Bounded solutions for an ordinary differential system from the Ginzburg-Landau theory

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Abstract.
In this paper, we look at a linear system of ordinary differential equations as derived from the two-dimensional Ginzburg-Landau equation. In two cases, it is known that this system admits bounded solutions coming from the invariance of the Ginzburg-Landau equation by translations and rotations. The specific contribution of our work is to prove that in the other cases, the system does not admit any bounded solutions. We show that this bounded solution problem is related to an eigenvalue problem.

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1 Introduction.

Let n and d be given integers, $n \geq 1$, $d \geq 1$. We define the following system

$$\left\{ \begin{array}{ll} a'' + \frac{a'}{r} - \frac{(n-d)^2}{r^2}a - f_d^2 b & = -(1-2f_d^2)a \\ b'' + \frac{b'}{r} - \frac{(n+d)^2}{r^2}b - f_d^2 a & = -(1-2f_d^2)b \end{array} \right. \tag{1.1}$$

and the following equations

$$a'' + \frac{a'}{r} - \frac{d^2}{r^2}a = -(1-f_d^2)a \tag{1.2}$$

and

$$a'' + \frac{a'}{r} - \frac{d^2}{r^2}a - 2af_d^2 = -(1-f_d^2)a. \tag{1.3}$$

with the variable $r > 0$, and for real valued functions $r \mapsto a(r)$ and $r \mapsto b(r)$.
Here f_d is the only solution of the differential equation

$$f_d'' + \frac{f_d'}{r} - \frac{d^2}{r^2}f_d = -f_d(1-f_d^2) \tag{1.4}$$

with the conditions $f_d(0) = 0$ and $\lim_{+\infty} f_d = 1$.

Let us consider the Ginzburg-Landau equation on a bounded connected domain Ω,

$$\begin{cases} -\Delta u = \frac{1}{\varepsilon^2} u(1-|u|^2) \text{ in } \Omega \\ u = g \text{ in } \partial\Omega \end{cases} \tag{1.5}$$

where $\varepsilon > 0$ is a small parameter, u and g have complex values and degree $(g, \partial\Omega) \geq 1$. Let us consider the following equation

$$-\Delta u = u(1-|u|^2) \text{ in } \mathbb{R}^2 \tag{1.6}$$

where u is a complex valued map. The study of the energy-minimizing solutions of equation (1.5) is in the book of Bethuel, Brezis Hélein, [3].
Let us explain how the system (1.1) and the equations (1.2) and (1.3) are derived from the equations (1.5) and (1.6).
We denote $\mathcal{N}_\varepsilon(u) = \Delta u + \frac{1}{\varepsilon^2} u(1-|u|^2)$. Let $u_0(x) = f_d(\frac{|x|}{\varepsilon})e^{id\theta}$. We have $\mathcal{N}_\varepsilon(u_0) = 0$. We will always denote

$$f(r) = f_d(\frac{r}{\varepsilon}).$$

We differentiate $\mathcal{N}_\varepsilon$ at u_0.

$$d\mathcal{N}_\varepsilon(u_0)(\omega) = \Delta\omega + \frac{\omega}{\varepsilon^2}(1-f^2) - \frac{2}{\varepsilon^2} f^2 e^{id\theta} e^{id\theta}.\omega,$$

where ω is any complex valued function and $2u.\omega = \overline{u}\omega + \overline{\omega}u$. We will use the operator $e^{-id\theta} d\mathcal{N}_\varepsilon(u_0) e^{id\theta}$ instead of $d\mathcal{N}_\varepsilon(u_0)$. We consider the Fourier expansion

$$\omega(x) = \sum_{n\geq 1}(a_n(r)e^{-in\theta} + b_n(r)e^{in\theta}) + a_0(r), \quad a_n(r) \in \mathbb{C}, \quad b_n(r) \in \mathbb{C}.$$

Letting $\omega_n(x) = a_n(r)e^{-in\theta} + b_n(r)e^{in\theta}$, we have

$$2e^{id\theta}.e^{id\theta}\omega_n = \omega_n + \overline{\omega}_n = (b_n + \overline{a}_n)e^{in\theta} + (\overline{b}_n + a_n)e^{-in\theta}.$$

$$\text{Moreover} \qquad e^{-id\theta}\Delta(e^{id\theta}\omega) = \Delta\omega - \frac{d^2}{r^2}\omega + i\frac{2d}{r^2}\frac{\partial\omega}{\partial\theta}.$$

Then

$$e^{-id\theta} d\mathcal{N}_\varepsilon(u_0) e^{id\theta}\omega = \sum_{n\geq 1} e^{-in\theta}(a_n'' + \frac{a_n'}{r} - \frac{(n-d)^2}{r^2}a_n + \frac{a_n}{\varepsilon^2}(1-f^2) - \frac{a_n}{\varepsilon^2}f^2 - \frac{\overline{b}_n}{\varepsilon^2}f^2)$$

$$+\sum_{n\geq 1} e^{in\theta}(b_n'' + \frac{b_n'}{r} - \frac{(n+d)^2}{r^2}b_n + \frac{b_n}{\varepsilon^2}(1-f^2) - \frac{b_n}{\varepsilon^2}f^2 - \frac{\overline{a}_n}{\varepsilon^2}f^2)$$

$$+a_0'' + \frac{a_0'}{r} - \frac{d^2}{r^2}a_0 + \frac{a_0}{\varepsilon^2}(1-f^2) - \frac{a_0 + \overline{a}_0}{\varepsilon^2}f^2.$$

Separating the Fourier components of $e^{-id\theta} d\mathcal{N}_\varepsilon(u_0) e^{id\theta}\omega$, we can consider the operators

$$\text{for } n \geq 1, \quad \mathcal{L}_n(a_n, b_n) = \begin{cases} a_n'' + \frac{a_n'}{r} - \frac{(n-d)^2}{r^2}a_n + \frac{a_n}{\varepsilon^2}(1-2f^2) - \frac{\overline{b}_n}{\varepsilon^2}f^2 \\ b_n'' + \frac{b_n'}{r} - \frac{(n+d)^2}{r^2}b_n + \frac{b_n}{\varepsilon^2}(1-2f^2) - \frac{\overline{a}_n}{\varepsilon^2}f^2 \end{cases}$$

$$\text{and, for } n=0, \quad \mathcal{L}_0(a_0) = a_0'' + \frac{a_0'}{r} - \frac{d^2}{r^2}a_0 + \frac{a_0}{\varepsilon^2}(1-f^2) - \frac{a_0+\overline{a}_0}{\varepsilon^2}f^2.$$

When we have to solve the system $(\mathcal{L}_n(a_n,b_n) = (\alpha_n,\beta_n), \mathcal{L}_0(a_0) = \alpha_0)$, for some given $(\alpha_n,\beta_n) \in \mathbb{C}\times\mathbb{C}$ and $\alpha_0 \in \mathbb{C}$, we are led to consider separately the real part and the imaginary part. So we consider the following operators, where a_n and b_n are real valued functions

$$\text{for } n \geq 1 \quad \mathcal{L}_{n,\mathcal{R}} : (a_n,b_n) \mapsto \begin{cases} a_n'' + \frac{a_n'}{r} - \frac{(n-d)^2}{r^2}a_n + \frac{a_n}{\varepsilon^2}(1-2f^2) - \frac{b_n}{\varepsilon^2}f^2 \\ b_n'' + \frac{b_n'}{r} - \frac{(n+d)^2}{r^2}b_n + \frac{b_n}{\varepsilon^2}(1-2f^2) - \frac{a_n}{\varepsilon^2}f^2 \end{cases} ;$$

$$\mathcal{L}_{n,\mathcal{I}} : (a_n,b_n) \mapsto \begin{cases} a_n'' + \frac{a_n'}{r} - \frac{(n-d)^2}{r^2}a_n + \frac{a_n}{\varepsilon^2}(1-2f^2) + \frac{b_n}{\varepsilon^2}f^2 \\ b_n'' + \frac{b_n'}{r} - \frac{(n+d)^2}{r^2}b_n + \frac{b_n}{\varepsilon^2}(1-2f^2) + \frac{a_n}{\varepsilon^2}f^2 \end{cases}$$

$$\text{and, for } n=0, \quad \mathcal{L}_{0,\mathcal{I}} : a_0 \mapsto a_0'' + \frac{a_0'}{r} - \frac{d^2}{r^2}a_0 + \frac{a_0}{\varepsilon^2}(1-f^2) \quad ;$$

$$\mathcal{L}_{0,\mathcal{R}} : a_0 \mapsto a_0'' + \frac{a_0'}{r} - \frac{d^2}{r^2}a_0 + \frac{a_0}{\varepsilon^2}(1-f^2) - \frac{2a_0}{\varepsilon^2}f^2.$$

Considering $(-a_n,b_n)$, we see that, for $n \geq 1$, only one of the operators $\mathcal{L}_{n,\mathcal{R}}$ or $\mathcal{L}_{n,\mathcal{I}}$ is of interest. The eigenvalue problems $\mathcal{L}_{n,\mathcal{R}}(a_n,b_n) = -\lambda(\varepsilon)(a_n,b_n)$, $(a_n,b_n)(1) = 0$, for all integers $d \geq 1$ and $n \geq 1$, as well as the problems $\mathcal{L}_{0,\mathcal{R}}(a_0) = -\lambda(\varepsilon)a_0$ and $\mathcal{L}_{0,\mathcal{I}}(a_0) = -\lambda(\varepsilon)a_0$, $a_0(1) = 0$, have been studied in several papers, including [6], [9], [7], [8], [1]. In the third chapter of their book [10], Pacard and Rivière study the system (1.1) and the equations (1.2) and (1.3) for $d = 1$. These authors' aim is to construct some solutions for (1.5).

Let us now bring together some of the results contained in the above studies.

Theorem 1.1 *For $d \geq 1$ and $n \geq 1$, the existence of an eigenvalue $\lambda(\varepsilon) \to 0$ as $\varepsilon \to 0$ is equivalent to the existence of a bounded solution of (1.1). For the equations (1.2) and (1.3) and for all $d \geq 1$, the results of [10] are valid for all $d \geq 1$, that is the real vector space of the bounded solutions of (1.2) is one-dimensional, spanned by f_d and there is no bounded solution of (1.3). For $n = 1$, the vector space of the bounded solutions of (1.1) is also a one dimensional vector space, spanned by $(f_d' + \frac{d}{r}f_d, f_d' - \frac{d}{r}f_d)$. For $d = 1$ and $n \geq 2$, there are no bounded solutions. For $d \geq 2$ and for $n \geq 2d-1$, there are no bounded solutions.*

A bounded solution is any solution defined at $r = 0$ and which has a finite limit as $r \to +\infty$. For all $d \geq 1$, the known bounded solutions, for $n = 0$ and $n = 1$, come from the invariance of the Ginzburg-Landau equation with respect to the translations and the rotations.

The present paper's aim is to prove the following

Theorem 1.2 *For all real numbers d and n such that $d \geq 1$ and $n > 1$, the system (1.1) has no bounded solution.*

We will consider n and d as real parameters, although the Ginzburg-Landau problem is about integers n and d. So we have to consider the functions f_d, for $d \in \mathbb{R}$, $d \geq 1$. But in [5], where all the solutions of (1.4) are studied, the authors consider only the case $d \in \mathbb{N}^\star$. However, this hypothesis is not essential in their paper. Here are the properties of f_d we need.

Theorem 1.3 *Let $d \in \mathbb{R}^{+\star}$. For all $a > 0$ there exists a unique solution of (1.4) such that $\lim_{r\to 0} \frac{1}{a} f(r) r^{-d} = 1$. There exists a unique value $A_d > 0$ such that this solution is defined in $\mathbb{R}^{+\star}$ and non decreasing. Denoting it by f_d, we have the expansions*

$$f_d(r) = 1 - \frac{d^2}{2r^2} + O(\frac{1}{r^4}) \text{ near } +\infty \tag{1.7}$$

and

$$f_d(r) = A_d(r^d - \frac{1}{4(d+1)} r^{d+2}) + O(r^{d+4}) \text{ near } 0. \tag{1.8}$$

Moreover, if we denote $g^{(d)}(r) = r^{-d} f_d(r)$, then, for all $\alpha > 0$, the map $d \mapsto g^{(d)}$ is continuous from $]0, +\infty[$ into $L^\infty([0, \alpha])$.

For $d \in \mathbb{R}^{+\star}$, $\gamma_1 \in \mathbb{R}^+$ and $\gamma_2 \in \mathbb{R}^+$, we define the following system

$$\begin{cases} a'' + \frac{a'}{r} - \frac{\gamma_1^2}{r^2} a - f_d^2 b - f_d^2 a & = -(1 - f_d^2) a \\ b'' + \frac{b'}{r} - \frac{\gamma_2^2}{r^2} b - f_d^2 a - f_d^2 b & = -(1 - f_d^2) b. \end{cases} \tag{1.9}$$

Letting $x = a + b$ and $y = a - b$, we are led to the system verified by (x, y), that is

$$\begin{cases} x'' + \frac{x'}{r} - \frac{\gamma^2}{r^2} x + \frac{\xi^2}{r^2} y - 2 f_d^2 x & = -(1 - f_d^2) x \\ y'' + \frac{y'}{r} - \frac{\gamma^2}{r^2} y + \frac{\xi^2}{r^2} x & = -(1 - f_d^2) y \end{cases} \tag{1.10}$$

with

$$\gamma^2 = \frac{\gamma_1^2 + \gamma_2^2}{2} \quad \text{and} \quad \xi^2 = \frac{\gamma_2^2 - \gamma_1^2}{2}.$$

In all the paper, when there is no other indication, we will suppose that $d \geq 1$ and $\frac{\gamma_1^2 + \gamma_2^2}{2} - d^2 \geq 1$. We will denote

$$n = \sqrt{\frac{\gamma_1^2 + \gamma_2^2}{2} - d^2}. \tag{1.11}$$

But Theorem 1.4, Theorem 1.5 and Theorem 1.6 will be valid for $d > 0$ and $\frac{\gamma_1^2+\gamma_2^2}{2} - d^2 > 0$. Let us cite a supposedly well-known principle : for any real number $R > 0$ and for any given Cauchy data $(a(R), a'(R), b(R), b'(R))$, the system (1.9) has a unique solution, defined in $\mathbb{R}^{+\star}$. This solution is continuous wrt the real positive parameters d, γ_1 and γ_2, because the coefficients of the system depend continuously on them. Moreover, when the Cauchy data depends continuously on the parameters, so does the solution $(a(r), a'(r), b(r), b'(r))$, which, consequently, is bounded independently of r and of the parameters, when r and the parameters stay in a given compact set. This principle comes from the Cauchy-Lipschitz Theorem, whose proof rests on an application of the Banach Fixed Point Theorem to a suitable integral equation. However, we don't know whether a given solution keeps the same behavior at 0 or at $+\infty$ for all the values of the parameters, even when this solution is continuous wrt the parameters. So we begin with the definition of some continuous solutions wrt to the parameters in a certain range, and whose behaviors remain inchanged, either at 0 or at $+\infty$.
To begin with, let us give the following definition

Definition 1.1 *We say that*

1. *$a = O(f)$ at 0 if there exists $R > 0$ and $C > 0$ such that*

$$\forall r \in]0, R], \quad |a(r)| \leq C|f(r)|.$$

2. *a has the behavior f at 0, and we denote $a \sim_0 f$, if there exists a map g, such that*

$$\lim_0 g = 0, \quad |a - f| = O(fg).$$

3. *$a = o(f)$ at 0 if there exists a map g, such that*

$$\lim_0 g = 0, \quad a = fg.$$

We will use the same convention at $+\infty$.

We will consider that (d, γ_1, γ_2) belongs to the set

$$\mathcal{D} = \{(d, \gamma_1, \gamma_2) \in (\mathbb{R}_+)^3; d \geq 1; \gamma_2 > 1; \quad 0 \leq \gamma_1 \leq \gamma_2 < \gamma_1 + 2d + 2\}.$$

Let us remark that $(d, |n-d|, n+d) \in \mathcal{D}$, whenever $d \geq 1$ and $n \geq 1$, $d \in \mathbb{R}$, $n \in \mathbb{R}$.
We will need the following subsets of $\mathcal{D}$

$$\mathcal{D}_1 = \{(d, \gamma_1, \gamma_2) \in \mathcal{D}; \gamma_1 > 0\}$$

and

$$\mathcal{D}_2 = \{(d, \gamma_1, \gamma_2) \in \mathcal{D}; 0 \leq \gamma_1 < \frac{1}{4}; -\gamma_1 - \gamma_2 + 2d + 2 > 0; -\gamma_2 + 2d + 1 > 0\}. \quad (1.12)$$

Whenever $d \geq 1$ and $n \geq 1$, $n \in \mathbb{R}$, $d \in \mathbb{R}$, we remark that $(d, |n-d|, n+d) \in \mathcal{D}_1$ when $n \neq d$ and that $(d, |n-d|, n+d) \in \mathcal{D}_2$ when $|n-d| < \frac{1}{4}$.
The following theorem is about a base of solutions defined near 0.

Theorem 1.4 *For all $(d, \gamma_1, \gamma_2) \in \mathcal{D}$, there exist four independent solutions (a, b) of (1.9) verifying the following conditions*

1. *$(a_1(r), b_1(r)) \sim_0 (O(r^{\gamma_2+2d+2}), r^{\gamma_2})$ and $(a_1'(r), b_1'(r)) \sim_0 (O(r^{\gamma_2+2d+1}), \gamma_2 r^{\gamma_2-1})$.*

2. $(a_2(r), b_2(r)) \sim_0 \begin{cases} (O(r^2\theta(r)), r^{-\gamma_2}) & \text{if} \quad (d, \gamma_1, \gamma_2) \in \mathcal{D}_1 \\ (O(r^{-\gamma_2+2d+2}), r^{-\gamma_2}) & \text{if} \quad (d, \gamma_1, \gamma_2) \in \mathcal{D}_2 \end{cases}$

   $(a_2'(r), b_2'(r)) \sim_0 \begin{cases} (O(r\theta(r)), -\gamma_2 r^{-\gamma_2-1}) & \text{if} \quad (d, \gamma_1, \gamma_2) \in \mathcal{D}_1 \\ (O(r^{-\gamma_2+2d+1}), -\gamma_2 r^{-\gamma_2-1}) & \text{if} \quad (d, \gamma_1, \gamma_2) \in \mathcal{D}_2 \end{cases}$

   where $\theta(r) = \begin{cases} \frac{-r^{\gamma_1-2}+r^{-\gamma_2+2d}}{\gamma_1+\gamma_2-2d-2} & \text{if} \quad \gamma_1 + \gamma_2 - 2d - 2 \neq 0 \\ -r^{\gamma_1-2}\log r & \text{if} \quad \gamma_1 + \gamma_2 - 2d - 2 = 0. \end{cases}$

3. *$(a_3(r), b_3(r)) \sim_0 (r^{\gamma_1}, O(r^{\gamma_1+2d+2}))$ and, if $\gamma_1 \neq 0$ $(a_3'(r), b_3'(r)) \sim_0 (\gamma_1 r^{\gamma_1-1}, O(r^{\gamma_1+2d+1}))$ while, if $\gamma_1 = 0$, $(a_3'(r), b_3'(r)) = (O(r), O(r^{2d+1}))$.*

4. $(a_4(r), b_4(r)) \sim_0 \begin{cases} (r^{-\gamma_1}, O(r^2\tilde{\theta}(r)) & \text{if} \quad (d,\gamma_1,\gamma_2) \in \mathcal{D}_1 \\ (\tau(r), O(\tau(r)r^{2d+2})) & \text{if} \quad (d,\gamma_1,\gamma_2) \in \mathcal{D}_2 \end{cases}$
   and $(a_4'(r), b_4'(r)) \sim_0 \begin{cases} (r^{-\gamma_1-1}, O(r\tilde{\theta}(r)) & \text{if} \quad (d,\gamma_1,\gamma_2) \in \mathcal{D}_1 \\ (\tau'(r), O(\tau'(r)r^{2d+2})) & \text{if} \quad (d,\gamma_1,\gamma_2) \in \mathcal{D}_2 \end{cases}$
   where
   $\tilde{\theta}(r) = \begin{cases} \frac{-r^{\gamma_2-2}+r^{-\gamma_1+2d}}{\gamma_1+\gamma_2-2d-2} & \text{if } \gamma_1+\gamma_2-2d-2 \neq 0 \\ -r^{\gamma_2-2}\log r & \text{if } \gamma_1+\gamma_2-2d-2 = 0 \end{cases}$; $\tau(r) = \begin{cases} \frac{r^{-\gamma_1}-r^{\gamma_1}}{2\gamma_1} & \text{if } \gamma_1 \neq 0 \\ -\log r & \text{if } \gamma_1 = 0. \end{cases}$

5. *For* $j = 1$ *and for* $j = 3$, *for all* $r > 0$, *the maps* $(d,\gamma_1,\gamma_2) \mapsto (a_j(r), a_j'(r), b_j(r), b_j'(r))$ *are continuous in* $\mathcal{D}$.

6. *For* $j = 1$ *and for* $j = 3$, *and for all* $r > 0$, $(a_j(r), a_j'(r), b_j(r), b_j'(r))$ *is differentiable wrt to* γ_1 *and wrt* γ_2, *whenever* $(d,\gamma_1,\gamma_2) \in \mathcal{D}$, *and* $\gamma_2 > \gamma_1$.
   Moreover the map $(d,\gamma_1,\gamma_2) \mapsto \frac{\partial}{\partial\gamma_i}(a_j(r), a_j'(r), b_j(r), b_j'(r))$ *is continous, for* $i = 1$ *and* $i = 2$. *And we have*
   $$\left(\frac{\partial a_1}{\partial\gamma_i}, \frac{\partial a_1'}{\partial\gamma_i}, \frac{\partial b_1}{\partial\gamma_i}, \frac{\partial b_1'}{\partial\gamma_i}\right)(r) \sim_0 \log r(O(r^{\gamma_2+2d+2}), O(r^{\gamma_2+2d+1}), r^{\gamma_2}, \gamma_2 r^{\gamma_2-1}) \quad (1.13)$$
   and, if $\gamma_1 \neq 0$
   $$\left(\frac{\partial a_3}{\partial\gamma_i}, \frac{\partial a_3'}{\partial\gamma_i}, \frac{\partial b_3}{\partial\gamma_i}, \frac{\partial b_3'}{\partial\gamma_i}\right)(r) \quad (1.14)$$
   $$\sim_0 \log r(r^{\gamma_1}, \gamma_1 r^{\gamma_1-1} + O(r^{\gamma_1+1}), O(r^{\gamma_1+2d+2}), O(r^{\gamma_1+2d+1}))$$

7. *For* $j = 2$ *or for* $j = 4$, *the same notation* (a_j, b_j) *is used for two solutions, one of them being defined for* $(d,\gamma_1,\gamma_2) \in \mathcal{D}_1$, *the other one being defined for* $(d,\gamma_1,\gamma_2) \in \mathcal{D}_2$.
   Moreover, for each domain $\mathcal{D}_i$, $i = 1, 2$ *and for all* $r > 0$ *the maps* $(d,\gamma_1,\gamma_2) \mapsto (a_j(r), a_j'(r), b_j(r), b_j'(r))$ *are continuous in* $\mathcal{D}_i$. *For each* $r > 0$, *the partial differentiability of* $(a_j(r), a_j'(r), b_j(r), b_j'(r))$ *wrt* γ_1 *or wrt* γ_2 *is also true separatly in each domain* $\mathcal{D}_i$, $i = 1, 2$.

The second theorem is about a base of solutions defined near $+\infty$.

Theorem 1.5 *1. We have a base of four solutions* (a, b) *of (1.9), with given behaviors at* $+\infty$. *In order to distinguish these solutions from the solutions defined in Theorem 1.4, we use the notation* (u_i, v_i), $i = 1, \ldots, 4$, *for these solutions. We have*

$$(u_1(r), v_1(r)) \sim_{r\to+\infty} (e^{\sqrt{2}r}/\sqrt{r}, e^{\sqrt{2}r}/\sqrt{r})(1 + O(r^{-2}));$$

$$(u_2(r), v_2(r)) \sim_{r\to+\infty} (e^{-\sqrt{2}r}/\sqrt{r}, e^{-\sqrt{2}r}/\sqrt{r})(1 + O(r^{-2}));$$

and

$$(u_3(r), v_3(r)) \sim_{r\to+\infty} (r^{-n}, -r^{-n})(1 + O(r^{-2}));$$

$$(u_4(r), v_4(r)) \sim_{r\to+\infty} (r^{n}, -r^{n})(1 + O(r^{-2})).$$

2. Except for $j = 2$, the construction of (u_j, v_j) is done separatly for each compact subset $\mathcal{K}$ of $\mathcal{D}$. For each of the four solutions and for all $r > 0$ the map $(d, \gamma_1, \gamma_2) \mapsto (u_j(r), u_j'(r), v_j(r)), v_j'(r))$ is continuous on $\mathcal{K}$. There partial derivatives wrt γ_1 and wrt γ_2 exist whenever $\gamma_1 < \gamma_2$ and are continuous. We have
$(\frac{\partial u_1}{\partial \gamma_i}, \frac{\partial u_1'}{\partial \gamma_i}, \frac{\partial v_1}{\partial \gamma_i}, \frac{\partial v_1'}{\partial \gamma_i})(r) \sim_{r \to +\infty} \frac{e^{\sqrt{2}r}}{\sqrt{r}} \log r (O(r^{-2}), O(r^{-3}), O(r^{-2}), O(r^{-3}))$;
$(\frac{\partial u_2}{\partial \gamma_i}, \frac{\partial u_2'}{\partial \gamma_i}, \frac{\partial v_2}{\partial \gamma_i}, \frac{\partial v_2'}{\partial \gamma_i})(r) \sim_{r \to +\infty} \frac{e^{-\sqrt{2}r}}{\sqrt{r}} \log r (O(r^{-2}), O(r^{-3}), O(r^{-2}), O(r^{-3}))$;
$(\frac{\partial u_3}{\partial \gamma_i}, \frac{\partial u_3'}{\partial \gamma_i}, \frac{\partial v_3}{\partial \gamma_i}, \frac{\partial v_3'}{\partial \gamma_i})(r) \sim_{r \to +\infty} \log r (r^n, O(r^{n-1}), -r^n, O(r^{n-1}))(1 + O(r^{-2}))$;
$(\frac{\partial u_4}{\partial \gamma_i}, \frac{\partial u_4'}{\partial \gamma_i}, \frac{\partial v_4}{\partial \gamma_i}, \frac{\partial v_4'}{\partial \gamma_i})(r) \sim_{r \to +\infty} \log r (r^{-n}, O(r^{-n-1}), -r^{-n}, O(r^{-n-1}))(1 + O(r^{-2}))$.

By our construction, the solution (u_j, v_j) depends on the given compact set $\mathcal{K}$, except for $j = 2$. For $j = 1$, this difficulty disappears after the proof of Theorem 1.6. For the other solutions, named (u_3, v_3) and (u_4, v_4), we will have to make sure that the parameter (d, γ_1, γ_2) stays in a compact set, as soon as we want and use the continuity and the differentiability of these solutions wrt the parameters.
In [1] we already gave the behaviors of a base of solutions at 0 and at $+\infty$. But, in the present paper, the continuity wrt to (d, γ_1, γ_2), especially of the five solutions (a_3, b_3) and (a_1, b_1) (defined at 0) and (u_2, v_2), (u_3, v_3), (u_4, v_4) (defined at $+\infty$) and there differentiability wrt γ_1 and γ_2, are essential.
The following theorem connects the least behavior at 0 to the exponentially blowing up behavior at $+\infty$ and the least behavior at $+\infty$ to the greater blowing up behavior at 0.

Theorem 1.6 *Suppose that $d > 0$ and that $\gamma_2 \geq \gamma_1 \geq 0$, $(\gamma_2^2 + \gamma_1^2)/2 > d^2$. Let (a_1, b_1) be the solution of (1.1) defined by $(a_1, b_1) \sim_0 (O(r^{\gamma_2+2d+2}), r^{\gamma_2})$. Then (a_1, b_1) blows up exponentially at $+\infty$. Let (u_2, v_2) be the solution of (1.1) defined by $(u_2, v_2) \sim_{+\infty} (\frac{e^{-\sqrt{2}r}}{\sqrt{r}}, \frac{e^{-\sqrt{2}r}}{\sqrt{r}})$. Then $(u_2, v_2) \sim_0 C(o(r^{-\gamma_2}), r^{-\gamma_2})$, for some $C \neq 0$.*

Now, let us relate the problem (1.9) to an eigenvalue problem, which is a little bit different from the one considered in the previous works on the subject, but, for our proof, we find it more suitable.
Let $0 \leq \gamma_1 < \gamma_2$, $\mu \in \mathbb{R}$ and $\varepsilon > 0$ be given and let us consider the following system

$$\left\{ \begin{array}{ll} a'' + \frac{a'}{r} - \frac{\gamma_1^2}{r^2} a - \frac{1}{\varepsilon^2} f^2 a - \frac{1}{\varepsilon^2} f^2 b & = -\frac{1}{\varepsilon^2} \mu (1 - f^2) a \\ b'' + \frac{b'}{r} - \frac{\gamma_2^2}{r^2} b - \frac{1}{\varepsilon^2} f^2 b - \frac{1}{\varepsilon^2} f^2 a & = -\frac{1}{\varepsilon^2} \mu (1 - f^2) b \end{array} \right. \tag{1.15}$$

for $r \in]0, 1]$, with the condition $a(1) = b(1) = 0$. Let us explain in which sense this can be considered as an eigenvalue problem.
Let $\gamma_1 \geq 0$ be given. We define

$$\mathcal{H}_{\gamma_1} = \{ r \mapsto (a(r), b(r)) \in \mathbb{R}^2; (a e^{i\gamma_1 \theta}, b e^{i\theta}) \in H_0^1(B(0,1), \mathbb{C}) \times H_0^1(B(0,1), \mathbb{C}) \},$$

where (r, θ) are the polar coordinates in $\mathbb{R}^2$. The dependence on γ_1 is needed to distinguish $\gamma_1 = 0$ and $\gamma_1 \neq 0$. We endow $\mathcal{H}_{\gamma_1}$ with the scalar product

$$< (a,b) | (u,v) >= \int_0^1 (r a' u' + r b' v' + \frac{\gamma_1^2}{r} a u + \frac{1}{r} b v) dr$$

and then $\mathcal{H}_{\gamma_1}$ is a Hibert space. Let $\mathcal{H}'_{\gamma_1}$ be the topological dual space of $\mathcal{H}_{\gamma_1}$.
We consider the $\mathcal{T}_{\gamma_1,\gamma_2} : \mathcal{H}_{\gamma_1} \to \mathcal{H}'_{\gamma_1}$ defined by

$$< \mathcal{T}_{\gamma_1,\gamma_2}(a,b),(u,v) >_{\mathcal{H}',\mathcal{H}} = \int_0^1 (ra'u' + rb'v' + \frac{\gamma_1^2}{r}au + \frac{\gamma_2^2}{r}bv + \frac{r}{\varepsilon^2}f^2(a+b)(u+v))dr.$$

We remark that

$$((a,b),(u,v)) \mapsto < \mathcal{T}_{\gamma_1,\gamma_2}(a,b),(u,v) >_{\mathcal{H}'_{\gamma_1},\mathcal{H}_{\gamma_1}}$$

is a scalar product on $\mathcal{H}_{\gamma_1}$. So, $\mathcal{T}_{\gamma_1,\gamma_2}$ is an isomorphism, by the Riesz Theorem.
Last, let us define the embedding

$$\begin{array}{rl} I: & \mathcal{H}_{\gamma_1} \to \mathcal{H}'_{\gamma_1} \\ & (a,b) \mapsto ((u,v) \mapsto \int_0^1 r(au+bv)dr) \end{array}$$

Since the embedding $H_0^1(B(0,1)) \times H_0^1(B(0,1)) \subset L^2(B(0,1)) \times L^2(B(0,1))$ is compact, then I is compact.
Let us define $\mathcal{C} = \frac{1}{\varepsilon^2}(1-f^2)I$. Since $\mathcal{C}$ is a compact operator and thanks to the continuity of $\mathcal{T}_{\gamma_1,\gamma_2}^{-1}$, then $\mathcal{T}_{\gamma_1,\gamma_2}^{-1}\mathcal{C}$ is a compact operator from $\mathcal{H}_{\gamma_1}$ into itself. By the standard theory of self adjoint compact operators, there exists a Hilbertian base of $\mathcal{H}$ formed of eigenvectors of $\mathcal{T}_{\gamma_1,\gamma_2}^{-1}\mathcal{C}$.
Now let us define $m_{\gamma_1,\gamma_2}(\varepsilon)$ as the first eigenvalue for the above eigenvalue problem in $\mathcal{H}_{\gamma_1}$, that is

$$m_{\gamma_1,\gamma_2}(\varepsilon) = \inf_{(a,b)\in\mathcal{H}^2_{\gamma_1}/\{(0,0)\}} \frac{\int_0^1 (ra'^2 + rb'^2 + \frac{\gamma_1^2}{r}a^2 + \frac{\gamma_2^2}{r}b^2 + \frac{r}{\varepsilon^2}f_d^2(\frac{r}{\varepsilon})(a+b)^2)dr}{\frac{1}{\varepsilon^2}\int_0^1 r(1-f_d^2(\frac{r}{\varepsilon}))(a^2+b^2)dr} \tag{1.16}$$

and let us define

$$m_0(\varepsilon) = \inf_{a\in\mathcal{H}_d/\{0\}} \frac{\int_0^1 (ra'^2 + \frac{d^2}{r}a^2)dr}{\frac{1}{\varepsilon^2}\int_0^1 r(1-f_d^2(\frac{r}{\varepsilon}))a^2dr} \tag{1.17}$$

It is a classical result that these infimum are attained. Considering the rescaling $(\tilde{a},\tilde{b})(r) = (a(\varepsilon r), b(\varepsilon r))$ and an extension by 0 outside $[0,1/\varepsilon]$, we see that $\varepsilon \mapsto m_{\gamma_1,\gamma_2}(\varepsilon)$ decreases when ε decreases. Then $\lim_{\varepsilon\to 0} m_{\gamma_1,\gamma_2}(\varepsilon)$ exists.
Moreover, $m_{\gamma_1,\gamma_2}(\varepsilon)$ is a simple eigenvalue and there exists an eigenvector (a,b) verifying

$$a(r) \geq -b(r) \geq 0 \text{ for all } r > 0.$$

Also, $m_0(\varepsilon)$ is realized by some function $a(r) \geq 0$.
In the previous works on the subject, the eigenvalue problem was $\mathcal{L}_{n\mathcal{R}}\omega = -\lambda(\varepsilon)\omega$. We have

$$(\exists \lambda < 0) \Leftrightarrow (\exists \mu < 1).$$

By examining the proof of Theorems 1.4, the possible behaviors at 0 of the solutions of the system

$$\left\{ \begin{array}{ll} a'' + \frac{a'}{r} - \frac{\gamma_1^2}{r^2}a - f_d^2 a - f_d^2 b & = -\mu(\varepsilon)(1-f_d^2)a \\ b'' + \frac{b'}{r} - \frac{\gamma_2^2}{r^2}b - f_d^2 b - f_d^2 a & = -\mu(\varepsilon)(1-f_d^2)b. \end{array} \right. \tag{1.18}$$

and those of the solutions of (1.1) are the same. More precisely, if $\mu(\varepsilon)$ is a bounded eigenvalue, it behaves as an additional bounded parameter and we construct two solutions denoted by (α_1, β_1) and (α_3, β_3), depending on ε and verifying, for $r \in [0, R]$,

$$|\alpha_1(r)| + |\beta_1(r) - r^{\gamma_2}| \leq Cr^{\gamma_2+2d+1}, \quad |\alpha_1'(r)| + |\beta_1'(r) - \gamma_2 r^{\gamma_2-1}| \leq Cr^{\gamma_2+2d}$$

and

$$|\alpha_3(r) - r^{\gamma_1}| + |\beta_3(r)| \leq Cr^{\gamma_1+2}, \quad |\alpha_3'(r) - \gamma_1 r^{\gamma_1-1}| + |\beta_3'(r)| \leq Cr^{\gamma_1+1},$$

where R and C are independent of ε, as in the proof of Theorem 1.4.
We can suppose that $\mu(\varepsilon) \to \mu$, as $\varepsilon \to 0$. Let $\omega_\varepsilon = (a_\varepsilon, b_\varepsilon)$ be an eigenvector associated to $\mu(\varepsilon)$. We define $\tilde{\omega}_\varepsilon(r) = \omega_\varepsilon(\varepsilon r)$, for $r \in [0, \frac{1}{\varepsilon}]$. For some constants A_ε and B_ε, $(\tilde{a}_\varepsilon, \tilde{b}_\varepsilon) = A_\varepsilon(\alpha_1, \beta_1) + B_\varepsilon(\alpha_2, \beta_2)$. We may suppose that $\max\{|A_\varepsilon|, |B_\varepsilon|\} = 1$. Thus, $(\tilde{a}_\varepsilon, \tilde{a'}_\varepsilon, \tilde{b}_\varepsilon, \tilde{b'}_\varepsilon)(R)$ is bounded independently of ε. Considering it as a Cauchy data, in the range $r \geq R$, we deduce that $(\tilde{a}_\varepsilon, \tilde{a'}_\varepsilon, \tilde{b}_\varepsilon, \tilde{b'}_\varepsilon)$ is bounded independently of ε, in every interval $[R, \alpha]$, $\alpha > 0$. Finally, we deduce the existence of some ω_0 such that

$$\tilde{\omega}_\varepsilon \to \omega_0, \quad \text{as } \varepsilon \to 0,$$

uniformly on each compact subset of $[0, +\infty]$, where $\omega_0 = (a_0, b_0)$ verifies

$$\left\{ \begin{array}{ll} a_0'' + \frac{a_0'}{r} - \frac{\gamma_1^2}{r^2} a_0 - f_d^2 a_0 - f_d^2 b_0 & = -\mu(1 - f_d^2) a_0 \\ b_0'' + \frac{b_0'}{r} - \frac{\gamma_2^2}{r^2} b_0 - f_d^2 b_0 - f_d^2 a_0 & = -\mu(1 - f_d^2) b_0 \end{array} \right. \tag{1.19}$$

Examining the proof of Theorem 1.5, the possible behaviors at $+\infty$ of the solutions of (1.19) are those given in Theorem 1.5, when we suppose that $\frac{\gamma_1^2+\gamma_2^2}{2} - \mu d^2 > 0$ and when we replace n by $\sqrt{\frac{\gamma_1^2+\gamma_2^2}{2} - \mu d^2}$.
Let us remark that the function f_d and the eigenvalue problem used here are not exatly the same as in the previous works [9], [7] and [8] and [1]. However, the proofs of the three following Theorems can be deduced from these works.

Theorem 1.7 *For all $d \geq 1$,*
(i) there exist $C > 0$ and $\varepsilon_0 > 0$ such that, for all $\varepsilon < \varepsilon_0$, $\frac{m_0(\varepsilon)-1}{\varepsilon^2} \geq C$; $m_0(\varepsilon) \to 1$ and there exists an associated eigenvector a_ε such that $\tilde{a}_\varepsilon \to f_d$, uniformly in each $[0, R]$, $R > 0$.
(ii) $m_{d-1,d+1}(\varepsilon) > 1$ and $\frac{m_{d-1,d+1}(\varepsilon)-1}{\varepsilon^2} \to 0$.
(iii) for $d > 1$ and $n \geq 2d - 1$, there exist $C > 0$ and $\varepsilon_0 > 0$ such that, for all $\varepsilon < \varepsilon_0$, $\frac{m_{|d-n|,d+n}(\varepsilon)-1}{\varepsilon^2} \geq C$.
(iv) There exists an eigenvector ω_ε associated to the eigenvalue $m_{d-1,d+1}(\varepsilon)$ such that $\|(1 - f_d^2)^{\frac{1}{2}}(\tilde{\omega}_\varepsilon - F_d)\|_{L^2(B(0,\frac{1}{\varepsilon}))} \to 0$, as $\varepsilon \to 0$, where $F_d = (f_d' + \frac{d}{r} f_d, f_d' - \frac{d}{r} f_d)$ appears in Theorem 1.1.

The interested reader can find a direct proof of Theorem 1.7 in the appendix of [2].
The following theorem is very important for our proof.

Theorem 1.8 *Let $d \in \mathbb{R}$, $d > 1$ be given. For all $n \in]1, d+1[$, there exists $C_n > 0$ independent of ε such that*

$$m_{|d-n|,d+n}(\varepsilon) \leq 1 - C_n.$$

For the sake of completeness, we give a proof of this theorem in Part VI of the present paper, following the proof of [9], given for $n = d = 2$.
The following theorem connects the eigenvalue problem to the existence of the bounded solutions.

Theorem 1.9 *(i) Let $d > 0$ and $\gamma_2 > \gamma_1 \geq 0$ be given. With the notation above, if $\mu(\varepsilon) \to \mu$, if $\tilde{\omega}_\varepsilon \to \omega_0$, if $\frac{\gamma_2^2+\gamma_1^2}{2} - \mu d^2 > 0$ and if ω_0 blows up at $+\infty$, then $\frac{\mu(\varepsilon)-1}{\varepsilon^2} \geq C$, where C is a given positive number, independent of ε.*
(ii) If there exists some bounded solution (a, b) of (1.9), then there exists an eigenvalue $\mu(\varepsilon)$ verifying $\mu(\varepsilon) - 1 \to 0$.

To make the paper as self contained as possible, we give the proof of Theorem 1.9 (i) in Part VI, following the proof of [8], given for $\mu = 1$ and for the eigenvalue $\lambda(\varepsilon)$. The interested reader can find a direct proof of Theorem 1.9 (ii) in [2].
The following theorems are new.

Theorem 1.10 *When $\frac{\gamma_1^2+\gamma_2^2}{\varepsilon^2} - d^2 > 0$, if there exists some bounded solution $\omega = (a, b)$ of (1.9), then we have $\lim_{\varepsilon\to 0} m_{\gamma_1,\gamma_2}(\varepsilon) \geq 1$.*

Combining Theorem 1.10 and Theorem 1.9 (ii), we get the following

Corollary 1.1 *If there exists some bounded solution $\omega = (a, b)$ of (1.9), then we have $\lim_{\varepsilon\to 0} m_{\gamma_1,\gamma_2}(\varepsilon) = 1$ and if ω_ε is some eigenvector associated to $m_{\gamma_1,\gamma_2}(\varepsilon)$, then $\tilde{\omega}_\varepsilon$ tends to ω, uniformly in all $[0, R]$, $R > 0$.*

The following theorem can be deduced at once from Theorem 1.10 and Theorem 1.8.

Theorem 1.11 *Let n and d be real numbers and $\gamma_1 = |n - d|$, $\gamma_2 = n + d$. There is no bounded solution of (1.9), when $d \geq 1$ and $1 < n < d + 1$.*

Using Theorems 1.1, 1.6 and 1.11, we will prove the following theorem.

Theorem 1.12 *There is no bounded solution of (1.9), whenever $d \geq 1$ and $n \geq d + 1$.*

Then Theorem 1.2 is proved. With Theorem 1.9 (i), we get

Theorem 1.13 *For $d \geq 1$, $n > 1$, $\gamma_1 = |n - d|$ and $\gamma_2 = n + d$, there is no eigenvalue $\mu(\varepsilon)$, with eigenvector in $\mathcal{H}_{|n-d|}$, such that $\frac{\mu(\varepsilon)-1}{\varepsilon^2} \to 0$, as $\varepsilon \to 0$.*

The paper is organised as follows. In Part II, we give a sketch of the proofs of Theorem 1.3, Theorem 1.4 and Theorem 1.5. Complete proofs of Theorem 1.4 and Theorem 1.5 are altogether long, technical and classical. The interested reader can consult Part II and Part III of [2], which is a long preliminary version of the present paper. In Part III, we prove Theorem 1.6. In Part IV, we prove Theorem 1.10. In Part V, we prove Theorem 1.12. In Part VI, we give the proof of Theorem 1.9 (i) and of Theorem 1.8, which is needed in the proof of Theorem 1.11. Theorem 1.9 (i) is needed to prove Theorem 1.10 and to deduce Theorem 1.13.

2 Proof of Theorem 1.3, proof of Theorem 1.4, proof of Theorem 1.5.

2.1 Proof of Theorem 1.3.

The existence of f_d, its expansion near 0 and $+\infty$ and its property of uniqueness are proved in [5]. However, these authors suppose that $d \in \mathbb{N}^\star$ and this is used only in the first step of their proof. Let us give an alternative proof for this first step, valid for all $d > 0$. We have to prove that for all $a > 0$ there exists some solution of (1.4) verifying $f \sim_0 ar^d$ and that f is defined in an interval $[0, R]$, with $R > 0$. We rewrite the equation (1.4) as

$$(r^{2d-1}(r^{-d}f)')' = -r^{2d-1}f(1-f^2).$$

For all $R > 0$ and all $a > 0$, f solves (1.4) in $[0, R]$, and $f \sim_0 ar^d$ if and only if the map $g : r \mapsto r^{-d}f(r)$ is a fixed point in $\mathcal{C}([0, R])$ of the function Φ defined by

$$\Phi(g)(r) = a + \int_0^r t^{-2d+1} \int_0^t -s^{3d-1}g(s)(1-s^{2d}g^2(s))ds. \tag{2.20}$$

Let us denote $\varphi(s, g) = -g(1-s^{2d}g^2)$. As in the proof of the Cauchy-Lipschitz Theorem, we remark first that for all $\alpha > 0$ and all $\beta > 0$ exist M and C such that

$$\big(s \in]0, \alpha], \quad \|g - a\|_{L^\infty([0,\alpha])} < \beta\big) \Rightarrow \big(\|\varphi(s, g)\|_{L^\infty([0,\alpha])} \leq M\big)$$

and

$$\big(s \in]0, \alpha], \quad \|g_1 - a\|_{L^\infty([0,\alpha])} < \beta, \quad \|g_2 - a\|_{L^\infty([0,\alpha])} < \beta\big) \\ \Rightarrow \left(|\varphi(s, g_1) - \varphi(s, g_2)|(s) \leq C|g_1 - g_2|(s)\right).$$

Moreover, M and C remain inchanged if α is replaced by a smallest positive number.
Now, we estimate, for $r \in [0, \alpha]$

$$|\Phi(g) - a|(r) \leq M\frac{r^{d+2}}{3d(d+2)} \text{ and } |\Phi(g_1) - \Phi(g_2)|(r) \leq C\frac{r^{d+2}}{3d(d+2)}\|g_1 - g_2\|_{L^\infty([0,\alpha])}.$$

Now, we choose some R such that

$$0 < R < \min\{1, \alpha, \frac{3d(d+2)\beta}{M}, \frac{3d(d+2)}{C}\}$$

and we denote $\overline{B}(a, \beta) = \{g \in \mathcal{C}[0, R]); \|g - a\|_{L^\infty([0,R])} \leq \beta\}$, in order Φ to be a contractant function from the closed subset $\overline{B}(a, R)$ of the Banach space $\mathcal{C}([0, R])$ into itself. Thus, by the Banach fixed Point Theorem, Φ has a unique fixed point g in $\overline{B}(a, R)$. Then $r \mapsto r^d g(r)$, defined in $[0, R]$, is the desired solution of (1.4).
The proof of [5] can be used to conclude to the existence of A_d.
Now, let us prove the continuity of $d \mapsto g^{(d)}$, where $g^{(d)}(r) = r^{-d}f_d(r)$. First, let us prove that the map $d \mapsto A_d$, defined in $\mathbb{R}^{+\star}$, increases.
As a first step, for $\delta \neq d$, we combine the equations of f_d and f_δ to obtain, for every (r_1, r_2), $0 < r_1 < r_2$,

$$[r(f_d' f_\delta - f_\delta' f_d)]_{r_1}^{r_2} = \int_{r_1}^{r_2} f_d f_\delta (f_d^2 - f_\delta^2)dt.$$

We derive two properties. The first one is that $f_d - f_\delta$ cannot keep the same sign in $[0,+\infty[$, otherwise, when $r_1 = 0$ and $r_2 \to +\infty$, the lrs would be 0 and the rhs would be non zero. The second one is that f_d and f_δ can be equal only for one value $r > 0$. Indeed, if $r_1 < r_2$ are such that $f_d(r_i) = f_\delta(r_i)$, for $i = 1, 2$, we get that $r_2 f_d(r_2)(f_d - f_\delta)'(r_2) - r_1 f_d(r_1)(f_d - f_\delta)'(r_1)$ has the same sign as $f_d^2 - f_\delta^2$ in $[r_1, r_2]$, and this is a contradiction.
Now, let $0 < \delta < d$ be given. Near $+\infty$ we have the expansion $f_d(r) - f_\delta(r) = \frac{\delta^2 - d^2}{2r^2} + 0(\frac{1}{r^4})$ and consequently, there exists $R > 0$ such that $f_d(r) < f_\delta(r)$, for all $r \in [R, +\infty[$. But we have also $r^d < r^\delta$ for $0 < r < 1$. Since the sign of $f_d - f_\delta$ has to change once in $[0, +\infty[$, and in view of the expansions near 0, we deduce that $A_d > A_\delta$.
Now, we denote $\lim_{d \to \delta, d > \delta} A_d = B$. But f_d is defined in $[0, +\infty[$. We have, for all $r > 0$, $g^{(d)} = \Phi(g^{(d)})$, where Φ is defined in (2.20), but with A_d instead of a. Using in addition $0 \le f_d(1 - f_d^2) \le 1$, we get that for all $\alpha > 0$, there exists $\beta > 0$ independent of d in an interval containing δ, such that $|g^{(d)}|(r) \le \beta$ and $|(g^{(d)})'|(r) \le \beta$, for all $r \in [0, \alpha]$. So, for all $r > 0$, $g^{(d)}(r)$ has a limit, denoted by g, as $d \to \delta$, uniformly in every $[0, \alpha]$, $\alpha > 0$ and we have $\Phi(g)(r) = g(r)$, for all $r > 0$, where Φ is defined in (2.20), but with B instead of a and δ instead of d. Consequently, if we denote $f(r) = r^d g(r)$, then $f \sim_0 B r^\delta$, f is a solution of (1.4) (with δ in place of d), f is non decreasing in $[0, +\infty[$. In view of the uniqueness of such a solution of (1.4) ([5]), we deduce that $B = A_\delta$ and that $f = f_\delta$. The same result remains true when $d \to \delta$, $d < \delta$. We have proved that $d \mapsto g^{(d)}$ is continuous from $[1, +\infty[$ into $L^\infty([0, \alpha])$, for all $\alpha > 0$.

2.2 Sketch of the proof of Theorem 1.4.

The pattern of proof is the same for the four solutions. Let us give an idea of the proof.

1. We construct some solution (a_1, b_1) such that for all compact subset $\mathcal{K}$ of $\mathcal{D}$, there exist some $R > 0$, depending only on $\mathcal{K}$ and some $C > 0$, also depending only on $\mathcal{K}$, such that for all $r \in]0, R]$ and all $(d, \gamma_1, \gamma_2) \in \mathcal{K}$, we have

$$|a_1(r)| + |b_1(r) - r^{\gamma_2}| \le C r^{\gamma_2 + 2d + 1} \quad \text{and} \quad |a_1'(r)| + |b_1'(r) - \gamma_2 r^{\gamma_2 - 1}| \le C r^{\gamma_2 + 2d}.$$

and such that, for all $r \in]0, R]$, $(d, \gamma_1, \gamma_2) \mapsto (a_1(r), a_1'(r), b_1(r), b_1'(r))$ is continuous on $\mathcal{K}$, and differentiable wrt γ_1 and wrt γ_2. First, the construction is done for $r \in]0, R]$. Then the definition of this solution in $[0, +\infty[$ and the continuity wrt $(d, \gamma_1, \gamma_2) \in \mathcal{K}$, for all $r > 0$, follows from the Cauchy-Lipschitz Theorem. Let us remark the importance for the constants C and R to be independent of the parameters.
We use a constructive method, similar to the proof of the Banach fixed point Theorem. We define a fixed point problem of the form $(a, b) = \Phi(a, b)$, that is

$$\left\{ \begin{array}{ll} a & = r^{\gamma_1} + r^{\gamma_1} \int_0^r t^{-2\gamma_1 - 1} \int_0^t s^{\gamma_1 + 1} (f_d^2 b - (1 - 2f_d^2) a) ds dt \\ b & = r^{\gamma_2} \int_0^r t^{-2\gamma_2 - 1} \int_0^t s^{\gamma_2 + 1} (f_d^2 a - (1 - 2f_d^2) b) ds dt. \end{array} \right. \tag{2.21}$$

whose solutions verify the differential system that we have to solve.

2. We construct some solution (a_3, b_3), such that, for any compact subset $\mathcal{K} \in \mathcal{D}$, exist some real numbers R and C verifying, for all $0 < r < R$,

$$|a_3(r) - r^{\gamma_1}| \le C r^{\gamma_1 + 2}, |b_3(r)| \le C r^{\gamma_1 + 2d + 2}, |a_3'(r) - \gamma_1 r^{\gamma_1 - 1}| \le C r^{\gamma_1 + 1}, |b_3'(r)| \le C r^{\gamma_1 + 2d + 1}.$$

For this purpose, we consider the fixed point problem

$$\left\{\begin{array}{ll} a & = r^{\gamma_1}\int_0^r t^{-2\gamma_1-1}\int_0^t s^{\gamma_1+1}(f_d^2 b-(1-2f_d^2)a)dsdt \\ b & = r^{\gamma_2}+r^{\gamma_2}\int_0^r t^{-2\gamma_2-1}\int_0^t s^{\gamma_2+1}(f_d^2 a-(1-2f_d^2)b)dsdt. \end{array}\right. \tag{2.22}$$

3. For the construction of (a_2, b_2), in the case when $(d, \gamma_1, \gamma_2) \in \mathcal{D}_1$, we consider the fixed point problem

$$\left\{\begin{array}{ll} a & = r^{-\gamma_1}\int_0^r t^{2\gamma_1-1}\int_1^t s^{-\gamma_1+1}(f_d^2 b-(1-2f_d^2)a)dsdt \\ b & = r^{-\gamma_2}+r^{-\gamma_2}\int_0^r t^{2\gamma_2-1}\int_1^t s^{-\gamma_2+1}(f_d^2 a-(1-2f_d^2)b)dsdt. \end{array}\right. \tag{2.23}$$

while, when $(d, \gamma_1, \gamma_2) \in \mathcal{D}_2$, we consider the fixed point problem

$$\left\{\begin{array}{ll} a & = r^{\gamma_1}\int_0^r t^{-2\gamma_1-1}\int_0^t s^{\gamma_1+1}(f_d^2 b-(1-2f_d^2)a)dsdt \\ b & = r^{-\gamma_2}+r^{-\gamma_2}\int_0^r t^{2\gamma_2-1}\int_1^t s^{-\gamma_2+1}(f_d^2 a-(1-2f_d^2)b)dsdt. \end{array}\right. \tag{2.24}$$

4. In order to construct a solution (a_4, b_4), when $(d, \gamma_1, \gamma_2) \in \mathcal{D}_1$, we solve the following fixed point problem

$$\left\{\begin{array}{ll} a & = r^{-\gamma_1}+r^{-\gamma_1}\int_0^r t^{2\gamma_1-1}\int_1^t s^{-\gamma_1+1}(f_d^2 b-(1-2f_d^2)a)dsdt \\ b & = r^{-\gamma_2}\int_0^r t^{2\gamma_2-1}\int_1^t s^{-\gamma_2+1}(f_d^2 a-(1-2f_d^2)b)dsdt. \end{array}\right. \tag{2.25}$$

and, when $(d, \gamma_1, \gamma_2) \in \mathcal{D}_2$, we solve the following fixed point problem

$$\left\{\begin{array}{l} a = \tau(r)+\tau(r)\int_0^r \frac{1}{t}\tau^{-2}(t)\int_0^t s\tau(s)(f_d^2 b-(1-2f_d^2)a)dsdt \\ b = r^{-\gamma_2}\int_0^r t^{2\gamma_2-1}\int_0^t s^{-\gamma_2+1}(f_d^2 a-(1-2f_d^2)b)dsdt \end{array}\right. \tag{2.26}$$

2.3 Sketch of the proof of Theorem 1.5.

We use the system (1.10) and we construct a base of four solutions, (x_j, y_j), $j = 1, \ldots, 4$, characterized by their behaviors at $+\infty$. The solutions (u_j, v_j) announced in Theorem 1.5 are obtained by $u_j = \frac{x_j+y_j}{2}$ and $v_j = \frac{x_j-y_j}{2}$.
We denote

$$J_+ = \frac{e^{\sqrt{2}r}}{\sqrt{r}}, \quad J_- = \frac{e^{-\sqrt{2}r}}{\sqrt{r}}, \quad \gamma^2 = \frac{\gamma_1^2+\gamma_2^2}{2}, \quad n = \sqrt{\gamma^2-d^2}, \quad \xi^2 = \frac{\gamma_2^2-\gamma_1^2}{2}.$$

We can replace the first equation of (1.10) by

$$(e^{2\sqrt{2}r}(xe^{-\sqrt{2}r})')' = e^{\sqrt{2}r}q(r)x - \frac{\xi^2}{r^2}y \text{ or } (e^{-2\sqrt{2}r}(xe^{\sqrt{2}r})')' = e^{-\sqrt{2}r}q(r)x - \frac{\xi^2}{r^2}y,$$

where

$$q(r) = \frac{-\gamma^2-3d^2}{r^2} + 3(1-f_d^2+\frac{d^2}{r^2}).$$

The second equation of the system (1.10) can be written as

$$(r^{2n+1}(r^{-n}y)')' = r^{n+1}(\frac{\xi^2}{r^2}x - (1-f_d^2-\frac{d^2}{r^2})y)$$

or

$$(r^{-2n+1}(r^n y)')' = r^{-n+1}(\frac{\xi^2}{r^2}x - (1 - f_d^2 - \frac{d^2}{r^2})y).$$

Finally, the system (1.10) can be written as

$$\begin{cases} (e^{\pm 2\sqrt{2}r}(r^{\frac{1}{2}}e^{\mp\sqrt{2}r}x)')' = r^{\frac{1}{2}}e^{\pm\sqrt{2}r}q(r)x - \frac{\xi^2}{r^2}y \\ (r^{\pm 2n+1}(r^{\mp n}y)')' = r^{\pm n+1}(\frac{\xi^2}{r^2}x - (1 - f_d^2 - \frac{d^2}{r^2})y) \end{cases} \tag{2.27}$$

In order to construct four solutions of (2.27), we give $R_0 > 0$ and we define fixed points problems of the form $(x, y) = \Phi(x, y)$, for (x, y) defined in $[R_0, +\infty[$, and whose solutions are solutions of (2.27). The function Φ will depend on R_0, except for one solution denoted by (x_2, y_2) (vanishing exponentially at $+\infty$). The present construction does not allow us to construct the solutions (x_j, y_j), $j \neq 2$ without taking into account a given compact subset

$$\mathcal{K} \subset \{(d, \gamma_1, \gamma_2); 0 \le \gamma_1 < \gamma_2; \xi^2 - d^2 > 0\}. \tag{2.28}$$

Indeed, R_0 depends on $\mathcal{K}$. Let us list the different fixed point problems we need.

1. The exponential blowing up behavior at $+\infty$: the solution (x_1, y_1). For $R_0 > 0$

$$\begin{cases} x &= J_+ + J_+ \int_{+\infty}^r (J_+)^{-2}\frac{1}{t}\int_{R_0}^t sJ_+(\frac{\xi^2}{s^2}y - 3(1 - f_d^2 - \frac{d^2}{s^2})x)dsdt \\ y &= r^n \int_{R_0}^r t^{-2n-1}\int_{R_0}^t s^{n+1}(\frac{\xi^2}{s^2}x - (1 - f_d^2 - \frac{d^2}{s^2})y)dsdt. \end{cases}$$

2. The intermediate blowing up behavior at $+\infty$: the solution (x_3, y_3). For $R_0 > 0$

$$\begin{cases} x &= J_+ \int_{+\infty}^r (J_+)^{-2}\frac{1}{t}\int_{R_0}^t sJ_+(\frac{\xi^2}{s^2}y - 3(1 - f_d^2 - \frac{d^2}{s^2})x)dsdt \\ y &= r^n + r^n \int_{+\infty}^r t^{-2n-1}\int_{R_0}^t s^{n+1}(\frac{\xi^2}{s^2}x - (1 - f_d^2 - \frac{d^2}{s^2})y)dsdt \end{cases}$$

3. The least behavior at $+\infty$: the solution (x_2, y_2). We consider

$$\begin{cases} x &= J_- + J_- \int_{+\infty}^r (J_-)^{-2}\frac{1}{t}\int_{+\infty}^t sJ_-(\frac{\xi^2}{s^2}y - 3(1 - f_d^2 - \frac{d^2}{s^2})x)dsdt \\ y &= r^{-n} \int_{+\infty}^r t^{2n-1}\int_{+\infty}^t s^{-n+1}(\frac{\xi^2}{s^2}x - 3(1 - f_d^2 - \frac{d^2}{s^2})y)dsdt \end{cases}$$

4. The intermediate vanishing behavior at $+\infty$: the solution (x_4, y_4). For $R_0 > 0$

$$\begin{cases} x &= J_- \int_{R_0}^r (J_-)^{-2}\frac{1}{t}\int_{+\infty}^t sJ_-(\frac{\xi^2}{s^2}y - 3(1 - f_d^2 - \frac{d^2}{s^2})x)dsdt \\ y &= r^{-n} + r^{-n} \int_{+\infty}^r t^{2n-1}\int_{+\infty}^t s^{-n+1}(\frac{\xi^2}{s^2}x - 3(1 - f_d^2 - \frac{d^2}{s^2})y)dsdt \end{cases}$$

We need the following estimate, which is not difficult to prove, by an integration by part. Let $\alpha \in \mathbb{R}$ and $\beta > 0$ be given. Then

$$\int_t^{+\infty} s^\alpha e^{-\beta s}ds \le \frac{2}{\beta}t^\alpha e^{-\beta t} \quad \text{for all } t \ge \frac{2\alpha}{\beta} \tag{2.29}$$

and

$$\int_R^t s^\alpha e^{\beta s}ds \le \frac{2}{\beta}t^\alpha e^{\beta t} \quad \text{for all } t \ge R \ge \frac{-2\alpha}{\beta} \tag{2.30}$$

3 The smallest behavior at zero is connected with the greatest behavior at infinity.

Proof of Theorem 1.6.
Let $(d, \gamma_1, \gamma_2) \in \mathcal{D}$. Let us prove first that (a_1, b_1) blows up exponentially at $+\infty$.
Let us define $x = a_1 + b_1$ and $y = a_1 - b_1$. We have $x(r) \sim_0 r^{\gamma_2}$ and $y(r) \sim_0 -r^{\gamma_2}$. Thus, we have $x(r) > 0$ and $y(r) < 0$ near $r = 0$. Let us suppose that $x(r) > 0$ and $y(r) < 0$ in $]0, R[$. Combining the first equation of the system (1.10) and the equation (1.4), we get, for all $r \geq 0$

$$[rx' f_d - r f_d' x]_0^r + \int_0^r \frac{-\gamma^2 + d^2}{s} x f_d ds + \mu^2 \int_0^r \frac{y}{s} f_d ds - 2 \int_0^r s f_d^3 x ds = 0.$$

For $0 < r \leq R$, we deduce that

$$r f_d^2 (\frac{x}{f_d})'(r) \geq 2 \int_0^r s f_d^3 x ds. \tag{3.31}$$

This proves that $\frac{x}{f_d}$ increases in $]0, R]$ and therefore $x(R) > 0$.
Moreover, combining the second equation of the system (1.10) and (1.4), we get

$$[ry' f_d - r f_d' y]_0^r + \int_0^r \frac{-\gamma^2 + d^2}{s} y f_d ds + \xi^2 \int_0^r \frac{x}{s} f_d ds = 0.$$

For $0 < r \leq R$, we deduce that

$$r f_d^2 (\frac{-y}{f_d})'(r) \geq \int_0^r \frac{-\gamma^2 + d^2}{s} y f_d ds. \tag{3.32}$$

This proves that $\frac{-y}{f_d}$ increases in $]0, R]$ and therefore $-y(R) > 0$. Finally, we have proved that $x(r) > 0$ and $y(r) < 0$ for all $r > 0$. Now (3.31) and (3.32) are valid for all $r > 0$ and we know that $f_d \sim_{+\infty} 1$. Thus, the behavior of x at $+\infty$ cannot be a polynomial increasing behavior. We return to Theorem 1.5 that gives all the possible behaviors at $+\infty$ and we deduce that x and y have an exponentially increasing behavior at $+\infty$. So a and b have an exponentially increasing behavior at $+\infty$, too.
Let us prove now that $(u_2, v_2) \sim_0 D(o(r^{\gamma_1}), r^{-\gamma_2})$, for some $D \neq 0$. Multiplying (1.9) and integrating by parts, we get easily, for all $r_1 > 0$ and $r > 0$

$$[s(a_1' u_2 - u_2' a_1 + v_2 b_1' - v_2' b_1)(s)]_r^{r_1} = 0.$$

Using $(a_1, b_1) \sim_{+\infty} C(\frac{e^{\sqrt{2}r}}{\sqrt{r}}, \frac{e^{\sqrt{2}r}}{\sqrt{r}})$, for some $C \neq 0$, and $(u_2, v_2) \sim_{+\infty} (\frac{e^{-\sqrt{2}r}}{\sqrt{r}}, \frac{e^{-\sqrt{2}r}}{\sqrt{r}})$, we get

$$\lim_{r \to +\infty} r(a_1' u_2 - u_2' a_1 + v_2 b_1' - v_2' b_1)(r) = 4C\sqrt{2}.$$

Consequently

$$\lim_{r \to 0} r(a_1' u_2 - u_2' a_1 + v_2 b_1' - v_2' b_1)(r) = 4C\sqrt{2}.$$

We know that $(a_1, b_1) \sim_0 (o(r^{\gamma_2}), r^{\gamma_2})$. According to Theorem 1.4, that gives all the possible behaviors at 0, we conclude that the only fitting behavior at 0 for (u_2, v_2) is $(u_2, v_2) \sim_0 D(o(r^{\gamma_1}), r^{-\gamma_2})$, for $D = \frac{2C\sqrt{2}}{\gamma_2}$.
This ended the proof of Theorem 1.6.

4 The proof of Theorem 1.10 and of Corollary 1.1.

Let $d > 1$. We can rewrite the system (1.9) as

$$X' = MX \text{ with } X = (a, ra', b, rb')^{\mathrm{t}} \tag{4.33}$$

with

$$M = \begin{pmatrix} 0 & \frac{1}{r} & 0 & 0 \\ -r(1-2f_d^2) + \frac{\gamma_1^2}{r} & 0 & rf_d^2 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ rf_d^2 & 0 & -r(1-2f_d^2) + \frac{\gamma_2^2}{r} & 0 \end{pmatrix}.$$

Lemma 4.1 *Let us suppose that there exists a bounded solution of (1.9) and let us chose a base of solutions, X_1, X_2, X_3, X_4, for (4.33), whose third vector is a bounded solution. Let us name $R(s)$ the resolvant matrix, whose columns are the vectors X_i, $i = 1, \ldots, 4$. Let us name $\mathcal{C}_2$ and $\mathcal{C}_4$ the second and the fourth column of $R^{-1}(s)$. We have*

at 0 and when $(d, \gamma_1, \gamma_2) \in \mathcal{D}_1$ and $\gamma_1 + \gamma_2 - 2d - 2 < 0$

$$\mathcal{C}_2 = \begin{pmatrix} O(s^{\gamma_1}) \\ O(s^{\gamma_1+2\gamma_2}) \\ O(s^{-\gamma_1}) \\ O(s^{\gamma_1}) \end{pmatrix} \quad \textit{and} \quad \mathcal{C}_4 = \begin{pmatrix} O(s^{-\gamma_2}) \\ O(s^{\gamma_2}) \\ O(s^{\gamma_2}) \\ O(s^{2\gamma_1+\gamma_2}) \end{pmatrix}$$

and

at 0 and when $(d, \gamma_1, \gamma_2) \in \mathcal{D}_1$ and $\gamma_1 + \gamma_2 - 2d - 2 > 0$

$$\mathcal{C}_2 = \begin{pmatrix} O(s^{-\gamma_2+2d+2}) \\ O(s^{\gamma_2+2d+2}) \\ O(s^{-\gamma_1}) \\ O(s^{\gamma_1}) \end{pmatrix} \quad \textit{and} \quad \mathcal{C}_4 = \begin{pmatrix} O(s^{-\gamma_2}) \\ O(s^{\gamma_2}) \\ O(s^{-\gamma_1+2d+2}) \\ O(s^{\gamma_1+2d+2}) \end{pmatrix}$$

and

at 0 and when $(d, \gamma_1, \gamma_2) \in \mathcal{D}_2$

$$\mathcal{C}_2 = \begin{pmatrix} O(\tau(s)s^{-\gamma_2+\gamma_1+2d+2}) \\ O(\tau(s)s^{\gamma_1+\gamma_2+2d+2}) \\ O(\tau(s)) \\ O(s^{\gamma_1}) \end{pmatrix} \quad \textit{and} \quad \mathcal{C}_4 = \begin{pmatrix} O(s^{\gamma_1-\gamma_2}\tau(s)) \\ O(s^{\gamma_1+\gamma_2}\tau(s)) \\ O(s^{2d+2}\tau(s)) \\ O(s^{\gamma_1+2d+2}) \end{pmatrix}$$

and in any case, at $+\infty$

$$\mathcal{C}_2 \sim_{+\infty} \frac{1}{-16n\sqrt{2}} \begin{pmatrix} 4nJ_- \\ 4nJ_+ \\ -4\sqrt{2}s^n \\ -4\sqrt{2}s^{-n} \end{pmatrix} \quad \textit{and} \quad \mathcal{C}_4 \sim_{+\infty} \frac{1}{-16n\sqrt{2}} \begin{pmatrix} 4nJ_- \\ 4nJ_+ \\ 4\sqrt{2}s^n \\ 4\sqrt{2}s^{-n} \end{pmatrix}$$

where $-16n\sqrt{2}$ is the determinant of $R(s)$.

Proof $R(s)$ is chosen as follows

$$R(s) \sim_{+\infty} \begin{pmatrix} J_+ & J_- & s^{-n} & s^n \\ s(J_+)' & s(J_-)' & -ns^{-n} & ns^n \\ J_+ & J_- & -s^{-n} & -s^n \\ s(J_+)' & s(J_-)' & ns^{-n} & -ns^n \end{pmatrix}$$

where, as usual, the notation J_+ stands for $\frac{e^{\sqrt{2}s}}{\sqrt{s}}$ and the notation J_- stands for $\frac{e^{-\sqrt{2}s}}{\sqrt{s}}$. To give the behaviors at 0, we return to Theorem 1.4. We have, for some $c_i \neq 0$, $i = 1, \dots, 4$

$$\text{If } (d, \gamma_1, \gamma_2) \in \mathcal{D}_1, \quad R(s) \sim_0 \begin{pmatrix} O(s^{\gamma_2+2d+2}) & O(s^{\tilde{\gamma}_1}) & c_3 s^{\gamma_1} & c_4 s^{-\gamma_1} \\ O(s^{\gamma_2+2d+2}) & O(s^{\tilde{\gamma}_1}) & c_3\gamma_1 s^{\gamma_1} & -c_4\gamma_1 s^{-\gamma_1} \\ c_1 s^{\gamma_2} & c_2 s^{-\gamma_2} & O(s^{\gamma_1+2d+2}) & O(s^{\tilde{\gamma}_2}) \\ c_1\gamma_2 s^{\gamma_2} & -c_2\gamma_2 s^{-\gamma_2} & O(s^{\gamma_1+2d+2}) & O(s^{\tilde{\gamma}_2}) \end{pmatrix}$$

where we use the notation

$$\tilde{\gamma}_1 = \min\{\gamma_1, -\gamma_2 + 2d + 2\} \text{ and } \tilde{\gamma}_2 = \min\{\gamma_2, -\gamma_1 + 2d + 2\} \quad \text{if } \gamma_1 + \gamma_2 - 2d - 2 \neq 0$$

(if $\gamma_1 + \gamma_2 - 2d - 2 = 0$, we have to replace $O(s^{\tilde{\gamma}_1})$ by $O(s^{\gamma_1} \log s)$ and $O(s^{\tilde{\gamma}_2})$ by $O(s^{\gamma_2} \log s)$) and

$$\text{If } (d, \gamma_1, \gamma_2) \in \mathcal{D}_2, \quad R(s) \sim_0 \begin{pmatrix} O(s^{\gamma_2+2d+2}) & O(s^{-\gamma_2+2d+2}) & c_3 s^{\gamma_1} & c_4 \tau(s) \\ O(s^{\gamma_2+2d+2}) & O(s^{-\gamma_2+2d+2}) & c_3\gamma_1 s^{\gamma_1} & -c_4 s\tau'(s) \\ c_1 s^{\gamma_2} & c_2 s^{-\gamma_2} & O(s^{\gamma_1+2d+2}) & O(\tau(s)s^{2d+2}) \\ c_1\gamma_2 s^{\gamma_2} & -c_2\gamma_2 s^{-\gamma_2} & O(s^{\gamma_1+2d+2}) & O(\tau(s)s^{2d+2}) \end{pmatrix}$$

where

$$\tau(s) = \begin{cases} \frac{s^{-\gamma_1} - s^{\gamma_1}}{2\gamma_1} \text{ if } \gamma_1 \neq 0 \\ -\log s \text{ if } \gamma_1 = 0 \end{cases}$$

The determinant W of $R(s)$ is independent of s, due to the fact that the matrix M of the differential system has a null trace. Moreover, $J_+ J_- = \frac{1}{s}$. Using the behavior at $+\infty$ of $R(s)$, given above, we deduce that W is the principal term, as $s \to +\infty$ of

$$\frac{1}{s} \begin{vmatrix} 1 & 1 & 1 & 1 \\ s\sqrt{2} & -s\sqrt{2} & -n & n \\ 1 & 1 & -1 & -1 \\ s\sqrt{2} & -s\sqrt{2} & n & -n \end{vmatrix}$$

that is

$$W = -16n\sqrt{2}.$$

A direct calculation of the suitable determinants gives the estimate of $\mathcal{C}_2$ and $\mathcal{C}_4$.

The proof of Theorem 1.10 completed.

Let $m = \lim_{\varepsilon \to 0} m_{\gamma_1,\gamma_2}(\varepsilon)$. We can define $\omega_\varepsilon \in \mathcal{H}_{\gamma_1}$ an eigenvector associated to $m_{\gamma_1,\gamma_2}(\varepsilon)$ and $\omega_0 = (a_0, b_0)$ such that $\tilde{\omega}_\varepsilon \to \omega_0$ on each compact subset of $[0, +\infty[$. In what follows, let us suppose that $m < 1$. Then $\frac{\gamma_1^2+\gamma_2^2}{2} - md^2 > 0$.
Since $a_0 \geq -b_0 \geq 0$, the possible behaviors at $+\infty$ for (a_0, b_0) are $(r^{-n_0}, -r^{-n_0})$ and $(r^{n_0}, -r^{n_0})$ where

$$n_0 = \sqrt{\frac{\gamma_1^2 + \gamma_2^2}{2} - md^2}. \tag{4.34}$$

Since $m < 1$, we have by Theorem 1.9 (i), that ω_0 has a bounded behavior at $+\infty$ and consequently

$$(a_0, b_0) \sim_{+\infty} (r^{-n_0}, -r^{-n_0}) \quad \text{and} \quad a_0 + b_0 = O(r^{-n_0-2}) \quad \text{at } +\infty.$$

At 0, in view of $a_0 \geq -b_0 \geq 0$, the only possible behavior is

$$(a_0, b_0) \sim_0 (cr^{\gamma_1}, O(r^{\gamma_1+2d+2})), \qquad \text{for some } c > 0.$$

Let us prove that the hypothesis $m < 1$ leads to a contradiction.
Since $n = \sqrt{\frac{\gamma_1^2+\gamma_2^2}{2} - d^2}$, we have, by (4.34)

$$(m < 1) \Leftrightarrow (n_0 > n).$$

Let us denote $X_0 = (a_0, ra_0', b_0, rb_0')^t$, the vector corresponding to ω_0. We have

$$X_0' = MX_0 - (m-1)(1 - f_d^2)(0, ra_0, 0, rb_0)^{\mathrm{t}}.$$

let us define X_1, X_2, X_3 and X_4 as in Lemma 4.1. We are going to prove that there exist some constants C_i such that

$$X_0 = \sum_{i=1}^{4} C_i X_i - (m-1) \sum_{i=1}^{4} \hat{X}_i,$$

with

$$\hat{X}_i \text{ bounded at } 0,\ i = 1, 2, 3, 4 \tag{4.35}$$

and

$$\text{at } +\infty \begin{cases} \hat{X}_1 = X_1 O(r^{-n_0-3} J_-) \quad ; \quad \hat{X}_2 = X_2 O(r^{-n_0-3} J_+) \\ \hat{X}_3 = X_3 O(1) \quad ; \quad \hat{X}_4 = X_4 O(1). \end{cases} \tag{4.36}$$

In order to prove (4.35) and (4.36), we write

$$X_0 = \sum_{i=1}^{4} A_i(r) X_i \tag{4.37}$$

with

$$i = 1, \ldots, 4, \quad A_i(r) = A_i - (m-1) \int_1^r [R^{-1}(s) s (1 - f_d^2) \begin{pmatrix} 0 \\ a_0 \\ 0 \\ b_0 \end{pmatrix} ds]_i \tag{4.38}$$

where the notation $[\ ]_i$ means the i^{th} line of the vector, and where A_i is a constant.
Let us examine the behavior of each term $A_i(r)X_i$ at $+\infty$ and at 0, using Lemma 4.1.
For the first term, we use the first terms of $\mathcal{C}_2$ and $\mathcal{C}_4$, given in Lemma 4.1, to obtain

$$[R^{-1}(s)s(1-f_d^2)\begin{pmatrix} 0 \\ a_0 \\ 0 \\ b_0 \end{pmatrix}]_1 \sim_{+\infty} O(\frac{1}{s}J_-(a_0+b_0))$$

$$\text{and } \sim_0 \left\{ \begin{array}{c} s(O(s^{\gamma_1}a_0 + O(s^{-\gamma_2}b_0)) \text{ if } (d,\gamma_1,\gamma_2) \in \mathcal{D}_1, \gamma_1+\gamma_2-2d-2<0 \\ s(O(s^{-\gamma_2+2d+2}a_0 + O(s^{-\gamma_2}b_0)) \text{ if } (d,\gamma_1,\gamma_2) \in \mathcal{D}_1, \gamma_1+\gamma_2-2d-2>0 \\ s(O(\tau(s)s^{\gamma_1-\gamma_2+2d+2}a_0 + O(\tau(s)s^{\gamma_1-\gamma_2}b_0)) \text{ if } (d,\gamma_1,\gamma_2) \in \mathcal{D}_2. \end{array} \right.$$

Let us define

$$B_1 = -(m-1)\int_1^{+\infty}[R^{-1}(s)s(1-f_d^2)\begin{pmatrix} 0 \\ a_0 \\ 0 \\ b_0 \end{pmatrix}]_1 ds \quad \text{and } \hat{X}_1 = X_1\int_{+\infty}^{r}[R^{-1}(s)s(1-f_d^2)\begin{pmatrix} 0 \\ a_0 \\ 0 \\ b_0 \end{pmatrix}]_1 ds$$

We can write

$$A_1(r)X_1 = (A_1+B_1)X_1 - (m-1)\hat{X}_1$$

We see that $\hat{X}_1 = X_1 O(1)$ at 0. Using (2.29), we get $\hat{X}_1 = X_1 O(r^{-n_0-3}J_-)$ at $+\infty$.

For the second term, we obtain

$$[R^{-1}(s)s(1-f_d^2)\begin{pmatrix} 0 \\ a_0 \\ 0 \\ b_0 \end{pmatrix}]_2 \sim_{+\infty} O(\frac{1}{s}J_+(a_0+b_0))$$

$$\text{and } \sim_0 \left\{ \begin{array}{c} s(O(s^{\gamma_1+2\gamma_2}a_0) + O(s^{\gamma_2}b_0)) \text{ if } (d,\gamma_1,\gamma_2) \in \mathcal{D}_1 \text{ and } \gamma_1+\gamma_2-2d-2<0 \\ s(O(s^{\gamma_2+2d+2}a_0) + O(s^{\gamma_2}b_0)) \text{ if } (d,\gamma_1,\gamma_2) \in \mathcal{D}_1 \text{ and } \gamma_1+\gamma_2-2d-2>0 \\ s\tau(s)(O(s^{\gamma_1+\gamma_2+2d+2})a_0 + O(s^{\gamma_1+\gamma_2})b_0) \text{ if } (d,\gamma_1,\gamma_2) \in \mathcal{D}_2 \end{array} \right.$$

Denoting

$$B_2 = -(m-1)\int_1^{0}[R^{-1}(s)s(1-f_d^2)\begin{pmatrix} 0 \\ a_0 \\ 0 \\ b_0 \end{pmatrix}]_2 ds \quad \text{and } \hat{X}_2 = X_2\int_{0}^{r}[R^{-1}(s)s(1-f_d^2)\begin{pmatrix} 0 \\ a_0 \\ 0 \\ b_0 \end{pmatrix}]_2 ds$$

we get

$$A_2(r)X_2 = (A_2+B_2)X_2 - (m-1)\hat{X}_2$$

with, by (2.30)

$$\hat{X}_2 = X_2 O(r^{-n_0-3}J_+) \text{ at } +\infty\ .$$

Moreover, $\hat{X}_2$ is bounded at 0.

For the third term, we obtain

$$[R^{-1}(s)s(1-f_d^2)\begin{pmatrix}0\\a_0\\0\\b_0\end{pmatrix}]_3 \sim_{+\infty} \frac{-1}{16n\sqrt{2}}\frac{4\sqrt{2}d^2}{s}s^n(-a_0+b_0). \tag{4.39}$$

Since $-a_0+b_0 \sim_{+\infty} -2r^{-n_0}$, then this term is integrable at $+\infty$. At 0, it is

$$\sim_0 \begin{cases} s(O(s^{-\gamma_1}a_0)+O(s^{\gamma_2}b_0)) \text{ if } (d,\gamma_1,\gamma_2)\in\mathcal{D}_1 \text{ and } \gamma_1+\gamma_2-2d-2<0 \\ s(O(s^{-\gamma_1}a_0)+O(s^{-\gamma_1+2d+2}b_0)) \text{ if } (d,\gamma_1,\gamma_2)\in\mathcal{D}_1 \text{ and } \gamma_1+\gamma_2-2d-2>0 \\ s(O(\tau(s)a_0)+O(\tau(s)s^{2d+2}b_0)) \text{ if } (d,\gamma_1,\gamma_2)\in\mathcal{D}_2 \end{cases}$$

and this is bounded at 0.
Letting

$$B_3 = -(m-1)\int_1^0 [R^{-1}(s)s(1-f_d^2)\begin{pmatrix}0\\a_0\\0\\b_0\end{pmatrix}]_3 ds \text{ and } \hat{X}_3 = X_3\int_0^r [R^{-1}(s)s(1-f_d^2)\begin{pmatrix}0\\a_0\\0\\b_0\end{pmatrix}]_3 ds,$$

we find

$$A_3(r)X_3 = (A_3+B_3)X_3-(m-1)\hat{X}_3$$

with

$$\hat{X}_3 = X_3 O(1) \text{ at } +\infty$$

and $\hat{X}_3$ is bounded at 0.
For the fourth term,

$$[R^{-1}(s)s(1-f_d^2)\begin{pmatrix}0\\a_0\\0\\b_0\end{pmatrix}]_4 \sim_{+\infty} \frac{-1}{16n\sqrt{2}}\frac{4d^2\sqrt{2}}{s}s^{-n}(-a_0+b_0)$$

and

$$\sim_0 \begin{cases} s(O(s^{\gamma_1}a_0)+O(s^{2\gamma_1+\gamma_2}b_0)) \text{ if } (d,\gamma_1,\gamma_2)\in\mathcal{D}_1 \text{ and } \gamma_1+\gamma_2-2d-2<0 \\ s(O(s^{\gamma_1}a_0)+O(s^{\gamma_1+2d+2}b_0)) \text{ if } (d,\gamma_1,\gamma_2)\in\mathcal{D}_1 \text{ and } \gamma_1+\gamma_2-2d-2>0 \\ s\tau(s)(O(s^{\gamma_1})a_0+O(s^{\gamma_1+2d+2})b_0) \text{ if } (d,\gamma_1,\gamma_2)\in\mathcal{D}_2 \end{cases}$$

Letting

$$B_4 = -(m-1)\int_1^0 [R^{-1}(s)s(1-f_d^2)\begin{pmatrix}0\\a_0\\0\\b_0\end{pmatrix}]_4 ds \text{ and } \hat{X}_4 = X_4\int_0^r [R^{-1}(s)s(1-f_d^2)\begin{pmatrix}0\\a_0\\0\\b_0\end{pmatrix}]_4 ds$$

we find

$$A_4(r)X_4 = (A_4+B_4)X_4-(m-1)\hat{X}_4.$$

Then $\hat{X}_4 = X_4 O(1)$ at $+\infty$ and $\hat{X}_4$ is bounded at 0.
Now, summing the four terms, and letting $C_i = A_i + B_i$, we find (4.35) and (4.36).
Since X_0 is bounded at 0, we have $C_2 = C_4 = 0$.
But X_0 is bounded at $+\infty$ and $\hat{X}_i$ is bounded at $+\infty$, $i = 1, 2, 3$. Since we have also $a_1 >> \hat{a}_4$ at $+\infty$, we infer that $C_1 = 0$ and that $\hat{X}_4$ must be bounded at $+\infty$. Returning to the definition of $\hat{X}_4$, we must have

$$\int_0^{+\infty} [R^{-1}(s)s(1-f_d^2)\begin{pmatrix} 0 \\ a_0 \\ 0 \\ b_0 \end{pmatrix}]_4 ds = 0,$$

therefore

$$\hat{X}_4 = X_4 \int_{+\infty}^{r} [R^{-1}(s)s(1-f_d^2)\begin{pmatrix} 0 \\ a_0 \\ 0 \\ b_0 \end{pmatrix}]_4 ds,$$

that gives

$$\hat{X}_4 = X_4 \int_{+\infty}^{r} s(1-f_d^2)[a_0\mathcal{C}_2 + b_0\mathcal{C}_4]_4 \sim_{+\infty} X_4 \int_{+\infty}^{r} \frac{-8\sqrt{2}}{-16n\sqrt{2}} s^{-n_0-n}\frac{d^2}{s} ds.$$

Thus,

$$\text{at } +\infty \quad \hat{a}_4 = a_4 \frac{-1}{16n\sqrt{2}} \frac{8d^2\sqrt{2}}{n+n_0} r^{-n} a_0 + o(r^{-n_0}). \tag{4.40}$$

Since we have now

$$X_0 = C_3 X_3 - (m-1)\sum_{i=1}^{4} \hat{X}_i$$

and since $\hat{a}_1 = O(r^{-n_0-4})$ and $\hat{a}_2 = O(r^{-n_0-4})$, then $\hat{a}_1 = o(a_0)$ and $\hat{a}_2 = o(a_0)$ at $+\infty$.
Consequently

$$a_0 + (m-1)\hat{a}_4 \sim_{+\infty} C_3 a_3 - (m-1)\hat{a}_3 \tag{4.41}$$

Recalling (4.40) and recalling $n < n_0$, this implies that

$$C_3 - (m-1)\int_0^{+\infty} [R^{-1}(s)s(1-f_d^2)\begin{pmatrix} 0 \\ a_0 \\ 0 \\ b_0 \end{pmatrix}]_3 ds = 0$$

and then

$$C_3 a_3 - (m-1)\hat{a}_3 = -(m-1)a_3 \int_{+\infty}^{r} [R^{-1}(s)s(1-f_d^2)\begin{pmatrix} 0 \\ a_0 \\ 0 \\ b_0 \end{pmatrix}]_3 ds.$$

Using (4.39), we get

$$\text{at } +\infty \quad C_3 a_3 - (m-1)\hat{a}_3 = -(m-1)a_3 \frac{-1}{16\sqrt{2}} \frac{-8d^2\sqrt{2}}{n(n-n_0)} r^{n-n_0} + o(r^{-n_0}). \tag{4.42}$$

Finally, we sum (4.40) and (4.42) to get, by (4.41)

$$a_0(1+(m-1)\frac{-8d^2}{16n}\frac{1}{n+n_0}) \sim_{+\infty} -(m-1)\frac{8d^2}{16n}(\frac{1}{n-n_0})r^{-n_0}$$

and thus

$$(m-1)\frac{8d^2}{16n}(\frac{-1}{n-n_0}+\frac{1}{n+n_0}) = 1.$$

But we have by (4.34)

$$n_0^2 - n^2 = (-m+1)d^2.$$

After simplification by $m-1$, we get $n_0 = n$, that gives $m = 1$, that is in contradiction with the hypothesis $m < 1$. So we deduce that $m = 1$.
The proof of (4.35) and (4.36) for $(d, \gamma_1, \gamma_2) \in \mathcal{D}_1$ and $\gamma_1 + \gamma_2 - 2d - 2 = 0$ is left to the reader.
Proof of Corollary 1.1.
By Theorem 1.9 (ii), if there exists a bounded solution ω, then there exists some eigenvalue tending to 1. So $m = 1$. It remains to prove that $\omega_0 = c\omega$, for some $c \neq 0$. But ω cannot have the least behavior at 0, otherwise it would blow up exponentially at $+\infty$. So, there exists $c \neq 0$ such that $\omega \sim_0 c\omega_0$. If $\omega \neq c\omega_0$, then $\omega - c\omega_0$ has the least behavior at 0, and consequently blows up exponentially at $+\infty$. This cannot be true, because ω is bounded at $+\infty$ and, since $a_0 \geq b_0 \geq 0$, the possible blowing up behavior at $+\infty$ for ω_0 can only be polynomial. We can conclude that $\omega = c\omega_0$.

5 The case $n \geq d+1$: the proof of Theorem 1.12.

Let $\omega_1 = (a_1, b_1)$ be the solution defined in Theorem 1.4 and $\eta_2 = (u_2, v_2)$ be the solution defined in Theorem 1.5. According to Theorem 1.6, $\omega_1 \sim_{+\infty} (J_+, J_+)$ and η_2 has the greater blowing up behavior at 0. Let η_3 and η_4 be defined in Theorem 1.5 and having the intermediate behaviors at $+\infty$. Let $\omega_3 = (a_3, b_3)$ be defined in Theorem 1.4. With these definitions, we can write

$$\omega_3 = C_1(n,d)\omega_1 + C_2(n,d)\eta_2 + C_3(n,d)\eta_3 + C_4(n,d)\eta_4.$$

Let us remark that ω_1 and $\omega_3 - C_1(n,d)\omega_1$ form a base of the bounded solutions at 0, and that $\omega_3 - C_1(n,d)\omega_1 = o(\omega_1)$ at $+\infty$. So the problem of the existence of some bounded solution is reduced to the problem $C_3(n,d) = 0$.
Supposing that there exists a bounded solution for (n_0, d_0), $d_0 > 1$, $n_0 \geq d_0 + 1$, we have, by Theorem 1.1, $n_0 \leq 2d_0 - 1$. From now on, (n,d) is such that $1 \leq d \leq d_0 + 1$ and $d \leq n \leq 2d$. Clearly, $(d, |n-d|, n+d)$ stays in a compact subset of $\mathcal{D}$. This is sufficient for the solutions η_3 and η_4 to be defined without ambiguity. The real numbers $C_i(n,d)$ defined above can be computed by means of determinants involving the four components $(a, a', b, b')(r)$ of the five solutions present, for a given $r > 0$. Thus, C_i is continuous wrt (d, γ_1, γ_2) and consequently is continuous wrt (d, n). C_i is also differentiable wrt γ_1 and wrt γ_2 and therefore wrt n, since $n \geq d$.

Lemma 5.2 *With the notation above, if there exists* (n_0, d_0)*,* $d_0 \geq 1$*,* $n_0 \geq d_0 + 1$ *such that* $C_3(n_0, d_0) = 0$*, then there exists a continuous map* $d \mapsto n(d)$*, defined for* $d < d_0$*, closed to* d_0 *and verifying* $C_3(n(d), d) = 0$*.*

Proof Let us prove that $\frac{\partial C_3}{\partial n}(n_0, d_0) \neq 0$. If $\frac{\partial C_3}{\partial n}(n_0, d_0) = 0$, then $\frac{\partial}{\partial n}(\omega_3 - C_1(n, d)\omega_1)(n_0, d_0)$ is bounded at $+\infty$. Let us denote $(a, b) = \omega_3 - C_1(n, d)\omega_1$. Then (a, b) verifies the system (1.1), with (n_0, d_0) in place of (n, d), and $(\frac{\partial a}{\partial n}, \frac{\partial b}{\partial n})(n_0, d_0)$ verifies also a system, obtained by differentiation wrt n, at (n_0, d_0), that is

$$\left\{ \begin{array}{ll} \frac{\partial a}{\partial n}'' + \frac{1}{r}\frac{\partial a}{\partial n}' - \frac{(n-d)^2}{r^2}\frac{\partial a}{\partial n} - 2\frac{(n-d)}{r^2}a - f_d^2\frac{\partial b}{\partial n} & = -(1 - 2f_d^2)\frac{\partial a}{\partial n} \\ \frac{\partial b}{\partial n}'' + \frac{1}{r}\frac{\partial b}{\partial n}' - \frac{(n+d)^2}{r^2}\frac{\partial b}{\partial n} - 2\frac{(n+d)}{r^2}b - f_d^2\frac{\partial a}{\partial n} & = -(1 - 2f_d^2)\frac{\partial b}{\partial n} \end{array} \right. \tag{5.43}$$

By combining the systems (1.1) and (5.43), for (n_0, d_0), an integration by parts gives

$$\int_0^{+\infty} -2\frac{n_0 - d_0}{r}a^2 - 2\frac{n_0 + d_0}{r}b^2 dr = 0$$

and we conclude that $a = b = 0$, that is false.
So, we have proved that $\frac{\partial C_3}{\partial n}(n_0, d_0) \neq 0$. The Implicit Functions Theorem gives a continuous map $d \mapsto n(d)$ such that $C_3(n(d), d) = 0$, and defined in a neighborhood of d_0, with values in a neighborhood of n_0.
The proof of Theorem 1.12 completed.
With the definitions given above, let us define the set

$$\mathcal{E} = \{d \geq 1; \quad d \leq d_0 + 1; \quad \exists n \geq d + \frac{1}{2}, \quad C_3(n, d) = 0\}.$$

If $d \in \mathcal{E}$, then $n \leq 2d - 1$, by Theorem 1.1. Thus, $\mathcal{E}$ is a closed subset of $[1, +\infty[$, thanks to the continuity of C_3 wrt (n, d). Since $d_0 \in \mathcal{E}$, $\mathcal{E} \neq \emptyset$ and we let $d_1 = \inf \mathcal{E}$. Given that $d_1 \in \mathcal{E}$, there exists $n_1 \geq d_1 + \frac{1}{2}$ such that $C_3(n_1, d_1) = 0$. According to Theorem 1.11, $n_1 \geq d_1 + 1$. If $d_1 > 1$, we deduce from Lemma 5.2 that there exists $d < d_1$, sufficiently closed to d_1 in order to have $n(d) > d_1 + \frac{1}{2}$. Therefore $n(d) \geq d + \frac{1}{2}$, which is in contradiction with $d_1 = \inf \mathcal{E}$. This proves that $d_1 = 1$. But $1 \notin \mathcal{E}$, by Theorem 1.1. This contradiction proves the non existence of (n_0, d_0) such that $n_0 \geq d_0 + 1$ and $C_3(n_0, d_0) = 0$.
The proof of Theorem 1.12 is complete.

6 The proof of Theorem 1.9 (i) and of Theorem 1.8.

Proof of Theorem 1.9 (i). Let us define $n_0 = \sqrt{\frac{\gamma_2^2 + \gamma_1^2}{2} - \mu d^2}$.
Let $\omega_\varepsilon = (a_\varepsilon, b_\varepsilon) \in \mathcal{H}_{\gamma_1}$ be an eigenvector associated to $\mu(\varepsilon)$. Using (1.15), we write

$$\frac{\mu(\varepsilon)}{\varepsilon^2}\int_0^1 r(1 - f^2)(a_\varepsilon^2 + b_\varepsilon^2)dr = \int_0^1 (ra_\varepsilon'^2 + rb_\varepsilon'^2 + \frac{\gamma_1^2}{r}\, a_\varepsilon^2 + \frac{\gamma_2^2}{r}\, b_\varepsilon^2 + \frac{r}{\varepsilon^2}f^2(a_\varepsilon + b_\varepsilon)^2)dr.$$

We use the definition (1.17) of $m_0(\varepsilon)$ to get

$$\frac{\mu(\varepsilon)}{\varepsilon^2}\int_0^1 r(1 - f^2)(a_\varepsilon^2 + b_\varepsilon^2)dr$$

$$\geq \frac{m_0(\varepsilon)}{\varepsilon^2}\int_0^1 r(1-f^2)(a_\varepsilon^2+b_\varepsilon^2)dr + \int_0^1 (\frac{\gamma_1^2-d^2}{r}a_\varepsilon^2 + \frac{\gamma_2^2-d^2}{r}b_\varepsilon^2 + \frac{r}{\varepsilon^2}f^2(a_\varepsilon+b_\varepsilon)^2)dr.$$

Now, we use the trick of TC Lin (see [7]). Letting $\tilde{b}_\varepsilon = \tau\tilde{a}_\varepsilon$, we consider the map

$$H : \tau \mapsto \frac{\gamma_1^2-d^2}{r} + \frac{\gamma_2^2-d^2}{r}\tau^2 + r{f_d}^2(1+\tau)^2 \tag{6.44}$$

and we minimize this map. The minimum is attained for τ_0 verifying

$$\tau_0(\frac{\gamma_2^2-d^2}{r} + rf_d^2) + rf_d^2 = 0 \quad \text{and} \quad 1+\tau_0 = \frac{\gamma_2^2-d^2}{r} / (\frac{\gamma_2^2-d^2}{r} + rf_d^2)$$

and consequently

$$H(\tau_0) = \frac{\gamma_1^2-d^2}{r} + (\frac{rf_d^2}{\frac{\gamma_2^2-d^2}{r}+rf_d^2})^2(\frac{\gamma_2^2-d^2}{r}) + rf_d^2(\frac{\frac{\gamma_2^2-d^2}{r}}{\frac{\gamma_2^2-d^2}{r}+rf_d^2})^2.$$

We have

$$H(\tau_0) \sim_{r\to+\infty} (\gamma_1^2+\gamma_2^2-2d^2)/r. \quad \text{Moreover, for all } \tau > 0, \quad H(\tau) \geq H(\tau_0).$$

Since $\frac{\gamma_1^2+\gamma_2^2}{2} - d^2 > 0$, there exists some constants $C_1 > 0$ and $R_0 > 0$, independent of τ, such that for all $\tau > 0$

$$H(\tau) \geq \frac{C_1}{r} \text{ for all } r > R_0.$$

Then, for all $R > R_0$ and all $\varepsilon < \frac{1}{R}$, we write

$$\int_0^{\frac{1}{\varepsilon}} H(r)\tilde{a}_\varepsilon^2(r)dr \geq \int_0^{R_0} H(r)\tilde{a}_\varepsilon^2(r)dr + \int_{R_0}^R H(r)\tilde{a}_\varepsilon^2(r)dr.$$

Now a_0 blows up exponentially at $+\infty$, or as r^{n_0}. We can choose R_0 large enough and a constant $C_2 > 0$ to have also

$$a_0^2(r) \geq C_2(\frac{e^{\sqrt{2}r}}{\sqrt{r}})^2 \text{ or } C_2 r^{2n_0} \text{ for all } r > R_0.$$

Since $\tilde{a}_\varepsilon \to a_0$ as $\varepsilon \to 0$, uniformly in $[0, R_0]$, we can chose ε_0 such that for all $\varepsilon < \varepsilon_0$

$$\int_0^{R_0} H(r)\tilde{a}_\varepsilon^2(r)dr \geq \frac{1}{2}\int_0^{R_0} H(r)a_0^2(r)dr.$$

Moreover, for all $R > R_0$, $\tilde{a}_\varepsilon \to a_0$ as $\varepsilon \to 0$, uniformly in $[R_0, R]$. Then, there exists $\varepsilon(R)$ such that for all $\varepsilon < \varepsilon(R)$ we have

$$\int_{R_0}^R H(r)\tilde{a}_\varepsilon^2(r)dr \geq \frac{C_2}{2}\int_{R_0}^R \frac{1}{r}r^{2n_0}dr \quad \text{or} \quad \int_{R_0}^R H(r)\tilde{a}_\varepsilon^2(r)dr \geq \frac{C_2}{2}\int_{R_0}^R \frac{1}{r}(\frac{e^{\sqrt{2}r}}{\sqrt{r}})^2 dr.$$

And finally, for $\varepsilon < \varepsilon(R)$, we have

$$(\frac{\mu(\varepsilon)-m_0(\varepsilon)}{\varepsilon^2})\int_0^1 r(1-f^2)(a_\varepsilon^2+b_\varepsilon^2)dr \geq \frac{1}{2}\int_0^{R_0} H(r)a_0^2(r)dr +$$

$$+\begin{cases} \frac{C_1C_2}{2}\int_{R_0}^{R}\frac{1}{r}r^{2n_0}, & \text{if } (a_0,b_0)\sim_{+\infty}(r^{n_0},-r^{n_0}) \\ \frac{C_1C_2}{2}\int_{R_0}^{R}\frac{1}{r}(\frac{e^{\sqrt{2}r}}{\sqrt{r}})^2dr, & \text{if } (a_0,b_0)\sim_{+\infty}(J^+,J^+) \end{cases}$$

where C_1 and C_2, given above, are independent of R and ε. But we can choose R such that the lhs is positive.

We deduce that $\mu(\varepsilon)-m_0(\varepsilon)>0$. Then we use Theorem 1.7 (i), that gives $\frac{m_0(\varepsilon)-1}{\varepsilon^2}\geq C$ and consequently $\frac{\mu(\varepsilon)-1}{\varepsilon^2}\geq C$. The lemma is proved.

The proof of Theorem 1.8. The proof for $n=2$ and $d=2$ is originally in [9].

For real numbers $d\geq 1$ and $n\geq 1$, let $x=\frac{f_d'}{r^{n-1}}$ and $y=d\frac{f_d}{r^n}$. A calculation gives

$$\begin{cases} -(rx')'+\frac{\gamma^2}{r}x-\frac{\xi^2}{r}y-r(1-3f_d^2)x &= -2\frac{n-1}{r^{n-1}}f_d(1-f_d^2) \\ -(ry')'+\frac{\gamma^2}{r}y-\frac{\xi^2}{r}x-r(1-f_d^2)y &= 0 \end{cases} \tag{6.45}$$

For $a=\frac{x+y}{2}$ and $b=\frac{x-y}{2}$, we deduce that

$$\begin{cases} -(ra')'+\frac{\gamma_1^2}{r}a+f_d^2b-r(1-2f_d^2)a &= -\frac{n-1}{r^{n-1}}f_d(1-f_d^2) \\ -(rb')'+\frac{\gamma_2^2}{r}b+f_d^2a-r(1-2f_d^2)b &= -\frac{n-1}{r^{n-1}}f_d(1-f_d^2) \end{cases} \tag{6.46}$$

where, as usual, $\gamma_1=|n-d|$, $\gamma_2=n+d$, $\gamma^2=\frac{\gamma_1^2+\gamma_2^2}{2}$ and $\xi^2=\frac{\gamma_2^2-\gamma_1^2}{2}$.

We verify that

$$x\sim_0 y\sim_0 dr^{d-n}+O(r^{d-n+2}) \text{ and, at } +\infty,\ x=O(r^{-n}),\quad y=O(r^{-n}),$$

and consequently that

$$a\sim_0 2dr^{d-n}+O(r^{d-n+2}) \text{ and } b\sim_0 O(r^{d-n+2}).$$

let us suppose that $d\geq 1$ and that $1<n<d+1$. We can multiply the system (6.46) and integrate by parts. We obtain that

$$\int_0^{+\infty}(ra'^2+rb'^2+\frac{\gamma_1^2}{r}a^2+\frac{\gamma_2^2}{r}b^2+rf_d^2(a+b)^2-r(1-f_d^2)(a^2+b^2))dr$$
$$=\int_0^{+\infty}\frac{-(n-1)}{r^{n-1}}f_d(1-f_d^2)(a+b)dr$$

This gives

$$\frac{\int_0^{+\infty}(ra'^2+rb'^2+\frac{\gamma_1^2}{r}a^2+\frac{\gamma_2^2}{r}b^2+rf_d^2(a+b)^2)dr}{\int_0^{+\infty}r(1-f_d^2)(a^2+b^2)dr}=1-C_n$$

with

$$C_n=\frac{\int_0^{+\infty}\frac{(n-1)}{r^{n-1}}f_d(1-f_d^2)(a+b)dr}{\int_0^{+\infty}r(1-f_d^2)(a^2+b^2)dr}>0.$$

Now we use an approximation argument, valid as soon as $n>0$. For example for a given constant $0<N<1$ we define

$(a_\varepsilon, b_\varepsilon)(r) = \left\{ \begin{array}{c} (a,b)(\frac{r}{\varepsilon}) \text{ in } [0,N] \\ = (a(r)\frac{(1-r)^2}{(1-N)^2}, b(r)\frac{(1-r)^2}{(1-N)^2}) \text{ in } [N,1] \end{array} \right.$. We have that $(a_\varepsilon, b_\varepsilon) \in \mathcal{H}_{|n-d|}$ and that

$$\frac{\int_0^1 (ra_\varepsilon'^2 + rb_\varepsilon'^2 + \frac{\gamma_1^2}{r}a_\varepsilon^2 + \frac{\gamma_2^2}{r}b_\varepsilon^2 + r\frac{1}{\varepsilon^2}f^2(a_\varepsilon + b_\varepsilon)^2)dr}{\frac{1}{\varepsilon^2}\int_0^1 r(1-f^2)(a_\varepsilon^2 + b_\varepsilon^2)dr}$$

$$= \frac{\int_0^{\frac{N}{\varepsilon}} (ra'^2 + rb'^2 + \frac{\gamma_1^2}{r}a^2 + \frac{\gamma_2^2}{r}b^2 + rf_d^2(a+b)^2)dr + O(\varepsilon^{2n})}{\int_0^{\frac{N}{\varepsilon}} r(1-f_d^2)(a^2+b^2)dr + O(\varepsilon^{2n})} \to 1 - C_n, \text{ as } \varepsilon \text{ tends to } 0.$$

We deduce that, if $1 < n < d+1$, $m_{d-n,d+n}(\varepsilon) < 1 - \frac{C_n}{2}$, for ε small enough and the proof of Theorem 1.8 is complete.

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