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Unscented Kalman Observer*

Assia Daid¹, Eric Busvelle², and Mohamed Aidene³

Abstract—The extended Kalman filter is an exponentially converging observer as soon as it is written in a canonical form of observability and in its high-gain form. It is shown that unlike extended Kalman filter, unscented Kalman filter can not be an exponentially converging observer. We propose a slight modification of the unscented Kalman filter to build an exponentially converging observer called unscented Kalman observer. Performances of this new observer are illustrated on an example of geolocation problem.

I. INTRODUCTION

In this paper, we consider the observation problem of a nonlinear dynamic system in continuous time, which have the following form

$$\begin{cases} \frac{dx(t)}{dt} = f(x(t), t) \\ y(t) = h(x(t), t) \end{cases} \quad (1)$$

$x(t) \in R^n$, $y(t) \in R^m$. The extended Kalman filter (EKF) is an important and widely used tool for the state estimation in such nonlinear dynamic systems (see [4], [12] in a deterministic setting and [18], [19] in the stochastic case, or [13] for a Matlab implementation including a complete discussion concerning EKF and its variants).

Generally, we distinguish two approaches for nonlinear state estimation. The first one is nonlinear filtering, which consists in studying the probability law of the state of a stochastic dynamic system conditionally to the observations. This is the usual setting for using EKF. The second approach is purely deterministic, it consists of constructing another dynamic system (the observer) that uses as input the available measurements and whose state converges asymptotically (and generally exponentially).

The extended Kalman filter is based on a linearization of the system along the estimated trajectory, the actual one being unknown, and using the equations of the linear Kalman filter. Although this biased linearization of the system prevents from analytically proving the observer's convergence for any initial error, such proofs exist when initial estimation errors are small enough (see [2], [7], [20], [21], [22]).

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However, first-order linearization can introduce significant errors when estimating the mean and covariance of the state. In addition, the derivation of Jacobian matrices is not easy in many applications (see, for instance, [16] and a way to work around the problem). This means that the EKF function is difficult to implement, to tune and its reliability is limited.

An other approach has been developed (introduced by Julier and Uhlmann in 1995, [15]) to improve the noise propagation through non-linearities of the system, called unscented Kalman filter (UKF). In fact, the only difference between EKF and UKF is how each filter handles mean and covariance propagation through the dynamic of the system. The UKF algorithm is based on the unscented transformation (UT) which is used for forming a Gaussian approximation of the joint distribution of gaussian random variables [23]. This transformation is briefly explain Section II-A.

Compared to extended Kalman filter, few works have been done to analyze the convergence of unscented Kalman filter. Some proof are established for nonlinear stochastic discrete-time systems (see [25], [17], [10]) and in the stochastic continuous-time case (see [14]). Despite its interest (due to the fact that it does not require a priori Jacobian computation), UKF convergence has not being studied in a deterministic setting. Two reasons may explain this neglecting : the first one is that UKF is an improvement of EKF only because it tackles gaussian noise more efficiently, the second one because without modification, it is not convergent as we will see in Section II-B.

A successful approach to study the exponential convergence of the EKF is to use the high gain theory (see [12], [5]). It can be proved that, provided that the nonlinear system is put into a normal forms of observability (which is always possible if the system is observable in some sense, see [12]), and if gain is large enough, EKF converges for any initial error. With this structure, EKF is called high-gain extended Kalman filter (HG-EKF) and it is an exponential observer. The high-gain is computed by a modification of the covariance matrices R and Q (which are interpreted as cost matrices in deterministic case), by the use of a fixed scalar parameter (usually denoted θ). The convergence is effective when θ is large enough (see for more explanations [4], [3], [12]).

Motivated by the use of the HG-EKF as a deterministic observer for nonlinear continuous time systems, and its good properties of convergence (both local and global), we study the convergence of an high-gain and non high-gain version of UKF for deterministic observer in continuous-time systems.

In Section II, we recall the unscented transformation, the unscented Kalman filter, and show that the unscented Kalman

filter will not converge (in a deterministic setting) as soon as the system is nonlinear.

In Section III, we propose a modification which avoid the bias of UKF and we call our observer the unscented Kalman observer. We build the high-gain version of the UKO (called high-gain unscented Kalman observer, HG-UKO) and we give some convergence results. In fact, HG-UKO is a local exponentially converging observer (see also [9]) and it becomes a global converging observer if we assume that the solution of the Riccati equation is bounded.

A simulation, very similar to the example presented in [11], has been done for the comparison of performance of each observers in Section IV.

II. CONTINUOUS-TIME UNSCENTED KALMAN FILTER

In this section, we will briefly present the unscented Kalman filter in matrix form (see [23]). This filter uses σ -points (denoted by X defined below) which capture the statistics of a Gaussian random variable. These σ -points prevent to compute the Jacobian matrices, which is very interesting, especially when non-linear functions are complicated functions (this is the case in the high-gain framework since these functions result from a non-linear change of coordinates, because of many Lie derivatives).

The UKF is based on the unscented transformation which is detailed in the following subsection.

A. Unscented transformation

The Unscented transformation is a non-linear method for calculating the statistics of a random variable, which undergoes a non linear transformation. The procedure to implement the UKF in continuous-time is summarized as follows [23], [24]

- 1) Choose $2n + 1$ σ -points:

$$X = [m \cdots m] + \sqrt{c} [0 \quad \sqrt{P} \quad -\sqrt{P}] \quad (2)$$

X is the matrix of σ -points, $c = \alpha^2(n+k)$, with $k \geq 0$, $\alpha \in (0, 1]$. c , k and α are setting parameters. Contrary to [23] where $\sqrt{P}$ is under defined (and arbitrarily chosen as the Cholesky factorization of P), in (2), the matrix square root $\sqrt{P}$ of the positive definite matrix P is the unique positive definite matrix such that $P = \sqrt{P}^2$. This point is required for the proof of convergence.

- 2) Compute the associated weights of the σ -points, $w_m = (W_m^0, W_m^1, \dots, W_m^{2n})^t$ where

$$W_m^{(0)} = \frac{\lambda}{n + \lambda},$$

$$W_m^{(i)} = \frac{1}{2(n + \lambda)}, \quad i = 1, \dots, 2n$$

and $w_c = (W_c^0, W_c^1, \dots, W_c^{2n})^t$ where

$$W_c^{(0)} = \frac{\lambda}{n + \lambda} + (1 - \alpha^2 + \beta),$$

$$W_c^{(i)} = \frac{1}{2(n + \lambda)}, \quad i = 1, \dots, 2n$$

λ is a scalar parameter defined by $\lambda = c - n$.

- 3) Convert each σ -points by the nonlinear transformation g

$$Y = g(X) = [g(X_0) \cdots g(X_{2n})] \quad (3)$$

where $X \in R^{n \times 2n+1}$, X_i denote the i^{th} column of X , $g : R^n \rightarrow R^m$, and (3) means that the i^{th} column Y_i of the matrix $Y \in R^{m \times 2n+1}$ is formed as follows:

$$Y_i = g(X_i)$$

- 4) The mean $E[g(X)]$ and covariance $\text{Cov}(g(X))$ of $g(X)$ are estimated respectively by

$$\mu_u = g(X)w_m = \sum_{i=0}^{2n} W_m^{(i)} g(X_i)$$

and

$$S_u = \sum_{i=0}^{2n} W_c^{(i)} (g(X_i) - m)(g(X_i) - m)^t$$

- 5) The cross-covariance of X and $g(X)$ is estimated as

$$C_u = \sum_{i=0}^{2n} W_c^{(i)} (X_i - m)(g(X_i) - \mu_u)^t \quad (4)$$

The matrix W is defined as

$$W = (I - [W_m \cdots W_m]) \times \text{diag}(W_c^0 \cdots W_c^{2n}) \times (I - [W_m \cdots W_m])^t \quad (5)$$

B. Algorithm of UKF in continuous-time

The equations corresponding to the UKF in the continuous-time are given by (see [23])

$$\begin{cases} K(t) = X(t)W h(X(t), t)^t R^{-1}(t) \\ \frac{dm(t)}{dt} = f(X(t), t)w_m + K(t)(y(t) - h(X(t), t)w_m) \\ \frac{dP(t)}{dt} = X(t)W f(X(t), t)^t + f(X(t), t)W X(t)^t \\ \quad + Q(t) - K(t)R(t)K(t)^t \end{cases} \quad (6)$$

In this algorithm $Q(t)$ and $R(t)$ represent covariance matrix of the state noise and covariance matrix of measurement noise respectively. They are both symmetric positive definite. In the deterministic case, these two matrices will be considered as tuning parameters (and usually independent from t). See [23], [24] for the transition from the unscented transformation to the UKF equations.

The unscented Kalman filter is not a good candidate as an exponential observer. Indeed, assuming that the function h is linear, and suppose $x(t) = 0$ is an equilibrium state. If $m(t) = 0$, one expect that the observer remains at equilibrium state 0 (since the innovation term $y(t) - h(X(t), t)w_m$ is equal to zero).

The state equation in (6) becomes

$$\begin{aligned} \frac{dm(t)}{dt} &= f(X(t), t)w_m \\ &= \frac{1}{2(n+\lambda)} \sum_{i=1}^n f((\sqrt{cP})_i) + f((-\sqrt{cP})_i) \end{aligned} \quad (7)$$

where $(\sqrt{cP})_i$ is the i^{th} column of $\sqrt{cP}$. The right term of (7) is equal to zero if f is linear or only odd but if not, it is different from zero and therefore, 0 is not an equilibrium state of (6).

III. HIGH-GAIN UNSCENTED KALMAN OBSERVER

A. Unscented Kalman Observer

The solution of the previous problem is quite obvious. Indeed, using the state equation that describes the evolution of EKF in (6), we obtain the following new equation describing our unscented Kalman observer (UKO) :

$$\begin{cases} K(t) = X(t)W h^t(X(t), t) \\ \frac{dm(t)}{dt} = f(m(t), t) + K(t)[y(t) - h(m(t), t)] \\ \frac{dP(t)}{dt} = X(t)W f^t(X(t), t) + f(X(t), t)W X^t(t) \\ \quad + Q(t) - K(t)R(t)K^t(t) \end{cases} \quad (8)$$

In the stochastic case, this filter add nothing more to EKF (unlike UKF), but when used as an observer, UKO allows to construct an unbiased high-gain observer (HG-UKO) that converges exponentially. We will now study this observer as well as its high-gain version.

B. High-gain extended Kalman filter

The high-gain observer presented in this section is similar to the one developed in [12]. It is shown in this book that if a system has the property of being observable for all inputs, then there is a coordinate transformation such that a multi-input single-output system can be written in canonical form of observability. The high-gain observer consists to write the observer in these new coordinates and to redefine the matrices Q and R so that they depend on a parameter that is supposed to be "high enough" (the property "high enough" being explicited in [12]). Although less canonical, this method can be apply to multi-input multi-output systems. From now, we suppose that our system (1) is written in the new coordinates:

$$\begin{cases} \dot{x}(t) = Ax(t) + b(x(t), t) \\ y(t) = Cx(t) \end{cases} \quad (9)$$

where $x(t) \in R^n$ and $y(t) \in R$ are the state and the measured outputs respectively. For simplicity, we assumed that the dimension of the output is 1, to facilitates the definition of the normal form. For the general case, see for

example [3]. The matrices A , C are defined as follows:

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & & & 0 \end{pmatrix} \text{ and } C = (1 \ 0 \ \cdots 0)$$

Vector $b(x(t), t)$ is assumed to be triangular and compactly supported as follows :

$$b(x(t), t) = \begin{pmatrix} b_1(x_1(t), t) \\ b_2(x_1(t), x_2(t), t) \\ \vdots \\ b_n(x(t), t) \end{pmatrix}$$

L_b is the bound of the Jacobian matrix $b^*(x(t), t)$ of $b(x(t), t)$ with respect to x , that is to say $\|b^*(x(t), t)\| \leq L_b$.

Function $b(x(t), t)$ is supposed to be uniformly Lipschitz in x independently of t :

$$\|b(x_1, t) - b(x_2, t)\| \leq L_b \|x_1(t) - x_2(t)\|$$

It is well known that system (9) admits the following observer (called high-gain extended Kalman filter, HG-EKF:

$$\begin{cases} \frac{dm}{dt} = Am + b(m, t) + PC^t R^{-1}(y(t) - Cm) \\ \frac{dP}{dt} = (A + b^*(m, t))P + P(A + b^*(m, t))^t \\ \quad + Q^\theta - PC^t R^{-1}CP \end{cases} \quad (10)$$

where i^{th} line and j^{th} column of Q^θ are equal to $Q_{i,j}^\theta = \theta^{i+j+1}Q_{i,j}$. This observer is described, among others, in [4]. This is a classical EKF, but written in canonical coordinates and with a particular Q matrix Q^θ .

C. High-gain unscented Kalman observer in final form

The equations corresponding to HG-UKO in the continuous-time for the system (9) are given by

$$\begin{cases} \frac{dm}{dt} = Am + b(m, t) + P(t)C^t R^{-1}(y(t) - Cm) \\ \frac{dP}{dt} = XW(AX + b(X, t)^t + (AX + b(X))WX^t \\ \quad + Q^\theta - XW X^t C^t R^{-1}CXW X^t \end{cases} \quad (11)$$

where X is the same matrix as defined in (2).

The matrix Q^θ is defined as in (10). These equations are obtained just as [23], [24], but using the equation of unscented Kalman Observer (UKO) in its high-gain version.

D. Main results

Theorem 1: Let us consider the Riccati equation of the UKO (8), and let us suppose that $Q(t)$ and $R(t)$ are bounded symmetric definite matrix (with lower bound greater than 0), f and h are C^2 functions with bounded second derivatives, assuming that the system is completely uniformly observable (standard hypothesis, see [1]), then for any $T > 0$, there

exists c small enough such that for all $t > T$, $P(t)$ (as well as $S(t) = P^{-1}(t)$) remains bounded from below (by a strictly positive number) and from above.

Proof: Bounds on the solution of the Riccati equation in the Kalman filtering context are classical. In the following, we will use the approach developed in [12]. This proof is rather long but simple. It applies to the following Riccati equation written in $S(t)$:

$$\frac{dS(t)}{dt} = -A(t)^t S(t) - S(t) A(t) + C(t)^t R(t)^{-1} C(t) - S(t)(Q(t) + \Omega(t))S(t)$$

where $A(t)$ and $C(t)$ are Jacobian matrix of f and h respectively, and where $R(t)$, $Q(t)$ and $\Omega(t)$ are bounded symmetric matrix, $Q(t) + \Omega(t)$ and $R(t)$ being symmetric definite positive matrix. More precisely, we assume that there exists $\underline{r}$, $\bar{r}$, $\underline{q}$, $\bar{q}$, and ω such that

- $0 < \underline{r} Id \leq R(t) \leq \bar{r} Id$
- $0 < \underline{q} Id \leq Q(t) \leq \bar{q} Id$
- $-\omega Id \leq \Omega(t) \leq \omega Id$ with $\omega < \underline{q}$ such that $Q(t) + \Omega(t)$ is symmetric definite positive.

Under these conditions, using a slight modification of the proof in [12], there exists $0 < \alpha < \beta$ such that

$$\alpha Id \leq S(t) \leq \beta Id \quad (12)$$

(in [12], Q and R are supposed to be constant matrix but it is not a big deal to generalize to time-dependant bounded matrix). The role of $\Omega(t)$ will be explain later.

Let us rewrite the Riccati equation in (8) in $S(t)$ rather than $P(t)$ (using $\frac{d}{dt}(P(t)S(t)) = 0$), it becomes after Taylor series expansion up to order 2:

$$\begin{aligned} \frac{dS}{dt} = & -A^t S - SA + C^t R^{-1} C - S Q S \\ & + \frac{\sqrt{c}}{4} \sqrt{S} \mathcal{H}_h^t R^{-1} C + \frac{\sqrt{c}}{4} C^t R^{-1} \mathcal{H}_h \sqrt{S} \\ & - \frac{\sqrt{c}}{4} \sqrt{S} \mathcal{H}_f^t S - \frac{\sqrt{c}}{4} S \mathcal{H}_f^t \sqrt{S} \\ & - \frac{c}{16} \sqrt{S} \mathcal{H}_h^t R^{-1} \mathcal{H}_h \sqrt{S} \end{aligned} \quad (13)$$

where $\mathcal{H}_f$ and $\mathcal{H}_h$ are large matrix involving second derivatives of f and h respectively, at some points depending from the position of σ -points. Let us rewrite (13) more simply

$$\frac{dS}{dt} = -A^t S - SA + C^t R^{-1} C - S(Q + \Omega)S \quad (14)$$

where $S(t)\Omega(t)S(t)$ are the five last terms in (13). A key point is that

$$\|\Omega(t)\| \leq \sqrt{c} K(f, g, \underline{r}, \bar{r}, \underline{q}, \bar{q}, \min(S), \max(S)) \quad (15)$$

where K is a bound depending from several parameters, including lower and upper bounds on S ($\min(S)$ and $\max(S)$ are scalar bounds in the sense of symmetric matrices partial ordering). Aside its dependence from bounds on S , the bound K is a consequence of the hypothesis in the theorem (f and h have bounded second derivatives) and the existence of bounds on R and Q .

To prove that $S(t)$ remains bounded in the Riccati equation (13), we use an *ad absurdum* argument. Given $\omega > 0$, let α and β given by (12). Let c small enough such that

$$\sqrt{c} K(f, g, \underline{r}, \bar{r}, \underline{q}, \bar{q}, \frac{\alpha}{2}, 2\beta) < \omega \quad (16)$$

Let $S(0)$ be a matrix such that $\alpha Id < S(0) < \beta Id$. So, let us suppose that there exists t_0 being the largest time such that

$$\alpha Id \leq S(t) \leq \beta Id \quad \forall t \leq t_0 \quad (17)$$

which is our *ad absurdum* hypothesis. By continuity of $S(t)$, there exists $t_1 > t_0$ such that $\frac{\alpha}{2} Id \leq S(t) \leq 2\beta Id$ for all $t \leq t_1$. Therefore, for all $t \leq t_1$, $\|\Omega(t)\| \leq \omega$ from (15) and (16). Hence, applying classical bounds on Riccati equations (12), for all $t \leq t_1$, $\alpha Id \leq S(t) \leq \beta Id$. This is a contradiction since $t_1 > t_0$ and t_0 is supposed to be the largest time satisfying (17). ■

As in [6] and [12] for instance, the following theorem is a direct consequence of the previous one.

Theorem 2: Under hypotheses of Theorem 1, the unscented Kalman observer (8) is a locally exponentially converging observer.

The high-gain construction permits us to extend this result from local to global. The bounds on the solution of the Riccati equation in (11) remains the main bottleneck for a mathematical result. Following the classical approach (see [12]), we guess that HG-UKO will be an exponentially converging observer as soon as θ is large enough and c small enough, under the same conditions as in Theorem 1. Actually, we have the following result:

Theorem 3: Let us suppose that for a given $T > 0$, there exists two constants m_1 and m_2 such that for any value of θ , $0 < m_1 Id \leq S(t) \leq m_2 Id$, then there exists θ and c such that the high-gain unscented Kalman observer (11) is a globally exponentially converging observer for of the system (9).

In Theorem 3, $S(t)$ is supposed to be bounded independently from θ for $t > T$. This hypothesis may looks very restrictive. In fact, it is verified for HG-EKF, see [6], [12]).

Complete proofs of these theorems are very technical and rather long. They consist mainly in straightforward adaptations of the proof in the book [12] and some analytic computations (such as Taylor series expansion to obtain (13) which use also some identities from the unscented transformation, see [23]). Theorems 1 and 2 are proved in details in [9]. Theorem 3 is a very simple result and it is proved in the Ph.D. of Assia Daid, in preparation. But Theorem 1 is clearly the "angular stone" of our new result.

IV. ILLUSTRATIVE EXAMPLE

Let us consider a boat moving in the half-plane $x_2 > 0$ (the half-plane $x_2 < 0$ corresponds to the land), at an unknown speed in an orthonormal frame (x_1, x_2) . In order to estimate its position, two beacons are used (denoted A and B), and the boat continuously measures the directions of these beacons in relation to the north, given by an electronic compass. By using the angle between the position of the

boat and each beacon, we want to estimate its instantaneous position.

The state of the system is denoted $x(t) = (x_1(t), x_2(t), \alpha(t))$, α being the angle between the direction of the boat and the north. The geolocation problem is expressed as the following Dubin's car dynamic model:

$$\begin{cases} \dot{x}_1(t) = u(t) \cos(\theta(t)) \\ \dot{x}_2(t) = u(t) \sin(\theta(t)) \\ \dot{\alpha}(t) = v(t) \end{cases} \quad (18)$$

Measurements are given by the following equations, assuming without loss of generality that the two beacons have respective positions $(0,0)$ for A and $(1,0)$ for B .

$$\begin{cases} y_1(t) = \arctan \frac{x_2(t)}{x_1(t)} \\ y_2(t) = \arctan \frac{x_2(t)}{(x_1(t) - 1)} \end{cases} \quad (19)$$

To apply the HG-UKO, we put (18) and (19) into a canonical form of observability, by using an appropriate change of variable as follows

$$\phi(x(t)) = \begin{pmatrix} \arctan \frac{x_2(t)}{x_1(t)} \\ \arctan \frac{x_2(t)}{(x_1(t) - 1)} \\ \frac{x_1(t) \sin \alpha(t) - x_2(t) \cos \alpha(t)}{x_1(t)^2 + x_2(t)^2} \\ \frac{(x_1(t) - 1) \sin \alpha(t) - x_2(t) \cos \alpha(t)}{(x_1(t) - 1)^2 + x_2(t)^2} \end{pmatrix} = \xi(t)$$

In canonical coordinates the system (18) becomes

$$\begin{pmatrix} \dot{\xi}_1(t) \\ \dot{\xi}_2(t) \\ \dot{\xi}_3(t) \\ \dot{\xi}_4(t) \end{pmatrix} = \begin{pmatrix} u \xi_3(t) \\ u \xi_4(t) \\ (v - 2u \xi_3(t)) \frac{x_1(t) \cos \alpha(t) + x_2(t) \sin \alpha(t)}{x_1(t)^2 + x_2(t)^2} \\ (v - 2u \xi_4(t)) \frac{(x_1(t) - 1) \cos \alpha(t) + x_2(t) \sin \alpha(t)}{(x_1(t) - 1)^2 + x_2(t)^2} \end{pmatrix}$$

with measures $y_1(t) = \xi_1(t)$ and $y_2(t) = \xi_2(t)$.

As already said, even with this simple system, the change of coordinates involves quite complicated functions and we are quite happy to avoid Jacobian matrices computations.

A. Comparative study

For testing, we will assume that the boat follows a circular trajectory at a constant speed V , by taking the two constant controls $u(t) = V$ and $v(t) = \omega = \frac{V}{R}$, and by choosing $x_1(0) = c_1$, $x_2(0) = c_2 - R$, $\alpha(0) = 0$, the solution is

$$\begin{cases} x_1(t) = R \sin(\omega t) + c_1 \\ x_2(t) = -R \cos(\omega t) + c_2 \\ \alpha(t) = \omega t \end{cases} \quad (20)$$

and we have $\sqrt{\dot{x}_1(t)^2 + \dot{x}_2(t)^2} = V$.

First, we compare UKO and HG-UKO. Both are used as observers, but the second is applied in the canonical form of observability and with a high-gain θ . This high-gain is taken equal to 2 in all simulations.

The initial state of the system is $x_0 = (3, 1.5, 0)$, and the initial state of the observer is $m = (3.5, 3, 1)$.

Estimated states from UKO and HG-UKO are compared to actual state on Figure 1. Figure 2 represents the same

Fig. 1. Estimated states with UKO and HG-UKO

Fig. 2. Trajectory estimation with UKO and HG-UKO

simulation result in the state space. Not surprisingly, UKO and HG-UKO converge to the actual state, but the convergence of the HG-UKO is faster than UKO (in fact, it can be arbitrarily fast but in this case, it will be very sensitive to noise and we chose a value of θ which is a compromise between convergence and robustness).

We seen in Section II-B, that the HG-UKF can not converge as an observer. Figures 3 and 4 clearly shows the inevitable static error with HG-UKF, while HG-UKO has no bias. The error is particularly important and visible on Figure 4, as expected.

Next, we compared HG-UKO with HG-EKF, Figure 5, and Figure 6 shows that the two observers have quite similar performance : they converge at the same time with similar amplitude error. This is not a surprise since in this context, HG-UKO is more or less a numerical approximation of HG-EKF, in the sense that formal Jacobian matrix computation in HG-EKF is replaced by a kind of numerical approximation, based on difference quotients in HG-UKO.

It should be noted that we didn't put noise into these simulations, as we only test the performance of (HG-)UKO

Fig. 3. Estimated states with HG-UKO and HG-UKF

Fig. 6. Trajectory estimation with HG-UKO and HG-EKF

Fig. 4. Trajectory estimation with HG-UKO and HG-UKF

Fig. 7. Actual and noisy output

Fig. 5. Estimated states with HG-UKO and HG-EKF

Fig. 8. Trajectory estimation with additive noise on the measurement

as an observer. Nevertheless, in order to test the robustness of HG-UKO, we introduce unknown perturbations on outputs. More precisely, we add some trigonometric functions of the

form $0.01 \left(\cos(\omega_i^{(1)}t) + \cos(\omega_i^{(2)}t) \right)$ to each output y_i , $i = 1, 2$ corresponding to a perturbation with standard deviation equal to 1% for y_1 and 0.5% for y_2 (see Figure 7). The value

of the high-gain parameter has not been adapted. Results are shown Figure 8. Although the noise induce some lack of performance, we see that convergence remains and is very satisfactory.

V. CONCLUSION

Several nonlinear observers were presented, among them only HG-EKF and HG-UKO are global nonlinear converging observers, while HG-UKF is not an observer (and has never been presented as is in literature).

The advantage of HG-UKO is a relative simplicity of writing since it is not necessary to calculate any Jacobian matrices. This property may be particularly interesting when the system is written in its canonical form, since it is usually a rather complicated form involving high order Lie derivatives. These results have been established for continuous systems, and should be generalized to continuous-discrete systems.

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