
</tr>
<tr>
<td>Const. Empir</td>
<td>0.5</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>3.62</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>4.54</td>
</tr>
<tr>
<td>Inst. Empir</td>
<td>$h_t(0.5)$</td>
<td>0.5</td>
</tr>
<tr>
<td></td>
<td>$h_t(1)$</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>$h_t(2)$</td>
<td>2</td>
</tr>
</tbody>
</table>

*Comparable to Wald's but with faster detection.*
### Results under $\mathbb{P}_1: ADD$

**Table** – $B = 100000$, $n = 100$, $v = 50$, $\alpha = 0.02$

<table>
<thead>
<tr>
<th>Threshold</th>
<th>$\delta$</th>
<th>$\widehat{ADD}$</th>
<th>$\widehat{Mdn}$</th>
<th>No-detect</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Wald</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>9.45</td>
<td>9</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>7.44</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>9.37</td>
<td>7</td>
<td>212</td>
</tr>
<tr>
<td><strong>Const. Empir</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>3</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>3.62</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>4.54</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td><strong>Inst. Empir</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$h^T_{(0.5)}$</td>
<td>0.5</td>
<td>8.47</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>$h^T_{(1)}$</td>
<td>1</td>
<td>6.22</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>$h^T_{(2)}$</td>
<td>2</td>
<td>6.23</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>$h^T_{(0.5)}$</td>
<td>1</td>
<td>5.1</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>$h^T_{(1)}$</td>
<td>2</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>$h^T_{(2)}$</td>
<td>3</td>
<td>5.8</td>
<td>4</td>
</tr>
</tbody>
</table>

- It detects more quickly than the fixed threshold and that of Wald.
Summary

- **Fixed $\alpha = 0.01$**
- **Fixed $\alpha = 0.02$**
- **Fixed $\alpha = 0.03$**

Legend:
- **O** Wald
- **•** Empir. Const
- **+** Empir. Inst
- **★** Empir. Inst dynamic

Parameters: 0.5, 1, 2
Theoretical study on the behavior of detection statistics (understand results given by the thresholds);
The case where the parameters of the pre-change regime are unknown: estimation methods;
Use the detection methods in the multivariate case;
Thesis framework: prediction of a dreaded event during online monitoring of lung transplant patients.
Thank you!
Annex
**New stopping rule**

**Classical stopping rule**

signals the existence of a changepoint when the detection statistic exceeds the instantaneous threshold.

**Corrected stopping rule**

signals the existence of a change-point when the detection statistic exceeds the instantaneous threshold during a time $c \geq 1$. 

---

**Diagram**

- **Classical (c = 1) stopping rule**
- **Corrected (c=3) stopping rule**

The graph shows the $W$-statistics over time, with $T_1$ and $T_2$ indicating specific points in time. The dashed line represents the classical rule, and the solid line represents the corrected rule. The threshold $h$ is marked on the graph.
### Results: corrected stop rule

**TABLE** – \( B = 100000, n = 100, v = 50, \alpha = 0.02 \)

<table>
<thead>
<tr>
<th>Threshold</th>
<th>( \delta )</th>
<th>Sous ( P_0 : \hat{\alpha} )</th>
<th>Sous ( P_1 : \overline{ADD} )</th>
<th>Stop rule “c”</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Wald</td>
<td>0.5</td>
<td>0.001</td>
<td>0.0009</td>
<td>0.0007</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0.003</td>
<td>0.002</td>
<td>0.001</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.004</td>
<td>0.001</td>
<td>0.0004</td>
</tr>
<tr>
<td>Const. empir</td>
<td>1.42</td>
<td>0.5</td>
<td>0.028</td>
<td>0.018</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0.028</td>
<td>0.014</td>
<td>0.008</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.027</td>
<td>0.007</td>
<td>0.002</td>
</tr>
<tr>
<td>Inst. empir.</td>
<td>( h_t(0.5) )</td>
<td>0.5</td>
<td>0.003</td>
<td>0.002</td>
</tr>
<tr>
<td></td>
<td>( h_t(1) )</td>
<td>1</td>
<td>0.007</td>
<td>0.003</td>
</tr>
<tr>
<td></td>
<td>( h_t(2) )</td>
<td>2</td>
<td>0.014</td>
<td>0.004</td>
</tr>
<tr>
<td>Inst. Empir Dynam</td>
<td>( h_t(0.5) )</td>
<td>0.5</td>
<td>0.013</td>
<td>0.007</td>
</tr>
<tr>
<td></td>
<td>( h_t(1) )</td>
<td>1</td>
<td>0.015</td>
<td>0.006</td>
</tr>
<tr>
<td></td>
<td>( h_t(2) )</td>
<td>2</td>
<td>0.017</td>
<td>0.004</td>
</tr>
</tbody>
</table>

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