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RECURRENCE OF 2-DIMENSIONAL QUEUEING PROCESSES, AND RANDOM WALK EXIT TIMES FROM THE QUADRANT

MARC PEIGNÉ AND WOLFGANG WOESS

Abstract. Let $X = (X_1, X_2)$ be a 2-dimensional random variable and $X(n), n \in \mathbb{N}$ a sequence of i.i.d. copies of X. The associated random walk is $S(n) = X(1) + \cdots + X(n)$. The corresponding absorbed-reflected walk $W(n), n \in \mathbb{N}$ in the first quadrant is given by $W(0) = x \in \mathbb{R}_+^2$ and $W(n) = \max\{0, W(n-1) - X(n)\}$, where the maximum is taken coordinate-wise. This is often called the Lindley process and models the waiting times in a two-server queue. We characterize recurrence of this process, assuming suitable, rather mild moment conditions on X. It turns out that this is directly related with the tail asymptotics of the exit time of the random walk $x + S(n)$ from the quadrant, so that the main part of this paper is devoted to an analysis of that exit time in relation with the drift vector, i.e., the expectation of X.

1. Introduction

The waiting times in a single server queue are modeled as a Markov chain $W(n) = W^x(n)$, $n \geq 0$, on the non-negative half-axis $\mathbb{R}_+$. Here, $x \geq 0$ is the initial value, $W(0) = x$, and $W(n) = \max\{0, W(n-1) - X(n)\}$, where the $X(n)$, $n \geq 1$, are i.i.d. real random variables. We think of this process as an absorbing-reflecting random walk on $\mathbb{R}_+$. Namely, setting $S(n) = X(1) + \cdots + X(n)$, the process evolves as the random walk $x - S(n)$ as long as it stays non-negative. Only when it attempts to cross 0 and become negative, the new value is reset to 0 before continuing. This is often called the *Lindley process,* see Lindley [21]. There is an extensive literature on this Markov chain, such as the seminal paper by Kendall [17] plus the references therein, and the monographs by Feller [12], Borovkov [6] and Asmussen [1].

In the present work, we are interested in the multi-dimensional case, and more precisely, the 2-dimensional one, where the $X(n) = \big(X_1(n), X_2(n)\big)$ are i.i.d. copies of a 2-dimensional random variable $X = (X_1\,, X_2)$, the starting point x lies in the non-negative quadrant of $\mathbb{R}^2$, and the maximum in $W(n) = W^x(n) = \max\{0, W(n-1) - X(n)\}$ is taken coordinate-wise. This multi-server queueing process was first studied rigorously by Kiefer and Wolfowitz [18]. Regarding its recurrence, see e.g. the recent note of Cygan and Kloas [8].

Another viewpoint is to consider the 2-dimensional absorbing-reflecting random walk $W^x(n)$ as a stochastic dynamical system (SDS) evolving in $\mathbb{R}_+^2$, as in the work of Leguesdron [20], Peigné [22], Benda [3], [4], [5]. The viewpoint of the last references – inspired by important work of Babillot, Bougerol and Elie [2] – has been to exploit the fact that this SDS is obtained by iterating a sequence of i.i.d. random contractions of $\mathbb{R}_+^2$, and

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that it is *locally contractive.* See Peigné and Woess [23], [24] for the precise definition, an outline of relevant parts of Benda's (not easily accessible) work, and further results concerning conservativity, ergodicity, and invariant measures.

Indeed, when the one-dimensional marginal processes are recurrent – a well-understood situation, and the one we are interested in here – then our SDS is *strongly contractive,* that is

$$|W_n^x - W_n^y| \to 0 \quad \text{almost surely, for all} \ \ x, y \in (0\,,\,\infty)^2\,.$$

This implies that the process is either *transient,* that is, $W^x(n) \to \infty$ almost surely, or else it is recurrent in the following sense: there is a closed, non-empty *limit set* $\mathsf{L} \subset \mathbb{R}_+^2$ such that for any open set U which intersects L and any starting point x, we have

$$\mathbb{P}[W^x(n) \in U \text{ for infinitely many } n] = 1. \tag{1.1}$$

A main goal of this paper is to understand when the 2-dimensional Lindley process is recurrent.

(1.2) Basic Assumption. Throughout this paper, we assume that

(i) X is not constrained to a hyperplane, and that
(ii) $\mathbb{P}[X \in (0\,,\,\infty)^2] > 0\,.$

This is a quite natural assumption on the support of the distribution of $X = (X_1\,,X_2)$; see the discussion in §4. Based on this assumption and the preceding observations, the following is our main result concerning the two-dimensional Lindley process, with quite general moment assumptions for the random variable X which models the increments, and its positive and negative parts $X_i^+ = \max\{X_i\,,0\}$ and $X_i^- = \max\{-X_i\,,0\}$, $i = 1, 2$.

(1.3) Theorem. (a) *If for at least one $i \in \{1, 2\}$,*

$$\mathsf{E}(X_i^+) < \mathsf{E}(X_i^-) \le \infty\,,$$

then $W(n)$ is transient.

(b) *If for both $i \in \{1, 2\}$,*

$$\mathsf{E}(X_i^-) < \mathsf{E}(X_i^+) < \infty\,,$$

then $W(n)$ is positive recurrent.

(c) *If for both $i \in \{1, 2\}$,*

$$\mathsf{E}(X_i) = 0 \quad \textit{and} \quad \mathsf{E}\big(|X_i|^{\max\{2+\delta\,,\,\pi/\arccos(-\rho)\}}\big) < \infty\,,$$

where $\delta > 0$ (arbitrary) and $\rho = \rho(X_1\,,X_2)$ is the correlation coefficient of X_1 and X_2, then $W(n)$ is null recurrent if $\rho \ge 0$, and transient, otherwise.

(d) *If for some $\delta > 0$, up to a possible exchange of the two coordinates,*

$$\mathsf{E}(|X_i|^{2+\delta}) < \infty \quad \textit{for} \ \ i = 1, 2, \quad \mathsf{E}(X_1) = 0, \quad \mathsf{E}(X_2) > 0, \quad \textit{and} \quad \mathsf{E}((X_2^-)^{3+\delta}) < \infty\,,$$

then $W(n)$ is null recurrent.

The two interesting cases are (c) and (d). In order to prove the theorem, we use the following.

(1.4) Definition. For $x \in (0\,,\,\infty)^2$, the exit time of the random walk from the positive quadrant is

$$\tau_x = \inf\{n \geq 1 : x + S(n) \notin (0\,,\,\infty)^2\}\,.$$

In cases (c) and (d), τ_x is almost surely finite. We have the following relation of the Lindley process with these exit times.

(1.5) Lemma. *For $x = (x_1\,,x_2) \in (0\,,\,\infty)^2$,*

$$\mathbb{P}\big[W^0(n) \in [0,x_1) \times [0,x_2)\big] = \mathbb{P}[\tau_x > n].$$

Thus, we are led to another topic of its own big interest, currently the object of very active work: the tail asymptotics of exit times of random walks from cones, in our case, the positive quadrant. Our main source is the profound paper of Denisov and Wachtel [9] on exit times from cones for centered random walks. As we shall see, this leads to statement (c) of Theorem 1.3. Our main focus is on the case when one coordinate is centered and the second one has positive drift. Exit times from cones for random walks with drift were considered by Duraj [11] and Garbit and Raschel [13], who however do not provide the tail asymptotics which we need. The central body of the present paper concerns that case. We summarize.

(1.6) Theorem. *Under the same moment conditions as in Theorem 1.3, we have the following for any $x \in (0\,,\,\infty)^2$, as $n \to \infty$.*

In case (a), *if* $\mathsf{E}(X_i) < 0$ *and* $r \geq 1$ *then* $\mathsf{E}\big((X_i^+)^r\big) < \infty$ *implies* $\mathsf{E}(\tau_x^r) < \infty$, *whence* $\mathbb{P}[\tau_x > n] = \mathfrak{o}(n^{-r})$.

In case (b), $\lim\limits_{n\to\infty} \mathbb{P}[\tau_x > n] = \mathbb{P}[\tau_x = \infty] > 0$

In case (c), $\mathbb{P}[\tau_x > n] \sim v(x)\,n^{-1/p}$, *where* $p = 2\arccos(-\rho)/\pi\,.$

In case (d), $\mathbb{P}[\tau_x > n] \sim \kappa\,h(x)\,n^{-1/2}\,.$

In (c) and (d), v and h are positive harmonic functions for the respective random walk $x + S(n)$ killed when exiting $(0\,,\,\infty)^2$, and $\kappa > 0$.

(The reader will see below why we prefer not to incorporate the constant κ into $h(x)$.)

These are the main results of the present paper. The main body of the work consists in the proof of Theorem 1.6(d), preceded by (a)–(c).

In §2, we collect some preliminary facts concerning the one-dimensional centered case. Furthermore, the asymptotic behaviour of the corresponding harmonic function on $\mathbb{R}^+$ is crucial for us. V. Wachtel has comunicated to us an improved version, due to himself and D. Denisov, of the outline in [9, 2.4]. With their kind permission , this is explained in the Appendix of this paper.

In §3, we provide the proofs of Theorem 1.3. Only in §4, we come back to the queueing process, i.e., the proofs of Lemma 1.5 and Theorem 1.3 plus additional observations and a discussion.

2. Dimension one: exit times from the half-line

In this section only, X is a one-dimensional random variable, $X(n)$, $n \in \mathbb{N}$, are i.i.d. copies of X, and $S(n) = X(1) + \cdots + X(n)$ (with $S(0) = 0$). For $x \in (0\,,\,\infty)$,

$$\tau_x = \inf\{n \in \mathbb{N} : x + S(n) \le 0\}. \tag{2.1}$$

We collect several facts concerning τ_x and related random variables, such as

$$\underline{M}(n) = \min\{S(k) : 0 \le k \le n\} \quad \text{and} \quad \overline{M}(n) = \max\{S(k) : 0 \le k \le n\}. \tag{2.2}$$

For the following, see the monograph by Gut [14, Thm. 3.1].

(2.3) Lemma. *If X is integrable and $\mathsf{E}(X) < 0$ then $\mathsf{E}(\tau_x) < \infty$. In addition, for any $r \ge 1$,*

$$\mathsf{E}\big((X^+)^r\big) < \infty \iff \mathsf{E}(\tau_x^r) < \infty\,.$$

For the following, [14] is again a good source, as well as the very nice exposition by Janson [16].

(2.4) Lemma. *If X is integrable and $\mathsf{E}(X) > 0$ then*

$$\mathbb{P}[\tau_x = \infty] > 0 \quad \textit{and} \quad \underline{M} = \inf\{S(n) : n \ge 0\} > -\infty \quad \textit{almost surely,}$$

so that $T_{\min} = \inf\{n \ge 0 : S(n) = \underline{M}\} < \infty$ almost surely. Furthermore, for any $r > 0$,

$$\mathsf{E}\big((X^-)^{r+1}\big) < \infty \iff \mathsf{E}\big(|\underline{M}|^r\big) < \infty \iff \mathsf{E}(\tau_x^r\,;\,\tau_x < \infty) < \infty \iff \mathsf{E}\big(T_{\min}^r\big) < \infty\,.$$

For the remainder of this section, we assume that X is centered.

(2.5) Proposition. *Suppose that $\mathsf{E}(X^2) < \infty$ and $\mathsf{E}(X) = 0$. Then the function*

$$h_1(x) = x - \mathsf{E}\big(x + S(\tau_x)\,;\,\tau_x < \infty\big)\,, \quad x > 0,$$

is finite and harmonic for the random walk killed when exiting $(0\,,\,\infty)$, that is,

$$h_1(x) = \mathsf{E}\Big(h_1\big(x + S(1)\big)\,;\,\tau_x > 1\Big).$$

We have

$$\lim_{x\to\infty} \frac{h_1(x)}{x} = 1. \tag{2.6}$$

Proof. As $\mathsf{E}(X^2) < \infty$, finiteness of $g_1(x) = \mathsf{E}\big(x + S(\tau_x)\,;\,\tau_x < \infty\big)$ follows from classical results, see e.g. [12]. Note that $g_1(x) \le 0$. Harmonicity is easily proved: Since $\mathsf{E}(X) = 0$,

$$\begin{aligned} x &= \int_{y>-x} (x+y)\,\mathbb{P}[X = dy] + \int_{y\le -x} (x+y)\,\mathbb{P}[X = dy] \\ &= \mathsf{E}\big(x + S(1)\,;\,\tau_x > 1\big) + \int_{y\le -x} (x+y)\,\mathbb{P}[X = dy]. \end{aligned}$$

On the other hand, decomposing with respect to the first step of the walk,

$$\begin{aligned} g_1(x) &= \int_{y>-x} \mathsf{E}\big(x + y + S(\tau_{x+y})\,;\,\tau_{x+y} < \infty\big)\,\mathbb{P}[X = dy] + \int_{y\le -x} (x+y)\,\mathbb{P}[X = dy] \\ &= \mathsf{E}\big(g_1\big(x + S(1)\big)\,;\,\tau_x > 1\big) + \int_{y\le -x} (x+y)\,\mathbb{P}[X = dy]\,. \end{aligned}$$

Taking the difference, we get harmonicity of h_1 as required.

Finally, a proof by D. Denisov and V. Wachtel that $h_1(x)/x \to 1$ as $x \to \infty$ was comunicated to us by V. Wachtel. It is provided in the Appendix. $\square$

In the next section, we shall need the following Lemma. The first part is in principle known; the second part is adapted from Pham [25, Lemma 4.5]. For the sake of completeness we provide a proof.

(2.7) Lemma. *Suppose that $\mathsf{E}(X^2) < \infty$ and $\mathsf{E}(X) = 0$. Then for any $p > 2$ there is a constant $c_p > 0$ such that when $\mathsf{E}(|X|^p) < \infty$, for each $t > 0$ and $n \in \mathbb{N}$*

$$\mathsf{E}\big(|S(n)|^p\big) \le c_p\, n^{p/2}\, \mathsf{E}\big(|X|^p\big) \quad \textit{and} \quad \mathbb{P}[\overline{M}(n) > t] \le c_p\, \mathsf{E}\big(|X|^p\big)\, n^{p/2}\, t^{-p}\,.$$

Furthermore, for any $\alpha > 0$,

$$\mathsf{E}\big(n^{1/2} + \overline{M}(n)\,;\, \overline{M}(n) > n^{1/2+\alpha}\big) \le c_p\, \mathsf{E}\big(|X|^p\big)\, \frac{2p-1}{p-1}\, n^{1/2-(p-1)\alpha}\,.$$

Proof. Since $\big(S(n)\big)$ is a martingale, Doob's maximal inequality implies

$$\mathbb{P}[\overline{M}(n) > t] \le \mathsf{E}\big(|S(n)|^p\big)\, t^{-p}\,.$$

By a well-known inequality which one finds e.g. in Burkholder [7, Proof of Thm. 3.2],

$$\mathsf{E}\big(|S(n)|^p\big) \le c_p\, \mathsf{E}\bigg(\Big(\sum_{k=1}^{n} X(k)^2\Big)^{p/2}\bigg).$$

Now Hölder's inequality concludes the proof of the first two inequalities. For the third inequality,

$$\begin{aligned}
&\mathsf{E}\big(n^{1/2} + \overline{M}(n)\,;\, \overline{M}(n) > n^{1/2+\alpha}\big) \\
&\qquad = \big(n^{1/2} + n^{1/2+\alpha}\big)\, \mathbb{P}\big[\overline{M}(n) > n^{1/2+\alpha}\big] + \mathsf{E}\Big(\big(\overline{M}(n) - n^{1/2+\alpha}\big)^+\Big) \\
&\qquad \le 2n^{1/2+\alpha}\, \mathbb{P}\big[\overline{M}(n) > n^{1/2+\alpha}\big] + \int_{n^{1/2+\alpha}}^{\infty} \mathbb{P}\big[\overline{M}(n) > t\big]\, dt \\
&\qquad \le 2\, c_p\, \mathsf{E}\big(|X|^p\big)\, n^{1/2-(p-1)\alpha} + c_p\, \mathsf{E}\big(|X|^p\big) \int_{n^{1/2+\alpha}}^{\infty} n^{p/2}\, t^{-p}\, dt \\
&\qquad = c_p\, \mathsf{E}\big(|X|^p\big)\, \frac{2p-1}{p-1}\, n^{1/2-(p-1)\alpha}\,,
\end{aligned}$$

as proposed. $\square$

Uniform convergence in the following proposition was proved by Doney [10].

(2.8) Proposition. *If $\mathsf{E}\big(|X|^2\big) < \infty$ and $\mathsf{E}(X) = 0$, then*

$$\mathbb{P}[\tau_x > n] \sim \kappa\, h_1(x)\, n^{-1/2}\,, \quad \textit{where} \quad \kappa = \big(\pi\ \mathsf{Var}(X)/2\big)^{-1/2}\,,$$

uniformly as $n \to \infty$ and $0 < x < \theta_n\, n^{1/2}$, where (θ_n) is an arbitrary positive sequence that converges to 0.

3. Dimension two: exit times from the quadrant

This section is devoted to the proof of Theorem 1.6. We return to the situation of the Introduction, where $X = (X_1\,, X_2)$ and the $X(n)$ are i.i.d. copies of X. Thus, $S(n) = \big(S_1(n), S_2(n)\big)$, and if $x = (x_1\,, x_2) \in (0\,,\,\infty)^2$ then

$$\tau_x = \min\{\tau_{x_1}\,, \tau_{x_2}\}\,, \quad \text{where} \quad \tau_{x_i} = \tau_{i,x_i} = \inf\{n \in \mathbb{N} : x_i + S_i(n) \le 0\,,\ i = 1,2\}\,. \tag{3.1}$$

Negative drift in at least one coordinate

Proof of Theorem 1.6(a). It is well known that if $\mathsf{E}(X_i) < 0$ then $\mathsf{E}\big((X_i^+)^r\big) < \infty$ if and only if $\mathsf{E}(\tau_{x_i}^r) < \infty$, see e.g. [12], [14] and [16]. In view of (3.1), this implies $\mathsf{E}(\tau_x^r) < \infty$. □

Positive drift in both coordinates

Proof of Theorem 1.6(b). We use Lemma 2.4 and set $\underline{M}_i = \inf\{S_i(n) : n \ge 0\}$. These random variables are almost surely finite. Therefore, for every n,

$$\mathbb{P}[\tau_x > n] = \mathbb{P}[\tau_{x_1} > n\,,\, \tau_{x_2} > n] \ge \mathbb{P}[\underline{M}_1 < -x_1\,,\, \underline{M}_2 < -x_2] \to 1\,, \quad \text{as } x_1\,, x_2 \to \infty.$$

Thus, there is $b > 0$ such that $\mathbb{P}[\tau_x = \infty] \ge 1/2$ if $x_1\,, x_2 > b$. If $x \in (0\,,\,\infty)^2$ is arbitrary then the Basic Assumption 1.2, namely that $\mathbb{P}[X \in (0\,,\,\infty)^2] > 0$ yields that with positive probability, the random walk $x + S(n)$ starting at x can reach $(b\,,\,\infty) \times (b\,,\,\infty)$ without exiting $(0\,,\,\infty)^2$. Therefore $\mathbb{P}[\tau_x = \infty] > 0$. □

The centered case

Proof of Theorem 1.6(c). We assume that both coordinates of X have finite moment of order $\max\{2+\delta\,,\, \pi/\arccos(-\rho)\}$ and are centered, with some $\delta > 0$ and $\rho = \rho(X_1\,, X_2) \in (-1\,,\,1)$.

The result is in principle stated in [9, Example 2]. For the sake of completeness, we give a few hints.

Let $\mathbb{K}_\alpha$ be a standard closed cone in $\mathbb{R}^2$: the sides of the cone are two half-lines issuing from the origin, which is the cone's vertex. The opening angle of the cone is $\alpha \in (0\,,\,\pi)$. On $\mathbb{K}_\alpha$, the paper [9] considers a random walk $\tilde{x} + \widetilde{S}(n)$ with $\tilde{x} \in \mathbb{K}_\alpha^o$ and $\widetilde{S}(n) = \widetilde{X}(1) + \cdots + \widetilde{X}(n)$, where the $\widetilde{X}(n)$ are i.i.d. copies of $\widetilde{X} = (\widetilde{X}_1\,, \widetilde{X}_2)$. The assumptions are that the coordinates of $\widetilde{X}$ have finite moment of order $\max\{2+\delta\,,\, \pi/\alpha\}$, are centered, with $\mathsf{E}(\widetilde{X}_i^2) = 1$ and $\rho(\widetilde{X}_1\,, \widetilde{X}_2) = \mathsf{E}(\widetilde{X}_1\widetilde{X}_2) = 0$. For the associated exit time $\widetilde{\tau}_{\tilde{x}}$, the main results of [9] yield that

$$\mathbb{P}[\widetilde{\tau}_{\tilde{x}} > n] \sim V(\tilde{x})\, n^{-1/p}\,, \quad \text{where} \quad p = 2\alpha/\pi\,.$$

Thus, the method is to decorrelate X_1 and X_2 and thereby to pass from $\mathbb{R}^2_+$ to a possibly modified cone. This can for example be achieved by the matrix transformation

$$\begin{pmatrix} \widetilde{X}_1 \\ \widetilde{X}_2 \end{pmatrix} = M \begin{pmatrix} X_1 \\ \widetilde{X}_2 \end{pmatrix}, \quad \text{where} \quad M = \begin{pmatrix} \frac{1}{\sigma_1} & 0 \\ \frac{-\rho}{\sigma_1\sqrt{1-\rho^2}} & \frac{1}{\sigma_2} \end{pmatrix}$$

Here, $\sigma_i^2 = \mathsf{E}(X_i^2)$. Then $(\widetilde{X}_1\,,\widetilde{X}_2)$ are centered, with variances $=1$, and non correlated. The mapping $x \mapsto Mx$ transforms $\mathbb{R}_+^2$ in a cone $\mathbb{K}_\alpha$ with $\alpha = \arccos(-\rho)$. Thus, we have

$$\mathbb{P}[\tau_x > n] = \mathbb{P}[\widetilde{\tau}_{Mx} > n] \sim V(Mx)\, n^{-1/p}\,, \quad \text{where} \quad p = 2\arccos(-\rho)/\pi\,.$$

Since $V(\tilde{x})$ is positive harmonic for the random walk killed when exiting $\mathbb{K}_\alpha^o$, the function $v(x) = V(Mx)$ is positive harmonic for the original random walk killed when exiting $(0\,,\infty)^2$. □

The mixed case positive drift - zero drift

We finally come to the main body of this paper, namely the proof of statement (d) of Theorem 1.6. We repeat that we assume that X_1 is centered with moment of order $2+\delta$ and that X_2 has finite second moment, $\mathfrak{m} = \mathsf{E}(X_2) > 0$ and in addition $\mathsf{E}\big((X_2^-)^{3+\delta}\big) < \infty$ for some $\delta > 0$.

The first step consists in finding the right harmonic function for the random walk restricted to the quadrant. For this, we were inspired by the work of Ignatiouk-Robert and Loree [15], who however assume exponential moment conditions.

(3.2) Proposition. *Under the assumptions (d), the function*

$$h(x) = x_1 - \mathsf{E}\big(x_1 + S_1(\tau_x)\,;\, \tau_x < \infty\big)\,, \quad x \in (0\,,\infty)^2\,,$$

is strictly positive and harmonic for the random walk killed upon exiting the positive quadrant. It satisfies

$$h(x) \le h_1(x_1) \quad \textit{and} \quad \lim_{x_1\,,\,x_2\to\infty} \frac{h(x)}{x_1} = 1, \tag{3.3}$$

where h_1 is the function of Proposition 2.5.

Proof. Write $g(x) = \mathsf{E}\big(x_1 + S_1(\tau_x)\,;\,\tau_x < \infty\big)$. Then by (3.1),

$$\begin{aligned}
g(x) &= \tilde{g}(x) + f(x) = g_1(x_1) - \tilde{f}(x) + f(x)\,, \quad \text{where}\\
\tilde{g}(x) &= \mathsf{E}\big(x_1 + S_1(\tau_{x_1})\,;\, \tau_{x_1} \le \tau_{x_2}\,,\ \tau_{x_1} < \infty\big)\,,\\
f(x) &= \mathsf{E}\big(x_1 + S_1(\tau_{x_2})\,;\, \tau_{x_2} < \tau_{x_1} < \infty\big)\,, \quad \text{and}\\
\tilde{f}(x) &= \mathsf{E}\big(x_1 + S_1(\tau_{x_1})\,;\, \tau_{x_2} < \tau_{x_1} < \infty\big)\,,
\end{aligned}$$

and g_1 is the function from Proposition 2.5. Since $\tilde{f}(x) \le 0$ and $f(x) \ge 0$, we see that $g(x) \ge g_1(x_1)$, whence $h(x) \le h_1(x_1)$.

Since $0 \le -\tilde{g}(x) \le -g_1(x_1)$, we get from Proposition 2.5, resp. the Appendix, that

$$\lim_{x_1\to\infty} \frac{\tilde{g}(x)}{x_1} = 0 \quad \text{uniformly in } \ x_2\,.$$

<u>Claim 1.</u> $$\lim_{x_2\to\infty} f(x) = 0 \quad \text{uniformly for } \ 0 < x_1 \le x_2^{2+\delta}\,.$$

To prove this, we use Lemma 2.4: since $\mathsf{E}\big((X_2^-)^{3+\delta}\big) < \infty\,,$

$$\mathsf{E}(\tau_{x_2}^{2+\delta}\,;\,\tau_{x_2} < \infty) < \infty \quad \text{and} \quad \mathsf{E}(\underline{M}_2^{2+\delta}\,;\,\tau_{x_2} < \infty) < \infty\,,$$

where $\underline{M}_2 = \inf\{S_2(n) : n \ge 0\}$. Write $\sigma_1^2 = \mathsf{E}(X_1^2)$, and recall that $\tau_{x_1} < \infty$ almost surely. For every $k \in \mathbb{N}$, using the Cauchy-Schwarz inequality,

$$\mathsf{E}\big(x_1 + S_1(k)\,;\,\tau_{x_1} > \tau_{x_2} = k\big) \le x_1\,\mathbb{P}[\tau_{x_1} > \tau_{x_2} = k] + \underbrace{\mathsf{E}\big(S_1(k)^2\big)^{1/2}}_{\sigma_1\,k^{1/2}}\,\mathbb{P}[\tau_{x_2} = k]^{1/2}$$

We again use the Cauchy-Schwarz inequality in the following estimate.

$$\begin{aligned} f(x) &= \sum_{k=1}^{\infty} \mathsf{E}\big(x_1 + S_1(k)\,;\,\tau_{x_1} > \tau_{x_2} = k\big) \\ &\le x_1\,\mathbb{P}[\tau_{x_1} > \tau_{x_2}] + \sigma_1 \sum_{k=1}^{\infty} k^{-(1+\delta)/2} \cdot k^{(2+\delta)/2}\,\mathbb{P}[\tau_{x_2} = k]^{1/2} \qquad (3.4) \\ &\le x_1\,\mathbb{P}[\tau_{x_1} > \tau_{x_2}] + C_\delta\,\mathsf{E}(\tau_{x_2}^{2+\delta}\,;\,\tau_{x_2} < \infty)^{1/2}, \quad \text{where} \quad C_\delta = \sigma_1 \left(\sum_{k=1}^{\infty} k^{-(1+\delta)}\right)^{1/2}. \end{aligned}$$

Now, our moment assumption implies that

$$\mathbb{P}[\tau_{x_2} < \infty] \le \mathbb{P}[\underline{M}_2 \le -x_2] = \mathfrak{o}(x_2^{-(2+\delta)}) \quad \text{as} \quad x_2 \to \infty\,. \tag{3.5}$$

Therefore

$$\lim_{x_2 \to \infty} x_1\,\mathbb{P}[\tau_{x_1} > \tau_{x_2}] = 0 \quad \text{uniformly for} \quad 0 < x_1 \le x_2^{2+\delta}\,.$$

For the second term, we use Lemma 2.4, with $T_{\min}$ referring to the second coordinate:

$$\tau_{x_2}^{2+\delta}\,\mathbf{1}_{[\tau_{x_2}<\infty]} \le T_{\min}^{2+\delta}\,\mathbf{1}_{[\tau_{x_2}<\infty]} \le T_{\min}^{2+\delta}\,,$$

and the middle term tends to 0 almost surely, as $x_2 \to \infty$. By dominated convergence,

$$\lim_{x_2 \to \infty} \mathsf{E}(\tau_{x_2}^{2+\delta}\,;\,\tau_{x_2} < \infty) = 0. \tag{3.6}$$

This concludes the proof of Claim 1 as well as of (3.3).

Claim 2. $\qquad h(x) \ge 0 \quad \text{for all} \quad x \in (0\,,\,\infty)^2\,.$

To see this, recall that $0 \le -\tilde{g}(x) \le -g_1(x_1) < \infty$, while we know from Claim 1 that $0 \le f(x) < \infty$. Therefore

$$\mathsf{E}\big(|x_1 + S_1(\tau_x)|\,;\,\tau_x < \infty\big) < \infty\,,$$

and by dominated convergence and the martingale stopping lemma,

$$\begin{aligned} g(x) &= \lim_{n\to\infty} g_n(x)\,, \qquad \text{where} \\ g_n(x) &= \mathsf{E}\big(x_1 + S_1(\tau_x)\,;\,\tau_x \le n\big) \\ &= \underbrace{\mathsf{E}\big(x_1 + S_1(\min\{n, \tau_x\})\big)}_{=x_1} - \mathsf{E}\big(\underbrace{x_1 + S_1(n)}_{>0}\,;\,\tau_x > n\big) \le x_1\,. \end{aligned}$$

This proves Claim 2.

It remains to prove that $h > 0$ strictly on $(0\,,\,\infty)^2$. Since $\tilde{g}(x) \le 0$, the estimate (3.4) plus the first inequality in (3.5) lead to

$$h(x) \ge x_1 - f(x) \ge x_1\,\mathbb{P}[\,\underline{M}_2 > -x_2] - C_\delta\,\mathsf{E}(\tau_{x_2}^{2+\delta}\,;\,\tau_{x_2} < \infty)^{1/2}.$$

By (3.6), this yields that for every $b_1 > 0$ (possibly small) there is $b_2 > 0$ (possibly large) such that $h > 0$ on $(b_1\,,\,\infty) \times (b_2\,,\,\infty)$. Now, by the Basic Assumption 1.2, there is $(a_1\,,a_2) \in (0\,,\,\infty)^2$ such that $\mathbb{P}[X \in (a_1\,,\,\infty) \times (a_2\,,\,\infty)] > 0$. Inductively, we get

$$\mathbb{P}[S(k) \in (k\,a_1\,,\,\infty) \times (k\,a_2\,,\,\infty) \text{ for } k = 1, \dots, n] > 0$$

for every $n \in \mathbb{N}$. Now let $x \in (0\,,\,\infty)^2$. Given x_1, by the above there is b_2 such that $h > 0$ on $(x_1\,,\,\infty) \times (b_2\,,\,\infty)$. Then there is n such that $x_2 + n\,a_2 \ge b_2$. But then

$$\mathbb{P}[x + S(n) \in (x_1\,,\,\infty) \times (b_2\,,\,\infty)\,, \tau_x > n] > 0\,.$$

Therefore, since we already know that $h \ge 0$,

$$\begin{aligned} h(x) &= \mathsf{E}\Big(h\big(x + S(n)\big)\,;\,\tau_x > n\big)\Big) \\ &\ge \mathsf{E}\Big(h\big(x + S(n)\big)\,;\,x + S(n) \in (x_1\,,\,\infty) \times (b_2\,,\,\infty)\,, \tau_x > n\big)\Big) > 0\,. \end{aligned}$$

This completes the proof of the proposition. □

We now choose $\varepsilon \in (0\,,\,1/2)$ and define for $x = (x_1\,,x_2) \in (0\,,\,\infty)^2$

$$\nu_{x_2}(n) = \inf\{k \in \mathbb{N} : x_2 + S_2(k) \ge \mathfrak{m}\,n^{1-3\varepsilon/2}\}.$$

(Recall that $\mathfrak{m} = \mathsf{E}(X_2)$.)

(3.7) Lemma. $$\mathbb{P}[\nu_{x_2}(n) > n^{1-\varepsilon}] \le C\,n^{-(1-\varepsilon)(1+\delta/2)}\,.$$

Proof. Let $\sigma_2^2 = \mathsf{Var}(X_2)$. Setting $p = 2 + \delta$, we apply the first inequality in Lemma 2.7 to $X_2 - \mathfrak{m}$ and use Markov's inequality[1]:

$$\begin{aligned} \mathbb{P}\big[\nu_{x_2}(n) > n^{1-\varepsilon}\big] &\le \mathbb{P}\big[x_2 + S_2(n^{1-\varepsilon}) < \mathfrak{m}\,n^{1-3\varepsilon/2}\big] \\ &\le \mathbb{P}\big[S_2(n^{1-\varepsilon}) - \mathfrak{m}\,n^{1-\varepsilon} < -\mathfrak{m}\,(1 - n^{-\varepsilon/2})\,n^{1-\varepsilon}\big] \\ &\le \mathbb{P}\big[\,|S_2(n^{1-\varepsilon}) - \mathfrak{m}\,n^{1-\varepsilon}|^p > \big(\mathfrak{m}\,(1 - n^{-\varepsilon/2})\big)^p\,n^{(1-\varepsilon)p}\big] \\ &\le \frac{c_p\,n^{(1-\varepsilon)p/2}\,\mathsf{E}\big(|X_2 - \mathfrak{m}|^p\big)}{\big(\mathfrak{m}\,(1 - n^{-\varepsilon/2})\big)^p\,n^{(1-\varepsilon)p}}\,. \end{aligned}$$

This proves the lemma. □

Proof of Theorem 1.6(d). We choose s, ε and ϕ_n as follows.

$$\varepsilon = \frac{1}{2} - \frac{1}{s} \quad \text{and} \quad \phi_n = n^{-\varepsilon/s} + n^{-\varepsilon}\,, \quad s > 2\,. \tag{3.8}$$

For the starting point $x \in (0\,,\,\infty)^2$, we assume that $x_1 < n^{1/2-\varepsilon} = n^{1/s}$ and $x_2 < \mathfrak{m}\,n^{1-3\varepsilon/2}$.

$$\mathbb{P}[\tau_x > n] = \mathbb{P}[\tau_x > n\,;\,\nu_{x_2}(n) \le n^{1-\varepsilon}] + \mathbb{P}[\nu_{x_2}(n) \ge \tau_x > n] + \mathbb{P}[\tau_x > \nu_{x_2}(n) > n^{1-\varepsilon}]$$

this is only an inequality here. NO! Disjoint decomposition.

$$= \mathbb{P}[\tau_x > n\,;\,\nu_{x_2}(n) \le n^{1-\varepsilon}] + \mathfrak{o}(n^{-1/2}),$$

because by Lemma 3.7,

$$\mathbb{P}[\nu_{x_2}(n) \ge \tau_x > n] + \mathbb{P}[\tau_x > \nu_{x_2}(n) > n^{1-\varepsilon}] \le \mathbb{P}[\nu_{x_2}(n) > n^{1-\varepsilon}] \le C\,n^{-(1-\varepsilon)(1+\delta/2)}\,.$$

[1]Here and in many subsequent instances, values such as $n^{1-\varepsilon}$ should be rounded to the next lower, resp. upper integer. This is omitted in the notation, since it will be clear from the context.

Now we write

$$\begin{aligned}
&\mathbb{P}[\tau_x > n\,;\, \nu_{x_2}(n) \le n^{1-\varepsilon}] \\
&\qquad = \sum_{k=1}^{n^{1-\varepsilon}} \int_{y\,:\,y_2 \ge \mathfrak{m}\, n^{1-3\varepsilon/2}} \mathbb{P}[\tau_x > k\,,\ \nu_{x_2}(n) = k\,,\ x + S(k) = dy]\; \mathbb{P}[\tau_y > n-k]\,.
\end{aligned}$$

We note that $\mathbb{P}[\tau_{y_1} > n-k] - \mathbb{P}[\tau_{y_2} \le n] \le \mathbb{P}[\tau_y > n-k] \le \mathbb{P}[\tau_{y_1} > n-k]$. By (3.5), we have for $y_2 \ge \mathfrak{m}\, n^{1-3\varepsilon/2}$ that

$$\mathbb{P}[\tau_y > n-k] = \mathbb{P}[\tau_{y_1} > n-k] + \mathfrak{o}(n^{-(1-3\varepsilon/2)(2+\delta)}) = \mathbb{P}[\tau_{y_1} > n-k] + \mathfrak{o}(n^{-1/2})$$

by our choices of ε and s. Therefore, independently of the choice of x,

$$\begin{aligned}
&\mathbb{P}[\tau_x > n\,;\, \nu_{x_2}(n) \le n^{1-\varepsilon}] \\
&\quad = \sum_{k=1}^{n^{1-\varepsilon}} \int_{y\,:\,y_2 \ge \mathfrak{m}\, n^{1-3\varepsilon/2}} \mathbb{P}[\tau_x > k\,,\ \nu_{x_2}(n) = k\,,\ x + S(k) = dy]\; \mathbb{P}[\tau_{y_1} > n-k]\ + \mathfrak{o}(n^{-1/2}) \\
&\quad = \mathbf{A} + \mathbf{B} + \mathfrak{o}(n^{-1/2})\,, \qquad \text{where}
\end{aligned}$$

$$\begin{aligned}
\mathbf{A} &= \sum_{k=1}^{n^{1-\varepsilon}} \int_{\substack{y\,:\,y_2 \ge \mathfrak{m}\, n^{1-3\varepsilon/2} \\ y_1 \le \phi_n\, n^{1/2}}} \mathbb{P}[\tau_x > k\,,\ \nu_{x_2}(n) = k\,,\ x + S(k) = dy]\; \mathbb{P}[\tau_{y_1} > n-k] \quad \text{and} \\
\mathbf{B} &= \sum_{k=1}^{n^{1-\varepsilon}} \int_{\substack{y\,:\,y_2 \ge \mathfrak{m}\, n^{1-3\varepsilon/2} \\ y_1 > \phi_n\, n^{1/2}}} \mathbb{P}[\tau_x > k\,,\ \nu_{x_2}(n) = k\,,\ x + S(k) = dy]\; \mathbb{P}[\tau_{y_1} > n-k]\,.
\end{aligned}$$

By propositions 2.5, 2.8 and 3.2, we have with $\kappa = (\pi\, \sigma_2^2/2)^{-1/2}$

$$\mathbb{P}[\tau_{y_1} > n-k] \sim \kappa\, h_1(y_1)\, n^{-1/2} \sim \kappa\, h(y)\, n^{-1/2}$$

uniformly for y in the range of integration of term $\mathbf{A}$ and $k \le n^{1-\varepsilon}$. Therefore

$$\begin{aligned}
\mathbf{A} &\sim \kappa\, n^{-1/2}\, \mathsf{E}\Bigl(h\Bigl(x + S\bigl(\nu_{x_2}(n)\bigr)\Bigr)\,;\ \begin{matrix} \tau_x > \nu_{x_2}(n)\,,\ \nu_{x_2}(n) \le n^{1-\varepsilon}\,, \\ x_1 + S_1\bigl(\nu_{x_2}(n)\bigr) \le \phi_n\, n^{1/2} \end{matrix} \Bigr) \\
&= \kappa\, n^{-1/2} (\mathbf{C} - \mathbf{D})\,, \qquad \text{where}
\end{aligned}$$

$$\begin{aligned}
\mathbf{C} &= \mathsf{E}\Bigl(h\Bigl(x + S\bigl(\nu_{x_2}(n)\bigr)\Bigr)\,;\ \tau_x > \nu_{x_2}(n)\,,\ \nu_{x_2}(n) \le n^{1-\varepsilon} \Bigr) \qquad \text{and} \\
\mathbf{D} &= \mathsf{E}\Bigl(h\Bigl(x + S\bigl(\nu_{x_2}(n)\bigr)\Bigr)\,;\ \tau_x > \nu_{x_2}(n)\,,\ \nu_{x_2}(n) \le n^{1-\varepsilon}\,,\ x_1 + S_1\bigl(\nu_{x_2}(n)\bigr) > \phi_n\, n^{1/2} \Bigr).
\end{aligned}$$

Also, $\mathbb{P}[\tau_{y_1} > n-k] \le C\, y_1\, n^{-1/2}$ for y in the range of integration of term $\mathbf{B}$. Therefore we get that both $n^{1/2}\, \mathbf{B}$ and $\mathbf{D}$ are bounded above by $C\, \mathbf{D}'$, where

$$\begin{aligned}
\mathbf{D}' &= \mathsf{E}\Bigl(x_1 + S_1\bigl(\nu_{x_2}(n)\bigr)\,;\ \tau_x > \nu_{x_2}(n)\,,\ \nu_{x_2}(n) \le n^{1-\varepsilon}\,,\ x_1 + S_1\bigl(\nu_{x_2}(n)\bigr) > \phi_n\, n^{1/2} \Bigr) \\
&\le C\, \mathsf{E}\bigl(n^{1/2-\varepsilon} + \overline{M}_1(n^{1-\varepsilon})\,;\ n^{1/2-\varepsilon} + \overline{M}_1(n^{1-\varepsilon}) > \phi_n\, n^{1/2} \bigr)\,.
\end{aligned}$$

We now apply Lemma 2.7 to the first marginal. We have $\phi_n\, n^{1/2} - n^{1/2-\varepsilon} = n^{1/2-\varepsilon/s}$. Next, we choose $p = 2 + \delta$, where $\mathsf{E}(|X|^{2+\delta}) < \infty$. Furthermore, we substitute $n^{1-\varepsilon} = k = k_n$, whence

$$n^{1/2-\varepsilon/s} = k^{1/2+\alpha} \quad \text{and} \quad n^{1/2-\varepsilon} = k^{1/2-\beta}\,, \quad \text{where}$$

$$\alpha = \frac{\varepsilon}{1-\varepsilon}\Big(\frac{1}{2} - \frac{1}{s}\Big) \quad \text{and} \quad \beta = \frac{\varepsilon}{2(1-\varepsilon)}.$$

Thus, using k in the place of n in Lemma 2.7,

$$\begin{aligned}
&\mathsf{E}\big(n^{1/2-\varepsilon} + \overline{M}_1(n^{1-\varepsilon})\,;\, n^{1/2-\varepsilon} + \overline{M}_1(n^{1-\varepsilon}) > \phi_n\, n^{1/2}\big) \\
&\qquad = \mathsf{E}\big(k^{1/2-\beta} + \overline{M}_1(k)\,;\, \overline{M}_1(k) > k^{1/2+\alpha}\big) \le c_p\, \mathsf{E}\big(|X|^p\big)\, \frac{2p-1}{p-1}\, k^{1/2-(p-1)\alpha}\,.
\end{aligned}$$

We need that $1/2 - (p-1)\alpha < 0$, where $p = 2+\delta$. This can be achieved by choosing s sufficiently large (*). Then $\mathbf{D}' \to 0$ uniformly for $x_1 \le n^{1/2-\varepsilon} = n^{1/s}$, independently of x_2.

We come to the estimation of the principal term $\mathbf{C}$. With $\gamma_{x_2}(n) = \min\{\nu_{x_2}(n), n^{1-\varepsilon}\}$,

$$\mathbf{C} = \mathsf{E}\Big(h\Big(x + S\big(\gamma_{x_2}(n)\big)\Big)\,;\, \tau_x > \gamma_{x_2}(n)\,,\ \nu_{x_2}(n) \le n^{1-\varepsilon}\Big) = h(x) - \mathbf{F}\,, \quad \text{where}$$

$$\mathbf{F} = \mathsf{E}\Big(h\big(x + S(n^{1-\varepsilon})\big)\,;\, \tau_x > n^{1-\varepsilon}\,,\ \nu_{x_2}(n) > n^{1-\varepsilon}\Big)$$

At last, once more since $h(x) \le C\,x_1$ by propositions 3.2 and 2.5,

$$\begin{aligned}
\mathbf{F} &\le C\,\mathsf{E}\big(x_1 + S_1(n^{1-\varepsilon})\,;\, \tau_x > n^{1-\varepsilon}\,,\ \nu_{x_2}(n) > n^{1-\varepsilon}\big) \\
&\le C\,\Big(\mathsf{E}\big(x_1 + S_1(n^{1-\varepsilon})\big)^2\Big)^{1/2}\,\Big(\mathbb{P}[\tau_x > n^{1-\varepsilon}\,,\ \nu_{x_2}(n) > n^{1-\varepsilon}]\Big)^{1/2} \\
&\le C\,\big(2\,x_1^2 + 2\,n^{1-\varepsilon}\sigma_1^2\big)^{1/2}\,\Big(\mathbb{P}[\nu_{x_2}(n) > n^{1-\varepsilon}]\Big)^{1/2},
\end{aligned}$$

which tends to 0 as $n \to \infty$ by Lemma 3.7 and our assumption that $x_1 \le n^{1/2-\varepsilon} = n^{1/s}$. Recall from (*) above that s depends on the δ of the moment condition for X_1. □

4. The 2-dimensional Lindley process

We first explain how the queueing process is related with the exit times.

Proof of Lemma 1.5. In a good number of references, the increments in the definition of $W(n)$ come with a "plus" sign. We have chosen the "minus" because this is more convenient when relating the process with the exit times. In Feller's second volume [12, VI.9], the "plus" is used, and the one-dimensional case is considered. See in particular [12, Theorem on p. 198]. That theorem applies without changes also to higher dimensions, and rewritten in terms of the "minus" sign, it says the following for $x = (x_1\,, x_2) \in \mathbb{R}_+^2$.

$$\begin{aligned}
\mathbb{P}\big[W^0(n) \in [0\,, x_1) \times [0\,, x_2)\big] &= \mathbb{P}\big[-\underline{M}_1(n) < x_1\,,\ -\underline{M}_2(n) < x_2\big] \\
&= \mathbb{P}\big[x + S(k) \in (0\,,\infty)^2 \ \text{ for all }\ k \le n\big] = \mathbb{P}[\tau_x > n].
\end{aligned}$$

This concludes the proof. □

We now consider recurrence versus transience of the 2-dimensional Lindley process.

Negative drift in at least one coordinate

Proof of Theorem 1.3(a). Suppose that $\mathsf{E}(X_1^+) < \mathsf{E}(X_1^-) \le \infty$. Then $S_1(n) \to -\infty$ almost surely. The times when the first marginal process $W_1^0(n)$ starting at 0 visits 0 are the non-strictly increasing ladder epochs of $S_1(n)$, see [21] and [12, VI.9] (and keep in mind the "plus/minus" sign issue mentioned above). It is well known that the latter terminates almost surely, so that $W_1^0(n) \to \infty$ and thus also $W_1^{x_1}(n) \to \infty$ almost surely for every $x_1 \ge 0$: the first marginal process is transient, whence also the 2-dimensional process is transient. $\square$

Zero drift and negative correlation

We now consider the case when both marginals have zero drift, but the correlation is negative.

Proof of Theorem 1.3(c), transient case. Under the moment conditions of Theorem 1.3(c), when $\rho(X_1\,, X_2) < 0$ then the combination of Lemma 1.5 with Theorem 1.6(c) shows that

$$\sum_{n=0}^{\infty} \mathbb{P}[W^0(n) \in [0\,,\,x_1) \times [0\,,\,x_2)] < \infty \quad \text{for all } \; x_1\,,\,x_2 > 0.$$

This is the expected number of visits of the process to each of those rectangles. Therefore, with probability 1, each rectangle is visited only finitely often, so that $|W^0(n)| \to \infty$ and the process is transient. $\square$

Recurrence

We now turn our attention the the remaining cases, that is, statements (b) and (d) of Theorem 1.3, as well as (c) with $\rho(X_1\,, X_2) \ge 0$.

Conclusion of the proof of Theorem 1.3. In all of those remaining cases, we have by Theorem 1.6

$$\sum_{n=0}^{\infty} \mathbb{P}\big[W^0(n) \in [0\,,\,x_1) \times [0\,,\,x_2)\big] = \infty \quad \text{for all } \; x_1\,,\,x_2 > 0.$$

In the case when the distribution of $X = (X_1\,, X_2)$ is non-lattice, this alone does not guarantee that the process is topologically recurrent. But thanks to Assumption 1.2(ii), there is $x = (x_1\,, x_2) \in (0\,,\,\infty)^2$ such that for each n,

$$\mathbb{P}\big[X(n+1) \in [x_1\,,\,\infty) \times [x_2\,,\infty)\big] = \mathbb{P}\big[X \in [x_1\,,\,\infty) \times [x_2\,,\infty)\big] > 0\,.$$

But if $W^0(n) \in [0\,,\,x_1) \times [0\,,\,x_2)$ and $X(n+1) \in [x_1\,,\,\infty) \times [x_2\,,\infty)$ then $W^0(n+1) = 0$. Therefore, for this choice of $(x_1\,, x_2)$, and since $W^0(n)$ and $X(n+1)$ are independent,

$$\begin{aligned}\sum_{n=0}^{\infty} \mathbb{P}[W^0(n+1) = 0] &\ge \sum_{n=0}^{\infty} \mathbb{P}\big[W^0(n) \in [0\,,\,x_1)\times[0\,,\,x_2)\,,\; X(n+1) \in [x_1\,,\,\infty)\times[x_2\,,\infty)\big]\\ &= \mathbb{P}\big[X \in [x_1\,,\,\infty)\times[x_2\,,\infty)\big] \sum_{n=0}^{\infty} \mathbb{P}\big[W^0(n) \in [0\,,\,x_1)\times[0\,,\,x_2)\big] = \infty.\end{aligned}$$

Since 0 is a single state of our Markov process, this implies via basic Markov chain theory that it is a recurrent state:

$$\mathbb{P}[W^0(n) = 0 \text{ infinitely often}] = 1\,.$$

Now let us look at this under the viewpoint of stochastic dynamical systems (SDS): the random mappings $F_k(x) = \max\{0, x - X(k)\}$ are elements of the semigroup $\mathfrak{L}$ of contractions of $\mathbb{R}^2_+$ with Liptschitz constants ≤ 1. We have

$$W^x(n) = F_n \circ \cdots \circ F_1(x)\,.$$

In particular, $|W^x(n) - W^y(n)|$ is decreasing in n, whence $|W^x(n) - W^0(n)| \le |x|$. Consequently, our SDS is non-transient:

$$\mathbb{P}\big[|W^x(n)| \to \infty\big] = 0 \quad \text{for every} \ \ x \in \mathbb{R}^2_+\,.$$

$\mathfrak{L}$ carries the topology of uniform convergence on compact sets. Denote by $\widetilde{\mu}$ the (common) distribution of the i.i.d. random mappings F_n. Then the above arguments which led to recurrence of the state 0 entail that the constant mapping $x \mapsto 0$ can be approximated in $\mathfrak{L}$ by a sequence $f_n \circ \cdots \circ f_1$, $n \in \mathbb{N}$, where each function f_k is in the support of $\widetilde{\mu}$. At this point, we can invoke a result going back to [20], developped further in [3] and [23]; see Kloas and Woess [19, Prop. 2.5] for a compact formulation. It implies the existence of a limit set L and recurrence as stated in the Introduction in (1.1) and the preceding paragraph. □

Discussion

Invariant measures. In the cases where the two-dimensional Lindley process is recurrent, it follows from [23] that it has a unique invariant measure ν up to constant factors. It is supported on the limit set L, and as a starting measure for the process, it makes the time shift ergodic. For details, see [23, Thm. 2.13]. In particular, its marginals ν_1 and ν_2 are the unique invariant measures for the respective marginal processes. It is well understood that the invariant measure for a recurrent one-dimensional Lindley process has finite total mass when the increment has positive expectation. (Recall that we *substract* the increment.) Also, that measure has infinite total mass in the drift-free case. This is the reason why for the two-dimensional process, ν has finite total mass (positive recurrence) in Theorem 1.3(b), while it has infinite total mass (null recurrence) in Theorem 1.3(c) and (d).

The roles of Assumption 1.2(ii) and of the condition $\rho(X_1, X_2) \ge 0$. Consider the typical example of a clerk at a counter serving a queue of customers. In that case, the waiting time of the n^{th} customer is modelled by a one-dimensional Lindley process, and recurrence means that the process returns to state 0 with probability one - the queue will be empty and the clerk at the counter will be able to have a break. In the two-dimensional case, we have two counters and two queues, and in general we assume that the clerks do not operate independently. Then both Assumption 1.2(ii) and the non-negative correlation can be interpreted in the sense that the two clerks are co-operative, which will give them the possibility to have a joint break.

Assumption 1.2(ii) was used in the proof of Proposition 3.2 in order to show that $h > 0$ strictly on all of $(0\,,\,\infty)^2$. Without it, our proof does not allow us to go beyond the statement that for every $b_1 > 0$ (possibly small) there is $b_2 > 0$ (possibly large) such that $h > 0$ on $(b_1\,,\,\infty) \times (b_2\,,\,\infty)$.

In the proof of Theorem 1.3(d) this would lead to the restriction that we only get

$$\sum_{n=0}^{\infty} \mathbb{P}\big[W^0(n) \in [0\,,\,x_1) \times [0\,,\,x_2)\big] = \infty$$

for sufficiently large x_1 and x_2.

Now, in the case when the process is discrete (in which case we assume without loss of generality that the increments are in $\mathbb{Z}^2$ and the process evolves within $\mathbb{N}_0^2$), this does imply recurrence in the sense that the process visits each point of the (integer) limit set infinitely often with probability one. However, the origin is then not necessarily part of the limit set: the two clerks will not be able to have a joint break.

On the other hand, when the process is non-discrete then it is by no means clear that divergence of the above series implies topological recurrence – here, recurrence was deduced because on the basis of Assumption 1.2(ii) we could show that $\sum_n \mathbb{P}\big[W^0(n) = 0\big] = \infty$.

The limit set. L depends on the support of the distribution of the increments. In our case, most importantly, it contains the origin. It appears quite hard to determine the full limit set explicitly, compare with [19].

Higher dimensions. We can consider the same issues on $\mathbb{R}_+^d$ for any $d \geq 2$. The model random variable for the increments is then $X = (X_1\,, \ldots, X_d)$, and the definition of $W(n)$ remains the same, as well as Assumption 1.2 with the obvious adaptation.

If there is one coordinate with $\mathsf{E}(X_i^+) < \mathsf{E}(X_i^-) \leq \infty$ then $W(n)$ is transient, while if all coordinates satisfy $\mathsf{E}(X_i^-) < \mathsf{E}(X_i^+) < \infty$, then $W(n)$ is positive recurrent.

Also, if one coordinate is centered and all others have positive expectiation, then under the same moment conditions as in Theorem 1.3(d) one gets null recurrence; the proof remains practically the same.

For recurrence, one cannot have more than two centered coordinates. So the main case still to be studied is the analogue of Theorem 1.3(d) in the case when there are two centered coordinates and all the other coordinates have positive expectation. There are several substantial additional issues to be tackled in this situation, which we reserve for future work.

Remarks on reflected random walk

A model which is very similar to the queueing process is reflected random walk. In dimension 1, with starting point $x > 0$, both processes evolve like a random walk $x - S(n)$ as long as it stays non-negative. When the walk enters the negative half-axis, the value is reset to zero for the queueing process $W^x(n)$, while for reflected random walk $R(n) = R^x(n)$, the sign is changed. In arbitrary dimension, this means that reflected random walk is given by

$$R(0) = x \in \mathbb{R}^+\,, \quad R(n) = |R(n-1) - X(n)|\,,$$

where (attention!) the absolute value is taken coordinate-wise. The one-dimensional model is very well studied; instead of re-displaying all references, we refer to [23], as well as to [22] and [19], where the recurrence of the multidimensional reflected random walk is studied in the case where $\mathsf{E}(X_i) > 0$ for all coordinates.

In general, it is easy to see that transience of the multi-dimensional queueing process implies transience of the corresponding reflected random walk. In other words, the recurrence issue is somewhat harder for reflected random walk. In the following case, the two processes are recurrent, resp. transient simultaneously.

(4.1) Lemma. *If there is $\Delta > 0$ such that $X_i \geq -\Delta$ almost surely for $i = 1, \dots, d$ then for the two processes starting at the same point,*

$$R_i(n) \leq X_i(n) + \Delta \quad \textit{almost surely for } i = 1, \dots, d \textit{ and all } n.$$

The proof is straightforward. Thus, when the multidimensional Lindley process visits the origin infinitely often with probability 1, then the corresponding multidimensional reflected random walk visits the set $[0\,,\,\Delta]^d$ infinitely often with probability 1. By the contractivity properties of $\big(R^x(n)\big)$, this implies that the reflected random walk is topologically recurrent on its limit set, compare with [23] and [19].

In dimension 2, this applies, in particular, to the situation of Theorem 1.3.

5. Appendix: a proof due to D. Denisov and V. Wachtel

In this subsection, the proof of Proposition 2.5 is completed. That is, we present the elaboration of an argument of Denisov and Wachtel which proves

$$\lim_{x\to\infty} \frac{h_1(x)}{x} = 1. \tag{2.6}$$

Given the centered real random variable X with $\mathsf{E}(X^2) < \infty$, we consider the following functions $a, b, m : \mathbb{R}_+ \to \mathbb{R}_+$ associated with the negative part X^- of X.

$$\begin{aligned} a(x) &= -\mathsf{E}(x + X\,;\, x + X \leq 0) = \int_x^\infty \mathbb{P}[X \leq -y]\,dy\,, \\ b(x) &= \int_x^\infty a(y)\,dy\,, \quad \text{and} \quad m(x) = \int_0^x b(y)\,dy\,, \quad x \geq 0. \end{aligned}$$

Similarly, we set $\bar a(x) = \displaystyle\int_x^\infty \mathbb{P}[X > y]\,dy$.

One easily verifies that $2b(x) = \mathsf{E}\big((\max\{X^- - x, 0\})^2\big) = \displaystyle\int_{-\infty}^x (x+y)^2 dF(y) \to 0$ as $x \to \infty$, whence also

$$\lim_{x\to\infty} \frac{m(x)}{x} = 0. \tag{5.1}$$

(5.2) Proposition. *There are positive constants R (sufficiently large) and A such that the function*

$$V(x) = x + A\,m(x) + R\,, \quad x \geq 0\,,$$

is superharmonic for the one-dimensional random walk $S(n)$ killed when exiting $\mathbb{R}_+$, that is,

$$\mathsf{E}\Big(V\big(x+S(1)\big)\,;\,\tau_x>1\Big)\le V(x)\quad \textit{for all}\;\; x\ge 0.$$

Let us first show how this implies the desired result.

Proof of (2.6). Proposition 5.2 implies that for all $x\ge 0$ and $n\in\mathbb{N}$

$$\begin{aligned} V(x) &\ge \mathsf{E}\Big(V\big(x+S(n)\big)\,;\,\tau_x>n\Big)\\ &\ge \mathsf{E}\big(x+S(n)\,;\,\tau_x>n\big)\\ &= \mathsf{E}\big(x+S(\min\{n,\tau_x\})\,;\,\tau_x>n\big)\\ &= x-\mathsf{E}\big(x+S(\tau_x)\,;\,\tau_x\le n\big). \end{aligned}$$

The last identity holds by the martingale property, since $S(n)$ is a centered random walk. We infer that

$$-\mathsf{E}\big(x+S(\tau_x)\,;\,\tau_x\le n\big)\le V(x)-x=A\,m(x)+R$$

We can let $n\to\infty$, and by monotone convergence

$$0\le -\mathsf{E}\big(x+S(\tau_x)\,;\,\tau_x<\infty\big)\le A\,m(x)+R$$

By (5.1),

$$\lim_{x\to\infty}\frac{1}{x}\,\mathsf{E}\big(x+S(\tau_x)\,;\,\tau_x<\infty\big)=0\,.$$

Since $h_1(x)=x-\mathsf{E}\big(x+S(\tau_x)\,;\,\tau_x<\infty\big)$, the result follows. $\square$

Proof of Proposition 5.2. We want to show that

$$\Delta(x)=\mathsf{E}\Big(V\big(x+S(1)\big)\,;\,\tau_x>1\Big)-V(x)\le 0\quad\text{for}\;\; x\ge 0.$$

We write $F(x)=\mathbb{P}[X\le x]$ and $\overline{F}(x)=1-F(x)$. Using that $x=\mathsf{E}(x+X)$,

$$\begin{aligned} \Delta(x) &= \mathsf{E}\big(x+X+A\,m(x+X)+R\,;\,X>-x\big)-x-A\,m(x)-R\\ &= -\mathsf{E}(x+X\,;\,X\le -x\big)-R\,F(-x)-A\,m(x)\,F(-x)\\ &\qquad\qquad\qquad\qquad +A\,\mathsf{E}\big(m(x+X)-m(x)\,;\,X>-x\big)\\ &= a(x)-R\,F(-x)-A\,m(x)\,F(-x)+A\int_{-x}^{\infty}\big(m(x+y)-m(x)\big)\,dF(y). \end{aligned}$$

We decompose the last integral and integrate twice by parts, recalling that $m'(x) = b(x), b'(x) = -a(x)$, $a'(x) = -F(-x)$ and $\bar{a}'(x) = -\overline{F}(x)$:

$$
\begin{aligned}
&\int_{-x}^{\infty} \big(m(x+y) - m(x)\big)\, dF(y) \\
&\quad = \int_{-x}^{0} \big(m(x+y) - m(x)\big)\, dF(y) - \int_{0}^{\infty} \big(m(x+y) - m(x)\big)\, d\overline{F}(y) \\
&\quad = m(x)\, F(-x) - \int_{-x}^{0} b(x+y)F(y)\, dy + \int_{0}^{\infty} b(x+y)\overline{F}(y)\, dy \\
&\quad = m(x)\, F(-x) + \int_{0}^{x} b(x-y)a'(y)\, dy - \int_{0}^{\infty} b(x+y)\overline{a}'(y)\, dy \\
&\quad = m(x)\, F(-x) + b(0)a(x) - b(x)a(0) - \int_{0}^{x} a(x-y)a(y)\, dy \\
&\qquad\qquad + b(x)\bar{a}(0) - \int_{0}^{\infty} a(x+y)\bar{a}(y)\, dy\,.
\end{aligned}
$$

We observe that $b(x)\bar{a}(0) - b(x)a(0) = b(x)\big(\mathsf{E}(X^+) - \mathsf{E}(X^-)\big) = 0$, and recall that $b(0) = \mathsf{E}\big((X^-)^2\big)/2$. Combining these computations,

$$
\begin{aligned}
\Delta(x) &= a(x) - R\,F(-x) + A\,\frac{\mathsf{E}\big((X^-)^2\big)}{2}\,a(x) - A\int_0^x a(x-y)a(y)\,dy - A\int_0^\infty a(x+y)\bar{a}(y)\,dy \\
&\le a(x) - R\,F(-x) + A\,\frac{\mathsf{E}\big((X^-)^2\big)}{2}\,a(x) - A\int_0^x a(x-y)a(y)\,dy\,.
\end{aligned}
$$

Using that $a(x)$ is monotone decreasing,

$$
\begin{aligned}
\int_0^x a(x-y)a(y)\,dy &= 2\int_0^{x/2} a(x-y)a(y)\,dy \\
&\ge 2a(x)\big(b(0) - b(x/2)\big) = a(x)\mathsf{E}\big((X^-)^2\big) - 2a(x)b(x/2).
\end{aligned}
$$

We now choose $A = 4\big/\mathsf{E}\big((X^-)^2\big)$. Then we get

$$
\Delta(x) \le -R\,F(-x) + a(x)\big(2A\,b(x/2) - 1\big) \le -R\,F(-x) + 3a(x).
$$

Since $b(x) \to 0$ as $x \to \infty$, there is $x_0 > 0$ such that $2A\,b(x_0/2) - 1 = 0$ and hence $\Delta(x) \le 0$ for all $x \ge x_0$.

Now suppose first that $F(-x_0) > 0$. Then we choose $R = 3\mathsf{E}(X^-)/F(-x_0)$, and for $x \le x_0$, we have $\Delta(x) \le -R\,F(-x_0) + 3a(0) = 0$.

Finally suppose that $F(-x_0) = 0$, that is, $\mathbb{P}[X > -x_0] = 1$. Then also $a(x_0) = 0$. This time, we choose $R = 3x_0$. For $0 \le x < x_0$, there is $\xi \in (x\,,\,x_0)$ such that $a(x) = a(x_0) - (x_0 - x)a'(\xi) = (x_0 - x)F(-\xi)$. Then

$$
\Delta(x) \le 3a(x) - R\,F(-x) = 3(x_0 - x)\big(F(-\xi) - F(-x)\big) - 3xF(-x) \le 0.
$$

This concludes the proof. $\square$

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