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Aymen Rahali

To cite this version:

Aymen Rahali. GENERIC REPRESENTATIONS AND FELL TOPOLOGY FOR SOME LIE GROUPS. Bulletin of the Belgian Mathematical Society - Simon Stevin, Belgian Mathematical Society, In press. hal-02282204
GENERIC REPRESENTATIONS AND FELL TOPOLOGY
FOR SOME LIE GROUPS

AYMEN RAHALI

March 27, 2019

To the memory of Majdi Ben Halima

Abstract. Let $N$ be a connected and simply connected nilpotent Lie group, and let $K$ be a subgroup of the automorphism group of $N$. We say that the pair $(K, N)$ is a nilpotent Gelfand pair if $L^1_K(N)$ is an abelian algebra. In this paper, we consider the semidirect $G = K \ltimes N$ with Lie algebra $\mathfrak{g}$. Let $\hat{G}$ be the unitary dual of $G$ and $(\hat{G})_{\text{gen}} \subset \hat{G}$ is the space of generic representations of $G$. By $\mathfrak{g}^1/G$ and $(\mathfrak{g}^1/G)_{\text{gen}}$ we means the space of the admissible coadjoint orbits and the generic orbits of $G$ respectively (see [21]). Under some assumption on the Gelfand pair $(K, N)$, we determine explicity the topology of $(\hat{G})_{\text{gen}}$ and we show that the topological space $(\hat{G})_{\text{gen}}$ equipped with the Fell topology is homeomorphic to $(\mathfrak{g}^1/G)_{\text{gen}}$ endowed with the quotient topology.

Key-words: Lie groups, semidirect product, unitary representations, coadjoint orbits, Fell topology.

AMS Subject 2010 Classification: 22D10-22E27-22E45

1. Introduction

Let $G$ be a second countable locally compact group and $\hat{G}$ its dual space, that is the set of all equivalence classes of irreducible unitary representations of $G$. An important tool for investigating the group algebra of $G$ is the so-called hull-kernel topology (Fell topology) of $\hat{G}$ which is a special case of the relation of weak containment [11,12]. The question arises: For a such group $G$, how do we determine $\hat{G}$ and its topology? For many groups $G$, Mackey’s theory of induced representations permits us to catalogue all the elements of $\hat{G}$. We recall the definition of weak containment. If $S$ is a family of unitary representations of $G$, and $\tau$ a unitary representation of $G$, $\tau$ is weakly contained in $S$ if all positive functionals on $G$ associated with $\tau$ can be weakly approximated by sums of positive functionals associated with representations in $S$. When restricted to $\hat{G}$, the relationship of weak containment gives the operation of closure in the hull-kernel topology. Then the description of the dual topology is a good candidate for some aspects of harmonic
analysis on $G$ (for example, see [1, 2, 7, 8]). In such a situation, the natural and important question arises of whether the bijection between the space of coadjoint orbits $\mathfrak{g}^*/G$ of $G$ ($\mathfrak{g}^*$ is the dual vector space of $\mathfrak{g} := \text{Lie}(G)$) and $\hat{G}$ is a homeomorphism. For a simply connected nilpotent Lie group and more generally for an exponential solvable Lie group $G = \text{exp}(\mathfrak{g})$, its dual space $\hat{G}$ is homeomorphic to the space of coadjoint orbits through the Kirillov mapping (see [17]). In the context of semidirect products $G = K \ltimes N$, where $K$ is a connected compact Lie group acting smoothly on simply connected nilpotent Lie group $N$. Then it was pointed out by Lipsman in [21], that we have again an orbit picture of the dual space of $G$. Let $G_{\psi}$ be the stabilizer in $G$ of a linear form $\psi \in \mathfrak{g}^*$. Then $\psi$ is called admissible if there exists a unitary character $\chi$ of the identity component of $G_{\psi}$ such that $d\chi = i\psi|_{\mathfrak{g}^\psi}$. Let $\mathfrak{g}^\updownarrow \subset \mathfrak{g}^*$ be the set of all the admissible linear forms on $\mathfrak{g}$. Here, we recall some results in the literature. The unitary dual space of Euclidean motion groups is homeomorphic to the admissible coadjoint orbits [8]. This result was generalized in [1], for a class of Cartan motion groups. Analogue results have been proved for which the so-called generalized motion groups (see [22]), and for the Heisenberg motion groups (see [2]). Recently, M. Elloumi, J. Kathrin Gunther and J. Ludwig have proved an analogue result for the compact extension of the Heisenberg groups (see [9]).

We turn to our setting. We say that the pair $(K, N)$ is a nilpotent Gelfand pair if $L_1^K(N)$ is an abelian algebra under convolution. For such pair $(K, N)$, let $\mathfrak{n} := \text{Lie}(N)$ and write $\mathfrak{n} = \mathcal{V} \oplus \mathfrak{z}$, where $\mathfrak{z}$ is the center of $\mathfrak{n}$ and $[\mathcal{V}, \mathcal{V}] \subset \mathfrak{z}$.

**Definition 1.1.** Let $(K, N)$ be a nilpotent Gelfand pair. We say that $(K, N)$ has spherical central orbits if generic orbits of the restricted action of $K$ on $\mathfrak{z}$ are of codimension one.

We endow $\mathfrak{n} = \mathcal{V} \oplus \mathfrak{z}$ with an inner product $\langle , \rangle$ such that $\mathfrak{z} = [\mathfrak{n}, \mathfrak{n}]$ and $\mathcal{V} \perp \mathfrak{z}$. If $(K, N)$ has a spherical central orbits, then we can fix a unit base point $A \in \mathfrak{z}$, and define a skew-symmetric form $(v, w) \mapsto [\langle [v, w], A \rangle]$ on $\mathcal{V}$ (for more details, see [10]).

**Definition 1.2.** We say that a nilpotent Gelfand pair $(K, N)$ is non-degenerate on $\mathcal{V}$ if the skew-symmetric form

$$(v, w) \mapsto [\langle [v, w], A \rangle]$$

is non-degenerate on $\mathcal{V}$. Here $v, w \in \mathcal{V}$ and $A \in \mathfrak{z}$ is the fixed unit base point.

In this document, $\text{Ad}_L$, $\text{Ad}_L^*$ denotes respectively the adjoint and the coadjoint representations for such Lie group $L$. Any action of Lie group we denote it by ".". Now, we describe exactly which pair $(K, N)$ our result applies to. Let $K$ be a connected compact abelian subgroup of automorphisms of $N$ such that $(K, N)$, is a nilpotent Gelfand pair satisfies Definitions 1.1 and 1.2. We shall assume that the subgroup $K$ is the product of $n \geq 1$ copies of the unit cercle.
$S^1$. Then we can form the semidirect product $G = K \ltimes N$ with group law

$$(k_1, (x_1, X_1))(k_2, (x_2, X_2)) = (k_1k_2, (x_1, X_1)(k_1, (x_2, X_2))).$$

Let $(\hat{G})_{gen} \subset \hat{G}$ be a subset of the unitary dual of $G$, which the so-called space of generic representations of $G$ (see Section 2 for this Definition). We denote by $g^\dagger/G$ the space of the admissible coadjoint orbits of $G$. In the spirit of the orbit method due to Kirillov, R. Lipsman established a bijection between $g^\dagger/G$ and the unitary dual $\hat{G}$ of $G$, in the following way. For every admissible linear form $\psi$ on the Lie algebra $g$ of $G$, we can construct an irreducible unitary representation $\pi_\psi$ by holomorphic induction and according to Lipsman (see [7, p. 23]) (compare [21]), every irreducible representation of $G$ arises in this manner. Then we get a map from the set $g^\dagger$ of the admissible linear forms onto the dual space $\hat{G}$ of $G$. Note that $\pi_\psi$ is equivalent to $\pi_\psi'$ if and only if $\psi$ and $\psi'$ are in the same $G$-orbit. That is, the association (Kirillov-Lipsman mapping)

$$g^\dagger/G \ni \mathcal{O}_\psi \longmapsto \pi_\psi \in \hat{G}$$

yields a bijection between admissible coadjoint orbits in $g^\dagger$ and irreducible unitary representations of $G$.

Let $(g^\dagger/G)_{gen}$ denotes the space of generic coadjoint orbits of $G$ arising from the generic representations of $G$ via the Kirillov-Lipsman orbit method. Our main results are the following.

**Theorem 1.3.** The convergence in $(\hat{G})_{gen}$ is given in terms of Mackey parameters.

According to Lipsman [21], we have the following bijection

$$(\hat{G})_{gen} \simeq (g^\dagger/G)_{gen}.$$ We arrive to.

**Theorem 1.4.** The topological space $(\hat{G})_{gen}$ equipped with the Fell topology is homeomorphic to $(g^\dagger/G)_{gen}$ endowed with the quotient topology.

Elements of $g = \mathfrak{k} \ltimes \mathfrak{n}$ ($\mathfrak{k} = \text{Lie}(K)$) will be written as $(U, (b, B))$ where $U \in \mathfrak{k}$ and $(b, B) \in \mathfrak{n} = \mathcal{V} \oplus \mathfrak{z}$. A direct computation, one obtains that the adjoint action of $G$ is

$$\text{Ad}_G(k, (x, X))(U, (b, B)) = \left(\text{Ad}_K(k)U, k.(b, B) - (\text{Ad}_K(k)U).(x, X)\right) + [([x, X], k.(b, B)] - \frac{1}{2}([x, X], (\text{Ad}_K(k)U).(x, X))].$$

Identify the Lie algebra $\mathfrak{n}$ with its vector dual space $\mathfrak{n}^*$ through the $K$-invariant scalar product $\langle , \rangle$. The coadjoint actions of $N$ and $\mathfrak{n}$ on $\mathfrak{n}^*$ are defined by

$$(\text{Ad}_N^*(x, X)(b, B))(y, Y) = (b, B)(\text{Ad}_N((x, X)^{-1})(y, Y))$$

$$(\text{ad}_N^*(x, X)(b, B))(y, Y) = -(b, B)(\text{ad}_N(x, X)(y, Y)).$$
Since $N$ is 2-step, the identification of $N$ with $n$ allows us to write

\[(1.1) \quad Ad^*_N(x, X)(b, B) = (b, B) + ad^*_N(x, X)(b, B).\]

Then, each linear functional $\psi \in \mathfrak{g}^*$ can be identified with an element $(U, (x, X)) \in \mathfrak{g}$ such that

$$\psi(V, (y, Y)) = \langle (U, (x, X)), (V, (y, Y)) \rangle_{\mathfrak{g}}$$

for $(V, (y, Y)) \in \mathfrak{g}$. Write points $\psi \in \mathfrak{g}^*$ as $\psi = (\nu, (b, B))$ where $\nu \in \mathfrak{k}^*$ and $(b, B) \in \mathfrak{n}^*$. That is

$$\psi(U, (y, Y)) = \nu(U) + b(y) + B(Y).$$

Following [4], we define a map $\times : \mathfrak{n} \times \mathfrak{n}^* \to \mathfrak{t}^*$ by

$$\left((x, X) \times (b, B)\right)(U) := (b, B)(U, (x, X)) = -(U, (b, B))(x, X)$$

for $U \in \mathfrak{t}$, $(x, X) \in \mathfrak{n}$ and $(b, B) \in \mathfrak{n}^*$. The map $\times : \mathfrak{n} \times \mathfrak{n}^* \to \mathfrak{t}^*$ satisfies the equivariance property

$$Ad^*_K(k)((x, X) \times (b, B)) = \left(k, (x, X)\right) \times \left(k, (b, B)\right).$$

Then the coadjoint action of $G$ on $\mathfrak{g}^*$ is given by

$$Ad^*_G(k)((x, X) \times (b, B))(\nu, (b, B)) = \left(Ad^*_K(k)\nu + (x, X) \times (k, (b, B))\right) + \frac{1}{2}(x, X) \times \left(ad^*_N(x, X)(k, (b, B))\right).$$

According to [4], one obtains the following description of the coadjoint orbits $O^G_{(\nu, (b, B))}$ of $G$.

$$O^G_{(\nu, (b, B))} = \left\{k.\left(\nu + (x, X) \times (b, B) + \frac{1}{2}(x, X) \times \left(ad^*_N(x, X)(b, B)\right), Ad^*_N(x, X)(b, B)\right)\right\} \big| k \in K, (x, X) \in \mathfrak{n}$$

where

$$k.\left(\nu, (b, B)\right) := \left(Ad^*_K(k)\nu, (b, B)\right) = Ad^*_G(k, 0)\left(\nu, (b, B)\right).$$

This paper is organized in the following way. In Section 1, we present some history and some definitions. Section 2, is devoted to the description of the unitary dual $\hat{G}$, in which we determine explicitly the topology of $(\hat{G})_{gen}$. We reserve the last Section to the admissible coadjoint orbits of $G$ and we prove Theorem 1.1.
2. Generic representations of $G = K \rtimes N$

We keep the notations of the previous Section. We begin this Section by recalling the unitary dual of the Heisenberg group. We cover this group because our proof of Theorem 3.1 uses the relationship between the infinite dimensional representations of the Heisenberg group (type I representations) and the infinite dimensional representations of $N$.

The Heisenberg group $\mathbb{H}_V$ is identified with $V \oplus \mathbb{R}$ (where $V \simeq \mathbb{C}^n$) with law

$$(z, t)(w, t') = (z + w, t + t' + \frac{1}{2} \text{Im}(z \cdot \overline{w}))$$

where $z \cdot w := \sum_{k=1}^n z_k w_k$ for all $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ in $V$.

A maximal compact connected group of automorphisms of $\mathbb{H}_V$ is given by the unitary group $U(n)$ acting via

$$k(z, t) = (kz, t).$$

Let $T^n$ be a maximal torus of $U(n)$. It is well known that the pairs $(U(n), \mathbb{H}_V)$ and $(T^n, \mathbb{H}_V)$ are Gelfand pairs. There are many proper closed subgroups $K \subset U(n)$ for which $(K, \mathbb{H}_V)$ is a Gelfand pair. A such pair was determined by Benson, Jenkins and Ratcliff.

Note that the group $T^n$ acts on $\mathbb{H}_V$ (by automorphisms) by

$$e^{i\theta}(z, t) = (e^{i\theta} z, t)$$

where $e^{i\theta} = \text{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n}) \in T^n$ (the torus $T^n$ is the product of $n$ copies of $S^1$). So one can form the semidirect product $G_n = T^n \rtimes \mathbb{H}_V$. The multiplication rule in this group is given by

$$(e^{i\theta}, z, t)(e^{i\theta'}, z', t') = (e^{i(\theta + \theta')}, z + e^{i\theta} z', t + t' - \frac{1}{2} \text{Im}(z \cdot \overline{e^{i\theta} z'})).$$

For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$, let $\chi_\lambda : T^n \to S^1$ be the character of $T^n$ defined by

$$\chi_\lambda(e^{i\theta}) = e^{i\lambda \cdot \theta},$$

where $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$.

The irreducible unitary representation of $\mathbb{H}_V$ are classified by Kirillov’s "orbit method" [17]. Then the unitary irreducible representations of $\mathbb{H}_V$ naturally split into tow types via their parametrization by the coadjoint orbits in $\mathfrak{h}_V^*$ ($\mathfrak{h}_V = \text{Lie}(\mathbb{H}_V)$). In particular, the type I representations are parametrized by real numbers $s \neq 0$, with associated coadjoint orbit

$$O_{s}^{\mathbb{H}_V} = V \oplus \{s\}.$$

The type II representations correspond to one-point orbits $O_{b}^{\mathbb{H}_V} = \{(b, 0)\}$ with $b \in V$. Recall that type I representations are infinite dimensional representations of the Heisenberg group $\mathbb{H}_V$. For $s \in \mathbb{R}^*$, we denote by $\pi_s$ the
type I representations of $\mathbb{H}_V$, which are realized on the Fock space

$$\mathcal{F}_s(V) := \left\{ f : V \rightarrow \mathbb{C} \text{ holomorphic} \mid \int_V |f(v)|^2 e^{-\frac{|v|^2}{2}} dv < \infty \right\}$$

by

$$\pi_s(z,t)f(w) = e^{ist-\frac{s}{4}|z|^2 - \frac{s}{2}(w,z)f(w+z) \quad if \ s > 0$$

and

$$\pi_s(z,t)f(\bar{w}) = e^{ist+\frac{s}{4}|z|^2 + \frac{s}{2}(w,z)f(\bar{w} + \bar{z}) \quad if \ s < 0.$$  

(See for example, [2, 8] for a discussion of the Fock space).

For each $e^{i\theta} \in T^n$, the operator $W_s(e^{i\theta}) : \mathcal{F}_s(V) \rightarrow \mathcal{F}_s(V)$ given by

$$W_s(e^{i\theta})f(z) = f(e^{-i\theta}z) \quad \forall f \in \mathcal{F}_s(V) \quad \forall z \in V$$

intertwines $\pi_s$ and $(\pi_s)_e^{i\theta}$. $W_s$ is called the projective intertwining representation of $T^n$ on the Fock space, then for each $s \in \mathbb{R}^*$ and each element $\chi_\lambda$ in $\hat{T}_n$,

$$\pi_\lambda(e^{i\theta}, z, t) := \chi_\lambda(e^{i\theta}) \otimes (\pi_s(z, t) \circ W_s(e^{i\theta})) \quad \forall (e^{i\theta}, z, t) \in G_n$$

is an irreducible unitary representation of $G_n$ realized on $\mathcal{F}_s(V)$.

We close this Section by describing the unitary dual of $G = K \ltimes N$. As a first step in this process, we review the representation theory of the nilpotent group $N$. Representations of simply connected, real nilpotent Lie groups are classified by Kirillov’s “orbit method” [17]. Given an element $\ell \in \mathfrak{n}^*$, one selects a subalgebra $m \subset \mathfrak{n}$ which is maximal in the sense that $\ell([m, m]) = 0$.

One then defines a character $\chi_{\ell}$ of $M = \exp(m)$ by

$$\chi_{\ell}(\exp(x, X)) = e^{i\ell(x, X)}$$

and $\sigma_{\ell} := ind_M^N \chi_{\ell}$. From Kirillov (see [17]), we know that each irreducible unitary representation of $N$ is of the form $\sigma_{\ell}$ for some $\ell$, and $\sigma_{\ell} \sim \sigma_{\ell'}$ if and only if $\ell$ and $\ell'$ are in the same coadjoint orbit in $\mathfrak{n}^*$. That is, the association

$$Ad_N^*(N)(\ell) =: N.\ell \mapsto \sigma_\ell$$

yields a bijection between coadjoint orbits in $\mathfrak{n}^*$ and irreducible unitary representations of $N$.

In our setting, $N$ is a two-step nilpotent Lie group. This structure allows us to choose an “aligned point” (see Definition below) in each coadjoint orbit [4]. We describe this process now. For a coadjoint orbit $\mathcal{O} \subset \mathfrak{n}^*$, we take $\ell \in \mathcal{O}$ so that $\mathcal{O}^\ell := Ad_N^*(N)\ell$, then we define a bilinear form on $\mathfrak{n}$ by

$$B_\mathcal{O}((x, X), (y, Y)) = \ell([[(x, X), (y, Y)]])$$

Let $a_\mathcal{O} := \{ v \in \mathcal{V}; \ell([v, \mathfrak{n}]) = 0 \}$. According to [5], this process gives us a decomposition

$$\mathfrak{n} = a_\mathcal{O} \oplus \mathfrak{w}_\mathcal{O} \oplus \mathfrak{z},$$
where \( w_\mathcal{O} = a_\mathcal{O} \cap \mathcal{V} \). The identification between \( w_\mathcal{O} \) and \( \mathcal{O} \) does depend on the choice of \( \ell \) (see [5], for a full discussion). However, in [5] it is shown that there is a canonical choice of \( \ell \) in the following sense.

**Definition 2.1.** A point \( \ell \in \mathcal{O} \) is said to be aligned if \( \ell|_{w_\mathcal{O}} = 0 \). Note that this gives us a canonical identification \( w_\mathcal{O} \cong \mathcal{O} \). The action of \( K \) on \( n^* \) sends aligned points to aligned points, which implies that the stabilizer

\[
K_\mathcal{O} = \{ k \in K : k.\mathcal{O} = \mathcal{O} \}
\]

of a coadjoint orbit coincide with the stabilizer

\[
K_\ell = \{ k \in K : k.\ell = \ell \}
\]

of its aligned point. (see Section 3.2 of [5] for more details).

Next, we use the Mackey machine to recall the process for describing the unitary dual \( \hat{G} \) in terms of representations of \( N \) and subgroups of \( K \). There is a natural action of \( K \) on \( \hat{N} \) by

\[
k.\sigma := \sigma \circ k^{-1},
\]

where \( k \in K \) and \( \sigma \in \hat{N} \). Let \( \sigma \) be an irreducible unitary representation of \( N \) corresponding to a coadjoint orbit \( \mathcal{O} \subset n^* \). Let

\[
K_\sigma = \{ k \in K : k.\sigma \cong \sigma \}
\]

the stabilizer of \( \sigma \) under the \( K \)-action (here \( \cong \) denotes unitary equivalence). According to Lemma 2.3 of [4], There is a unitary representation

\[
W_\sigma : K_\sigma \rightarrow U(\mathcal{H}_\sigma)
\]

(here \( \mathcal{H}_\sigma \) denote the space of \( \sigma \)) of \( K_\sigma \) that intertwines \( k.\sigma \) with \( \sigma \). For \( k \in K_\sigma \), the \( W_\sigma(k) \)'s are only characterized up to multiplicative constants in the unit circle \( \mathbb{S}^1 \) by the intertwining condition. Then Mackey theory ensures that

\[
\pi_{(\rho,\sigma)} := \text{Ind}_{K_\sigma \times N}^{K \times N} \left( (k, (x, X)) \mapsto \rho(k) \otimes \sigma(x, X)W_\sigma(k) \right)
\]

is an unitary irreducible representation of \( G \). Up to unitary equivalence, all unitary irreducible representations of \( G \) have this form. This means that

\[
\hat{G} = \{ \pi_{(\rho,\sigma)}, \rho \in \hat{K_\sigma}, \sigma \in \hat{N} \}.
\]

We say that \( \pi_{(\rho,\sigma)} \) has Mackey parameters \((\rho, \sigma)\).

Let \( \mathcal{O} \subset n^* \) be a coadjoint orbit with aligned point \( \ell \in \mathcal{O} \). The corresponding representation \( \sigma \in \hat{N} \) factors through

\[
N_\mathcal{O} = \exp(n/\ker(\ell|_n)).
\]

The group \( N_\mathcal{O} \) is the product of a Heisenberg group \( \mathbb{H} \) and the (possibly trivial) abelian group \( \mathfrak{n}_\mathcal{O} \). Using the inner product \( \langle \cdot, \cdot \rangle \) to construct an explicit isomorphism \( \varphi \) between \( \mathbb{H} \) and the standard Heisenberg group \( \mathbb{H}_V := V \oplus \mathbb{R} \), with \( V \) is a unitary \( K_\sigma \)-space (see Section 5.1 of [5]). According to this construction, an important worth mentioning here is that one can realize \( \pi \) as
the standard representation of $\mathbb{H}_V$ in the Fock space $\mathcal{F}_s(V)$ on $V$. Thus we realize $W_\sigma$ as the restriction to $K_\sigma$ of the standard representation of $U(V)$ on $\mathcal{F}_s(V)$.

Now, we fix $A \in \mathfrak{z}$ to be a unit base point (as in Section 1). For any $\ell = (b, B) \in \mathfrak{n}^*$ with $B \neq 0$, we have $B = sA$ with $s > 0$. The form

$$(v, w) \mapsto \langle [v, w], A \rangle$$

is non-degenerate on $V$. Hence the orbit through $\ell$ is $V \oplus sA$ with aligned point $\ell = (0, sA)$.

For $\ell = (b, 0)$ the coadjoint orbit through $\ell$ is a single point. From [10], we conclude that we have two types of coadjoint orbits.

- **Type I Orbits**: When the parameter $s > 0$, we have an aligned point of the form $\ell = (0, sA)$. We call the corresponding coadjoint orbits type I orbits and the corresponding representations $\sigma_s \in \widehat{N}$ type I representations. These orbits depend only on the positive real number $s > 0$, we denote them $O_{sA}^N$.

- **Type II Orbits**: For $s = 0$, the corresponding coadjoint orbits $O_b^N$ contains only the aligned point $\ell = (b, 0)$ where $b \in V$. We call such coadjoint orbits type II orbits and the corresponding representations $\chi_b \in \widehat{N}$ type II representations. Since these orbits depend only on the parameter $b \in V$, we denote them $O_b^N$.

We turn our attention to the type I coadjoint orbits $O_{sA}^N$ associated to the aligned point $\ell = (0, sA)$ and corresponding type I representations $\sigma_s \in \widehat{N}$. Recall that the coadjoint orbit $O_{sA}^N$ has the form

$$O_{sA}^N = V \oplus sA.$$

Note that the representation $\sigma_s$ has codimension 1 kernel in $\mathfrak{z}$, and factors through

$$N_{O_{sA}^N} = \text{exp}(\mathfrak{n} / \text{ker}(\ell_{1j})).$$

Here $N_{O_{sA}^N}$ is the Heisenberg group $\mathbb{H}_A := V \oplus \mathbb{R}A$. On $\mathbb{H}_A$, our representation is a type I representation $\pi_s$ of the standard Heisenberg group, which can be realized on Fock space (see [10] ). Following [10], we can describe the group isomorphism $\varphi$ from $\mathbb{H}_A$ to the standard Heisenberg group as follows: At first, note that the stabilizer $K_{\sigma_s}$ of $\sigma_s$ is equal to the stabilizer $K_A$ of $A$. Then we define a $K_A$-equivariant group isomorphism

$$\varphi : \mathbb{H}_A \longrightarrow \mathbb{H}_V := V \oplus \mathbb{R}$$

$$(v, tA) \mapsto (\varphi(v), t)$$

(see [10] for a full discussion about the expression of $\varphi$). This gives us the relationship between type I representations $\sigma_s \in \widehat{N}$ and the type I representations $\pi_s \in \widehat{\mathbb{H}}_V$:

$$\sigma_s = \pi_s \circ \varphi.$$
Then for \( s \in \mathbb{R}_{>0} \), the representation \( \sigma_s \) realized on the Fock space \( \mathcal{F}_s(\mathcal{V}) \) as follows
\[
\sigma_s(v, tA)f(w) = e^{ist - \frac{s}{22}w_0(t, f(\varphi(v))) - \frac{s}{22}w_0(t, \varphi(v))} f(\varphi(v) + w)
\]
where \( v, w \in \mathcal{V}, t \in \mathbb{R} \) and \( f \in \mathcal{F}_s(\mathcal{V}) \). Let \( \chi_\mu \) be an unitary irreducible representation of \( K_A \) with highest weight \( \mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n \). The one dimensional representation of \( K_A \) is given by
\[
\chi_\mu(e^{i\theta}) := e^{i\mu \theta} \quad \forall \theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n.
\]
From [10] and according to Mackey theory, one can see that the unitary irreducible representations of \( G \) arising from the type I representations \( \sigma_s \) of \( N \) are giving by
\[
\pi(\mu, s) := \text{Ind}_{K_A \times N}^G(\pi_\mu := \chi_\mu \otimes \sigma_s \circ W_s).
\]
A such representation \( \pi(\mu, s) \) is called generic representation and the set of all generic representations of \( G \) is denoted by \( (\hat{G})_{\text{gen}} \). More precisely the following identification
\[
(\hat{G})_{\text{gen}} := \{ \pi(\mu, s); \, \mu \in \overline{K_A}, \, s \in \mathbb{R}_{>0} \} \simeq \overline{K_A} \times \mathbb{R}_{>0}
\]
is a bijection.

Hereby, we give some results which are being used in the description of the dual topology of \( G \). These are required for our proof of Theorem 1.3.

Given \((\pi, \mathcal{H}_\pi)\) an irreducible unitary representation of \( G \) on the Hilbert space \( \mathcal{H}_\pi \), the functions of positive type of \( \pi \) are, by definition, given by the linear functionals
\[
C^\pi_\xi : G \rightarrow \mathbb{C}, \, g \mapsto \langle \pi(g)\xi, \xi \rangle,
\]
where \( \xi \) is a cyclic vector in \( \mathcal{H}_\pi \).

**Theorem 2.2.** ([6]) Let \((\pi_k, \mathcal{H}_{\pi_k})_{k \in \mathbb{N}}\) be a sequence of irreducible unitary representations of \( G \). Then \((\pi_k)_k\) converges to \( \pi \) in \( \hat{G} \) if and only if for some non-zero (resp. for every) vector \( \xi \) in \( \mathcal{H}_\pi \), there exist \( \xi_k \in \mathcal{H}_{\pi_k}, \, k \in \mathbb{N} \), such that the sequence \((C^\pi_\xi)_k\) of functions converges uniformly on compacta to \( C^\pi_\xi \).

The topology of \( \hat{G} \) can also be expressed by the weak convergence of the coefficient functions.

**Theorem 2.3.** ([6]) Let \((\pi_k, \mathcal{H}_{\pi_k})_{k \in \mathbb{N}}\) be a sequence of irreducible unitary representations of \( G \). Then \((\pi_k)_k\) converges to \( \pi \) in \( \hat{G} \) if and only if for some non-zero (resp. for every) vector \( \xi \) in \( \mathcal{H}_\pi \), there are \( \xi_k \in \mathcal{H}_{\pi_k} \) such that the sequence of linear functionals \((C^\pi_{\xi_k})_k\subset C^* (G)' \) converges weakly on some dense subspace of the \( C^* \)-algebra \( C^* (G) \) of \( G \) to the linear functional \( C^\pi_{\xi} \).

If \( G \) is a Lie group, then we denote respectively by \( g \) its Lie algebra and by \( \mathcal{U}(g) \) the enveloping algebra of \( g \). For a unitary representation \((\pi, \mathcal{H}_\pi)\) of \( G \), let \( \mathcal{H}_\pi^\infty \) be the subspace of \( \mathcal{H}_\pi \) consisting of the smooth vectors for \( \pi \).
Corollary 2.4. ([6]) Let \((\pi_k, \mathcal{H}_{\pi_k})_{k \in \mathbb{N}}\) be a sequence of irreducible unitary representations of the Lie group \(G\). If \((\pi_k)\) converges to \(\pi\) in \(\hat{G}\) then for every unit vector \(\xi\) in \(\mathcal{H}^\infty_{\pi_k}\), there exist \(\xi_k \in \mathcal{H}^\infty_{\pi_k}\), \(k \in \mathbb{N}\), such that the numerical sequence \((\langle d\pi_k(D)\xi_k, \xi_k \rangle_k)\) converges to \(\langle d\pi(D)\xi, \xi \rangle\), for each \(D \in \mathcal{U}(g)\).

Let \(B_s := \{h_{m,s}, m = (m_1, \ldots, m_n) \in \mathbb{N}^n\}\) be the orthonormal basis of the Fock space \(\mathcal{F}_s(V)\) defined by the Hermite functions

\[
h_{m,s}(z) = \frac{|s|^{\frac{n}{2}}}{\sqrt{2^m m!}} z^m
\]

with \(|m| = m_1 + \ldots + m_n\), \(m! = m_1! \ldots m_n!\) and \(z^m = z_1^{m_1} \ldots z_n^{m_n}\). We take as basis of the Lie algebra \(\mathfrak{h}_V\) the left invariant vectors fields

\[
\{Z_1, \ldots, Z_n, \overline{Z}_1, \ldots, \overline{Z}_n, T\}
\]

where

\[
Z_p = 2 \frac{\partial}{\partial z_p} + iz_p \frac{\partial}{\partial \bar{z}_p} \quad \forall p \in \{1, \ldots, n\}, \quad T = \frac{\partial}{\partial t}.
\]

The differential operator

\[
L_p := \frac{1}{2}(Z_p \overline{Z}_p + \overline{Z}_p Z_p)
\]

will play a key role in our proof of Theorem *. We fix the elements

\[
E_p = \text{diag}(0, \ldots, 0, i, 0 \ldots, 0) \in \mathfrak{k}_A := \text{Lie}(K_A) \quad \forall p \in \{1, \ldots, n\}
\]

where the complex number \(i\) is the \(p^{th}\) diagonal entry. We refer to [3], we give the following Lemma.

Lemma 2.5. For every irreducible representation \(\tilde{\pi}_{(\mu,s)}\) of \(K_A \ltimes N\), we have

\[
d\tilde{\pi}_{(\mu,s)}(L_p)h_{m,s} = -s(2m_p + 1)h_{m,s}
\]

for each \(m = (m_1, \ldots, m_n) \in \mathbb{N}^n\).

We easily obtain the following Lemma.

Lemma 2.6. For any irreducible representation \(\tilde{\pi}_{(\mu,s)}\) of \(K_A \ltimes N\), we have

\[
\langle d\tilde{\pi}_{(\mu,s)}(T)\xi, \xi \rangle = is
\]

where \(\xi\) is a unit vector in \(\mathcal{F}_s(V)\).

With the above notations, we have.

Theorem 2.7. The sequence \((\pi_{(\mu^j,s^j)})_j\) converges to the irreducible representation \(\pi_{(\mu,s)}\) in \((\hat{G})^{\text{gen}}\), if and only if \(\lim_{j \to +\infty} s^j = s\) and \(\mu^j = \mu\) for large \(j\).
Proof. We note that $K_A \ltimes N$ is a normal subgroup of $G = K \ltimes N$. Then Proposition 1.1 in [12], tell us that the sequence $(\pi_{(\mu^j,s_j)})_j$ converges to $\pi_{(\mu,s)}$ in $\hat{G}$, if and only if

$$\pi_{(\mu,s)} \text{ is weakly contained in } (\pi_{(\mu^j,s_j)})_j.$$  

By Theorem 4.4 in [11], this is equivalent to the following condition

$$(2.1) \quad \tilde{\pi}_{(\mu,s)} \text{ is weakly contained in } \{\tilde{\pi}_{(\mu^j,s_j)}\}_j.$$  

Mackey’s theory of induced representations tells us that $\tilde{\pi}_{(\mu,s)}$ is an unitary irreducible representation of the subgroup $K_A \ltimes N$. Then (2.1) is equivalent to the following convergence

$$(2.2) \quad \lim_{j \to +\infty} \tilde{\pi}_{(\mu^j,s_j)} = \tilde{\pi}_{(\mu,s)}.$$  

The representation $\tilde{\pi}_{(\mu,s)}$ of $K_A \ltimes N$ realized on the Fock space $\mathcal{F}_s(\mathcal{V})$, as follows:

$$\tilde{\pi}_{(\mu,s)}(e^{i\theta}, v, tA)f(w) = e^{i\theta}e^{ist-\frac{t}{4}(|\varphi(v)|^2 - \frac{1}{2}w \varphi(v))}f(e^{-i\theta}\varphi(v) + e^{-i\theta}w)$$

where $e^{i\theta} \in K_A$, $v, w \in \mathcal{V}$ and $f \in \mathcal{F}_s(\mathcal{V})$. Now, let $\xi = \sum_{m \in \mathbb{N}^n} c_m h_{m,s}$ be a smooth unit vector in $\mathcal{F}_s(\mathcal{V})$. Then by Corollary 2.4, the fact (2.2) implies that there exists a sequence of smooth unit vectors $\xi_j = \sum_{m \in \mathbb{N}^n} c_{m,j} h_{m,s_j}$ in $\mathcal{F}_{s_j}(\mathcal{V})$, such that

$$(2.3) \quad \lim_{j \to +\infty} \langle d\tilde{\pi}_{(\mu^j,s_j)}(T)\xi_j, \xi_j \rangle = \langle d\tilde{\pi}_{(\mu,s)}(T)\xi, \xi \rangle$$

and for all $p \in \{1, \ldots, n\}$,

$$(2.4) \quad \lim_{j \to +\infty} \langle d\tilde{\pi}_{(\mu^j,s_j)}(E_p)\xi_j, \xi_j \rangle = \langle d\tilde{\pi}_{(\mu,s)}(E_p)\xi, \xi \rangle$$

$$(2.5) \quad \lim_{j \to +\infty} \langle d\tilde{\pi}_{(\mu^j,s_j)}(\mathcal{L}_p)\xi_j, \xi_j \rangle = \langle d\tilde{\pi}_{(\mu,s)}(\mathcal{L}_p)\xi, \xi \rangle.$$  

Taking into account Lemmas 2.5, and 2.6 together with (2.5), we obtain

$$(2.6) \quad \lim_{j \to +\infty} s_j = s$$

and for all $p \in \{1, \ldots, n\}$,

$$(2.7) \quad \lim_{j \to +\infty} s_j \sum_{m \in \mathbb{N}^n} m_p |c_{m,j}|^2 = s \sum_{m \in \mathbb{N}^n} m_p |c_m|^2$$

$$(2.8) \quad \lim_{j \to +\infty} \left(\mu^j_p - \sum_{m \in \mathbb{N}^n} m_p |c_{m,j}|^2\right) = \mu_p - \sum_{m \in \mathbb{N}^n} m_p |c_m|^2.$$  

From the above convergence, we easily see that $\lim_{j \to +\infty} \mu^j_p = \mu_p$ for all $p \in \{1, \ldots, n\}$, i.e.,

$$\mu^j = \mu.$$
for \( j \) large enough.

Conversely, let us assume that \( \lim_{j \to +\infty} s_j = s \) and \( \mu^j = \mu \) for large \( j \). Let \( f \in C_c^\infty (K_A \ltimes N) \), we have

\[
\langle C_{\tilde{h}_0,s_j}^{\pi}(\mu^j, \mu^j), f \rangle = \int_{K_A} \int_N f(u, v, tA) e^{i\mu^j \theta} e^{is_j t - \frac{s_j}{4} |\varphi(v)|^2} \left( \frac{s_j}{2\pi} \right)^n \times \int_V e^{-\frac{s_j}{4} (w, \varphi(v) + |w|^2)} dw d\theta dv dt.
\]

Using Lebesgue’s theorem, one can see that \( \langle C_{\tilde{h}_0,s_j}^{\pi}(\mu^j, \mu^j), f \rangle \) converges to \( \langle C_{h_0,s}^{\pi}(\mu, \mu), f \rangle \). According to Theorem 2.3, we conclude that

\[
\lim_{j \to +\infty} \tilde{\pi}(\mu^j, \mu^j) = \tilde{\pi}(\mu, \mu).
\]

Finally, we refer to the relation (2.2), we can see that the last convergence implies that

\[
\lim_{j \to +\infty} \pi(\mu^j, \mu^j) = \pi(\mu, \mu).
\]

This completes the proof of the theorem. \( \square \)

3. GENERIC ADMISSIBLE COADJOURT ORBITS OF \( G \)

As usual, we continue to use the notations of the previous Sections. We put \( \psi_s := (\mu, (0, sA)) \in g^* \) and we denote by \( G_{\psi_s} \) the stabilizer of \( \psi_s \) in \( G \) under the coadjoint action of \( G \). Then we have

\[
G_{\psi_s} = \{ (k, x, X) \in G, \ Ad^* (k, x, X)(\mu, 0, sA) = (\mu, 0, sA) \} = \{ (k, x, X) \in G, \ k \in K_A \ and \ Ad^*(k)\mu = \mu \}.
\]

Then \( G_{\psi_s} = K_{\psi_s} \ltimes N_{\psi_s} \), then \( \psi_s \) is aligned (see [21]). Note that \( \psi_s \) is an admissible linear form in the sense of Lipsman. A linear form \( \psi \in g^* \) is called admissible if there exists a unitary character \( \chi \) of the identity component of \( G \) such that \( d\chi = i\psi|g^* \). According to Lipsman (by [7, p. 23]) (compare [21]), the representation of \( G \) obtained by holomorphic induction from \( (\mu, 0, sA) \) is equivalent to the representation \( \pi(\mu, \mu) \). We denote by \( g^* \subset g^* \) the set of all admissible linear forms on \( g \). The quotient space \( g^*/G \) is called the space of admissible coadjoint orbits of \( G \). The admissible coadjoint orbits passing through the linear form \( \psi_s \) are called generic orbits of \( G \) and we denote by \( (g^*/G)_{gen} \) the space of all the generic orbits of \( G \). Let \( O_{(\pi, \mu)} \) be the admissible coadjoint orbit of \( G \) contains the linear form \( \psi_s \). Recall that \( O_{(\pi, \mu)} \) is given by

\[
O_{(\pi, \mu)} = \left\{ k.(\mu+(x, X) \times (0, sA) + \frac{1}{2} (x, X) \times ad^*_N(x, X)(0, sA), Ad^*_N(x, X)(0, sA)) \bigg| k \in K, (x, X) \in N \right\} \text{ and } (g^*/G)_{gen} \text{ is the union of all the generic orbits } O_{(\pi, \mu)}. \]
A useful result is now given.

**Lemma 3.1.** Let \( p_G : \mathfrak{g}^* \to \mathfrak{g}^*/G \) be the canonical projection. We equip \( \mathfrak{g}^*/G \) with the quotient topology, i.e., a subset \( V \) in \( \mathfrak{g}^*/G \) is open if and only if \( p_G^{-1}(V) \) is open in \( \mathfrak{g}^* \). Therefore, a sequence \((\mathcal{O}_n^G, n)\) of elements in \( \mathfrak{g}^*/G \) converges to the orbit \( \mathcal{O}_G \) in \( \mathfrak{g}^*/G \) if and only if for any \( l \in \mathcal{O}_G \), there exist \( l_n \in \mathcal{O}_n^G, n \in \mathbb{N} \), such that \( l = \lim_{n \to +\infty} l_n \).

A proof of this Lemma can be found in [7, p. 17]. Now, we are able to prove the following.

**Theorem 3.2.** Let \((\mathcal{O}_n^{G(\mu^n,s_n)}))_{n} \) be a sequence in \( \mathfrak{g}^*/G \). Then \((\mathcal{O}_n^{G(\mu^n,s_n)}))_{n} \) converges to \( \mathcal{O}_G^{\mu,s} \) in \( \mathfrak{g}^*/G \), if and only if \((s_n)_{n} \) converges to \( s \) and \( \mu^n = \mu \) for \( n \) large enough.

**Proof.** We assume that \((s_n)_{n} \) converges to \( s \) and \( \mu^n = \mu \) for \( n \) large enough. Applying Lemma 3.1 we easily see that

\[
\lim_{n \to +\infty} \mathcal{O}_n^{G(\mu^n,s_n)} = \mathcal{O}_G^{\mu,s}.
\]

Conversely, assume that \((\mathcal{O}_n^{G(\mu^n,s_n)}))_{n} \) converges to \( \mathcal{O}_G^{\mu,s} \) in \( \mathfrak{g}^*/G \). Using Lemma 3.1, we can say that there exist two sequences \( \{k_n\}_n \subset K \) and \( \{(x_n,X_n)\}_n \subset N \), such that the sequence \( \left(k_n, (\mu^n + (x_n,X_n) \times (0,s_nA) + \frac{1}{2}(x_n,X_n) \times ad_N^*(x_n,X_n)(0,s_nA), Ad_N^*(x_n,X_n)(0,s_nA))\right)_{n} \) converges to \((\mu, (0,sA))\). Then we obtain

\[
\left(k_n, Ad_N^*(x_n,X_n)(0,s_nA)\right)_{n} \converges{\mu}
\]

and

\[
\lim_{n \to +\infty} Ad_N^*(x_n,X_n)(0,s_nA) = k^{-1}.(0,sA)
\]

(3.1)

From (3.1), we deduce that

(3.4) \[ \lim_{n \to +\infty} s_n A = sk^{-1} A \]

then

(3.5) \[ \lim_{n \to +\infty} s_n = \|s_n A\| = \|sk^{-1} A\| = s \]
and \( k \in K_A \). Hence there exists \( w \) in the Weyl group \( W_{K_A} \) associated to \( K_A \), such that \( Ad^{*}_K(k^{-1})\mu = w.\mu \). According to (3.2), we get
\[
\lim_{n \to +\infty} \left( \mu^n(U) + \frac{1}{2}(x_n, X_n) \times ad^{*}_U(x_n, X_n)(0, s_nA)(U) \right) = (w.\mu)(U)
\]
for all \( U \in \mathfrak{k}_A \). By an easy calculation we obtain the following
(3.6) \[
\lim_{n \to +\infty} \left( \mu^n(U) + \frac{s_n}{2} A([U.(x_n, X_n), (x_n, X_n)]) \right) = (w.\mu)(U)
\]
for all \( U \in \mathfrak{k}_A \). Passing to subsequence if necessary, we may assume without loss of generality that for each \( U \in \mathfrak{k}_A \), the numerical sequence
(3.5) \[
\lim_{n \to +\infty} \left( A([U.(x_n, X_n), (x_n, X_n)]) \right)_n
\]
converges to an element \( t_U \in [-1, 1] \). From (3.5) and (3.6), we deduce that the sequence \( (\mu^n)_n \) converges and we can put
\[
\nu := \lim_{n \to +\infty} \mu^n
\]
i.e., \( \mu^n = \nu \) for \( n \) large enough. The mapping \( \mathfrak{k}_A \ni U \mapsto t_U \in [-1, 1] \) must be identically zero, otherwise we obtain from (3.5) that
(3.7) \[
\lim_{n \to +\infty} \frac{s_n}{2} A([U.(x_n, X_n), (x_n, X_n)]) = |(w.\mu - \nu)(U)| = \frac{s}{2} t_U
\]
for some non-zero \( U \in \mathfrak{k}_A \). Let
\[
t_0 := \sup_{\|U\|=1} |t_U|.
\]
Then we can write
(3.8) \[
\|w.\mu - \nu\| = \sup_{\|U\|=1} |(w.\mu - \nu)(U)| = \frac{s}{2} t_0.
\]
If \( t_0 = 1 \), we obtain \( \|w.\mu - \nu\| = \frac{s}{2} \). This implies that \( s \in 2\mathbb{N} \), which is absurd. It remains to take \( t_0 \in [0, 1] \), hence we obtain from (3.7), that
(3.9) \[
2\|w.\mu - \nu\| \left( \frac{1}{s} \right) \leq \lfloor t_0 \rfloor = 0.
\]
Using (3.10), we deduce that \( s > 1 \), which is also contradicted the fact that \( s \in \mathbb{R}_{>0} \). From (3.5), we conclude that \( \mu^n = w.\mu \) for \( n \) large enough. Since the weights \( \mu^n \) and \( \mu \) are contained in the set \( iC^+_K \) (here \( C^+_K \) denotes the positive Weyl chamber associated to the Weyl group \( W_{K_A} \) [16]) and since each \( W_{K_A} \)-orbit in \( \mathfrak{k}_A^* \) intersects the closure \( \overline{iC^+_K} \) in exactly one point, it follows that
\[
\mu^n = \mu
\]
for \( n \) large enough. \( \square \)

From Theorem 3.1 together with Theorem 3.2, we obtain immediately the following consequence.

Corollary 3.3. The topological space \( (\hat{G})_{gen} \) equipped with the Fell topology is homeomorphic to \( (g^+/G)_{gen} \) endowed with the quotient topology.
On examining Corollary 3.3, one naturally asks the following question: Is that the orbit mapping
\[ g^\dagger/G \ni \mathcal{O}_\pi \mapsto \pi \in \hat{G} \]
is a homeomorphism?

Unfortunately, we are not able to answer this question at present. So we can state the following Conjecture.

**Conjecture 3.4.** Let \( G = K \ltimes N \), such that \((K,N)\) is a nilpotent Gelfand pair satisfies Definitions 1.1 and 1.2. Then the orbit mapping
\[ g^\dagger/G \ni \mathcal{O}_\pi \mapsto \pi \in \hat{G} \]
is a homeomorphism.

**References**


Université de Sfax, Faculté des Sciences Sfax, BP 1171, 3038 Sfax, Tunisie. Laboratoire de recherche LAMHA: LR 11 ES 52.

E-mail address: aymenrahali@yahoo.fr