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POLAR DECOMPOSITION OF SEMIGROUPS GENERATED BY NON-SELFADJOINT QUADRATIC DIFFERENTIAL OPERATORS AND REGULARIZING EFFECTS

PAUL ALPHONSE AND JOACKIM BERNIER

Abstract. We study semigroups generated by accretive non-selfadjoint quadratic differential operators. We give a description of the polar decomposition of the associated evolution operators as products of a selfadjoint operator and a unitary operator. The selfadjoint parts turn out to be also evolution operators generated by time-dependent real-valued quadratic forms that are studied in details. As a byproduct of this decomposition, we give a geometric description of the regularizing properties of semigroups generated by accretive non-selfadjoint quadratic operators. Finally, by using the interpolation theory, we take advantage of this smoothing effect to establish subelliptic estimates enjoyed by quadratic operators.

1. Introduction

1.1. Motivation. Given \( q : \mathbb{R}^{2n} \to \mathbb{C} \) a complex-valued quadratic form with a non-negative real part \( \text{Re} q \geq 0 \), we consider the maximal realization on \( L^2(\mathbb{R}^n) \) of the quadratic operator \( q^w(x, D_x) \) defined by the Weyl quantization of the quadratic form \( q \), that is the pseudodifferential operator

\[
q^w(x, D_x)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{i(x-y, \xi)} q\left( \frac{x+y}{2}, \xi \right) u(y) dy d\xi,
\]
equipped with the domain

\[
D(q^w) = \{ u \in L^2(\mathbb{R}^n) : q^w(x, D_x)u \in L^2(\mathbb{R}^n) \}.
\]

This non-selfadjoint operator is only a differential operator since the Weyl quantization of the quadratic symbols \( x^\alpha \xi^\beta \), with \( (\alpha, \beta) \in \mathbb{N}^{2n} \) such that \( |\alpha + \beta| = 2 \), is given by

\[
(x^\alpha \xi^\beta)^w = \operatorname{Op}^w(x^\alpha \xi^\beta) = \frac{1}{2} (x^\alpha D_x^2 + D_x^3 x^\alpha),
\]
with \( D_x = i^{-1} \partial_x \). Since the real part of the quadratic form \( q \) is non-negative \( \text{Re} q \geq 0 \), the quadratic operator \( q^w(x, D_x) \) is shown in [20] (pp. 425-426) to be maximal accretive and to generate a strongly continuous contraction semigroup \( (e^{-tq^w})_{t \geq 0} \) on \( L^2(\mathbb{R}^n) \). We aim in this work at studying the evolution operators \( e^{-tq^w} \) and to make explicit their polar decompositions as bounded operators on the Hilbert space \( L^2(\mathbb{R}^n) \) as defined in Subsection 6.1. As an application of this decomposition, we study the regularizing effects of the semigroup \( (e^{-tq^w})_{t \geq 0} \) in any positive times \( t > 0 \) and we take advantage of this smoothing features to establish subelliptic estimates enjoyed by the quadratic operator \( q^w(x, D_x) \).

As an example, let us consider \( Q \) and \( B \) some \( n \times n \) real matrices with \( Q \) symmetric positive semidefinite, and the Ornstein-Uhlenbeck operator \( L \) defined by

\[
L = -\frac{1}{2} \text{Tr}(Q \nabla_x^2) + \langle Bx, \nabla_x \rangle,
\]
and equipped with the domain

\[
D(L) = \{ u \in L^2(\mathbb{R}^n) : Lu \in L^2(\mathbb{R}^n) \}.
\]

Since the Weyl symbol of \( L \) is

\[
p(x, \xi) = \frac{1}{2} \langle Q \xi, \xi \rangle + i \langle Bx, \xi \rangle - \frac{1}{2} \text{Tr}(B),
\]

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\[ L + \frac{1}{2} \text{Tr}(B) \] is an accretive quadratic operator, see Example 1.17. Therefore, the operator \( L \) generates a strongly continuous semigroup \((e^{-tL})_{t \geq 0}\) on \(L^2(\mathbb{R}^n)\) and the two authors proved in [3] (Theorem 1.1) that this semigroup is explicitly given by the Kolmogorov formula
\[
\forall t \geq 0, \quad e^{-tL} = \exp \left( -\frac{1}{2} \int_0^t (Q_2 e^{R^T} D_x)^2 \, d\tau \right) e^{-t(B_x, \nabla_x)}.
\]
This remarkable formula actually provides the polar decomposition of the evolution operator \(e^{-tL}\) since a real-valued Fourier multiplier is a \(L^2\)-selfadjoint operator and that the transport operator \((B_x, \nabla_x)\) generates a unitary group on \(L^2(\mathbb{R}^n)\). Formula (1.3) has been extended for general fractional Ornstein-Uhlenbeck semigroups defined in Example 1.15 and under a suitable algebraic condition on the matrices \(Q\) and \(B\), namely the Kalman rank condition, this formula has allowed the two authors of the present work to study their regularizing effects and to establish subelliptic estimates enjoyed by their infinitesimal generators, see Theorem 1.2 and Theorem 1.14 in [3].

1.2. Hamilton map and Singular space. Before stating the main results contained in this paper, we need to introduce the Hamilton map and the singular space associated to the quadratic form \(q\), which will play a key role in the following. According to [19] (Definition 21.5.1), the Hamilton map \(F\) of the quadratic form \(q\) is defined as the unique matrix \(F \in M_{2n}(\mathbb{C})\) satisfying the identity
\[
\forall X, Y \in \mathbb{R}^{2n}, \quad q(X, Y) = \sigma(X, FY),
\]
with \(q(\cdot, \cdot)\) the polarized form associated to \(q\) and \(\sigma\) the standard symplectic form given by
\[
\sigma((x, \xi), (y, \eta)) = \langle \xi, y \rangle - \langle x, \eta \rangle, \quad (x, y), (\xi, \eta) \in \mathbb{C}^{2n},
\]
where \(\langle \cdot, \cdot \rangle\) denotes the inner product on \(\mathbb{C}^n\) defined by
\[
\langle x, y \rangle = \sum_{j=0}^{n} x_j y_j, \quad x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{C}^n.
\]
Note that this inner product \(\langle \cdot, \cdot \rangle\) is linear in both variables but not sesquilinear. By definition, the matrix \(F\) is given by
\[
F = JQ,
\]
where \(Q \in S_{2n}(\mathbb{C})\) is the symmetric matrix associated to the bilinear form \(q(\cdot, \cdot)\),
\[
\forall X, Y \in \mathbb{R}^{2n}, \quad q(X, Y) = \langle QX, Y \rangle = \langle X, QY \rangle,
\]
and \(J \in \text{GL}_{2n}(\mathbb{R})\) stands for the symplectic matrix defined by
\[
J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix} \in \text{Sp}_{2n}(\mathbb{R}).
\]

The notion of singular space was introduced in [14] (formula (1.1.14)) by M. Hitrik and K. Pravda-Starov by pointing out the existence of a particular vector subspace \(S\) in the phase space \(\mathbb{R}^{2n}\), which is intrinsically associated to the quadratic symbol \(q\), and defined as the following intersection of kernels
\[
S = \bigcap_{j=0}^{+\infty} \text{Ker}(\text{Re} F(\text{Im} F)^j) \cap \mathbb{R}^{2n},
\]
where the notations \(\text{Re} F\) and \(\text{Im} F\) stand respectively for the real part and the imaginary part of the Hamilton map \(F\) associated to \(q\). The subspace \(S\) readily satisfies the two following properties
\[
(\text{Re} F)S = \{0\} \quad \text{and} \quad (\text{Im} F)S \subset S.
\]
Notice that the Cayley-Hamilton theorem applied to the matrix \(\text{Im} F\) shows that
\[
\forall k \in \mathbb{N}, \forall X \in \mathbb{R}^{2n}, \quad (\text{Im} F)^k X \in \text{Span}(X, \ldots, (\text{Im} F)^{2n-1} X),
\]
where \(\text{Span}(X, \ldots, (\text{Im} F)^{2n-1} X)\) is the vector space spanned by the vectors \(X, \ldots, (\text{Im} F)^{2n-1} X\), and therefore the singular space is actually equal to the following finite intersection of the kernels
\[
S = \bigcap_{j=0}^{2n-1} \text{Ker}(\text{Re} F(\text{Im} F)^j) \cap \mathbb{R}^{2n}.
\]
According to \((1.12)\), we may consider \(0 \leq k_0 \leq 2n - 1\) the smallest integer satisfying
\[(1.13)\]
\[
S = \bigcap_{j=0}^{k_0} \text{Ker}(\text{Re } F(\text{Im } F)^j) \cap \mathbb{R}^{2n}.
\]
This integer \(k_0\) will play a key role in the following. Since the quadratic symbol has a non-negative real part \(\text{Re } q \geq 0\), the singular space can be defined in an equivalent way as the subspace in the phase space where all the Poisson brackets
\[
H_{\text{Im } q}^k \text{Re } q = (\partial_x \text{Im } q \cdot \partial_x - \partial_x \text{Im } q \cdot \partial_x)^k \text{Re } q, \quad k \geq 0,
\]
are vanishing
\[
S = \{ X \in \mathbb{R}^{2n} : (H_{\text{Im } q}^k \text{Re } q)(X) = 0, \ k \geq 0 \}.
\]
This dynamical definition shows that the singular space corresponds exactly to the set of points \(X \in \mathbb{R}^{2n}\), where the real part of the symbol \(\text{Re } q\) under the flow of the Hamilton vector field \(H_{\text{Im } q}\) associated with its imaginary part
\[(1.14)\]
\[
t \mapsto (\text{Re } q)(e^{iH_{\text{Im } q} t} X),
\]
vansishes to infinite order at \(t = 0\). This is also equivalent to the fact that the function \((1.14)\) is identically zero on \(\mathbb{R}\). As pointed out in \(15, 16, 30, 33\), the singular space is playing a basic role in understanding the spectral and hypoelliptic properties of non-elliptic quadratic operators, as well as the spectral and pseudospectral properties of certain classes of degenerate doubly characteristic pseudodifferential operators \(15, 16\).

1.3. Polar decomposition of quadratic semigroups. We begin by giving a sharp description of the polar decomposition of the evolution operators \(e^{-tq^w}\). More precisely, we aim at establishing that for any \(t \geq 0\), the operator \(e^{-tq^w}\) admits the decomposition
\[(1.15)\]
\[
e^{-tq^w} = e^{-ta^w} e^{-ibt}.
\]
where \(a_t, b_t : \mathbb{R}^{2n} \to \mathbb{R}\), with \(t \geq 0\), are real-valued time-dependent quadratic forms, \(a_t\) being non-negative. In formula \((1.15)\), the linear operators \(e^{-ta^w}\) and \(e^{-ibt}\) are defined as follows: for some fixed \(t \geq 0\), the quadratic operators \(a^w_t(x, D_x)\) and \(b^w_t(x, D_x)\) respectively generate a semigroup \((e^{-s a^w_t})_{s \geq 0}\) and a group \((e^{-s b^w_t})_{s \in \mathbb{R}}\) of contraction operators on \(L^2(\mathbb{R}^n)\) (since the quadratic form \(a_t\) is non-negative and the quadratic form \(b_t\) is purely imaginary) and the operators \(e^{-ta^w}\) and \(e^{-ibt}\) are respectively defined by
\[(1.16)\]
\[
e^{-ta^w} = e^{-sa^w_t}
\]
\[
e^{-ibt} = e^{-ibt}.
\]
Notice that if the quadratic operators \((\text{Re } q)^w\) and \((\text{Im } q)^w\) commute, then the relation \((1.15)\) is satisfied with \(a_t = \text{Re } q\) and \(b_t = \text{Im } q\). Let us check that formula \((1.15)\) is the polar decomposition of the evolution operator \(e^{-tq^w}\) as defined in the end of Subsection 6.1. The operator \(e^{-ta^w}\) is injective from Corollary 6.3. In order to check that this operator is also non-negative and selfadjoint on \(L^2(\mathbb{R}^n)\), we recall that the adjoint of any evolution operator \(e^{-sq^w}\) generated by the accretive operator \(q^w(x, D_x)\), with \(q : \mathbb{R}^{2n} \to \mathbb{C}\) a quadratic form with a non-negative real part \(\text{Re } q \geq 0\), is given by \((e^{-sq^w})^* = e^{-s\overline{q^w}}\), see e.g. \(28\) (Chapter 1, Corollary 10.6) and \(20\) (p. 426). This formula implies that \((e^{-ta^w})^* = e^{-ta^w}\), since the quadratic form \(a_t\) is real-valued. The operator \(e^{-ta^w}\) is therefore selfadjoint on \(L^2(\mathbb{R}^n)\). By using this selfadjointness together with the semigroup property of the family \((e^{-sa^w_t})_{s \geq 0}\), we deduce that
\[
\forall u \in L^2(\mathbb{R}^n), \quad \langle e^{-ta^w_t} u, u \rangle_{L^2(\mathbb{R}^n)} = \| e^{-\overline{q^w}} u \|^2_{L^2(\mathbb{R}^n)} \geq 0,
\]
which proves that the operator \(e^{-ta^w}\) is also non-negative. Finally, the operator \(e^{-ibt}\) is unitary on \(L^2(\mathbb{R}^n)\) since the quadratic form \(b_t\) is real-valued. In fact, the estimate \((1.15)\) will be proven only for small times \(0 \leq t \ll 1\). In the case where \(t \gg 1\), an estimate similar to \((1.15)\) will be established with the operator \(e^{-ibt}\) replaced by a unitary operator \(U_t\) which \(a\ priori\) cannot be written as an operator defined in \((1.15)\). The main result contained in this article is the following:

**Theorem 1.1.** Let \(q : \mathbb{R}^{2n} \to \mathbb{C}\) be a complex-valued quadratic form with a non-negative real part \(\text{Re } q \geq 0\). Then, there exist a family \((a_t)_{t \in \mathbb{R}}\) of non-negative quadratic forms \(a_t : \mathbb{R}^{2n} \to \mathbb{R}_+\) depending analytically on the time-variable \(t \in \mathbb{R}\) and a family \((U_t)_{t \in \mathbb{R}}\) of metaplectic operators such that
\[
\forall t \geq 0, \quad e^{-tq^w} = e^{-ta^w_t} U_t.
\]
Moreover, there exists a positive constant $T > 0$ and a family $(b_t)_{T < t < T}$ of real-valued quadratic forms $b_t : \mathbb{R}^{2n} \to \mathbb{R}$ also depending analytically on the time-variable $-T < t < T$, such that
\[
\forall t \in [0, T), \quad e^{-tq^w} = e^{-ta^w} e^{-tb^w}.
\]

We refer the reader to Definition 6.5 in Appendix where the metaplectic operators (and more generally the Fourier integral operators associated to non-negative complex symplectic transformations) are defined.

The principal application of this decomposition will be to describe the regularizing effects of the semigroup $(e^{-tq^w})_{t \geq 0}$, which requires a precise knowledge of the selfadjoint part $e^{-ta^w}$ given by Theorem 1.1. More precisely, we will need a bound from below of the time-dependent quadratic form $a_t$. This is the purpose of the following theorem:

**Theorem 1.2.** Let $q : \mathbb{R}^{2n} \to \mathbb{C}$ be a complex-valued quadratic form with a non-negative real part $\text{Re } q \geq 0$. We consider $F$ the Hamilton map associated to $q$ and $S$ its singular space. Let $(a_t)_{t \in \mathbb{R}}$ be the family of non-negative quadratic forms given by Theorem 1.1. Then, there exist some positive constants $0 < T < 1$ and $c > 0$ such that for all $0 \leq t \leq T$ and $X \in \mathbb{R}^{2n}$,
\[
a_t(X) \geq c \sum_{j=0}^{k_0} t^{2j} \text{Re } q((\text{Im } F)^j X),
\]
where $0 \leq k_0 \leq 2n - 1$ is the smallest integer such that (1.13) holds.

Theorem 1.2 implies in particular that for all $0 \leq t \ll 1$, the quadratic form $a_t$ enjoys degenerate anisotropic coercive estimates in the phase space. This corollary is proven in Lemma 1.1. In the particular case when $S = \{0\}$, this lemma implies that the quadratic form $a_t$ is positive definite for all $0 \leq t \ll 1$. Moreover, it highlights the role of the singular space $S$ in the polar decomposition given by Theorem 1.1 through the index $0 \leq k_0 \leq 2n - 1$ which is intrinsically related to its structure.

The calculation of the quadratic forms $a_t$ and $b_t$ is quite difficult in practice, except for example for the Ornstein-Uhlenbeck operators $L$ defined in (1.1). Let $K$ be the Kramers-Fokker-Planck operator without external potential also makes an exception as illustrated in the following example:

**Example 1.3.** Let $K$ be the Kramers-Fokker-Planck operator without external potential defined by
\[
K = -\Delta_v + |v|^2 + (v, \nabla_x), \quad (x, v) \in \mathbb{R}^{2n},
\]
and equipped with the domain
\[
D(K) = \{u \in L^2(\mathbb{R}^{2n}) : Ku \in L^2(\mathbb{R}^{2n})\}.
\]
The operator $K$ is quadratic since its Weyl symbol is the quadratic form $q : \mathbb{R}^{4n} \to \mathbb{C}$ given by
\[
q(x, v, \xi, \eta) = |\eta|^2 + |v|^2 + i(v, \xi), \quad (x, v, \xi, \eta) \in \mathbb{R}^{4n}.
\]
Moreover, for all $t \geq 0$, the evolution operator $e^{-tK}$ can be written as
\[
e^{-tK} = e^{-ta^w} e^{-tb^w},
\]
where the time-dependent quadratic operators $a^w_t$ and $b^w_t$ are defined for all $t \geq 0$ by
\[
a^w_t = -\Delta_v + |v|^2 - \frac{\sinh(2t)}{\cosh(2t) + 1} \langle \nabla_x, \nabla_v \rangle - \frac{2t \cosh(2t) - \sinh(2t)}{4t(\cosh(2t) + 1)} \Delta_x,
\]
and
\[
b^w_t = \frac{\sinh t}{it} (v, \nabla_x).
\]
Indeed, as we will see in the proof of Theorem 1.1, establishing the relation (1.19) is equivalent to proving the following equality between matrices:
\[
e^{-2tJQ} = e^{-2tJA_t} e^{2tJB_t},
\]
where $J \in \text{Sp}_{4n}(\mathbb{R})$ is the symplectic matrix defined in (1.9), $Q \in \text{Sp}_{4n}(\mathbb{C})$ is the matrix of the quadratic form $q$ in the canonical basis of $\mathbb{R}^{4n}$, and the time-dependent matrices $A_t, B_t \in \text{Sp}_{4n}(\mathbb{R})$
are respectively defined for all \( t \geq 0 \) by
\[
A_t = \begin{pmatrix}
0_n & 0_n & 0_n & 0_n \\
0_n & I_n & 0_n & 0_n \\
0_n & 0_n & 2t \cosh(2t) - \sinh(2t) & 0_n \\
0_n & 0_n & -\frac{2t \cosh(2t) - \sinh(2t)}{4t} & I_n
\end{pmatrix},
\]
and
\[
B_t = \begin{pmatrix}
0_n & 0_n & 0_n & 0_n \\
0_n & 0_n & 0_n & 0_n \\
0_n & -\frac{\sinh t}{t} I_n & 0_n & 0_n \\
0_n & 0_n & 0_n & 0_n
\end{pmatrix}.
\]
Moreover, (1.20) follows from a direct calculus.

**Remark 1.4.** The technics used to derive the polar decompositions of semigroups generated by accretive non-selfadjoint quadratic differential operators can also be used to obtain other splitting formulas. For example, let us consider the harmonic oscillator \( \mathcal{H} = -\Delta_x + |x|^2 \), with \( x \in \mathbb{R}^n \). We prove in Proposition 6.3 (in dimension 1, but the proof works the same in any dimension by tensorization) with the same arguments as the ones used in the proof of Theorem 1.1 that for all \( t \geq 0 \), the evolution operator \( e^{-tH} \) generated by \( \mathcal{H} \) writes as
\[
e^{-tH} = e^{-\frac{1}{2} (\tanh t)|x|^2} e^{\frac{1}{2} \sinh(2t)\Delta_x} e^{-\frac{1}{2} (\tanh t)|x|^2}.
\]
The method can be generally used for all semigroups generated by accretive non-selfadjoint quadratic differential operators.

**Remark 1.5.** The polar decomposition provided by Theorem 1.1 for the semigroups generated by accretive non-selfadjoint quadratic differential operators is as well valid for an other general class of semigroups called fractional Ornstein-Uhlenbeck semigroups defined as follows: given \( s > 0 \) a positive real number, \( B \) and \( Q \) real \( n \times n \) matrices, with \( Q \) symmetric positive semidefinite, we define the fractional Ornstein-Uhlenbeck operator \( L_s \) as
\[
L_s = \frac{1}{2} \text{Tr}^s(-Q \nabla_x^2) + \langle Bx, \nabla_x \rangle,
\]
and equipped with the domain
\[
D(L_s) = \left\{ u \in L^2(\mathbb{R}^n) : L_s u \in L^2(\mathbb{R}^n) \right\}.
\]
The operator \( \text{Tr}^s(-Q \nabla_x^2) \) stands for the Fourier multiplier with symbol \( (Q \xi, \xi)^s \). Notice that \( L_1 \) is the Ornstein-Uhlenbeck operator defined in (1.1) and (1.2). The two authors proved in Theorem 1.1 that the operator \( L_s \) generates a strongly continuous semigroup \( (e^{-tL_s})_{t \geq 0} \) on \( L^2(\mathbb{R}^n) \) and that for all \( t \geq 0 \), the evolution operator \( e^{-tL_s} \) is explicitly given by the following formula which is an extension of the Kolmogorov formula (1.3):
\[
(1.21) \quad \forall t \geq 0, \quad e^{-tL_s} = \exp \left( -\frac{1}{2} \int_0^t \langle Q \frac{1}{2} e^{\tau B^T D_x} |^{2s} d\tau \right) e^{-t(Bx, \nabla_x)}.
\]
For all \( t \geq 0 \), the relation (1.21) is the polar decomposition of the operator \( e^{-tL_s} \).

### 1.4. Regularizing effects of semigroups generated by accretive non-selfadjoint quadratic differential operators.

As an application of the splitting formula given by Theorem 1.1 and the estimate given by Theorem 1.2 we investigate the regularizing properties of the evolution operators \( e^{-tq^u} \) for all \( t \geq 0 \). As pointed out in the works [2, 14, 17, 18, 32], the understanding of this smoothing effect is closely related to the structure of the singular space \( S \). Indeed, the notion of singular space allows to study the propagation of Gabor singularities for the solutions of the quadratic differential equations
\[
\begin{cases}
\partial_t u + q^u(x, D_x) u = 0, \\
u(0) = u_0 \in L^2(\mathbb{R}^n).
\end{cases}
\]
We recall from (Section 5) that the Gabor wave front set \( WF(u) \) of a tempered distribution \( u \in \mathcal{S}'(\mathbb{R}^n) \) measures the directions in the phase space in which a tempered distribution does not behave like a Schwartz function. In particular, when \( u \in \mathcal{S}'(\mathbb{R}^n) \), its Gabor wave front set \( WF(u) \)
is empty if and only if \( u \in \mathcal{F}(\mathbb{R}^n) \). The following microlocal inclusion was proven in [32] (Theorem 6.2):

\[
(1.22) \quad \forall u \in L^2(\mathbb{R}^n), \forall t > 0, \quad WF(e^{-tq^2} u) \subset e^{tH_{imq}}(WF(u) \cap S) \subset S,
\]

where \((e^{tH_{imq}})_{t \in \mathbb{R}}\) is the flow generated by the Hamilton vector field associated to the imaginary part of the quadratic form \( q \).

This result points out that the possible Gabor singularities of the solution \( e^{-tq^2} u \) can only come from Gabor singularities of the initial datum \( u \) localized in the singular space \( S \) and are propagated along the curves given by the flow of the Hamilton vector field \( H_{imq} \) associated to the imaginary part of the symbol. The microlocal inclusion (1.22) was shown to hold as well for other types of wave front sets, as Gelfand-Shilov wave front sets [9] or polynomial phase space wave front sets [35].

Drawing our inspiration from the work [17], we consider the vector subspaces \( V_0, \ldots, V_{k_0} \subset \mathbb{R}^{2n} \) defined by

\[
(1.23) \quad V_k = \bigcap_{j=0}^k \text{Ker}(Re F(\text{Im} F)^j) \cap \mathbb{R}^{2n}, \quad 0 \leq k \leq k_0,
\]

where \( 0 \leq k_0 \leq 2n - 1 \) is the smallest integer such that (1.13) holds. According to (1.13), the family of vector subspaces \( V_0, \ldots, V_{k_0} \) is increasing for the inclusion and satisfies

\[
(1.24) \quad V_0^\perp \subset \ldots \subset V_{k_0}^\perp = S^\perp,
\]

where the orthogonality is taken with respect to the canonical Euclidean structure of \( \mathbb{R}^{2n} \). This stratification allows one to define the index with respect to the singular space of any point \( X_0 \in S^\perp \) as

\[
(1.25) \quad k_{X_0} = \min \{ 0 \leq k \leq k_0 : X_0 \in V_k^\perp \}.
\]

When the singular space of \( q \) is reduced to zero \( S = \{0\} \), the microlocal inclusion (1.22) implies that the semigroup \((e^{-tq^2})_{t \geq 0}\) is smoothing in any positive time \( t > 0 \) in the Schwartz space \( \mathcal{F}(\mathbb{R}^n) \), but this result does not provide any control of the blow-up of the associated seminorms as \( t \to 0^+ \). However, the notion of index was shown in [17] to allow to determine the short-time asymptotics of the regularizing effect induced by the semigroup \((e^{-tq^2})_{t \geq 0}\) in the phase space direction given by the vector \( X_0 \in \mathbb{R}^{2n} \). More precisely, [17] (Theorem 1.1) states that when the singular space is trivial \( S = \{0\} \), there exists a positive constant \( C > 1 \) such that for all \( X_0 \in \mathbb{R}^{2n} = S^\perp, 0 < t \leq 1 \) and \( u \in L^2(\mathbb{R}^n) \),

\[
(1.26) \quad \| (X_0,X)^w e^{-tq^2} u \|_{L^2(\mathbb{R}^n)} \leq \frac{C|X_0|}{t^{k_{X_0} + \frac{1}{2}}} \| u \|_{L^2(\mathbb{R}^n)},
\]

where \( 0 \leq k_{X_0} \leq k_0 \) denotes the index of the point \( X_0 \in \mathbb{R}^{2n} = S^\perp \) with respect to the singular space and where the pseudodifferential operator \( \langle X_0, X \rangle^w \) is defined as the differential operator whose Weyl symbol is given by the linear form \( \langle X_0, X \rangle \), that is

\[
(1.27) \quad \langle X_0, X \rangle^w = \langle x_0, x \rangle + \langle \xi_0, D_x \rangle, \quad X_0 = (x_0, \xi_0) \in \mathbb{R}^{2n}.
\]

This result shows that the structure of the singular space accounting for the family of vector subspaces \( (V_k)_{0 \leq k \leq k_0} \), allows one to sharply describe the short-time asymptotics of the regularizing effect induced by the semigroup \((e^{-tq^2})_{t \geq 0}\). The degeneracy degree of the phase space direction \( X_0 \in \mathbb{R}^{2n} = S^\perp \) given by the index with respect to the singular space directly accounts for the blow-up upper bound \( t^{-k_{X_0} - \frac{1}{2}} \), for small times \( t \to 0^+ \). As a corollary, the same three authors proved in [17] (Corollary 1.2) that still under the assumption \( S = \{0\} \), there exists a positive constant \( C > 1 \) such that for all \( m \geq 1 \) and \( X_1, \ldots, X_m \in \mathbb{R}^{2n} = S^\perp, 0 < t \leq 1 \) and \( u \in L^2(\mathbb{R}^n) \),

\[
(1.28) \quad \| (X_1,X)^w \ldots (X_m,X)^w e^{-tq^2} u \|_{L^2(\mathbb{R}^n)} \leq \frac{C^m}{t^{[k_0 + \frac{1}{2}]m}} \prod_{j=1}^m |X_j| \left( m! \right)^{k_0 + \frac{1}{2}} \| u \|_{L^2(\mathbb{R}^n)}.
\]

This implies in particular that when \( S = \{0\} \), the semigroup \((e^{-tq^2})_{t \geq 0}\) is smoothing in any positive time \( t > 0 \) in the Gelfand-Shilov space \( S_{k_0 + 1/2}^{k_0 + 1/2}(\mathbb{R}^n) \). We recall that when \( \mu \) and \( \nu \) are...
two positive real numbers satisfying $\mu + \nu \geq 1$, the Gelfand-Shilov space $S^\mu_0(\mathbb{R}^n)$ consists in all the Schwartz functions $f \in \mathcal{S}(\mathbb{R}^n)$ satisfying that

$$
\exists C > 1, \forall (\alpha, \beta) \in \mathbb{N}^{2n}, \quad \|x^\alpha \partial_x^\beta f(x)\|_{L^2(\mathbb{R}^n)} \leq C^{1+|\alpha|+|\beta|} (\alpha)!^\nu (\beta)!^\mu.
$$

We refer to [25] (Chapter 6) for an extensive discussion about the Gelfand-Shilov spaces. This result was sharpened by the same three authors in [18] (Theorem 1.2) with a different approach based on FBI techniques, where they proved that the semigroup $(e^{-tq\nu})_{t \geq 0}$ is actually smoothing in any positive time $t > 0$ in the Gelfand-Shilov space $S^{1/2}(\mathbb{R}^n)$ with a control of the blow-up of the associated seminorms in the asymptotics $t \to 0^+$. Moreover, estimates similar to (1.23) in the asymptotics $t \to +\infty$ were obtained in the case where $S = \{0\}$, see again Theorem 1.1 and Corollary 1.2 in [17]. We also refer the reader to [26, 29] where quadratic semigroups are studied in long-time asymptotics.

On the other hand, when the singular space $S$ of $q$ is possibly non-zero but still has a symplectic structure, that is, when the restriction of the canonical symplectic form to the singular space $\sigma_S$ is non-degenerate, the above result can be easily extended but only when differentiating the semigroup in the directions of the phase space given by the symplectic orthogonal complement of the singular space

$$
S^{0,\perp} = \{ X \in \mathbb{R}^{2n} : \forall Y \in S, \quad \sigma(X, Y) = 0 \}.
$$

Indeed, when the singular space $S$ has a symplectic structure, it was proven in [17] (Subsection 2.5) that the quadratic form $q$ writes as $q = q_1 + q_2$ with $q_1$ a purely imaginary-valued quadratic form defined on $S$ and $q_2$ another one defined on $S^{0,\perp}$ with a non-negative real part and a zero singular space. The symplectic structures of $S$ and $S^{0,\perp}$ imply that the operators $q_1^*(x, D_x)$ and $q_2^*(x, D_x)$ commute as well as their associated semigroups

$$
\forall t > 0, \quad e^{-tq_1} = e^{-tq_1} e^{-tq_2} = e^{-tq_2} e^{-tq_1}.
$$

Moreover, since $\text{Re} q_1 = 0$, $(e^{-tq_1\nu})_{t \geq 0}$ is a contraction semigroup on $L^2(\mathbb{R}^n)$ and the partial smoothing properties of the semigroup $(e^{-tq_2\nu})_{t \geq 0}$ can be deduced from a symplectic change of variables and the result known for zero singular spaces applied to the semigroup $(e^{-tq_2\nu})_{t \geq 0}$. We refer the reader to [17] (Subsection 2.5) for more details about the reduction by tensorization of the non-zero symplectic case to the case when the singular space is zero.

In the case when the singular space $S$ is not necessary trivial nor symplectic but satisfies the condition $S \subset \text{Ker}(\text{Im } F)$, with $F$ the Hamilton map of the quadratic form $q$, some partial Gelfand-Shilov smoothing effects in any positive time $t > 0$ for the semigroup $(e^{-tq\nu})_{t \geq 0}$ were obtained by the first author in [2] (Theorem 1.4), with some control of the associated seminorms as $t \to 0^+$. Moreover, we mention that under an algebraic condition on the matrices $Q$ and $B$, the regularizing effects of the Ornstein-Uhlenbeck operator (1.1), whose singular space is not symplectic nor satisfies the condition $S \subset \text{Ker}(\text{Im } F)$, see (1.31) with $R = 0$, were studied by the two authors in [3] (Theorem 1.2).

In this paper, we investigate the smoothing properties of the evolution operators $e^{-tq\nu}$ for any positive times $t > 0$, and we aim at sharpening and generalizing the estimates (1.23) without making any assumption on the singular space $S$. As in the work [17], the notion of index plays a key role in understanding the blow-up of the seminorms associated to the smoothing effects of the semigroup $(e^{-tq\nu})_{t \geq 0}$:

**Theorem 1.6.** Let $q : \mathbb{R}^{2n} \to \mathbb{C}$ be a complex-valued quadratic form with a non-negative real part $\text{Re } q \geq 0$. We consider $S$ the singular space of $q$ and $0 \leq k_0 \leq 2n - 1$ the smallest integer such that (1.13) holds. Then, there exist some positive constants $c > 1$ and $t_0 > 0$ such that for all $m \geq 1$, $X_1, \ldots, X_m \in S^{0,\perp}$, $0 < t < t_0$ and $u \in L^2(\mathbb{R}^n)$,

$$
\| (X_1, X) \cdots (X_m, X)^{w} e^{-tq\nu} u \|_{L^2(\mathbb{R}^n)} \leq \frac{c^m}{k_{X_1} \cdots k_{X_m} + \frac{1}{2}} \prod_{j=1}^{m} \| X_j \|^{m!\nu} \| u \|_{L^2(\mathbb{R}^n)},
$$

where $0 \leq k_{X_j} \leq k_0$ stands for the index of the point $X_j \in S^{0,\perp}$ with respect to the singular space.

In the case when $m = 1$, Theorem 1.6 recovers the estimate (1.26). The short-time asymptotics given by (1.23) of $m$ differentiations of the semigroup $(e^{-tq\nu})_{t \geq 0}$, as for it, is sharpened in $O(t^{-k_{X_1} \cdots k_{X_m} - \frac{1}{2}})$, which was the bound conjectured by the three authors of [17] in page 622. This result discloses that these short-time asymptotics depend on the phase space directions of differentiations. Moreover, the power over $(m!)^{k_0 + \frac{1}{2}}$ is sharpened in $(m!)^{\frac{1}{2}}$, which in particular
allows one to recover the Gelfand-Shilov $S^{1/2}_1(\mathbb{R}^n)$ regularizing effect of the semigroup $(e^{-tq^n})_{t \geq 0}$ in any positive time $t > 0$ when $S = \{0\}$ already established in [13] (Theorem 1.2), with now a precise control in short-time of the associated seminorms.

**Example 1.7.** Let $Q, R$ and $B$ be real $n \times n$ matrices, $Q$ and $R$ being symmetric positive semi-definite. We consider the generalized Ornstein-Uhlenbeck operator

\[(1.29) \quad P = -\frac{1}{2} \text{Tr}(Q \nabla^2_x) + \frac{1}{2} (Rx, x) + \langle Bx, \nabla_x \rangle,
\]

equipped with the domain

\[(1.30) \quad D(P) = \{ u \in L^2(\mathbb{R}^n) : Pu \in \mathbb{R}^n \}.
\]

Notice that $P$ is a pseudodifferential operator whose Weyl symbol $p$ is given by

\[(1.31) \quad p(x, \xi) = \frac{1}{2} \langle Q \xi, \xi \rangle + \frac{1}{2} (Rx, x) + i \langle Bx, \xi \rangle - \frac{1}{2} \text{Tr}(B).
\]

The operator $\tilde{P} = P + \frac{i}{2} \text{Tr}(B)$ is therefore a quadratic operator and it follows from a straightforward computation, see e.g. [2] (Section 5), that the Hamilton map $F$ and the singular space $S$ of $\tilde{P}$ are respectively given by

\[(1.32) \quad F = \frac{1}{2} \begin{pmatrix} iB & Q \\ -R & -iB^T \end{pmatrix},
\]

and

\[(1.33) \quad S = \bigcap_{j=0}^{n-1} (\text{Ker}(RB^j) \times \text{Ker}(QB^{T^j})).
\]

We can consider $0 \leq k_0 \leq n - 1$ the smallest integer such that $S$ writes as

\[(1.34) \quad S = \bigcap_{j=0}^{k_0} (\text{Ker}(RB^j) \times \text{Ker}(QB^{T^j})).
\]

We notice that the singular space of $\tilde{P}$ has a decoupled structure in the phase space in the sense that $S$ writes as the cartesian product $S = S_x \times S_\xi$, where the two vector subspaces $S_x \subset \mathbb{R}^n_x$ and $S_\xi \subset \mathbb{R}^n_\xi$ are respectively defined by

\[S_x = \bigcap_{j=0}^{k_0} \text{Ker}(RB^j) \subset \mathbb{R}^n_x \quad \text{and} \quad S_\xi = \bigcap_{j=0}^{k_0} \text{Ker}(QB^{T^j}) \subset \mathbb{R}^n_\xi.
\]

For all $x \in S_x^1$ and $\xi \in S_\xi^1$, we can define the indexes $0 \leq k_x \leq k_0$ and $0 \leq k_\xi \leq k_0$ of the points $x$ and $\xi$ with respect to the spaces $S_x$ and $S_\xi$ respectively by

\[k_x = \min \left\{ 0 \leq k \leq k_0 : x \in \left( \bigcap_{j=0}^{k} \text{Ker}(RB^j) \right) \right\},
\]

and

\[k_\xi = \min \left\{ 0 \leq k \leq k_0 : \xi \in \left( \bigcap_{j=0}^{k} \text{Ker}(QB^{T^j}) \right) \right\}.
\]

Notice that the integer $k_x$ (resp. $k_\xi$) coincides with the index of the point $(x, 0) \in S_x^1 \times \{0\} \subset S^\perp$ (resp. of the point $(0, \xi) \in \{0\} \times S_\xi^1 \subset S^\perp$) with respect to the singular space. Theorem [14] implies in particular that there exist some positive constants $c > 1$ and $t_0 > 0$ such that for all $m, p \geq 0$, $x_1, \ldots, x_m \subset S_x^1$, $\xi_1, \ldots, \xi_p \subset S_\xi^1$, $0 < t < t_0$, and $u \in L^2(\mathbb{R}^n)$,

\[(1.35) \quad \| \langle x_1, x \rangle \cdots \langle x_m, x \rangle \langle \xi_1, \nabla_x \rangle \cdots \langle \xi_p, \nabla_x \rangle e^{-tp} u \|_{L^2(\mathbb{R}^n)} \leq \frac{c^{1+m+p}}{t^{k_{x_1} + \cdots + k_{x_m} + k_{\xi_1} + \cdots + k_{\xi_p} + |x| + |\xi|}} \prod_{j=1}^{m} |x_j| \prod_{j=1}^{p} |\xi_j| (m!)^{\frac{1}{2}} (p!)^{\frac{1}{2}} \| u \|_{L^2(\mathbb{R}^n)},
\]

where the integers $0 \leq k_{x_j} \leq k_0$ (resp. $0 \leq k_{\xi_j} \leq k_0$) denote the indexes of the points $x_j$ (resp. $\xi_j$) with respect to $S_x$ (resp. $S_\xi$). This proves that the semigroup $(e^{-tP})_{t \geq 0}$ enjoys partial Gelfand-Shilov regularity in any positive time $t > 0$. 
Theorem 1.8. Let \( q : \mathbb{R}^{2n} \rightarrow \mathbb{C} \) be a complex-valued quadratic form with a non-negative real part \( \text{Re} q \geq 0 \). We consider \( S \) the singular space of \( q \). If there exist \( t > 0 \) and \( X_0 \in \mathbb{R}^{2n} \) such that the linear operator \( \langle X_0, X \rangle^w e^{-tq^w} \) is bounded on \( L^2(\mathbb{R}^n) \), then \( X_0 \in S^\perp \).

Notice that if \( t > 0 \) and \( X_0 \in \mathbb{R}^{2n} \) are such that the operator \( \langle X_0, X \rangle^w e^{-tq^w} \) is bounded on \( L^2(\mathbb{R}^n) \), then \( X_0 \in S^\perp \) according to Theorem 1.8 and then Theorem 1.6 can be applied to obtain that for all \( m \geq 1 \), the operators \( \langle (X_0, X)^w \rangle^m e^{-tq^w} \) are also bounded on \( L^2(\mathbb{R}^n) \).

Remark 1.9. In the study of the null-controllability of quadratic differential equations, a key ingredient is to obtain some dissipation estimates for the semigroup \( (e^{-tq^w})_{t \geq 0} \) in order to use a Lebeau-Robbiano strategy, see e.g. \([2, 3, 5, 6, 7, 11]\). The regularizing effects given by Theorem 1.6 allow to give a sufficient geometric condition on the singular space \( S \) of \( q \) so that such dissipation estimates hold. More precisely, let \( \pi_k : L^2(\mathbb{R}^n) \rightarrow E_k \) be the frequency cutoff projection defined as the orthogonal projection onto the vector subspace \( E_k \subset L^2(\mathbb{R}^n) \) given by \( E_k = \{ u \in L^2(\mathbb{R}^n) : \text{Supp} \hat{u} \subset [-k, k]^n \} \), with \( k \geq 1 \) a positive integer. It can be proven while using Theorem 1.6 and the strategy used in \([4]\) (Section 4.2), that when the singular space \( S \) of \( q \) takes the form \( S = \Sigma \times \{ 0 \}^n \), with \( \Sigma \subset \mathbb{R}^n \) a vector subspace, there exist some positive constants \( c_1, c_2, c_3 > 0 \) and \( 0 < t_0 < 1 \) such that for all \( k \geq 1, 0 < t < t_0 \) and \( u \in L^2(\mathbb{R}^n) \),

\[
\left\| (1 - \pi_k) e^{-tq^w} u \right\|_{L^2(\mathbb{R}^n)} \leq c_1 e^{-c_2k^{2k^2+1}k^2t} \| u \|_{L^2(\mathbb{R}^n)}.
\]

When the singular space of \( q \) is reduced to zero \( S = \{ 0 \} \), dissipative estimates similar to (1.36) were obtained with \( \pi_k \) some cutoff projections with respect to the Hermite basis of \( L^2(\mathbb{R}^n) \), see e.g. \([6, 7]\).

1.5. Subelliptic estimates enjoyed by quadratic operators. Finally, we study the subelliptic estimates enjoyed by accretive non-selfadjoint quadratic differential operators. When the singular space of the quadratic form \( q \) is reduced to zero \( S = \{ 0 \} \), K. Pravda-Starov proved in \([30]\) that the quadratic operator \( q^w(x, D_x) \) satisfies specific subelliptic estimates with a loss of derivatives with respect to the elliptic case directly depending on the structural parameter of the singular space \( 0 \leq k_0 \leq 2n - 1 \) defined in (1.13). More precisely, \([30]\) (Theorem 1.2.1) states that when the singular space is equal to zero \( S = \{ 0 \} \), there exists a positive constant \( c > 0 \) such that for all \( u \in D(q^w) \),

\[
\left\| \langle (x, D_x) \rangle_{\frac{q^w}{2 + x}} u \right\|_{L^2(\mathbb{R}^n)} \leq c \left[ \| q^w(x, D_x) u \|_{L^2(\mathbb{R}^n)} + \| u \|_{L^2(\mathbb{R}^n)} \right],
\]

where \( 0 \leq k_0 \leq 2n - 1 \) is the smallest integer such that (1.13) holds, with

\[
\langle (x, D_x) \rangle_{\frac{q^w}{2 + x}} = (1 + x^2 + D_x^2)^{\frac{q^w}{2 + x}},
\]

being the operator defined by the functional calculus of the harmonic oscillator. The estimate (1.37) was first proven in \([30]\) with a technical multiplier method, and recovered in the two papers \([18]\) (Theorem 1.1) and \([17]\) (Corollary 1.3) respectively by using technics of FBI transforms and the interpolation theory. Moreover, the three authors of \([17]\) and \([18]\) sharpened this result by improving it in the directions of the phase space which are less degenerate, that is with smaller indices with respect to the singular space. In order to recall their result, we need to consider the following quadratic forms

\[
p_k(X) = \sum_{j=0}^{k} \text{Re} q((\text{Im} F)^j X), \quad 0 \leq k \leq k_0,
\]

where \( 0 \leq k_0 \leq 2n - 1 \) is the smallest integer such that (1.13) holds. We also consider the quadratic operators \( \Lambda_k^2 \) defined for all \( 0 \leq k \leq k_0 \) by

\[
\Lambda_k^2 = 1 + p_k^w(x, D_x),
\]

and equipped with the domains

\[
D(\Lambda_k^2) = \{ u \in L^2(\mathbb{R}^n) : \Lambda_k^2 u \in L^2(\mathbb{R}^n) \}.
\]

Since \( \text{Re} q \geq 0 \) is a non-negative quadratic form, it can be proven by using for example Lemma (1.3) that the operators \( \Lambda_k^2 \) are positive and as a consequence, we can consider the fractional powers of
those operators. When the singular space $S$ of $q$ is reduced to zero, Theorem 1.4 in [17] states that there exists a positive constant $c > 0$ such that for all $u \in D(q^w)$,

$$\|A_0 u\|_{L^2(\mathbb{R}^n)} + \sum_{k=1}^{k_0} \|A_k^{2k+1} u\|_{L^2(\mathbb{R}^n)} \leq c \left[ \|q^w(x, D_x) u\|_{L^2(\mathbb{R}^n)} + \|u\|_{L^2(\mathbb{R}^n)} \right].$$

The authors of [17] expected the powers $2/(2k + 1)$ over the operators $A_k$ to be sharp but also expected the power over the term $A_0$ to be equal to 2 and not to 1.

No general theory has been developed when the singular space $S$ is not necessarily equal to zero. However, let us mention that some subelliptic estimates were obtained for the Kramers-Fokker-Planck operator without external potential $K$ defined in [17] by F. Hérau and K. Pravda-Starov in [13] (Proposition 2.1) with a multiplier method and for the Ornstein-Uhlenbeck operator $L$ defined in [13] under an algebraic condition on the matrices $Q$ and $B$ (the Kalman rank condition) by the two authors in [13] (Corollary 1.15) while using the interpolation theory as in the work [17].

In this paper, we aim at extending and sharpening the subelliptic estimates (1.41) to all quadratic forms $q : \mathbb{R}^{2n} \to \mathbb{C}$ with non-negative real parts $\text{Re} \ q \geq 0$, without making any assumption on their singular spaces $S$.

**Theorem 1.10.** Let $q : \mathbb{R}^{2n} \to \mathbb{C}$ be a complex-valued quadratic form with a non-negative real part $\text{Re} \ q \geq 0$. We consider $S$ the singular space of $q$ and $0 \leq k_0 \leq 2n - 1$ the smallest integer such that (1.12) holds. Then, there exists a positive constant $c > 0$ such that for all $u \in D(q^w)$,

$$\sum_{k=0}^{k_0} \|A_k^{2k+1} u\|_{L^2(\mathbb{R}^n)} \leq c \left[ \|q^w(x, D_x) u\|_{L^2(\mathbb{R}^n)} + \|u\|_{L^2(\mathbb{R}^n)} \right].$$

As in the case when the singular space is trivial, this result shows that the quadratic operator $q^w(x, D_x)$ enjoys anisotropic subelliptic estimates, this anisotropy being directly related to the structure (1.13) of the singular space $S$. Moreover, Theorem 1.10 confirms that the power over the operator $A_0$ associated to the real part of the quadratic form $q$ is actually equal to 2.

**Example 1.11.** Let $P$ be the generalized Ornstein-Uhlenbeck operator defined in (1.29) and equipped with the domain (1.30). It follows from a straightforward calculation that for all $0 \leq k \leq k_0$, the operator $A_k^2$ associated to the quadratic operator $P + \frac{1}{2} \text{Tr}(B)$ is given by

$$A_k^2 = 1 + \sum_{j=0}^{k} \frac{1}{2j+1} |R^j B^j x|^2 + \sum_{j=0}^{k} \frac{1}{2j+1} (Q^j (B^T)^j D_x x)^2,$$

where $0 \leq k_0 \leq n - 1$ is the smallest integer such that (1.34) holds. It therefore follows from Theorem 1.10 that there exists a positive constant $c > 0$ such that for all $0 \leq k \leq k_0$ and $u \in D(P)$,

$$\left\| \left( 1 + \sum_{j=0}^{k} \frac{1}{2j+1} |R^j B^j x|^2 + \sum_{j=0}^{k} \frac{1}{2j+1} (Q^j (B^T)^j D_x x)^2 \right)^{1/2} u \right\|_{L^2(\mathbb{R}^n)} \leq c \left[ \|Pu\|_{L^2(\mathbb{R}^n)} + \|u\|_{L^2(\mathbb{R}^n)} \right].$$

**Outline of the work.** In Section 2, we establish the polar decomposition of quadratic semigroups in any positive times whereas Section 3 is devoted to the study of the selfadjoint part for small times. As a byproduct of this decomposition, we study the regularizing effects of semigroups generated by non-selfadjoint quadratic differential operators in Section 4 from which we derive subelliptic estimates enjoyed by accretive quadratic operators in Section 5. Section 6 is an appendix containing the proofs of some technical results.

**Convention.** Any complex-valued quadratic form $q : \mathbb{R}^{2n} \to \mathbb{C}$ will be implicitly extended to the complex phase space $\mathbb{C}^{2n}$ in the following way:

$$\forall X \in \mathbb{C}^{2n}, \qquad q(X) = X^T Q X = q(\text{Re} \ X) + q(\text{Im} \ X),$$

where $Q \in S_{2n}(\mathbb{C})$ denotes the matrix of the quadratic form $q$ in the canonical basis of $\mathbb{R}^{2n}$.

**Notations.** The following notations will be used all over the work:

1. For all complex matrix $M \in M_n(\mathbb{C})$, $M^T$ denotes the transpose matrix of $M$ while $M^* = \overline{M^T}$ denotes its adjoint.
2. $(\cdot, \cdot)$ denotes the inner product on $\mathbb{C}^n$ as defined in (1.6).
3. We set $| \cdot |$ the Euclidean norm on $\mathbb{R}^n$ extended to $\mathbb{C}^n$ as explained in the previous convention.
4. The notation $\| \cdot \|$ stands for the matrix norm on $M_{2n}(\mathbb{C})$ induced by the norm $\| \cdot \|_2$ on $\mathbb{C}^{2n}$. From there, we introduce the norm $\| \cdot \|_\infty$ on $M_{2n}(\mathbb{C}) \times M_{2n}(\mathbb{C})$ defined by

$$\|(M, N)\|_\infty = \max(\|M\|, \|N\|).$$

5. When $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we denote by $Sp_{2n}(\mathbb{K})$ the symplectic group whose definition is recalled at the beginning of Subsection 6.2.

6. We denote by $\mathbb{C}(X, Y)$ the ring of the non-commutative polynomials in $X$ and $Y$, as defined e.g. in [8] (Chapter 6). For all non-negative integer $k \geq 0$, we set $\mathbb{C}_{k,0}(X, Y)$ the subspace of $\mathbb{C}(X, Y)$ of non-commutative polynomials of degree smaller than or equal to $k$ vanishing in $(0, 0)$.

7. For all vector subspace $V \subset \mathbb{K}^n$, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the notation $V^\perp$ is devoted to the orthogonal complement of $V$ with respect to the canonical Euclidean (when $\mathbb{K} = \mathbb{R}$) or Hermitian (when $\mathbb{K} = \mathbb{C}$) structure of $\mathbb{K}^n$.

8. If $f : (-\alpha, \alpha) \to M_n(\mathbb{C})$ is an analytic function such that $f(0) = 0$, with $\alpha \in (0, +\infty]$, there exists an other analytic function $g : (-\alpha, \alpha) \to M_n(\mathbb{C})$ such that for all $t \in (-\alpha, \alpha)$, $f(t) = tg(t)$. With an abuse of notation, we will denote

$$\forall t \in (-\alpha, \alpha), \quad g(t) = f(t)/t.$$  

2. **Splitting of semigroups generated by non-selfadjoint quadratic differential operators**

This section is devoted to the proof of Theorem 1.1. Let $q : \mathbb{R}^{2n} \to \mathbb{C}$ be a complex-valued quadratic form with a non-negative real part $\text{Re} \ q \geq 0$. We consider $Q \in S_{2n}(\mathbb{C})$ the matrix of $q$ in the canonical basis of $\mathbb{R}^{2n}$. We also consider $J$ the symplectic matrix defined in (1.9). Our goal is first to construct a family $(a_t)_{t \in \mathbb{R}}$ of non-negative quadratic forms $a_t : \mathbb{R}^{2n} \to \mathbb{R}_+$ depending analytically on the time-variable $t \in \mathbb{R}$ and a family $(U_t)_{t \in \mathbb{R}}$ of metaplectic operators such that for all $t \geq 0$,

$$e^{-tq^w} = e^{-ta_t^w}U_t,$$

and then to prove that there exist a positive constant $T > 0$ and a family $(b_t)_{-T < t < T}$ of real-valued quadratic forms $b_t : \mathbb{R}^{2n} \to \mathbb{R}$ also depending analytically on the time-variable $-T < t < T$, such that for all $0 \leq t < T$,

$$e^{-tq^w} = e^{-ta_t^w}e^{-itb_t^w}.$$

To that end, we begin by establishing that proving (2.1) and (2.2) is actually equivalent to solving a finite-dimensional problem involving matrices. First of all, in order to give an intuition of this equivalence, let us formally prove that given some $t > 0$, the equality of bounded operators

$$e^{-tq^w} = e^{-ta_t^w}e^{-itb_t^w},$$

is equivalent to the finite dimensional matrix relation

$$e^{-2itJQ} = e^{-2itJA_t}e^{2itJb_t},$$

where $A_t$ (resp. $B_t$) is the matrix of the quadratic form $a_t$ (resp. $b_t$) in the canonical basis of $\mathbb{R}^{2n}$. The equivalence between (2.3) and (2.4) will be justified rigorously shortly later with the theory of Fourier integral operators. By applying the Baker-Campbell-Hausdorff formula introduced in [4] and [12], the relation (2.4) is formally equivalent to

$$-tq^w = \sum_{m=0}^{+\infty} \sum_{p \in \{\alpha_t, \beta_t\}^m} (\text{ad}_{p_{t_1}^w}) \cdots (\text{ad}_{p_{t_m}^w})(\alpha_tta_t^w + \beta_titb_t^w),$$

where $\alpha_t, \beta_t \in \mathbb{Q}$ are explicit rational coefficients and

$$\text{ad}_{p_1} p_2 := [p_1, p_2] = p_1p_2 - p_2p_1,$$

denotes the commutator between the operators $p_1$ and $p_2$. However, if $q_1, q_2 : \mathbb{R}^{2n} \to \mathbb{C}$ are two quadratic forms, elements of Weyl calculus, see e.g. [19] (Theorem 18.5.6), show that the commutator $[q_1^w, q_2^w]$ is also a differential operator given by

$$[q_1^w, q_2^w] = -i\{q_1, q_2\}^w,$$

where

$$\{q_1, q_2\} = \nabla_\xi q_1 \cdot \nabla_\xi q_2 - \nabla_\eta q_1 \cdot \nabla_\eta q_2.$$
is the Poisson bracket between the quadratic forms \( q_1 \) and \( q_2 \). We therefore deduce that (2.10) is equivalent to the equality between quadratic forms

\[
- t q = \sum_{m=0}^{+\infty} \sum_{p \in \{-\alpha, \beta\}^m} (ad_{p_1}) \ldots (ad_{p_m})(\alpha p_t a_t + \beta p_t b_t),
\]

where we set \( ad_{p_1} p_2 := \{p_1, p_2\} \). Moreover, we observe that if \( q_1, q_2 : \mathbb{R}^{2n} \rightarrow \mathbb{C} \) are two quadratic forms, the Hamilton map of the Poisson bracket \( \{q_1, q_2\} \) is \(-2[F_1, F_2] \) with \([F_1, F_2] \) the commutator of \( F_1 \) and \( F_2 \) the Hamilton maps of \( q_1 \) and \( q_2 \), see e.g. [29] (Lemma 3.2). As a consequence, we deduce while using (1.7) and multiplying by 2/ that (2.7) is equivalent to the matrix relation

\[
- 2t JQ = \sum_{m=0}^{+\infty} \sum_{p \in \{2t A_t, 2t B_t\}^m} (ad_{p_1}) \ldots (ad_{p_m})(\alpha p_{2t J A_t} + \beta p_{2t J B_t}).
\]

Thus, by applying once again the Baker-Campbell-Hausdorff formula, the relation (2.9) is equivalent to (2.4). Obtaining the quadratic forms \( a_t \) and \( b_t \) is then far easier henceforth the equivalence between (2.9) and (2.4) is established. Indeed, let us check that the relation (2.4) is equivalent to the following triangular system

\[
\begin{align*}
\alpha e^{-4it J A_t} &= e^{-2it J Q} e^{-2it J Q},
\beta e^{2it J B_t} &= e^{2it J A_t} e^{-2it J Q}.
\end{align*}
\]

Obviously, if (2.9) holds, then (2.4) is satisfied. On the other hand, when (2.4) holds, we observe that

\[
e^{-2it J Q} e^{-2it J Q} = e^{-4it J A_t} e^{-2it J B_t} e^{-2it J B_t} e^{-2it J A_t} = e^{-4it J A_t}.
\]

Moreover, the equality \( e^{2it J B_t} = e^{2it J A_t} e^{-2it J Q} \) is only a rewriting of (2.4) and hence, (2.9) holds. The first equation of (2.9) will be solved for any time \( t \in \mathbb{R} \) by using the holomorphic functional calculus. The second one will only be solved for short times \( |t| \ll 1 \).

In order to justify rigorously this reduction to a finite-dimensional problem, we shall use the Fourier integral operator representation of the evolution operators \( e^{-it \tilde{q}} \) proven in [20] (Theorem 5.12) and recalled in the following proposition:

**Proposition 2.1.** Let \( \tilde{q} : \mathbb{R}^{2n} \rightarrow \mathbb{C} \) be a complex-valued quadratic form with a non-negative real part \( \text{Re} \tilde{q} \geq 0 \). Then, for all \( t \geq 0 \), the evolution operator \( e^{-it \tilde{q}} = \mathcal{K}_{e^{-it \tilde{q}}} \) generated by the quadratic operator \( \tilde{q}^{\omega}(x, D_x) \) is a Fourier integral operator whose kernel is a Gaussian distribution associated to the non-negative complex symplectic linear bijection \( e^{-2it J Q} \in \mathfrak{sp}_{2n}(\mathbb{C}) \), with \( Q \in S_{2n}(\mathbb{C}) \) the matrix of \( \tilde{q} \) with respect to the canonical basis of \( \mathbb{R}^{2n} \).

We refer the reader to Subsection 6.3 in Appendix for the definition of the Fourier integral operators \( \mathcal{K}_T \) and their basic properties, where \( T \) is a non-negative complex symplectic linear bijection in \( \mathbb{C}^{2n} \). The key property satisfied by the operators \( \mathcal{K}_T \) that we will need here is that if \( T_1 \) and \( T_2 \) are two non-negative complex symplectic linear bijections in \( \mathbb{C}^{2n} \), then \( T_1 T_2 \) is also a non-negative complex symplectic linear bijection and

\[
\mathcal{K}_{T_1 T_2} = \pm \mathcal{K}_{T_1} \mathcal{K}_{T_2},
\]

see Proposition 6.4. The sign uncertainty in (2.10) will not be an issue in the following. As a consequence of (2.10) and Proposition 2.1, we shall on the one hand, to prove (2.11), obtain the existence of two families \( \{A_t\}_{t \in \mathbb{R}} \) and \( \{H_t\}_{t \in \mathbb{R}} \) of real symmetric positive semidefinite matrices \( A_t \in S_{2n}^+(\mathbb{R}) \) and real symplectic matrices \( H_t \in \mathfrak{sp}_{2n}(\mathbb{R}) \) respectively, whose coefficients depend analytically on the time variable \( t \in \mathbb{R} \), such that for all \( t \in \mathbb{R} \),

\[
e^{-2it J Q} = e^{-2it J A_t} H_t.
\]

On the other hand, to establish (2.2), we shall prove that there exist a positive constant \( T > 0 \) and a family \( \{B_t\}_{-T < t < T} \) of real symmetric matrices, whose coefficients also depend analytically on the time-variable \(-T < t < T\), such that for all \(-T < t < T\), the real symplectic matrix \( H_t \) is given by

\[
H_t = e^{2it B_t}.
\]

Indeed, let us first assume that (2.11) holds and let us prove (2.3). It follows from (2.10) that for all \( t \geq 0 \), up to sign,

\[
e^{-t \tilde{q}} = \mathcal{K}_{e^{-2it J Q}} = \mathcal{K}_{e^{-2it J A_t} H_t} = \pm \mathcal{K}_{e^{-2it J A_t}} \mathcal{K}_{H_t} = e^{-t \tilde{q}} \mathcal{U}_t,
\]
where $U_t = e_t K_{H_t}$ is a metaplectic operator on $L^2(\mathbb{R}^n)$, see Definition 6.3, with $e_t \in \{-1, 1\}$, and $a_t : \mathbb{R}^{2n} \to \mathbb{R}_+$ is the non-negative time-dependent quadratic form associated to the matrix $A_t$ in the canonical basis of $\mathbb{R}^{2n}$. This proves that (2.1) holds. On the other hand, to derive (2.2) from (2.12), we consider the time-dependent quadratic form $b_t : \mathbb{R}^{2n} \to \mathbb{R}$, with $0 \leq t < T$, associated to the time-dependent matrix $B_t$ in the canonical basis of $\mathbb{R}^{2n}$. Indeed, when (2.12) holds, it follows from the definition of the operators $U_t$ and Proposition 2.1 that for all $0 \leq t < T$,

$$U_t = e_t K_{H_t} = e_t K_{e^{2itb_t}} = e_t e^{-itb_t^w}.$$  

We then deduce from (2.1) and (2.13) that for all $0 \leq t < T$,

$$e^{-tq}w = e^{-tq}w - e^{-itb_t^w}.$$  

It only remains to check that $\varepsilon = 1$ for all $0 \leq t < T$. To that end, we consider $u \in \mathcal{F}(\mathbb{R}^n)$ a non-zero Schwartz function. We deduce from (2.14) that for all $0 \leq t < T$,

$$\langle e^{-tq}w, e^{-itb_t^w} u \rangle_{L^2(\mathbb{R}^n)} = \varepsilon_t \langle e^{-tq}w, e^{-itb_t^w} u \rangle_{L^2(\mathbb{R}^n)}.$$  

Since the quadratic form $a_t$ is non-negative for all $t \geq 0$, the operator $e^{-tq}$ is selfadjoint on $L^2(\mathbb{R}^n)$ and we therefore deduce by using the semigroup property of the family of operators $(e^{-sa_t^w})_{s \geq 0}$ that for all $t \geq 0$,

$$\langle e^{-tq}w, e^{-itb_t^w} u \rangle_{L^2(\mathbb{R}^n)} = \varepsilon_t \langle e^{-tq}w, e^{-itb_t^w} u \rangle_{L^2(\mathbb{R}^n)}.$$  

The operator $e^{2itb_t^w}$ is injective from Corollary 6.3 and the operator $e^{-itb_t^w}$ is unitary for all $t \geq 0$, since the quadratic form $b_t$ is real-valued. Thus, the Schwartz functions $e^{2itb_t^w}$ for all $t \geq 0$ and we have that for all $t \geq 0$,

$$\varepsilon_t = \langle e^{-tq}w, e^{-itb_t^w} u \rangle_{L^2(\mathbb{R}^n)} \frac{\|e^{-tq}w\|_{L^2(\mathbb{R}^n)}}{\|e^{-itb_t^w} u\|_{L^2(\mathbb{R}^n)}}.$$  

Moreover, it follows from (2.2) (Theorem 4.2) that the applications $t \mapsto e^{-tq}w$, $t \mapsto e^{-itb_t^w}$ and $t \mapsto e^{-tq}w e^{-itb_t^w}$ are continuous from $[0, T)$ to $\mathcal{F}(\mathbb{R}^n)$. It follows from (2.15) that the map $t \mapsto \varepsilon_t$ is also continuous from $[0, T]$ to $\{-1, 1\}$ and since $\varepsilon_0 = 1$, we have $\varepsilon_t = 1$ for all $0 \leq t < T$. This ends the proof of (2.2).

The present subsection is therefore devoted to the proof of (2.11) and (2.12). We first focus on the identity (2.11). As above, we can prove that this relation is equivalent to the following triangular system,

$$\begin{align*}
e^{-4itA_t} &= e^{-2itJQ} e^{-2itJQ}, \\
H_t &= e^{2itJQ} e^{-2itJQ}.
\end{align*}$$  

We begin by solving the first equation of (2.10):

**Theorem 2.2.** There exists a family $(A_t)_{t \in \mathbb{R}}$ of real symmetric positive semidefinite matrices $A_t \in \mathfrak{S}^+_2(\mathbb{R})$ whose coefficients depend analytically on the time-variable $t \in \mathbb{R}$ such that for all $t \in \mathbb{R}$,

$$e^{-4itA_t} = e^{-2itJQ} e^{-2itJQ}.$$  

To prove Theorem 2.2, we need some technical lemmas. The first of them investigates the spectrum of the symplectic matrices $e^{-2itJQ} e^{-2itJQ}$ appearing in Theorem 2.2.

**Lemma 2.3.** For all $t \in \mathbb{R}$, the eigenvalues of the matrix $e^{-2itJQ} e^{-2itJQ}$ are positive real numbers,

$$\sigma(e^{-2itJQ} e^{-2itJQ}) \subset \mathbb{R}^*_+. $$  

**Proof.** For all $t \in \mathbb{R}$, we define

$$K_t = e^{-2itJQ} e^{-2itJQ}.$$  

We first check that the following integral representation holds for all $t \in \mathbb{R}$,

$$K_t = I_{2n} - 4iJ\Gamma_t,$$

where the matrix $\Gamma_t$ is given by

$$\Gamma_t = \int_0^t (e^{-2isJQ})^* (\text{Re} Q) (e^{-2isJQ}) \, ds.$$  

It follows from a direct computation for all $t \in \mathbb{R}$,

$$\partial_t (e^{-2itJQ} e^{-2itJQ}) = -2ie^{-2itJQ} J(Q + JQ)e^{-2itJQ} = -4ie^{-2itJQ} J(\text{Re} Q)e^{-2itJQ}.$$
Since $Q$ is a symmetric matrix, it follows from Lemma 6.2 that for all $t \in \mathbb{R}$, $e^{-2itJQ} \in \text{Sp}_{2n}(\mathbb{C})$ is a symplectic matrix and as a consequence of the above estimate,

$$
\partial_t(e^{-2itJQ}e^{-2itJQ^T}) = -4iJ(e^{-2itJQ})^T(\text{Re}Q)e^{-2itJQ} = -4iJ(e^{-2itJQ})^T(\text{Re}Q)e^{-2itJQ}.
$$

This proves that (2.18) holds. Since the matrices $\Gamma_t \in \text{H}_{2n}(\mathbb{C})$ are Hermitian positive semidefinite when $t \geq 0$ and Hermitian negative semidefinite when $t \leq 0$, we deduce from Lemma 6.10 that for all $t \in \mathbb{R}$, the spectra of the matrices $\Gamma_t$ satisfy $\sigma(\Gamma_t) \subset i\mathbb{R}$. This combined with (2.18) and (2.19) shows that for all $t \in \mathbb{R}$, $\sigma(K_t) \subset \mathbb{R}$. The matrices $K_t \in \text{GL}_{2n}(\mathbb{C})$ are non singular and therefore, these inclusions can be refined to $\sigma(K_t) \subset \mathbb{R}^+$. Moreover, $\sigma(K_0) = \{1\}$ and the eigenvalues of $K_t$ are continuous with respect to the time-variable $t \in \mathbb{R}$ since the coefficients of the matrix $K_t$ are themselves continuous with respect to the time-variable $t \in \mathbb{R}$, see [22] (Theorem II.5.1). Since $\mathbb{R}$ is connected, this proves that $\sigma(K_t) \subset \mathbb{R}^+_t$ and ends the proof of Lemma 2.3.

In the following, we shall need to define some matrices through the holomorphic functional calculus. We refer the reader to [10] (VII - 3.) where this theory is presented. As a first application of this theory, we consider the matrix square root function $\sqrt{z}$ defined on the set of matrices whose spectrum is contained in $\mathbb{C} \setminus \mathbb{R}_-$, which is possible since the function $z \mapsto \sqrt{z} = e^{\frac{i}{2} \text{Log} z}$ is well-defined and holomorphic in $\mathbb{C} \setminus \mathbb{R}_-$, with $\text{Log}$ the principal determination of the logarithm in $\mathbb{C} \setminus \mathbb{R}_-$. For all $t \in \mathbb{R}$, since the spectrum of the matrix $K_t$ is only composed of positive real numbers, we can consider the matrix $G_t$ defined by

$$
G_t = \sqrt{e^{-2itJQ}e^{-2itJQ^T}}.
$$

We shall check that the matrices $G_t$ are symplectic:

**Lemma 2.4.** For all $t \in \mathbb{R}$, $G_t \in \text{Sp}_{2n}(\mathbb{C})$ is a complex symplectic matrix.

**Proof.** Let $t \in \mathbb{R}$. We consider $K_t$ the matrix defined in (2.17). We first observe that since both matrices $Q$ and $\overline{Q}$ are symmetric, Lemma 6.2 shows that the matrices $e^{-2itJQ}$ and $e^{-2itJQ^T}$ are symplectic and as a consequence, the matrices $K_t \in \text{Sp}_{2n}(\mathbb{C})$ are also symplectic. To prove that the matrix $G_t$ is also symplectic, we need to go back to the definition of the matrix square root given by the functional holomorphic calculus. Therefore, we consider $\Sigma_t \subset \mathbb{C}$ the following domain of the complex plane

$$
\Sigma_t = \left\{ re^{i\theta} : c_{1,t} < r < c_{2,t}, \theta \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \right\},
$$

where the positive constants $c_{1,t}, c_{2,t} > 0$ are chosen so that $\sigma(K_t) \subset (c_{1,t}, c_{2,t})$ and $\sigma(K_t^{-1}) \subset (c_{1,t}, c_{2,t})$. Notice that the existence of the constants $c_{1,t}, c_{2,t} > 0$ is given by Lemma 2.3. We assume that the boundary $\partial \Sigma_t$ of the domain $\Sigma_t$ is oriented counterclockwise. Then, it follows from (2.20) and the holomorphic functional calculus that the matrix $G_t$ is defined by

$$
G_t = \frac{1}{2i\pi} \int_{\partial \Sigma_t} \sqrt{z} (K_t - zI_{2n})^{-1} \, dz,
$$

with $\sqrt{z} = e^{\frac{i}{2} \text{Log} z}$, where $\text{Log}$ denotes the principal determination of the logarithm in $\mathbb{C} \setminus \mathbb{R}_-$. Moreover, since the matrix $K_t$ is symplectic, we deduce that

$$
\begin{align*}
JG_t &= \frac{1}{2i\pi} \int_{\partial \Sigma_t} \sqrt{z} J(K_t - zI_{2n})^{-1} \, dz = -\frac{1}{2i\pi} \int_{\partial \Sigma_t} \sqrt{z} (K_tJ - zJ)^{-1} \, dz \\
&= \frac{1}{2i\pi} \int_{\partial \Sigma_t} \sqrt{z} (J(K_t^T)^{-1} - zJ)^{-1} \, dz \\
&= \frac{1}{2i\pi} \int_{\partial \Sigma_t} \sqrt{z} ((K_t^T)^{-1} - zI_{2n})^{-1} J \, dz \\
&= \left( \frac{1}{2i\pi} \int_{\partial \Sigma_t} \sqrt{z} (K_t^{-1} - zI_{2n})^{-1} \, dz \right)^T J = \left( \sqrt{K_t^{-1}} \right)^T J.
\end{align*}
$$

Finally, since the function $z \mapsto (\sqrt{z})^{-1} = \frac{1}{\sqrt{z}}$ is holomorphic on $\mathbb{C} \setminus \mathbb{R}_- \setminus \{0\}$ and that the eigenvalues of the matrices $K_t$ are positive real numbers, it follows from the holomorphic functional calculus, see e.g. [10] (VII.3.12, Theorem 12), that

$$
\sqrt{K_t^{-1}} = (\sqrt{K_t})^{-1} = G_t^{-1}.
$$

This, combined with (2.22), proves that $JG_t = (G_t^T)^{-1}J$, that is $G_t \in \text{Sp}_{2n}(\mathbb{C})$ is a symplectic matrix. This ends the proof of Lemma 2.4. \qed
We can now construct the matrices $A_t$. Since the function $z \mapsto \text{atanh}((z - 1)(z + 1)^{-1})$ is holomorphic on a neighborhood of $\mathbb{R}^*_+$, where $\text{atanh}$ denotes the hyperbolic arctan function (whose definition and properties can be found in [1] (Section 4.6)), and that $\sigma(G_t) \subset \mathbb{R}^*_+$ for all $t \in \mathbb{R}$ from (2.20) and Lemma 2.3, the functional holomorphic calculus also allows to consider the family of matrices $(A_t)_{t \in \mathbb{R}}$ defined for all $t \in \mathbb{R}$ by

\begin{equation}
A_t = -(itJ)^{-1} \text{atanh} \left( \left( G_t - I_{2n} \right) \left( G_t + I_{2n} \right)^{-1} \right).
\end{equation}

By construction, the function $t \in \mathbb{R} \mapsto \text{atanh}((G_t - I_{2n})(G_t + I_{2n})^{-1})$ is real analytic and vanishes in $t = 0$, since $G_0 = I_{2n}$ from (2.20) and $\text{atanh}(0) = 0_2$. The matrix $A_t$ is therefore well-defined for all $t \in \mathbb{R}$ and the function $t \in \mathbb{R} \mapsto A_t$ is as well analytic according to (1.43). This family $(A_t)_{t \in \mathbb{R}}$ satisfies the algebraic part of Theorem (2.2) as proved in the

\begin{lemma}
For all $t \in \mathbb{R}$, the matrix $A_t$ satisfies

\begin{equation}
e^{-4itJ}A_t = e^{-2itJQ}e^{-2itJQ^*}.
\end{equation}

\end{lemma}

\begin{proof}
We first observe that

\begin{equation}
\forall x > 0, \quad \exp \left( 4 \text{atanh} \left( \frac{x - 1}{x + 1} \right) \right) = x^2.
\end{equation}

Indeed, if $x > 0$ a positive real number and $y \in \mathbb{R}$ is a real number such that $x = e^{2y}$, we have

\begin{equation}
\exp \left( 4 \text{atanh} \left( \frac{x - 1}{x + 1} \right) \right) = \exp \left( 4 \text{atanh} \left( e^{2y} - 1 \right) \right) = \exp \left( 4 \text{atanh} \left( \frac{e^{2y} - 1}{e^{2y} + 1} \right) \right) = \exp \left( 4 \left( \frac{e^{2y} - 1}{e^{2y} + 1} \right) \right) = e^{4y} = x^2.
\end{equation}

Moreover, both functions $z \mapsto \exp(4 \text{atanh}((z - 1)(z + 1)^{-1}))$ and $z \mapsto z^2$ are holomorphic on a connected open neighborhood of $\mathbb{R}_+^*$ and $\sigma(G_t) \subset \mathbb{R}_+^*$ from (2.20) and Lemma 2.3. We therefore deduce from (2.24), (2.25) and the holomorphic functional calculus that for all $t \in \mathbb{R}$,

\begin{equation}
e^{-4itJ}A_t = \exp \left( 4 \text{atanh} \left( \left( G_t - I_{2n} \right) \left( G_t + I_{2n} \right)^{-1} \right) \right) \circ G_t = e^{-2itJQ}e^{-2itJQ^*}.
\end{equation}

This ends the proof of Lemma 2.5.
\end{proof}

Notice that the matrices $A_t$ can therefore be expressed by taking the logarithm of the matrices $e^{-2itJQ}e^{-2itJQ^*}$. Indeed, since the spectra of these matrices is contained in $\mathbb{R}_+^*$ from Lemma 2.3, and that the function $\text{Log}$ (which still denotes the principal determination of the logarithm in $\mathbb{C} \setminus \mathbb{R}_-$) is holomorphic in a neighborhood of $\mathbb{R}_+^*$, Lemma 2.5 and the holomorphic functional calculus imply that for all $t \in \mathbb{R}$,

\begin{equation}
tA_t = -(4itJ)^{-1} \text{Log} \left( e^{-2itJQ}e^{-2itJQ^*} \right)
\end{equation}

Moreover, the function $t \in \mathbb{R} \mapsto \text{Log}(e^{-2itJQ}e^{-2itJQ^*})$ is analytic by construction and vanishes in $t = 0$. It therefore follows from (1.43) that the matrix $A_t$ is given for all $t \in \mathbb{R}$ by

\begin{equation}
A_t = -(4itJ)^{-1} \text{Log} \left( e^{-2itJQ}e^{-2itJQ^*} \right)
\end{equation}

This formula will be useful in Section 4.

Now, it only remains to prove that the matrices $A_t$ are real and symmetric positive semi-definite.

To that end, we introduce the family of matrices $(M_t)_{t \in \mathbb{R}}$ where $M_t$ is defined for all $t \in \mathbb{R}$ by

\begin{equation}
M_t = -(itJ)^{-1} \left( G_t - I_{2n} \right) \left( G_t + I_{2n} \right)^{-1}.
\end{equation}

Notice that the matrices $M_t$ are well-defined according to (1.43) since one the one hand, (2.20) and Lemma 2.3 imply that $-1$ is not an eigenvalue of any matrix $G_t$ and on the other hand, the function $t \in \mathbb{R} \mapsto (G_t - I_{2n})(G_t + I_{2n})^{-1}$ is real analytic by construction and vanishes in $t = 0$. Moreover, the function $t \in \mathbb{R} \mapsto M_t$ is analytic. We will prove in Lemma 2.7 that the matrices $A_t$ can be expressed in terms of the matrices $M_t$ which will turn out to be real and symmetric. Moreover, the next lemma will imply that the matrices $M_t$ are positive semi-definite. The properties required for the matrices $A_t$ will then arise from the ones of the matrices $M_t$.

\begin{lemma}
For all $t \in \mathbb{R}$, the matrix $M_t$ admits the following integral representation

\begin{equation}
M_t = \int_0^1 \left( e^{-2itJQ} \Phi_t \right)^* (\text{Re} Q) \left( e^{-2itJQ} \Phi_t \right) \, d\alpha,
\end{equation}

where the matrix $\Phi_t$ is given by

\begin{equation}
\Phi_t = \left( \frac{\sqrt{e^{-2itJQ}e^{-2itJQ^*} + I_{2n}}}{2} \right)^{-1}.
\end{equation}

\end{lemma}
In particular, the matrices $M_t$ are Hermitian positive semidefinite.

Proof. Let $t \in \mathbb{R}$. We begin by checking that the matrix $M_t$ satisfies the relation
\begin{equation}
(G_t + I_{2n})^* t M_t (G_t + I_{2n}) = i J (I_{2n} - G_t^2).
\end{equation}
We recall that the matrix $J$ satisfies $J^{-1} = J^T = -J$. On the one hand, the left-hand side of this equality can be computed with the definition (2.20) of $M_t$:
\begin{equation}
(G_t + I_{2n})^* t M_t (G_t + I_{2n}) = -i (G_t + I_{2n})^* J (G_t - I_{2n}).
\end{equation}
On the other hand, since the matrix square root given by the holomorphic functional calculus (which can be readily checked by using (2.21)) and with the invert function defined for all non-singular matrix whose spectrum is composed of positive real numbers, it follows from (2.20) that the matrix $G_t$ satisfies
\begin{equation}
\frac{d}{dt} \left( e^{2itJQ} e^{2itJQ} \right) = \left( e^{2itJQ} e^{2itJQ} \right)^{-1} = G_t^{-1}.
\end{equation}
Moreover, $G_t \in \text{Sp}_{2n}(\mathbb{C})$ is a symplectic matrix from Lemma 2.1 and we deduce that
\begin{equation}
(G_t + I_{2n})^* J = (G_t + I_{2n})^T J = -(J G_t^{-1} + J)^T = -(G_t J + J)^T = J (G_t^T + I_{2n})^T = J (G_t + I_{2n}).
\end{equation}
Hence, substituting this equality in (2.30), we get that
\begin{equation}
(G_t + I_{2n})^* t M_t (G_t + I_{2n}) = -i J (G_t + I_{2n}) (G_t - I_{2n}) = -i J (G_t^2 - I_{2n}).
\end{equation}
This proves that (2.29) holds. Then, we deduce from (2.18), (2.19) and (2.20) that the right-hand side of (2.29) writes as
\begin{equation}
i J (I_{2n} - G_t^2) = i J (I_{2n} - e^{-2itJQ} e^{-2itJQ}) = 4 \int_0^t (e^{-2i(s+t)JQ})^* (Re Q) (e^{-2isJQ}) \ ds.
\end{equation}
Therefore, we derive the following expression for the matrix $t M_t$:
\begin{equation}
t M_t = 4 \int_0^t (e^{-2i(s+t)JQ} (G_t + I_{2n})^{-1})^* (Re Q) (e^{-2isJQ}) (G_t + I_{2n})^{-1} \ ds.
\end{equation}
Since the matrix $\Phi_t$ defined in (2.23) also writes as
\begin{equation}
\Phi_t = \left( G_t + I_{2n} \right)^{-1},
\end{equation}
we deduce from (2.31) that the matrix $t M_t$ is given by
\begin{equation}
t M_t = \int_0^t (e^{-2isJQ} \Phi_t)^* (Re Q) (e^{-2isJQ} \Phi_t) \ ds.
\end{equation}
A change of variable in the integral ends the proof of Lemma 2.6. □

We can now derive the end of the proof of Theorem 2.2 from Lemma 2.6. This is done in the following Lemma which will also be key to prove Theorem 1.2 in Section 3.

**Lemma 2.7.** For all $t \in \mathbb{R}$, the matrix $A_t$ is real and symmetric positive semidefinite. Moreover, the matrices $A_t$ and $M_t$ satisfy the following estimate:
\begin{equation}
\forall t \in \mathbb{R}, \quad A_t \geq M_t \geq 0.
\end{equation}

Proof. To simplify the notations in the following, we consider the following matrices for all $t \in \mathbb{R}$,
\begin{equation}
\Psi_t = (G_t - I_{2n}) (G_t + I_{2n})^{-1}.
\end{equation}
We recall that the matrix atanh function admits the following Taylor expansion for all matrices $R$ whose norm satisfies $\|R\| < 1$,
\begin{equation}
\text{atanh} \ R = \sum_{k=0}^{+\infty} \frac{R^{2k+1}}{2k + 1}.
\end{equation}
We also recall from (2.21) that the matrices $A_t$ are defined for all $t \in \mathbb{R}$ with the convention (1.43) by
\begin{equation}
A_t = -(iiJ)^{-1} \text{atanh} \ \Psi_t.
\end{equation}
It follows from the inequality
\[
\forall x > 0, \quad \frac{\sqrt{x} - 1}{\sqrt{x} + 1} < 1,
\]
the definitions (2.20) and (2.35) of the matrices \(G_t\) and \(\Psi_t\), and Lemma 2.3 that the spectrum of the matrix \(t\) satisfies \(\sigma(\Psi_t) \subseteq (-1, 1)\) for all \(t \in \mathbb{R}\). It therefore follows from (2.21) (Lemma 5.6.10) that for all \(t \in \mathbb{R}\), there exists a norm \(\| \cdot \|_t\) on \(M_n(\mathbb{C})\) such that \(\|\Psi_t\|_t < 1\). This proves that the series \(\sum_{k=0}^{+\infty} \Psi_{2k+1}\) converge in \(M_n(\mathbb{C})\) for all \(t \in \mathbb{R}\) and we deduce from (2.30) and (2.31) that for all \(t \in \mathbb{R}\),
\[
(2.38) \quad A_t = -(itJ)^{-1} \sum_{k=0}^{+\infty} \frac{\Psi_{2k+1}}{2k+1}.
\]

To prove that the matrices \(A_t\) are real and symmetric, we need to derive a new expression for them. To that end, we compute the product \(t\Psi_t\) by using the relation (2.32) (which also holds when the matrix \(I_{2n}\) is replaced by \(-I_{2n}\):
\[
(2.39) \quad t\Psi_t = J(G_t - I_{2n})(G_t + I_{2n})^{-1} = (G_t - I_{2n})^* J(G_t + I_{2n})^{-1}
\]
\[
= (G_t - I_{2n})^* ((G_t + I_{2n})^{-1})^* J = \Psi_t^* J.
\]

We deduce from (2.27), (2.35), (2.36) and (2.39) that for all \(t \in \mathbb{R}\),
\[
(2.40) \quad A_t = \frac{1}{2k+1} (\Psi_k^*)^*(it)^{-1} \Psi_t (\Psi_k^*) = \sum_{k=0}^{+\infty} \frac{1}{2k+1} (\Psi_k^*)^* M_t(\Psi_k^*).
\]

We observe from (2.31) and (2.35) that for all \(t \in \mathbb{R}\),
\[
(2.41) \quad \Psi_t = (G_t - I_{2n})(G_t + I_{2n})^{-1} = (G_t^{-1} - I_{2n})(G_t^{-1} + I_{2n})^{-1}
\]
\[
= (I_{2n} - G_t)(I_{2n} + G_t)^{-1} = -\Psi_t.
\]

Consequently, from (2.27), (2.35), (2.39) and (2.41), the matrices \(M_t\) satisfy the two relations
\[
(2.42) \quad M_t = (itJ)^{-1} \Psi_t = M_t.
\]

and
\[
(2.43) \quad M_t^* = (it)^{-1} \Psi_t^* J = (it)^{-1} J \Psi_t = (it)^{-1} \Psi_t = M_t.
\]

It follows from (2.41), (2.22) and (2.43) that for all \(t \in \mathbb{R}\) and \(k \geq 0\), the matrix \((\Psi_k^*)^* M_t(\Psi_k^*)\) is real and symmetric. As sums of such matrices, the matrices \(A_t\) are also real and symmetric. Finally, we deduce from (2.40) and Lemma 2.6 that for all \(t \in \mathbb{R}\), \(A_t \geq M_t \geq 0\). This ends the proof of Lemma 2.7. \(\square\)

We recall from [20] (Theorem 4.2) that the evolution operators \(e^{-\sigma^2}\), with \(t \geq 0\), generated by an accretive quadratic operator \(\tilde{q}^2(x, D_x)\), with \(\tilde{q} : \mathbb{R}^{2n} \rightarrow \mathbb{C}\) a complex-valued quadratic form with a non-negative real-part \(\text{Re} \tilde{q} \geq 0\), are pseudodifferential operators whose symbols are tempered distributions \(p_t \in \mathcal{F}'(\mathbb{R}^{2n})\). More specifically, these symbols are \(L^\infty(\mathbb{R}^{2n})\) functions explicitly given by the Mehler formula
\[
(2.44) \quad p_t(X) = \frac{1}{\sqrt{\text{det}(\cos(t\tilde{F}))}} \exp(-\sigma(X, \tan(t\tilde{F}))X), \quad X \in \mathbb{R}^{2n},
\]

whenever the condition \(\text{det}(\cos(t\tilde{F})) \neq 0\) is satisfied, where \(\tilde{F}\) denotes the Hamilton map associated to the quadratic form \(\tilde{q}\). As a Corollary of Lemma 2.7, we can compute the Weyl symbol of the operator \(e^{-tq^2}\) for all \(t \geq 0\), with \(q : \mathbb{R}^{2n} \rightarrow \mathbb{R}\) the non-negative quadratic form whose matrix in the canonical basis of \(\mathbb{R}^{2n}\) is \(A_t\), in terms of \(m_t : \mathbb{R}^{2n} \rightarrow \mathbb{R}_+\) the non-negative quadratic form whose matrix in the canonical basis of \(\mathbb{R}^{2n}\) is \(M_t\). By the way, this is a justification \textit{a posteriori} of the introduction of the matrices \(M_t\).

\textbf{Corollary 2.8.} For all \(t \geq 0\), the operator \(e^{-tq^2}\) is a pseudodifferential operator whose Weyl symbol is given by
\[
X \in \mathbb{R}^{2n} \mapsto \frac{1}{\sqrt{\text{det}(\cos(tJA_t))}} e^{-tm_t(X)} \in L^\infty(\mathbb{R}^{2n}).
\]
Proof. Let $t \geq 0$. It follows from Lemma 2.7 that the matrix $A_t$ is real symmetric positive semidefinite and this combined with Lemma 6.10 show that the spectrum of the matrix $tJA_t$ is purely imaginary. As a consequence, the matrix $\cos(tJA_t)$ is non-singular and it follows from the Mehler formula (2.44) that the operator $e^{-tJQ}$ is a pseudodifferential operator whose Weyl symbol is a $L^\infty(\mathbb{R}^{2n})$-function given for all $X \in \mathbb{R}^{2n}$ by

$$
\frac{1}{\sqrt{\det \cos(tJA_t)}} \exp(-\sigma(X, \tan(tJA_t)X)).
$$

Moreover, we deduce from (2.45), (2.47) and Lemma 2.5 that

$$(tJ)^{-1} \tan(tJA_t) = -(itJ)^{-1}(e^{-2itJA_t} - I_{2n})(e^{-2itJA_t} + I_{2n})^{-1}
= -(itJ)^{-1}(G_t - I_{2n})(G_t + I_{2n})^{-1} = M_t.
$$

We deduce from (1.3) and the above equality that for all $X \in \mathbb{R}^{2n}$,

$$
\sigma(X, tJA_tX) = \sigma(X, tJM_tX) = t\langle X, M_tX \rangle = t\sigma t(X).
$$

This ends the proof of Corollary 2.8. $\square$

The study of the family $(A_t)_{t \in \mathbb{R}}$ is now ended. Still in order to prove (2.11) via (2.10), we consider the time-dependent matrices $H_t$ defined for all $t \in \mathbb{R}$ by

$$
H_t = e^{2itJA_t}e^{-2itJQ}.
$$

Notice that the analyticity of the function $t \in \mathbb{R} \mapsto H_t$ is induced by the ones of the functions $t \in \mathbb{R} \mapsto A_t$ and $t \in \mathbb{R} \mapsto e^{-2itJM_t}$ for all $M \in M_{2n}(\mathbb{C})$. We only need to check that each matrix $H_t$ is real and symplectic.

Lemma 2.9. For all $t \in \mathbb{R}$, $H_t$ is a real symplectic matrix.

Proof. Let $t \in \mathbb{R}$. Since both matrices $A_t$ and $Q$ are symmetric (from Lemma 2.7 concerning $A_t$), Lemma 6.2 shows that the matrices $e^{2itJA_t}$ and $e^{-2itJQ}$ are symplectic. As a consequence, the matrix $H_t$ is also symplectic. Moreover, it follows from Lemma 2.8 that

$$
H_t = e^{2itJA_t}e^{-2itJQ}
= e^{2itJA_t}e^{-4itJA_t}e^{2itJQ}
= e^{2itJA_t}e^{-2itJQ}e^{-2itJQ}e^{2itJQ}
= e^{2itJA_t}e^{-2itJQ} = H_t,
$$

which proves that $H_t$ is a real matrix. This ends the proof of Lemma 2.9. $\square$

This ends the proof of (2.11) and the splitting of the symplectic matrices $e^{-2itJQ}$ in any time $t \in \mathbb{R}$.

The rest of this section is then devoted to prove (2.12) which sharpens the decomposition (2.11) for small times $|t| \ll 1$. The strategy will be different than the one used until now, since the holomorphic functional calculus will not be used anymore to define the different matrices at play. The identity (2.12) is proved in the following lemma:

Lemma 2.10. There exist a positive constant $T > 0$ and a family $(B_t)_{-T < t < T}$ of real symmetric matrices $B_t \in S_{2n}(\mathbb{R})$ whose coefficients depend analytically on the time-variable $-T < t < T$ such that for all $-T < t < T$, the symplectic matrix $H_t$ writes as $H_t = e^{2itJQ}$. 

Proof. First, we recall that for all matrix $M \in M_{2n}(\mathbb{C})$ satisfying $\|M - I_{2n}\| < 1$, the matrix Log $M$ is given by the following sum

$$
\log M = \sum_{k=1}^{+\infty} \frac{(-1)^{k-1}}{k} (M - I_{2n})^k.
$$

Since the following limit holds

$$
\lim_{t \to 0} \|H_t - I_{2n}\| = 0,
$$

there exists a positive constant $T > 0$ such that for all $-T < t < T$,

$$
\|H_t - I_{2n}\| < 1.
$$

Since $H_t^{-1} = e^{2itJQ}e^{-2itJA_t}$, we can even assume that the constant $T > 0$ is chosen sufficiently small so that for all $-T < t < T$,

$$
\|H_t^{-1} - I_{2n}\| < 1.
$$
The estimate (2.18) allows to consider the matrix $B_t$ defined for all $-T < t < T$ by
\[(2.50) \quad B_t = (2tJ)^{-1} \Log H_t.\]
Notice that the function $t \in (-T, T) \mapsto \Log H_t$ is analytic by construction and vanishes in $t = 0$ since $H_0 = I_{2n}$. The matrix $B_t$ is therefore well-defined for all $-T < t < T$ according to (1.43). We deduce from (2.45), (2.48) and (2.50), that for all $-T < t < T$,
\[e^{2tJt} = \exp \left( \Log H_t \right) = H_t.\]
It remains to check that the matrices $B_t$ are real and symmetric. First we observe from (2.46) and (2.50), Lemma 2.9 and the binomial formula that for all $-T < t < T$,
\[B_t^T = (2t)^{-1} (\Log H_t)^T J = (2t)^{-1} \sum_{k=1}^{+\infty} \frac{(-1)^{k-1}}{k} ((H_t - I_{2n})^k)^T J = (2t)^{-1} \sum_{k=1}^{+\infty} \frac{(-1)^{k-1}}{k} \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{k-\ell} (H_t^\ell)^T J = (2t)^{-1} \sum_{k=1}^{+\infty} \frac{(-1)^{k-1}}{k} J(H_t^{k-1} - I_{2n}) = - (2tJ)^{-1} \Log(H_t^{-1}) = (2tJ)^{-1} \Log H_t = B_t.\]
The matrices $B_t$ are therefore symmetric. Moreover, the function $t \in (-T, T) \mapsto B_t$ is analytic by contraction. This ends the proof of Lemma 2.10. \qed

3. Study of the real part for short times

In this section, we prove Theorem 1.2. Let $q : \mathbb{R}^{2n} \to \mathbb{C}$ be a complex-valued quadratic form with a non-negative real part $\Re q \geq 0$. We consider $F$ the Hamilton map associated to $q$, $S$ its singular space and $0 < k_0 \leq 2n - 1$ the smallest integer such that (1.13) holds. Let $(a_t)_{t \in \mathbb{R}}$ be the family of non-negative quadratic forms $a_t : \mathbb{R}^{2n} \to \mathbb{R}_+$ given by Theorem 1.1 and $(m_t)_{t \in \mathbb{R}}$ be the family of non-negative quadratic forms $m_t : \mathbb{R}^{2n} \to \mathbb{R}_+$ whose matrices in the canonical basis of $\mathbb{R}^{2n}$ are the matrices $M_t$ defined in (2.27). We shall prove that the quadratic forms $m_t$ (and therefore the quadratic forms $a_t$) satisfy a sharp lower bound implying some degenerate anisotropic coercivity properties on the phase space. More precisely, we shall prove that there exist some positive constants $c > 0$ and $T > 0$ such that for all $0 \leq t \leq T$ and $X \in \mathbb{R}^{2n}$,
\[(3.1) \quad a_t(X) \geq m_t(X) \geq c \sum_{k=0}^{k_0} t^{2k} \Re q(\Im F)^k X.\]
Notice that the left inequality in (3.1) is a consequence of Lemma 2.7. We are therefore interested in proving the right one. To that end, we consider the time-dependent quadratic form $\kappa_t : \mathbb{C}^{2n} \to \mathbb{R}$ defined in accordance with the convention (1.32) for all $t \geq 0$ and $X \in \mathbb{C}^{2n}$ by
\[(3.2) \quad \kappa_t(X) = \sum_{k=0}^{k_0} t^{2k} \Re q(\Im F)^k X = \sum_{k=0}^{k_0} t^{2k} \sqrt{\Re Q(\Im F)^k X}^2.\]
We recall from Lemma 2.6 that for all $t \geq 0$, the matrix $M_t$ admits the following integral representation
\[M_t = \int_0^1 (e^{-2i\alpha t\overline{F}} \Phi_t)^* (\Re Q)(e^{-2i\alpha t\overline{F}} \Phi_t) \ d\alpha,\]
where the matrices $\Phi_t$ are given by
\[(3.3) \quad \Phi_t = \left( \frac{\sqrt{e^{-2itF}e^{-2it\overline{F}} + I_{2n}}}{2} \right)^{-1},\]
since $F = JQ$ from \((\ref{eq:1})\). We therefore deduce that for all $t \geq 0$ and $X \in \mathbb{R}^{2n}$,
\begin{equation}
(3.4) \quad m_t(X) = X^T M_t X = \int_0^1 (e^{-2i\alpha t}\Phi_t X)^*(\text{Re } Q)(e^{-2i\alpha t}\Phi_t X) \, d\alpha,
\end{equation}
and this equality can be written as
\begin{equation*}
m_t(X) = \int_0^1 |\sqrt{\text{Re } Q}e^{-2i\alpha t}\Phi_t X|^2 \, d\alpha = \|\sqrt{\text{Re } Q}e^{-2i\alpha t}\Phi_t X\|_{L^2(0,1)}^2.
\end{equation*}
By applying the Minkowski inequality, we therefore obtain that for all $t \geq 0$ and $X \in \mathbb{R}^{2n}$,
\begin{equation}
(3.5) \quad \sqrt{m_t(X)} \geq \left\| \sum_{k=0}^{k_0} \frac{(-2t\alpha)^k}{k!} \sqrt{\text{Re } Q}(iF)^k \Phi_t X \right\|_{L^2(0,1)} - \left\| \sum_{k > k_0} \frac{(-2t\alpha)^k}{k!} \sqrt{\text{Re } Q}(iF)^k \Phi_t X \right\|_{L^2(0,1)}.
\end{equation}
We then study separately the two terms of the right-hand side of the above estimate.

**1.** First, we focus on controlling the first term, namely
\begin{equation*}
\left\| \sum_{k=0}^{k_0} \frac{(-2t\alpha)^k}{k!} \sqrt{\text{Re } Q}(iF)^k \Phi_t X \right\|_{L^2(0,1)}.
\end{equation*}
On the finite-dimensional vector space $(\mathcal{C}_{k_0}[X])^{2n}$, the Hardy’s norm $\| \cdot \|_{\mathcal{H}}$, defined by
\begin{equation*}
\left\| \sum_{k=0}^{k_0} y_k X^k \right\|_{\mathcal{H}} = \sum_{k=0}^{k_0} k!^2 |y_k|, \quad y_0, \ldots, y_{k_0} \in \mathcal{C}^{2n},
\end{equation*}
is equivalent to the standard Lebesgue’s norm $\| \cdot \|_{L^2(0,1)}$ given by
\begin{equation*}
\left\| \sum_{k=0}^{k_0} y_k X^k \right\|_{L^2(0,1)}^2 = \int_0^1 \left\| \sum_{k=0}^{k_0} y_k \alpha^k \right\|^2 d\alpha, \quad y_0, \ldots, y_{k_0} \in \mathcal{C}^{2n}.
\end{equation*}
Thus, there exists a positive constant $c_1 > 0$ such that for all $t \geq 0$ and $X \in \mathbb{R}^{2n}$,
\begin{equation}
(3.6) \quad \left\| \sum_{k=0}^{k_0} \frac{(-2t\alpha)^k}{k!} \sqrt{\text{Re } Q}(iF)^k \Phi_t X \right\|_{L^2(0,1)} \geq c_1 \sum_{k=0}^{k_0} t^k \|\sqrt{\text{Re } Q}(iF)^k \Phi_t X\|.
\end{equation}
We develop the matrices $(iF)^k$ in the following way:
\begin{equation}
(3.7) \quad (iF)^k = (\text{Im } F)^k + B_k,
\end{equation}
where the matrices $B_k$ can be written as
\begin{equation}
(3.8) \quad B_k = \sum_{j=1}^{2^k - 1} \varepsilon_{j,k} M_{j,k}(\text{Re } F)(\text{Im } F)^{m_{j,k}},
\end{equation}
with $0 \leq m_{j,k} \leq k - 1$, $\varepsilon_{j,k} \in \{-1, 1, -i, i\}$ and the matrices $M_{j,k}$ are finite products of $\text{Re } F$ and $\text{Im } F$. Then, by putting $\ref{eq:1}$ in $\ref{eq:3}$ and using the triangle inequality, we obtain the following estimate for all $t \geq 0$ and $X \in \mathbb{R}^{2n}$,
\begin{equation}
(3.9) \quad \sum_{k=0}^{k_0} t^k \|\sqrt{\text{Re } Q}(iF)^k \Phi_t X\| \geq \sum_{k=0}^{k_0} t^k \|\sqrt{\text{Re } Q}(\text{Im } F)^k \Phi_t X\| - \sum_{k=0}^{k_0} t^k \|\sqrt{\text{Re } QB_k \Phi_t X}\|.
\end{equation}
We consider the two positive quantities
\begin{equation*}
c_2 = \max_{0 \leq k \leq k_0} \max_{1 \leq j \leq 2^k - 1} \|\sqrt{\text{Re } Q}M_{j,k} J \sqrt{\text{Re } Q}\| > 0,
\end{equation*}
and
\begin{equation*}
c_2' = \max_{0 \leq k \leq k_0} \max_{0 \leq m \leq k-1} \#\{1 \leq j \leq 2^k - 1 : m_{j,k} = m\} > 0,
\end{equation*}
where \( \# \) denotes the cardinality. Since \( F = JQ \) from \([17]\), it follows from \([38]\) that for all \( t \geq 0 \) and \( X \in \mathbb{R}^{2n} \),
\[
\sum_{k=0}^{k_0} t^k |\sqrt{\text{Re}Q B_k \Phi_t X}| \leq \sum_{k=0}^{k_0} t^k \sum_{j=1}^{2^{k-1}} |\sqrt{\text{Re}Q M_{j,k} (\text{Re} F)(\text{Im} F)^{m_{j,k}} \Phi_t X}|
= \sum_{k=0}^{k_0} t^k \sum_{j=1}^{2^{k-1}} |\sqrt{\text{Re}Q M_{j,k} J \sqrt{\text{Re}Q \sqrt{\text{Re}Q (\text{Im} F)^{m_{j,k}} \Phi_t X}}|
\leq c_2 \sum_{k=0}^{k_0} t^k \sum_{j=1}^{2^{k-1}} |\sqrt{\text{Re}Q (\text{Im} F)^{m_{j,k}} \Phi_t X}|.
\]
Then, we gather the integers \( 0 \leq m_{j,k} \leq k - 1 \) taking the same value, which shows that for all \( t \geq 0 \) and \( X \in \mathbb{R}^{2n} \),
\[
\sum_{k=0}^{k_0} t^k |\sqrt{\text{Re}Q B_k \Phi_t X}| \leq c_2 \sum_{k=0}^{k_0} t^k \sum_{m=0}^{k-1} \sum_{1 \leq j \leq 2^{k-1}} |\sqrt{\text{Re}Q (\text{Im} F)^{m} \Phi_t X}|
\leq c_2 c_2' \sum_{k=0}^{k_0} t^k \sum_{m=0}^{k-1} |\sqrt{\text{Re}Q (\text{Im} F)^{m} \Phi_t X}|.
\]
Since \( k - m \geq 1 \), we have that for all \( 0 \leq t \leq 1 \),
\[
t^k = t^{k-m} t^m \leq t^{1+m}.
\]
The following inequality therefore holds for all \( 0 \leq t \leq 1 \) and \( X \in \mathbb{R}^{2n} \),
\[
\sum_{k=0}^{k_0} t^k |\sqrt{\text{Re}Q B_k \Phi_t X}| \leq c_2 c_2' \sum_{k=0}^{k_0} \sum_{m=0}^{k-1} t^k |\sqrt{\text{Re}Q (\text{Im} F)^{m} \Phi_t X}|.
\]
As a consequence, there exists a positive constant \( c_3 > 0 \) such that for all \( 0 \leq t \leq 1 \) and \( X \in \mathbb{R}^{2n} \),
\[
(3.10) \quad \sum_{k=0}^{k_0} t^k |\sqrt{\text{Re}Q B_k \Phi_t X}| \leq c_3 \sum_{k=0}^{k_0} t^k |\sqrt{\text{Re}Q (\text{Im} F)^{k} \Phi_t X}|.
\]
It follows from \([36], [39] \) and \([310]\) that for all \( 0 \leq t \leq 1 \) and \( X \in \mathbb{R}^{2n} \),
\[
(3.11) \quad \left\| \frac{1}{k!} \sum_{k=0}^{k_0} \frac{(-2 \alpha)^k}{k!} \sqrt{\text{Re}Q (i F)^k \Phi_t X} \right\|_{L^2(0,1)} \geq c_1 (1 - c_3 t) \sum_{k=0}^{k_0} t^k \left| \sqrt{\text{Re}Q (\text{Im} F)^{k} \Phi_t X} \right|.
\]
We recall from the third inequality of \([528]\) (no assumption of smallness is required for \( t \geq 0 \) to apply this estimate) that for all \( 0 \leq t \leq 1 \) and \( X \in \mathbb{R}^{2n} \),
\[
(3.12) \quad \sqrt{\kappa_t(\Phi_t X)} \leq \sum_{k=0}^{k_0} t^k \left| \sqrt{\text{Re}Q (\text{Im} F)^{k} \Phi_t X} \right|.
\]
As a consequence of \([311] \) and \([312]\), there exist some positive constants \( t_1 > 0 \) and \( c_4 > 0 \) such that for all \( 0 \leq t \leq t_1 \) and \( X \in \mathbb{R}^{2n} \),
\[
(3.13) \quad \left\| \frac{1}{k!} \sum_{k=0}^{k_0} \frac{(-2 \alpha)^k}{k!} \sqrt{\text{Re}Q (i F)^k \Phi_t X} \right\|_{L^2(0,1)} \geq c_4 \sqrt{\kappa_t(\Phi_t X)}.
\]
In order to estimate from below the term \( \sqrt{\kappa_t(\Phi_t X)} \), we would like to apply Lemma \([612]\) to the function
\[
(3.14) \quad G(M, N) = \left( \frac{e^{-2 \alpha (M+iN)} + e^{-2 \alpha (M-iN)} + I_{2n}}{2} \right)^{-1},
\]
in view of the definition \([33]\) of the matrices \( \Phi_t \). We prove in Lemma \([613]\) of the Appendix that the function \( G \) actually satisfies the assumptions of Lemma \([612]\) and as a consequence, there exist some positive constants \( c_5 > 0 \) and \( 0 < t_2 < t_1 \) such that for all \( 0 \leq t \leq t_2 \) and \( X \in \mathbb{R}^{2n} \),
\[
\left\| \sum_{k=0}^{k_0} \frac{(-2 \alpha)^k}{k!} \sqrt{\text{Re}Q (i F)^k \Phi_t X} \right\|_{L^2(0,1)} \geq c_5 \sqrt{\kappa_t(X)}.
\]
This inequality, combined with (3.15), leads to the following estimate for all $0 \leq t \leq t_2$ and $X \in \mathbb{R}^{2n}$,

\[(3.15) \quad \sqrt{m_t(X)} \geq c_3 \sqrt{\kappa_t(X)} - \left\| \sum_{k > k_0} \frac{(-2t\alpha)^k}{k!} \sqrt{\text{Re} Q(i\mathcal{F})^k} \Phi_t X \right\|_{L^2(0,1)}.
\]

2. The end of the proof consists in controlling the remainder term

\[\left\| \sum_{k > k_0} \frac{(-2t\alpha)^k}{k!} \sqrt{\text{Re} Q(i\mathcal{F})^k} \Phi_t X \right\|_{L^2(0,1)}.
\]

The technique employed will be similar to the ones used in the end of the proof of Lemma 6.12. We begin by observing that for all $0 \leq t \leq t_2$ and $X \in \mathbb{R}^{2n}$,

\[\left\| \sum_{k > k_0} \frac{(-2t\alpha)^k}{k!} \sqrt{\text{Re} Q(i\mathcal{F})^k} \Phi_t X \right\|_{L^2(0,1)}^2 = t^{2k_0 + 2} \left\| \sum_{k > k_0} t^{k-k_0-1} \frac{(-2t\alpha)^k}{k!} \sqrt{\text{Re} Q(i\mathcal{F})^k} \Phi_t X \right\|_{L^2(0,1)}^2.
\]

The coefficients of the time-dependent quadratic form

\[\left\| \sum_{k > k_0} t^{k-k_0-1} \frac{(-2t\alpha)^k}{k!} \sqrt{\text{Re} Q(i\mathcal{F})^k} \Phi_t X \right\|_{L^2(0,1)}^2,
\]

are continuous with respect to the time-variable $0 \leq t \leq t_2$. As a consequence, there exists a positive constant $c_6 > 0$ such that for all $0 \leq t \leq t_2$ and $X \in \mathbb{R}^{2n}$,

\[(3.16) \quad \left\| \sum_{k > k_0} \frac{(-2t\alpha)^k}{k!} \sqrt{\text{Re} Q(i\mathcal{F})^k} \Phi_t X \right\|_{L^2(0,1)}^2 \leq c_6 t^{2k_0 + 2} |X|^2.
\]

On the other hand, it follows from Lemma 6.13 that there exists a positive constant $c_7 > 0$ such that for all $0 \leq t \leq 1$ and $X \in S^\perp$,

\[(3.17) \quad \kappa_t(X) \geq c_7 t^{2k_0} |X|^2.
\]

As a consequence of (3.16) and (3.17), we have that for all $0 \leq t \leq t_2$ and $X \in S^\perp$,

\[(3.18) \quad \left\| \sum_{k > k_0} \frac{(-2t\alpha)^k}{k!} \sqrt{\text{Re} Q(i\mathcal{F})^k} \Phi_t X \right\|_{L^2(0,1)}^2 \leq c_6 |X|^2 \kappa_t(X).
\]

We deduce from (3.15) and (3.18) that there exist some positive constants $c_8 > 0$ and $0 < t_3 < t_2$ such that for all $0 \leq t \leq t_3$ and $X \in S^\perp$,

\[(3.19) \quad m_t(X) \geq \left( c_4 - \sqrt{\frac{c_6}{c_7}} t \right)^2 \kappa_t(X) \geq c_8 \kappa_t(X).
\]

It remains to check that the estimate (3.19) holds for all $X \in \mathbb{R}^{2n}$. To that end, we will use the following elementary lemma of linear algebra:

**Lemma 3.1.** Let $E$ be a real finite-dimensional vector space and $q_1, q_2$ be two non-negative quadratic forms on $E$. If $E = F \oplus G$ is a direct sum of two vector subspaces such that $q_1 \leq q_2$ on $F$ and $q_1, q_2$ both vanish on $G$, then $q_1 \leq q_2$ on $E$.

Let $0 \leq t \leq t_3$. Since $\mathbb{R}^{2n} = S \oplus S^\perp$ and that (3.19) is valid on $S^\perp$, it is sufficient to prove that both non-negative quadratic forms $\kappa_t$ and $m_t$ vanish on the singular space $S$, according to Lemma 3.1. We first notice from (1.4), (1.13) and (3.2) that $\kappa_t$ is zero on the singular space $S$. We now prove that this property holds true as well for the quadratic form $m_t$, that is

\[(3.20) \quad \forall X \in S, \quad m_t(X) = 0.
\]

To that end, we use anew the integral representation of $m_t$ given by (3.3).

\[(3.21) \quad \forall X \in \mathbb{R}^{2n}, \quad m_t(X) = \int_0^1 (e^{-2i\alpha t\mathcal{F}} \Phi_t X)^*(\text{Re} Q)(e^{-2i\alpha t\mathcal{F}} \Phi_t X) \, d\alpha.
\]

According to (3.21), it is sufficient to prove that

\[(3.22) \quad \forall \alpha \in [0, 1], \quad (e^{-2i\alpha t\mathcal{F}} \Phi_t) S \subset S + iS,
\]

since $(\text{Re} Q)S = J^{-1}(\text{Re} F)S = \{0\}$ from (1.7) and (1.11). As a consequence of (3.61), the inclusion $\Phi_t S \subset S + iS$ holds, up to decrease the positive constant $t_3 > 0$. Moreover, we have already noticed from (1.11) that the space $S + iS$ is stable by the matrix $\mathcal{F}$, and therefore by the matrices $e^{-2i\alpha t\mathcal{F}}$ for all $0 \leq \alpha \leq 1$. This proves that the inclusion (3.22) actually holds. The estimate (3.19) can
therefore be extended to all $0 \leq t \leq t_3$ (up to decrease $t_3 > 0$) and $X \in \mathbb{R}^{2n}$. This ends the proof of the estimate (4.1).

4. Regularizing effects of semigroups generated by non-selfadjoint quadratic differential operators

The aim of this section is to prove Theorem 1.6 and Theorem 1.8 about the regularizing properties of the semigroups generated by non-selfadjoint quadratic differential operators. Let $q : \mathbb{R}^{2n} \rightarrow \mathbb{C}$ be a complex-valued quadratic form with a non-negative real part $\text{Re} q \geq 0$. We consider $Q \in S_{2n}(\mathbb{C})$ the matrix of $q$ in the canonical basis of $\mathbb{R}^{2n}$, $F \in M_{2n}(\mathbb{C})$ its Hamilton map and $S$ its singular space.

4.1. Regularizing effects. We begin by proving Theorem 1.6. Let $T > 0$ and $(a_t)_{t \in \mathbb{R}}$, $(b_t)_{-T < t < T}$ be the families of quadratic forms given by Theorem 1.1. We recall that the quadratic forms $a_t$ are non-negative, the quadratic forms $b_t$ are real-valued and $a_t$, $b_t$ depend analytically on the time-variable $t \in \mathbb{R}$ and $-T < t < T$ respectively. Moreover, the evolution operators $e^{-tq^w}$ can be factorized as

$$e^{-tq^w} = e^{-ta^w} e^{-tb^w}.$$  (4.1)

We can assume that the positive constant $0 < T < 1$ is the one given by Theorem 1.2, which implies that there exists a positive constant $c > 0$ such that for all $0 \leq t \leq T$ and $X \in \mathbb{R}^{2n}$,

$$a_t(X) \geq c_t \sum_{j=0}^{k_0} t^{2j} \text{Re} q((\text{Im } F)^j X),$$  (4.2)

where $0 \leq k_0 \leq 2n - 1$ is the smallest integer such that (1.13) holds. As in Section 2, we denote by $A_t$ and $B_t$ the respective matrices of $a_t$ and $b_t$ in the canonical basis of $\mathbb{R}^{2n}$. Moreover, we consider the time-dependent quadratic form $\kappa_t$ defined in accordance with the convention (1.42) for all $t \geq 0$ and $X \in \mathbb{C}^{2n}$ by

$$\kappa_t(X) = \sum_{j=0}^{k_0} t^{2j} \text{Re} q((\text{Im } F)^j X) = \sum_{j=0}^{k_0} t^{2j} | \sqrt{\text{Re} Q(\text{Im } F)^j X |^2.}$$  (4.3)

The estimate (4.2) reads as: for all $0 \leq t \leq T$ and $X \in \mathbb{C}^{2n}$,

$$a_t(X) \geq c \kappa_t(X).$$  (4.4)

The aim of this section is to understand the smoothing properties of the evolution operators $e^{-tq^w}$. Since the operators $e^{-tq^w}$ are unitary on $L^2(\mathbb{R}^n)$, we first notice from (4.1) that it is sufficient to study the regularizing properties of the operators $e^{-ta^w}$ to derive the ones of the operators $e^{-tq^w}$. Therefore, for some $m \geq 1$ and $X_1, \ldots, X_m \in S_{2n}^+$, we are interested in the following linear operators

$$\langle X_1, X \rangle^w \ldots \langle X_m, X \rangle^w e^{-ta^w},$$

where the operators $\langle X_j, X \rangle^w$ are defined in (1.27). To deal with them, we will use the Fourier integral operator representation of the operators $e^{-ta^w}$ and the Egorov formula (4.5). More precisely, it follows from (1.10) and Proposition 2.1 that the operator $e^{-ta^w}$ is a Fourier integral operator associated to the non-negative complex symplectic transformation $e^{-2itJ A_t}$, and the Egorov formula (4.5) implies that for all $0 \leq t \leq T$ and $X_0 \in \mathbb{R}^n$,

$$\langle X_0, X \rangle^w e^{-ta^w} = e^{-ta^w} \langle J^{-1} e^{2itJ A_t} JX_0, X \rangle^w = e^{-ta^w} \langle e^{2itA_t} JX_0, X \rangle^w.$$  (4.5)

By using (4.5), we obtain the following factorization

$$\langle X_1, X \rangle^w \ldots \langle X_m, X \rangle^w e^{-ta^w} = \langle X_1, X \rangle^w \ldots \langle X_m, X \rangle^w e^{-\frac{\pi}{2m} a^w_1} \ldots e^{-\frac{\pi}{2m} a^w_m}$$

$$= \langle Y_{1,t}, X \rangle^w e^{-\frac{\pi}{2m} a^w_1} \ldots \langle Y_{m,t}, X \rangle^w e^{-\frac{\pi}{2m} a^w_m},$$  (4.6)

where the time-dependent points $Y_{j,t} \in \mathbb{C}^{2n}$ are given by

$$Y_{j,t} = e^{\frac{2itX_j}{2m} A_t} J X_j, \quad 1 \leq j \leq m,$$

and where we used the semigroup property of the family of linear operators $(e^{-sa^w})_{s \geq 0}$. The initial problem is therefore reduced to the analysis of the operators

$$\langle Y_{j,t}, X \rangle^w e^{-\frac{\pi}{2m} a^w_j}.$$
The main instrumental result of this section is Lemma 4.1 which requires some technical results to be proven. The first of them investigates the anisotropic coercivity properties of the time-dependent quadratic form \( \kappa_t \) on \( S^\perp \), the canonical Euclidean orthogonal complement of the singular space \( S \). This is a refinement of Lemma 6.13.

**Lemma 4.1.** There exists a positive constant \( c > 0 \) such that for all \( 0 \leq t \leq 1 \), \( X_0 \in S^\perp \setminus \{0\} \) and \( X \in \mathbb{C}^{2n}, \)

\[
\kappa_t(X) \geq \frac{c}{|X_0|^2} t^{2k_{X_0}} \langle X_0, X \rangle^2,
\]

where \( 0 \leq k_{X_0} \leq k_0 \) denotes the index of the point \( X_0 \in S^\perp \) with respect to the singular space defined in (1.25).

**Proof.** For all \( 0 \leq k \leq k_0 \), let \( r_k \) be the non-negative quadratic form defined on the phase space by

\[
r_k(X) = \sum_{j=0}^k \text{Re} q((\text{Im } F)^j X) = \sum_{j=0}^k |\text{Re } Q(\text{Im } F)^j X|^2 \geq 0, \quad X \in \mathbb{R}^{2n}.
\]

Moreover, we consider \( V_k \) the vector subspace defined in (1.23). We begin by proving that there exists a positive constant \( c_k > 0 \) such that

\[
\forall X \in V_k^\perp, \quad r_k(X) \geq c_k|X|^2.
\]

If a point \( X \in V_k^\perp \) satisfies \( r_k(X) = 0 \), we deduce from (4.7) that

\[
\forall j \in \{0, \ldots, k\}, \quad \sqrt{\text{Re } Q(\text{Im } F)^j X} = 0,
\]

and since \( F = JQ \) from (1.17), this implies that \( (\text{Re } F)(\text{Im } F)^j X = 0 \) for all \( 0 \leq j \leq k \), that is \( X \in V_k \). It then follows that \( X = 0 \). The non-negative quadratic form \( r_k \) is therefore positive on the vector subspace \( V_k^\perp \). The estimate (4.8) is then proved.

Now, we consider \( X_0 \in S^\perp \setminus \{0\} \) and \( 0 \leq k_{X_0} \leq k_0 \) the index of the point \( X_0 \) with respect to the singular space defined in (1.25). For all \( X \in \mathbb{R}^{2n} \), we decompose \( X = X' + X'' \) with \( X' \in V_{k_{X_0}}^\perp \) and \( X'' \in V_{k_{X_0}}^\perp \). Since \( X_0 \in V_{k_{X_0}}^\perp \) and that \( r_{k_{X_0}} \) is a non-negative quadratic form which vanishes on the vector subspace \( V_{k_{X_0}}^\perp \) from (4.4), (1.23) and (4.7), we deduce from (4.8) that

\[
(X_0, X)^2 = (X_0, X')^2 \leq |X_0|^2 |X'|^2 \leq \frac{|X_0|^2}{c_{k_{X_0}}} r_{k_{X_0}}(X') = \frac{|X_0|^2}{c_{k_{X_0}}} r_{k_{X_0}}(X).
\]

Setting \( c_0 = \min_{0 \leq k \leq k_0} c_k > 0 \), we deduce from (4.3), (4.7) and (4.9) that for all \( 0 \leq t \leq 1 \), \( X_0 \in S^\perp \setminus \{0\} \) and \( X \in \mathbb{R}^{2n}, \)

\[
\kappa_t(X) \geq t^{2k_{X_0}} r_{k_{X_0}}(X) \geq \frac{c_0}{|X_0|^2} t^{2k_{X_0}} \langle X_0, X \rangle^2,
\]

since \( 0 \leq k_{X_0} \leq k_0 \). It follows that for all \( 0 \leq t \leq 1 \), \( X_0 \in S^\perp \setminus \{0\} \) and \( X \in \mathbb{C}^{2n}, \)

\[
\kappa_t(X) = \kappa_t(\text{Re } X) + \kappa_t(\text{Im } X) \geq \frac{c_0}{|X_0|^2} t^{2k_{X_0}} \langle \text{Re } X, X_0 \rangle^2 + \frac{c_0}{|X_0|^2} t^{2k_{X_0}} \langle \text{Im } X, X_0 \rangle^2
\]

\[
= \frac{c_0}{|X_0|^2} t^{2k_{X_0}} |\langle X, X_0 \rangle|^2.
\]

This ends the proof of Lemma 4.1. \( \square \)

The next result will be instrumental to prove Lemma 4.3. Its proof is based on the study of a time-dependent functional.

**Lemma 4.2.** For all \( s > 0 \), \( t \geq 0 \) and \( u \in \mathcal{F}(\mathbb{R}^n) \), the following estimate holds

\[
\langle u^w e^{-sa^w} u, e^{-sa^w} u \rangle_{L^2(\mathbb{R}^n)} \leq \frac{1}{2s} \| u \|^2_{L^2(\mathbb{R}^n)}.
\]

**Proof.** For fixed \( t \geq 0 \) and \( u \in \mathcal{F}(\mathbb{R}^n) \), we consider the following time-dependent functional defined for all \( s \geq 0 \) by

\[
G(s) = \langle sa^w t e^{-sa^w} u, e^{-sa^w} u \rangle_{L^2(\mathbb{R}^n)} + \frac{1}{2} \| e^{-sa^w} u \|_{L^2(\mathbb{R}^n)}^2.
\]

(4.10)
The function $G$ is differentiable on $(0, +\infty)$ and its derivative is given for all $s > 0$ by

$$G'(s) = -s\langle (a^w_t)^2 e^{-s a^w_t} u, e^{-s a^w_t} u \rangle_{L^2(\mathbb{R}^n)} - s\langle a^w_t e^{-s a^w_t} u, a^w_t e^{-s a^w_t} u \rangle_{L^2(\mathbb{R}^n)}$$

$$- \frac{1}{2} \langle a^w_t e^{-s a^w_t} u, e^{-s a^w_t} u \rangle_{L^2(\mathbb{R}^n)} - \frac{1}{2} \langle e^{-s a^w_t} u, a^w_t e^{-s a^w_t} u \rangle_{L^2(\mathbb{R}^n)}$$

$$+ \langle a^w_t e^{-s a^w_t} u, e^{-s a^w_t} u \rangle_{L^2(\mathbb{R}^n)}.$$ 

Since $a^w_t$ is a selfadjoint operator (as its Weyl symbol is real-valued), we obtain that for all $s > 0$,

$$G'(s) = -2s\| a^w_t e^{-s a^w_t} u \|_{L^2(\mathbb{R}^n)}^2 \leq 0.$$ 

We therefore deduce that for all $s \geq 0$, $t \geq 0$ and $u \in \mathcal{F}(\mathbb{R}^n)$,

$$G(s) = \langle s a^w_t e^{-s a^w_t} u, e^{-s a^w_t} u \rangle_{L^2(\mathbb{R}^n)} + \frac{1}{2} \| e^{-s a^w_t} u \|_{L^2(\mathbb{R}^n)}^2 \leq G(0) = \frac{1}{2} \| u \|_{L^2(\mathbb{R}^n)}^2.$$ 

This ends the proof of Lemma 4.4. \qed

We need the following lemma whose proof can be found e.g. in [17] (Lemma 2.6):

**Lemma 4.3.** Let $\tilde{q} : \mathbb{R}^{2n} \to \mathbb{R}_+$ be a non-negative quadratic form. Then, the quadratic operator $\tilde{q}^w(x, D_x)$ is accretive, that is

$$\forall u \in \mathcal{F}(\mathbb{R}^n), \quad \langle \tilde{q}^w(x, D_x)u, u \rangle_{L^2(\mathbb{R}^n)} \geq 0.$$ 

The anisotropic estimates given by Lemma 4.1 combined with Lemma 4.2 provide a first regularizing effect for the evolution operators $e^{-s a^w_t}$.

**Lemma 4.4.** There exist some positive constants $0 < t_1 < T$ and $c > 0$ such that for all $0 \leq \alpha \leq 1$, $0 < t \leq t_1$, $s > 0$, $X_0 \in \mathbb{S}^\perp$ and $u \in L^2(\mathbb{R}^n)$,

$$\| \langle e^{2i a t A_J} X_0, X \rangle^w e^{-s a^w_t} u \|_{L^2(\mathbb{R}^n)} \leq c |X_0| t^{-k X_0} s^{-\frac{1}{2}} \| u \|_{L^2(\mathbb{R}^n)},$$

where $0 \leq k X_0 \leq k_0$ denotes the index of the point $X_0 \in \mathbb{S}^\perp$ with respect to the singular space defined in (1.25).

**Proof.** We shall first prove that there exist some positive constants $c_0 > 0$ and $0 < t_0 < T$ such that for all $0 \leq \alpha \leq 1$, $0 < t \leq t_0$, $X_0 \in \mathbb{S}^\perp$ and $X \in \mathbb{R}^{2n}$,

$$\| \langle e^{2i a t A_J} X_0, X \rangle \|^2 \leq c_0 |X_0|^2 t^{-2k X_0} a_t(X),$$

where $0 \leq k X_0 \leq k_0$ denotes the index of the point $X_0 \in \mathbb{S}^\perp$ with respect to the singular space defined in (1.23). If the estimate (1.13) holds, the proof of Lemma 4.3 is done. Indeed, by denoting $M_{a, t} = \text{Re}(e^{2i a t A_J})$, we deduce from (1.13) that for all $0 \leq \alpha \leq 1$, $0 < t \leq t_0$, $X_0 \in \mathbb{S}^\perp$ and $X \in \mathbb{R}^{2n}$,

$$\langle M_{a, t} X_0, X \rangle^2 \leq c_0 |X_0|^2 t^{-2k X_0} a_t(X).$$

It then follows from (1.14) and Lemma 4.3 that for all $0 \leq \alpha \leq 1$, $0 < t \leq t_0$, $s > 0$, $X_0 \in \mathbb{S}^\perp$ and $u \in \mathcal{F}(\mathbb{R}^n)$,

$$\| \langle (M_{a, t} X_0, X)^w e^{-s a^w_t} u, e^{-s a^w_t} u \rangle_{L^2(\mathbb{R}^n)} \|_{L^2(\mathbb{R}^n)} \leq c_0 |X_0|^2 t^{-2k X_0} \langle a^w_t e^{-s a^w_t} u, e^{-s a^w_t} u \rangle_{L^2(\mathbb{R}^n)}.$$ 

Moreover, the Weyl calculus, see e.g. the composition formula (18.5.4) in [19], provides that for all $0 \leq \alpha \leq 1$ and $0 < t \leq t_0$,

$$\langle M_{a, t} X_0, X \rangle^2 \equiv \langle M_{a, t} X_0, X \rangle^w \langle (M_{a, t} X_0, X)^w \rangle.$$

since the symbol $\langle M_{a, t} X_0, X \rangle$ is a linear form, where $\cdot^w$ denotes the Moyal product defined for all $p_1$ and $p_2$ in proper symbol classes by

$$(p_1^w p_2)(x, \xi) = e^{\frac{\sigma(D_x D_\xi)}{2}} p_1(x, \xi) p_2(y, \eta) \big|_{(x, \xi) = (y, \eta)},$$

with $\sigma$ the symplectic form defined in (1.23). This implies that for all $0 \leq \alpha \leq 1$ and $0 < t \leq t_0$,

$$\langle (M_{a, t} X_0, X)^w \rangle = \langle M_{a, t} X_0, X \rangle^w \langle M_{a, t} X_0, X \rangle^w.$$ 

We deduce from (1.15) and (1.17) that for all $0 \leq \alpha \leq 1$, $0 < t \leq t_0$, $s > 0$, $X_0 \in \mathbb{S}^\perp$ and $u \in \mathcal{F}(\mathbb{R}^n)$,

$$\| \langle M_{a, t} X_0, X \rangle^w e^{-s a^w_t} u \|_{L^2(\mathbb{R}^n)} \leq c_0 |X_0|^2 t^{-2k X_0} \langle a^w_t e^{-s a^w_t} u, e^{-s a^w_t} u \rangle_{L^2(\mathbb{R}^n)}.$$
On the other hand, we recall from (2.26) that for all \(0 \leq \alpha \leq 1, 0 < t \leq t_0, s > 0, X_0 \in S^\perp\) and \(u \in L^2(\mathbb{R}^n)\),
\[
\|\langle N_{\alpha,t}X_0, X \rangle e^{-s \omega \alpha} u \|_{L^2(\mathbb{R}^n)}^2 \leq \frac{c_0}{2} |X_0|^2 t^{-2kX_0 s s^{-1}} \|u\|_{L^2(\mathbb{R}^n)}^2.
\]

Notice that the estimate (4.18) can be extended to all \(u \in L^2(\mathbb{R}^n)\) since the Schwartz space \(\mathcal{F}(\mathbb{R}^n)\) is dense in \(L^2(\mathbb{R}^n)\). Similarly, if we denote \(N_{\alpha,t} = \text{Im}(e^{2iatA_tJ})\), we have that for all \(0 \leq \alpha \leq 1, 0 < t \leq t_0, s > 0, X_0 \in S^\perp\) and \(u \in L^2(\mathbb{R}^n)\),
\[
\|\langle N_{\alpha,t}X_0, X \rangle e^{-s \omega \alpha} u \|_{L^2(\mathbb{R}^n)}^2 \leq \frac{c_0}{2} |X_0|^2 t^{-2kX_0 s s^{-1}} \|u\|_{L^2(\mathbb{R}^n)}^2.
\]

Finally, we deduce from the triangle inequality that for all \(0 \leq \alpha \leq 1, 0 < t \leq t_0, s > 0, X_0 \in S^\perp\) and \(u \in L^2(\mathbb{R}^n)\),
\[
\|\langle e^{2iatA_tJ}X_0, X \rangle e^{-s \omega \alpha} u \|_{L^2(\mathbb{R}^n)} \leq \|\langle M_{\alpha,t}X_0, X \rangle e^{-s \omega \alpha} u \|_{L^2(\mathbb{R}^n)} + \|\langle N_{\alpha,t}X_0, X \rangle e^{-s \omega \alpha} u \|_{L^2(\mathbb{R}^n)},
\]
and the estimates (4.18) and (4.19) imply that
\[
\|\langle e^{2iatA_tJ}X_0, X \rangle e^{-s \omega \alpha} u \|_{L^2(\mathbb{R}^n)} \leq \sqrt{2c_0 |X_0|^2 t^{-kX_0 s s^{-1}}} \|u\|_{L^2(\mathbb{R}^n)}.
\]

It therefore remains to prove that the estimate (4.13) actually holds. We shall actually prove that there exist some positive constants \(c_1 > 0\) and \(0 < t_1 < T\) such that for all \(0 \leq \alpha \leq 1, 0 < t \leq t_1, X_0 \in S^\perp\) and \(X \in \mathbb{C}^n\),
\[
|\langle e^{2iatA_tJ}X_0, X \rangle|^2 \leq c_1 |X_0|^2 t^{-2kX_0 \kappa_t}(X).
\]

The estimate (4.13) is then a straightforward consequence of (4.13) and (4.20). It follows from Lemma 4.11 that there exists a positive constant \(c_2 > 0\) such that for all \(0 \leq t \leq 1, X_0 \in S^\perp\) and \(X \in \mathbb{C}^n\),
\[
l_t^2kX_0 |\langle X_0, X \rangle|^2 \leq c_2 |X_0|^2 \kappa_t(X).
\]

On the other hand, we recall from (2.20) that for all \(0 \leq \alpha \leq 1\) and \(0 \leq t \leq T\),
\[
e^{2iatJ}\alpha_t = \exp \left(-\frac{\alpha}{2} \log (e^{-2itF e^{-2itT}})\right).
\]

We would like to deduce from Lemma 6.12 applied with the functions
\[
G_\alpha(M, N) = \exp \left(-\frac{\alpha}{2} \log (e^{-2i(M+iN) e^{-2i(M-iN)}})\right), \quad \alpha \in [0, 1],
\]
that there exist some positive constants \(0 < t_1 < T\) and \(c_3 > 0\) such that for all \(0 \leq \alpha \leq 1, 0 < t \leq t_1\) and \(X \in \mathbb{C}^n\),
\[
\kappa_t(X) \leq c_3 \kappa_t(e^{2iatA_tJ}X).
\]

This application of Lemma 6.12 is made rigorous in Lemma 6.16 of the Appendix, which implies that the estimate (4.23) actually holds. Combining (4.21) and (4.23), we obtain that for all \(0 \leq \alpha \leq 1, 0 \leq t \leq t_1, X_0 \in S^\perp\) and \(X \in \mathbb{C}^n\),
\[
l_t^2kX_0 |\langle X_0, X \rangle|^2 \leq c_2 c_3 |X_0|^2 \kappa_t(e^{2iatA_tJ}X),
\]
and a straightforward change of variable shows that for all \(0 \leq \alpha \leq 1, 0 \leq t \leq t_1, X_0 \in S^\perp\) and \(X \in \mathbb{R}^n\),
\[
l_t^2kX_0 |\langle e^{2iatA_tJ}X_0, X \rangle|^2 \leq c_2 c_3 |X_0|^2 \kappa_t(X).
\]

This proves that (4.20) holds and ends the proof of Lemma 4.4.

We can now derive the proof of Theorem 1.6. To that end, we implement the strategy presented in the beginning of this subsection. Let \(m \geq 1\) and \(X_1, \ldots, X_m \in S^\perp\). We denote by \(0 \leq k_{X_j} \leq k_0\) the index of the point \(X_j \in S^\perp\) with respect to the singular space. It follows from (1.6) that for all \(0 \leq t \leq T\),
\[
\langle X_1, X \rangle^w \cdots \langle X_m, X \rangle^w e^{-t \omega \alpha} = \langle Y_{1,t}, X \rangle^w e^{-\frac{t}{m} \omega \alpha} \cdots \langle Y_{m,t}, X \rangle^w e^{-\frac{m}{m} \omega \alpha},
\]
where the time-dependent points \(Y_{j,t} \in \mathbb{C}^n\) are given for all \(1 \leq j \leq m\) by
\[
Y_{j,t} = e^{2it(J^{-1})(A_tJ)}X_j.
\]

According to Lemma 4.4 there exist some positive constants \(0 < t_1 < T\) and \(c > 0\) such that for all \(0 \leq \alpha \leq 1, 0 < t \leq t_1, s > 0, X_0 \in S^\perp\) and \(u \in L^2(\mathbb{R}^n)\),
\[
\|\langle e^{2iatA_tJ}X_0, X \rangle^w e^{-s \omega \alpha} u \|_{L^2(\mathbb{R}^n)} \leq c |X_0| t^{-k_{X_0} s s^{-\frac{1}{2}}} \|u\|_{L^2(\mathbb{R}^n)},
\]
where \(0 \leq k_{X_0} \leq k_0\) denotes the index of the point \(X_0 \in S^1\) with respect to the singular space. We deduce from (4.25) and (4.26) that for all \(1 \leq j \leq m, 0 < t \leq t_1\) and \(u \in L^2(\mathbb{R}^n),\)

\[
\|\langle Y_j, t, X \rangle^u e^{-\frac{1}{2} t a^w_j} u\|_{L^2(\mathbb{R}^n)} \leq c|X_j| t^{-k_{X_j} - \frac{d}{2}} m^{\frac{1}{2}} \|u\|_{L^2(\mathbb{R}^n)}.
\]

Notice that the constant \(c > 0\) is independent on the integer \(m \geq 1\) and the points \(X_j \in S^1\). It now follows from (4.27), (4.28) and a straightforward induction that for all \(0 < t \leq t_1\) and \(u \in L^2(\mathbb{R}^n),\)

\[
\|\langle X_1, X \rangle^w \ldots \langle X_m, X \rangle^w e^{-t a^w} u\|_{L^2(\mathbb{R}^n)} \leq \frac{c_m}{t^{k_{X_1} + \ldots + k_{X_m} + \frac{d}{2}} \prod_{j=1}^{m} |X_j|} m^{\frac{1}{2}} \|u\|_{L^2(\mathbb{R}^n)}
\]

\[
\leq \frac{e^{\frac{d}{2}} c_m}{t^{k_{X_1} + \ldots + k_{X_m} + \frac{d}{2}}} \left(\prod_{j=1}^{m} |X_j|\right) (m!)^{\frac{1}{2}} \|u\|_{L^2(\mathbb{R}^n)},
\]

where we used that \(m^m \leq e^{m} m!\). We then deduce from (4.26) that for all \(0 < t \leq t_1\) and \(u \in L^2(\mathbb{R}^n),\)

\[
\|\langle X_1, X \rangle^w \ldots \langle X_m, X \rangle^w e^{-t q^w} u\|_{L^2(\mathbb{R}^n)} \leq \frac{e^{\frac{d}{2}} c_m}{t^{k_{X_1} + \ldots + k_{X_m} + \frac{d}{2}}} \left(\prod_{j=1}^{m} |X_j|\right) (m!)^{\frac{1}{2}} \|u\|_{L^2(\mathbb{R}^n)}
\]

\[
= \frac{e^{\frac{d}{2}} c_m}{t^{k_{X_1} + \ldots + k_{X_m} + \frac{d}{2}}} \left(\prod_{j=1}^{m} |X_j|\right) (m!)^{\frac{1}{2}} \|u\|_{L^2(\mathbb{R}^n)},
\]

since the operators \(e^{-t q^w}\) are unitary on \(L^2(\mathbb{R}^n)\). This ends the proof of Theorem 1.6.

### 4.2. Directions of regularity.

We now perform the proof of Theorem 1.8. The family \((a_t)_{t \in \mathbb{R}}\) still stands for the family given by Theorem 1.1 composed of non-negative quadratic forms \(a_t : \mathbb{R}^{2n} \to \mathbb{R}_+\) with coefficients depending analytically on the time-variable \(t \in \mathbb{R}\). As in the previous subsection, the matrix of the quadratic forms \(a_t\) in the canonical basis of \(\mathbb{R}^{2n}\) is denoted \(A_t\). Moreover, we consider \((U_t)_{t \in \mathbb{R}}\) the family of metaplectic operators also given by Theorem 1.1.

We recall that the evolution operators \(e^{-t q^w}\) split as

\[
\forall t \geq 0, \quad e^{-t q^w} = e^{-t a^w} U_t.
\]

Let \(t > 0, X_0 \in \mathbb{R}^{2n}\). We assume that the linear operator \(\langle X_0, X \rangle^w e^{-t q^w}\) is bounded on \(L^2(\mathbb{R}^n)\). We aim at proving that \(X_0 \in S^1\). We first notice that since the metaplectic operator \(U_t\) is unitary on \(L^2(\mathbb{R}^n)\), it follows from (4.28) that the linear operator \(\langle X_0, X \rangle^w e^{-t q^w}\) is also bounded on \(L^2(\mathbb{R}^n)\). As a consequence, there exists a positive constant \(c_{t, X_0}\) depending on \(t\) and \(X_0\) such that

\[
\forall u \in L^2(\mathbb{R}^n), \quad \|\langle X_0, X \rangle^w e^{-t a^w} u\|_{L^2(\mathbb{R}^n)} \leq c_{t, X_0} \|u\|_{L^2(\mathbb{R}^n)}.
\]

According to the decomposition \(\mathbb{R}^{2n} = S \oplus S^1\) of the phase space, the orthogonality being taken with respect to the euclidean structure of \(\mathbb{R}^{2n}\), we write \(X_0 = X_{0,S} + X_{0,S^1}\), with \(X_{0,S} \in S\) and \(X_{0,S^1} \in S^1\). For all \(\lambda \geq 0\), we consider \(X_\lambda \in S\) the point of the singular space defined by

\[
X_\lambda = \lambda X_{0,S} = (x_\lambda, \xi_\lambda) \in S \subset \mathbb{R}^{2n}.
\]

Moreover, we consider for all \(\lambda \geq 0\) the Gaussian function \(u_\lambda \in \mathcal{F}(\mathbb{R}^n)\) given for all \(x \in \mathbb{R}^n\) by

\[
u(\lambda)(x) = e^{i\langle \xi, x \rangle} e^{-|x-x_\lambda|^2}.
\]

The strategy will be to find upper and lower bounds for the term

\[
\langle \langle X_0, X \rangle^w e^{-t a^w} u_\lambda, u_\lambda \rangle_{L^2(\mathbb{R}^n)}\]

and to consider the asymptotics when \(\lambda\) tends to \(+\infty\) in order to conclude that the point \(X_0,S\) has to be equal to zero. An upper bound can be established readily since it follows from (4.29), (4.31) and the Cauchy-Schwarz inequality that for all \(\lambda \geq 0\),

\[
\|\langle X_0, X \rangle^w e^{-t a^w} u_\lambda, u_\lambda \|_{L^2(\mathbb{R}^n)} \leq c_{t, X_0} \|u_\lambda\|_{L^2(\mathbb{R}^n)} = c_{t, X_0} \|u_0\|_{L^2(\mathbb{R}^n)}.
\]

Notice that the right-hand side of the above estimate does not depend on the parameter \(\lambda \geq 0\). Now, we investigate a lower bound for the term (4.32) by a direct calculus. It follows from the Mehler formula (Corollary 2.2) that the operator \(e^{-t q^w}\) is a pseudodifferential operator whose symbol is given by

\[
e_{t} e^{-t q^w}(X) \in L^\infty(\mathbb{R}^{2n}), \quad \text{where} \quad c_{t} = \frac{1}{\sqrt{\det \cos(IJA_1)}} > 0.
\]
and where \( m_t : \mathbb{R}^{2n} \to \mathbb{R}_+ \) is the non-negative quadratic form whose matrix in the canonical basis of \( \mathbb{R}^{2n} \) is the matrix \( M_t \) defined in \((2.27)\). We therefore deduce from \((4.31)\) and \((4.34)\) that the term \((4.32)\) is given for all \( \lambda \geq 0 \) by

\[
\langle (X_0, X) w e^{-t\alpha X^2} u, u \rangle_{L^2(\mathbb{R}^n)} = c_t \langle (X_0, X) w (e^{-t\alpha})^w T_\lambda u_0, T_\lambda u_0 \rangle_{L^2(\mathbb{R}^n)},
\]

where the operator \( T_\lambda : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \) is defined for all \( u \in L^2(\mathbb{R}^n) \) by

\[
T_\lambda u = e^{i(\xi \cdot x) - x_\lambda)} u(\cdot - x_\lambda).
\]

We need compute the commutators between the operators \( T_\lambda \) and the operators \( (X_0, X) w \) and \((e^{-t\alpha})^w \) respectively. This is done in the following lemma:

**Lemma 4.5.** Let \( a \in \mathcal{F}'(\mathbb{R}^{2n}) \). We have that for all \( \lambda \geq 0 \) and \( u \in \mathcal{F}(\mathbb{R}^n) \),

\[
a^w T_\lambda u = T_\lambda (L_\lambda a)^w u \quad \text{in} \quad \mathcal{F}'(\mathbb{R}^n),
\]

where \( L_\lambda a \in \mathcal{F}'(\mathbb{R}_n) \) is given by \( L_\lambda a = a(\cdot + X_\lambda) \).

**Proof.** Let \( \lambda \geq 0 \) and \( u \in \mathcal{F}(\mathbb{R}^n) \) be a Schwartz function. For all \( v \in \mathcal{F}(\mathbb{R}^n) \), we consider the Wigner function \( \mathcal{H}_\lambda(u, v) \) associated to the functions \( T_\lambda u \) and \( T_\lambda v \) defined for all \( (x, \xi) \in \mathbb{R}^{2n} \) by

\[
\mathcal{H}_\lambda(u, v)(x, \xi) = \int_{\mathbb{R}^n} e^{-i(\xi \cdot x + \frac{y}{2})(\xi \cdot x - \frac{y}{2})} u(x + \frac{y}{2}) v(x - \frac{y}{2}) \, dy.
\]

It follows from \((4.36)\) and \((4.37)\) that for all \( \lambda \geq 0 \) and \( v \in \mathcal{F}(\mathbb{R}^n) \),

\[
\mathcal{H}_\lambda(\xi + \frac{y}{2} - x^\xi_x + \frac{y}{2} - x^\xi_x + \frac{y}{2}) u(x - x^\xi_x + \frac{y}{2}) v(x - x^\xi_x + \frac{y}{2}) \, dy
\]

\[
= \mathcal{H}_\lambda(u, v)(x, \xi - \lambda) = (L_\lambda^{-1} \mathcal{H}(u, v))(x, \xi),
\]

since \( T_\lambda \) is the identity operator. It then follows from \((4.38)\) and the definition of the Weyl calculus that for all \( v \in \mathcal{F}(\mathbb{R}^n) \),

\[
\langle T_\lambda a^w T_\lambda u, \mathcal{F}'(\mathbb{R}_n) \rangle = (a^w T_\lambda u, (L_\lambda a)^w u)_{\mathcal{F}'(\mathbb{R}_n)} = (a, \mathcal{H}_\lambda(u, v))_{\mathcal{F}'(\mathbb{R}_n)}
\]

\[
= (a, L_\lambda^{-1} \mathcal{H}_\lambda(u, v))_{\mathcal{F}'(\mathbb{R}_n)} = (L_\lambda a, \mathcal{H}_\lambda(u, v))_{\mathcal{F}'(\mathbb{R}_n)}
\]

\[
= ((L_\lambda a)^w u, \mathcal{F}'(\mathbb{R}_n)).
\]

Since the above estimate holds for all Schwartz functions \( v \in \mathcal{F}(\mathbb{R}^n) \), we proved that \( T_\lambda a^w T_\lambda u = (L_\lambda a)^w u \) in \( \mathcal{F}'(\mathbb{R}_n) \). As \( T_\lambda T_\lambda \) is the identity operator, we obtain that \( a^w T_\lambda u = T_\lambda (L_\lambda a)^w u \) in \( \mathcal{F}'(\mathbb{R}_n) \). This ends the proof of Lemma 4.5.

The quadratic form \( m_t \) vanishes on the singular space \( S \). Indeed, if \( X \in S \), we recall from \((4.20)\) that \( m_s(X) = 0 \) when \( 0 \leq s \ll 1 \) and since the function \( s \in \mathbb{R} \mapsto m_s(X) \) is analytic, see \((2.27)\) where the matrices \( M_s \) are constructed, we deduce that \( m_s(X) = 0 \) for all \( s \geq 0 \). Since the quadratic forms \( m_t \) are positive semidefinite from Lemma 2.7 and the points \( X_\lambda \) are elements of \( S \), we deduce that

\[
\forall \lambda \geq 0, \forall X \in \mathbb{R}^{2n}, \quad (L_\lambda m_t)(X) = m_t(X + X_\lambda) = m_t(X).
\]

We therefore deduce from \((4.34)\) and Lemma 4.5 that for all \( \lambda \geq 0 \) and \( u \in \mathcal{F}(\mathbb{R}^n) \),

\[
(e^{-t\alpha})^w T_\lambda u = T_\lambda (L_\lambda e^{-t\alpha})^w u = T_\lambda e^{-t\alpha} e^{-t\alpha} u, \quad \text{in} \quad \mathcal{F}'(\mathbb{R}_n).
\]

Moreover, \( \tilde{\varphi} \) (Theorem 4.2) states that for all \( s \geq 0 \), the evolution operator \( e^{-t\varphi} \) generated by an accretive quadratic operator \( \varphi : \mathbb{R}^{2n} \to \mathbb{C} \) with \( \varphi : \mathbb{R}^{2n} \to \mathbb{C} \) a complex-valued quadratic form with a non-negative real-part \( \text{Re} \varphi \geq 0 \), maps \( \mathcal{F}(\mathbb{R}_n) \) into \( \mathcal{F}(\mathbb{R}_n) \):

\[
\forall s \geq 0, \forall u \in \mathcal{F}(\mathbb{R}_n), \quad e^{-t\varphi} u \in \mathcal{F}(\mathbb{R}_n).
\]

This implies that \( T_\lambda e^{-t\varphi} u \in \mathcal{F}(\mathbb{R}_n) \) for all \( \lambda \geq 0 \) and \( u \in \mathcal{F}(\mathbb{R}_n) \) and that the equality \((4.39)\) holds in \( \mathcal{F}(\mathbb{R}_n) \). On the other hand, it follows from Lemma 4.5 anew that for all \( \lambda \geq 0 \) and \( u \in \mathcal{F}(\mathbb{R}_n) \),

\[
\langle X_0, X \rangle w T_\lambda u = T_\lambda \langle X_0, X + X_\lambda \rangle w u, \quad \text{in} \quad \mathcal{F}'(\mathbb{R}_n).
\]
Since the right-hand side of the above formula belongs to the Schwartz space $\mathcal{F}(\mathbb{R}^n)$ for all $\lambda \geq 0$ and $u \in \mathcal{F}(\mathbb{R}^n)$, the equality (4.10) holds in $\mathcal{F}(\mathbb{R}^n)$. As a consequence of (4.35), (4.39) and (4.40), we have that for all $\lambda \geq 0$,
\[
\langle (X_0, X)^w e^{-ta^w} u, \lambda \rangle_{L^2(\mathbb{R}^n)} = \langle (X_0, X)^w T(\lambda) e^{-ta^w} u, T(\lambda) u \rangle_{L^2(\mathbb{R}^n)} = \langle (X_0, X + \lambda)^w e^{-ta^w} u, T(\lambda) u \rangle_{L^2(\mathbb{R}^n)}
\]

and
\[
\langle (X_0, X + \lambda)^w e^{-ta^w} u, T(\lambda) u \rangle_{L^2(\mathbb{R}^n)} = \langle (X_0, X)^w e^{-ta^w} u, T(\lambda) u \rangle_{L^2(\mathbb{R}^n)},
\]

since the operators $T(\lambda)$ are unitary on $L^2(\mathbb{R}^n)$. Moreover, it follows from (4.30) that for all $\lambda \geq 0$ and $X \in \mathbb{R}^{2n}$,
\[
(X_0, X + \lambda)^w e^{-ta^w} = (X_0, X)^w e^{-ta^w} + \lambda |X_{0, S}|^2 e^{-ta^w}.
\]

This proves that for all $\lambda \geq 0$,
\[
\langle (X_0, X)^w e^{-ta^w} u, \lambda \rangle_{L^2(\mathbb{R}^n)} = \langle (X_0, X)^w e^{-ta^w} u, u \rangle_{L^2(\mathbb{R}^n)} + \lambda |X_{0, S}|^2 \langle e^{-ta^w} u, u \rangle_{L^2(\mathbb{R}^n)}.
\]

Combining the above estimate with (4.35), we obtain that for all $\lambda \geq 0$,
\[
\lambda |X_{0, S}|^2 \langle e^{-ta^w} u, u \rangle_{L^2(\mathbb{R}^n)} \leq \langle (X_0, X)^w e^{-ta^w} u, u \rangle_{L^2(\mathbb{R}^n)} + c_1, X_{0, S} \|u\|_{L^2(\mathbb{R}^n)}^2.
\]

We now only need to check that the term $\langle e^{-ta^w} u, u \rangle_{L^2(\mathbb{R}^n)}$ in not equal to zero to conclude that $X_{0, S} = 0$, since the right-hand side of the above estimate does not depend on the parameter $\lambda \geq 0$. Since $a_k$ is a non-negative quadratic form, it follows from Corollary 6.9 that the operator $e^{-\frac{ta^w}{2}}$ is injective. As the Gaussian function $u_0 \in \mathcal{F}(\mathbb{R}^n)$ is non-zero, we deduce that
\[
\langle e^{-ta^w} u, u \rangle_{L^2(\mathbb{R}^n)} = \|e^{-\frac{ta^w}{2}} u\|_{L^2(\mathbb{R}^n)}^2 \neq 0,
\]

while using the semigroup property of the family of linear selfadjoint operators $(e^{-ta^w})_{t \geq 0}$. It therefore follows that $X_{0, S} = 0$ and $X_0 \in S^{+}$. This ends the proof of Theorem 1.8.

5. Subelliptic estimates enjoyed by quadratic operators

This section is devoted to the proof of Theorem 1.10. Let $q : \mathbb{R}^{2n} \to \mathbb{C}$ be a complex-valued quadratic form with a non-negative real part $\text{Re} \, q \geq 0$. We consider $S$ the singular space of $q$ and $0 \leq k_0 \leq 2n - 1$ the smallest integer such that (1.13) holds. Let $p_k : \mathbb{R}^{2n} \to \mathbb{R}$ be the non-negative quadratic form given by (1.35) and $\Lambda^k$ be the operator defined in (4.39) and (4.40), with $0 \leq k \leq k_0$. To prove Theorem 1.10 we will use the interpolation theory as in [17] (Subsection 2.4) which allow to derive subelliptic estimates for the quadratic operator $q^w(x, D_x)$ from estimates for the evolution operators $e^{-tq^w}$. In the following, several estimates will involve the operators $\Lambda^k$ and we recall from the theory of positive operators, see e.g. [24] (Section 4), that they are positive operators whose domains are given by
\[
D(\Lambda^k) = \{ u \in L^2(\mathbb{R}^n) : \Lambda^k u \in L^2(\mathbb{R}^n) \}.
\]

First of all, we need to prove some additional estimates for the semigroup $(e^{-tq^w})_{t \geq 0}$.

**Lemma 5.1.** There exist some positive constants $c > 0$ and $\mu > 0$ such that for all $0 \leq k \leq k_0$, $t > 0$ and $u \in L^2(\mathbb{R}^n)$,
\[
\|\Lambda^k e^{-tq^w} u\|_{L^2(\mathbb{R}^n)} \leq \frac{c e^{\mu t}}{\lambda^{k+1/2}} \|u\|_{L^2(\mathbb{R}^n)}.
\]

**Proof.** Let $0 \leq k \leq k_0$. It follows from the Gauss decomposition of non-negative quadratic forms that there exist a positive integer $N_k \geq 1$ and some points $X^1_k, \ldots, X^N_k \in \mathbb{R}^{2n}$ such that for all $X \in \mathbb{R}^{2n}$,
\[
(5.1) \quad p_k(X) = \sum_{j=1}^{N_k} \langle X^1_k, X \rangle^2.
\]

We deduce from (1.23) and (1.38) that for all $X \in V_k$,
\[
p_k(X) = \sum_{j=1}^{N_k} \langle X^1_k, X \rangle^2 = 0.
\]
This proves that for all $1 \leq j \leq N_k$, $\langle X_j^k, X \rangle = 0$ for all $X \in V_k$. The points $X_j^k \in \mathbb{R}^{2n}$ are therefore elements of $V_j^k \subset S^j$ and their associated indexes $0 \leq k_{X_j^k} \leq k_0$ satisfy from (1.25) that for all $1 \leq j \leq N_k$,

$$0 \leq k_{X_j^k} \leq k.$$  

As we have already noticed, the Weyl calculus shows that for all $1 \leq j \leq N_k$,

$$\text{Op}^w ((X_j^k, X)^2) = \langle X_j^k, X \rangle^w \langle X_j^k, X \rangle^w,$$

and we deduce from (5.3) that

$$\begin{align*}
\Lambda_k^4 &= \left(1 + \sum_{j=1}^{N_k} \langle X_j^k, X \rangle^w \langle X_j^k, X \rangle^w \right)^2 \\
&= 1 + 2 \sum_{j=1}^{N_k} \langle X_j^k, X \rangle^w \langle X_j^k, X \rangle^w + \sum_{j=1}^{N_k} \sum_{\ell=1}^{N_k} \langle X_j^k, X \rangle^w \langle X_j^k, X \rangle^w \langle X_\ell^k, X \rangle^w \langle X_\ell^k, X \rangle^w.
\end{align*}$$

It follows from (5.7) and Theorem 5.6 that there exist some positive constants $c > 0$ and $0 < t_0 < 1$ such that for all $1 \leq j, \ell \leq N_k$, $0 < t \leq t_0$ and $u \in L^2(\mathbb{R}^n)$,

$$\left\| (X_j^k, X)^w \langle X_j^k, X \rangle^w e^{-tq^w} u \right\|_{L^2(\mathbb{R}^n)} \leq \sqrt{\frac{2 \sqrt{6} c}{t^{k+2}}} \left| X_j^k \right|^2 \left\| u \right\|_{L^2(\mathbb{R}^n)},$$

and

$$\left\| (X_j^k, X)^w \langle X_j^k, X \rangle^w e^{-tq^w} u \right\|_{L^2(\mathbb{R}^n)} \leq \frac{2 \sqrt{6} c}{t^{k+2}} \left| X_j^k \right|^2 \left\| u \right\|_{L^2(\mathbb{R}^n)},$$

since $X_j^k X_\ell^k \in S^j$. We deduce from (5.3), (5.4) and (5.5) that there exists a positive constant $c_k > 0$ such that for all $0 < t \leq t_0$ and $u \in L^2(\mathbb{R}^n)$,

$$\left\| \Lambda_k^4 e^{-tq^w} u \right\|_{L^2(\mathbb{R}^n)} \leq \frac{c_k}{t^{k+2}} \left\| u \right\|_{L^2(\mathbb{R}^n)}.$$

Furthermore, it follows from (5.6) and the contraction semigroup property of the family $(e^{-tq^w})_{t \geq 0}$ for all $t > t_0$ and $u \in L^2(\mathbb{R}^n)$,

$$\left\| \Lambda_k^4 e^{-tq^w} u \right\|_{L^2(\mathbb{R}^n)} = \left\| \Lambda_k^4 e^{-tq^w} e^{-(t-t_0)q^w} u \right\|_{L^2(\mathbb{R}^n)} \leq \frac{c_k}{t^{k+2}} \left\| e^{-(t-t_0)q^w} u \right\|_{L^2(\mathbb{R}^n)} \leq \frac{c_k}{t^{k+2}} \left\| u \right\|_{L^2(\mathbb{R}^n)}.$$

According to (5.6) and (5.7), there exists a positive constant $\mu_k > 0$ such that for all $t > 0$ and $u \in L^2(\mathbb{R}^n)$,

$$\left\| \Lambda_k^4 e^{-tq^w} u \right\|_{L^2(\mathbb{R}^n)} \leq \frac{c_k \mu_k}{t^{k+2}} \left\| u \right\|_{L^2(\mathbb{R}^n)}.$$
It follows from (5.10) and the strong continuity of the semigroup \((e^{-tp^w})_{t \geq 0}\) that for all \(u \in L^2(\mathbb{R}^n)\), \(t_0 > 0\) and \(t > 0\), we have
\[
\|e^{-(t+t_0)p^w}u - e^{-tp^w}u\|_{\mathcal{H}_k} = \|e^{-tp^w}(e^{-tp^w}u - u)\|_{\mathcal{H}_k} \leq \frac{e}{t_0^{4k+2}}\|e^{-tp^w}u - u\|_{L^2(\mathbb{R}^n)} \to 0.
\]
This proves that for all \(u \in L^2(\mathbb{R}^n)\), the function \(t \in (0, +\infty) \mapsto e^{-tp^w}u \in \mathcal{H}_k\) is continuous, and therefore measurable. Moreover, we deduce from (20) (pp. 425-426) that the operator \(p^w(x, D_x)\) equipped with the domain \(D(q^w)\) is maximal accretive. Corollary 5.13 in [24] therefore shows that the following continuous inclusion holds between the domain of the quadratic operator \(H \in L^2(\mathbb{R}^n)\) and \(L^2(\mathbb{R}^n)\), \(\mathcal{H}_k \supseteq D(q^w) \subset \left(L^2(\mathbb{R}^n), \mathcal{H}_k\right)_{1/(4k+2), 2}\) since \(\mathcal{H}_k\) is the domain of the operator \(\Lambda_k^2\) and that \(\Lambda_k^2\) is a positive selfadjoint operator, we deduce from Theorem 4.36 in [24] that
\[
(L^2(\mathbb{R}^n), \mathcal{H}_k)_{1/(4k+2), 2} = (D(\Lambda_k^2)_{0}), \Lambda_k^2)_{1/(4k+2), 2} = D((\Lambda_k^2)^\frac{2}{4k+2}) = D((\Lambda_k^2)^\frac{2}{4k+2})
\]
We therefore obtain from (5.11) and (5.12) that the following continuous inclusion holds
\[
D(q^w) \subset D((\Lambda_k^2)^\frac{2}{4k+2})
\]
This implies that there exists a positive constant \(c_k > 0\) such that
\[
\forall u \in D(q^w), \quad \|\Lambda_k^2 u\|_{L^2(\mathbb{R}^n)} \leq c_k \left[\|p^w(x, D_x)u\|_{L^2(\mathbb{R}^n)} + \|u\|_{L^2(\mathbb{R}^n)}\right],
\]
and we deduce from (5.10) that
\[
\forall u \in D(q^w), \quad \|\Lambda_k^2 u\|_{L^2(\mathbb{R}^n)} \leq c_k (1 + \mu) \left[\|q^w(x, D_x)u\|_{L^2(\mathbb{R}^n)} + \|u\|_{L^2(\mathbb{R}^n)}\right].
\]
This ends the proof of Theorem 4.10.

6. Appendix

6.1. About the polar decomposition. To begin this appendix, we recall the basics about the polar decomposition of a bounded operator on a Hilbert space. As a prerequisite, we recall that if \(H\) is an Hilbert space and \(T \in \mathcal{L}(H)\) is a non-negative selfadjoint bounded linear operator, there exists a unique non-negative selfadjoint bounded operator \(\sqrt{T} \in \mathcal{L}(H)\) such that \((\sqrt{T})^2 = T\), see e.g. [25] (Theorem 4.4.2). From there, we define the absolute value of any bounded operator \(T \in \mathcal{L}(H)\) as the selfadjoint operator defined by \(|T| = \sqrt{T^*T}\). The operator \(|T|\) satisfies \(\text{Ker}|T| = \text{Ker} T\). Moreover, we recall that a bounded operator \(U \in \mathcal{L}(H)\) is a partial isometry if \(\|Ux\|_H = \|x\|_H\) for all \(x \in (\text{Ker} U)^\perp\). We can now state the standard polar decomposition theorem whose proof can be found e.g. in [25] (Theorem 4.4.3).

**Theorem 6.1.** Let \(H\) be an Hilbert space and \(T \in \mathcal{L}(H)\) be a bounded linear operator. Then, there exist a unique non-negative selfadjoint bounded linear operator \(S \in \mathcal{L}(H)\) and a partial isometry \(U \in \mathcal{L}(H)\) such that \(T = US\) and \(\text{Ker} U = \text{Ker} T\). Moreover, the operator \(S\) is given by \(S = |T|\).

However, the decomposition given by Theorem 6.1 is not useful for us. We are more interested here with decompositions of the type \(T = |T|U\). Let us assume that \(T \in \mathcal{L}(H)\) writes as
\[
T = SU,
\]
with \(S \in \mathcal{L}(H)\) a non-negative selfadjoint injective bounded linear operator and \(U \in \mathcal{L}(H)\) be a unitary operator. By passing to the adjoint, we deduce that \(T^* = U^*S\). Since the operator \(U^* \in \mathcal{L}(H)\) remains unitary on \(H\) and that \(\text{Ker} U^* = \text{Ker} T^* = \{0\}\), the operator \(T^*\) being injective as a composition of two injective operators, we deduce from Theorem 6.1 that such a couple \((U, S)\) is uniquely defined and \(S = |T^*|\). With an abuse of terminology, we call the decomposition (6.1) when it exists (it will always be the case in this paper), with the bounded linear operators \(S\) and \(U\) respectively non-negative selfadjoint injective and unitary, the polar decomposition of the operator \(T\).
6.2. A symplectic lemma. We now prove that any matrix of the form $e^{JQ}$, with $J$ the real symplectic matrix defined in (1.9) and $Q$ a complex symmetric matrix, is symplectic. Before that, let us recall that when $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the symplectic group $Sp_{2n}(\mathbb{K})$ is the subgroup of $GL_{2n}(\mathbb{K})$ composed of all matrices $M \in GL_{2n}(\mathbb{K})$ such that $M^TJM = J$, or equivalently $JM = (M^T)^{-1}J$, where $J$ is again the matrix defined in (1.3).

Lemma 6.2. For all $Q \in S_{2n}(\mathbb{C})$, we have $e^{JQ} \in Sp_{2n}(\mathbb{C})$.

Proof. Since the matrix $J$ satisfies $J^2 = -I_{2n}$ and $JT = -J$, and the matrix $Q$ is symmetric, we first notice that for all $t \geq 0$,
\[
\partial_t[(e^{tJQ})^TJe^{tJQ}] = (JQe^{tJQ})^TJe^{tJQ} + (e^{tJQ})^TJJQe^{tJQ} = (e^{tJQ})^TQe^{tJQ} - (e^{tJQ})^TQe^{tJQ} = 0.
\]
Moreover, $(e^{tJQ})^TJe^{tJQ} = J$, which proves that for all $t \geq 0$, $(e^{tJQ})^TJe^{tJQ} = J$. In particular, the matrix $e^{JQ}$ is symplectic. This ends the proof of Lemma 6.2.

6.3. About Fourier integral operators. Fourier integral operators associated with non-negative complex linear transformations play a key role in this paper to manipulate the evolution operators $e^{-tq}$ generated by quadratic forms $q : \mathbb{R}^{2n} \to \mathbb{C}$ with non-negative real parts $\text{Re}q \geq 0$. In this subsection, we recall their definition and their basic properties following [20] (Section 5) and [31] (Section 2). Let $T \in Sp_{2n}(\mathbb{C})$ be a non-negative complex symplectic linear transformation, that is, a complex symplectic transformation satisfying
\[
\forall X \in \mathbb{C}^{2n}, \quad i(\sigma(TX, TX) - \sigma(X, X)) \geq 0,
\]
with $\sigma$ the canonical symplectic form on $\mathbb{C}^{2n}$ defined in (1.5). Associated to this non-negative symplectic linear transformation is its twisted graph
\[
\lambda_T = \{(TX, X') : X \in \mathbb{C}^{2n}\} \subset \mathbb{C}^{2n} \times \mathbb{C}^{2n},
\]
where $X' = (x, -\xi) \in \mathbb{C}^{2n}$ if $X = (x, \xi) \in \mathbb{C}^{2n}$, which defines a non-negative Lagrangian plane of $\mathbb{C}^{2n} \times \mathbb{C}^{2n}$ equipped with the symplectic form
\[
\sigma_1((z_1, z_2), (\zeta_1, \zeta_2)) = \sigma(z_1, \zeta_1) + \sigma(z_2, \zeta_2), \quad (z_1, z_2), (\zeta_1, \zeta_2) \in \mathbb{C}^{2n} \times \mathbb{C}^{2n}.
\]
The set
\[
\tilde{\lambda}_T = \{(z_1, z_2, \zeta_1, \zeta_2) : (z_1, \zeta_1, z_2, \zeta_2) \in \lambda_T\} \subset \mathbb{C}^{4n},
\]
is then a non-negative Lagrangian plane of $\mathbb{C}^{4n}$ equipped with the canonical symplectic form on $\mathbb{C}^{4n}$ (see [15]). According to [20] (Proposition 5.1 and Proposition 5.5), there exists a complex-valued quadratic form
\[
p(x, y, \theta) = \langle (x, y, \theta), P(x, y, \theta) \rangle, \quad (x, y) \in \mathbb{R}^{2n}, \theta \in \mathbb{R}^N,
\]
where
\[
P = \begin{pmatrix} P_{x,y,x,y} & P_{x,y,\theta} \\ P_{\theta,x,y} & P_{\theta,\theta} \end{pmatrix} \in M_{2n+N}(\mathbb{C}),
\]
is a symmetric matrix satisfying the conditions:
1. $\text{Im}P \geq 0$.
2. The row vectors of the submatrix $(P_{x,y,\theta} \quad P_{\theta,\theta}) \in \mathbb{C}^N \times (2n+N)$ are linearly independent over $\mathbb{C}$, parametrizing the non-negative Lagrangian plane
\[
\tilde{\lambda}_T = \left\{(x, y, \frac{\partial p}{\partial x}(x, y, \theta), \frac{\partial p}{\partial y}(x, y, \theta)) : \frac{\partial p}{\partial \theta}(x, y, \theta) = 0\right\}.
\]
By using some integrations by parts as in [20] (p. 442), this quadratic form $p$ allows to define the tempered distribution
\[
K_T = \frac{1}{(2\pi)^{2n}} \sqrt{\det \begin{pmatrix} \frac{\partial^2 p}{\partial x^2} & \frac{\partial^2 p}{\partial x \partial \theta} \\ \frac{\partial^2 p}{\partial y \partial x} & \frac{\partial^2 p}{\partial y^2} \end{pmatrix}} \int_{\mathbb{R}^N} e^{ip(x,y,\theta)}d\theta \in \mathcal{F}'(\mathbb{R}^{2n}),
\]
as an oscillatory integral. Notice here that we do not prescribe the sign of the square root so the tempered distribution $K_T$ is defined up to its sign. Appert form this sign uncertainty, it is checked in [20] (p. 444) that this definition only depends on the non-negative complex symplectic transformation $T$, and not on the choice of the parametrization of the non-negative Lagrangian $\tilde{\lambda}_T$ by the quadratic form $p$. Associated to the non-negative complex symplectic linear transformation $T$ is therefore the Fourier integral operator
\[
\mathcal{K}_T : \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}'(\mathbb{R}^n),
\]
Theorem 6.7. Let 

satisfying the Egorov formula

Schwartz space

L

is a bounded operator on

T

plectic transformation

Definition 6.5. defined by the kernel

K

∀

The Egorov formula is presented in the following way in [31] (Proposition 2.1):

Proposition 6.6. Let

trated in the following proposition which comes from [20] (Theorem 5.12):

Fourier integral operators associated to non-negative complex symplectic linear transformations

C

2

n

J

where

which extends by duality as a continuous map on the space of tempered distributions

K

S

Due to the fact that the Schwartz distributions

K

are called metaplectic. Then, the following identity holds for all tempered distributions

K

is the tempered distribution

K

Due to the fact that the Schwartz distributions

K

is also a non-negative complex symplectic transformation and

2

are two non-negative complex symplectic linear transformations in

C

2

n

If

T

1

and

T

2

are two non-negative complex symplectic linear transformations in

C

2

n

, then

T

1

T

2

is also a non-negative complex symplectic linear transformation and

K

T

1

T

2

= ±K

T

1

K

T

2

Finally, we are interested in the real case:

Definition 6.5. A Fourier integral operator

K

T

associated to a real symplectic linear transformation

T

t is called metaplectic.

The metaplectic operators stand out among the other Fourier integral operators

K

T

as illustrated in the following proposition which comes from [20] (Theorem 5.12):

Proposition 6.6. Let

K

T

be a Fourier integral operator associated a non-negative complex symplectic transformation

T

. The operator

K

T

: L

2

(R

n

) → L

2

(R

n

) is invertible if and only if

K

T

is a metaplectic operator, that is, if and only if

T

is a real symplectic transformation. In this case, the operator

K

T

: L

2

(R

n

) → L

2

(R

n

) defines a bijective isometry on

L

2

(R

n

).

To finish, let us recall the metaplectic invariance of the Weyl calculus:

Theorem 6.7. Let

T

be a real symplectic transformation and

K

T

the associated metaplectic operator. Then, the following identity holds for all tempered distributions

a

∈

L

1

(R

n

),

K

T

−1

a

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(x

D

x

)K

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= (a

T

w

(x

D

x

)).
The general result of metaplectic invariance of the Weyl calculus can be found e.g. in [19] (Theorem 18.5.9). Notice that the Egorov formula (6.5) is a particular case of this Theorem for linear forms since (6.5) can also be written in the following way
\[ \forall x_0 \in \mathbb{C}^{2n}, \quad \mathcal{K}_T^{-1}(X_0, X)^w \mathcal{K}_T = \langle X_0, TX \rangle^w, \]
by using that \((J^{-1}T^{-1}J)^T = T\), which is a straightforward property of real symplectic matrices.

6.4. Splitting of the harmonic oscillator semigroup. In this subsection, we give a decomposition of the harmonic oscillator semigroup. To obtain this splitting, we will make use once again of the theory of Fourier integral operators in the very same way as in Section 2. Let us mention as an anecdote that the identity (6.5) involved in the proof of the following proposition has played a major role and has been widely used in image processing in order to make rotations, see e.g. [27].

This identity is also key here for our purpose. As a byproduct of this splitting, we obtain the injectivity property of the evolution operators generated by accretive quadratic operators associated to non-negative quadratic forms.

**Proposition 6.8.** Let \( \mathcal{H} = -\partial_x^2 + x^2 \), with \( x \in \mathbb{R} \), be the harmonic oscillator. Then, the semigroup \( (e^{-t\mathcal{H}})_{t \geq 0} \) generated by the operator \( \mathcal{H} \) admits the following decomposition:
\[ \forall t \geq 0, \quad e^{-t\mathcal{H}} = e^{-\frac{t}{2}(\tanh t)x^2} e^{\frac{t}{2} \sinh(2t)\partial_x^2} e^{-\frac{t}{2}(\tanh t)x^2}. \]
This implies in particular that the evolution operators \( e^{-t\mathcal{H}} \) are injective.

**Proof.** We begin by observing that for all \( t \in (-\pi, \pi) \),
\[ \left( \begin{array}{cc} 1 \tan \frac{t}{2} & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} 1 & -\sin t \\ 0 & 1 \end{array} \right) = \left( \begin{array}{cc} \cos t & -\sin t \\ \sin t & \cos t \end{array} \right). \]
Since the functions \( \cos, \sin \) and \( \tan \) are analytic on \( \{ z \in \mathbb{C} : |\operatorname{Im} z| < \frac{\pi}{2} \} \), the formula (6.8) can be extended to all \( t \in i\mathbb{R} \). As a consequence, we have that for all \( t \in \mathbb{R} \),
\[ \left( \begin{array}{cc} 1 \tanh \frac{t}{2} & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} 1 & -i \sinh t \\ 0 & 1 \end{array} \right) = \left( \begin{array}{cc} \cosh t & -i \sinh t \\ i \sinh t & \cosh t \end{array} \right). \]
On the other hand, it follows from a readily computation that for all \( t \in \mathbb{R} \),
\[ \left( \begin{array}{cc} 1 \tanh \frac{t}{2} & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), \]
and
\[ \left( \begin{array}{cc} \cosh t & -i \sinh t \\ i \sinh t & \cosh t \end{array} \right) = \exp \left( -it \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \right). \]
It follows from (1.7), (1.9), (2.10), (6.9), (6.10), (6.11) and Proposition 2.1 that for all \( t \geq 0 \),
\[ e_t e^{-\frac{t}{2}(\tanh t)x^2} e^{\frac{t}{2} \sinh(2t)\partial_x^2} e^{-\frac{t}{2}(\tanh t)x^2} = e^{-t(x^2-\partial_x^2)}, \]
with \( \varepsilon_t \in \{-1, 1\} \) for all \( t \geq 0 \). It only remains to prove that \( \varepsilon_t = 1 \) for all \( t \geq 0 \) to establish (6.7).
To that end, we consider \( u_0 \in \mathcal{F}(\mathbb{R}) \) the Gaussian function defined for all \( x \in \mathbb{R} \) by \( u_0(x) = e^{-x^2} \).
We first notice that for all \( t \geq 0 \),
\[ e^{-\frac{t}{2}(\tanh t)x^2} e^{\frac{t}{2} \sinh(2t)\partial_x^2} e^{-\frac{t}{2}(\tanh t)x^2} u_0 > 0. \]
Indeed, this estimate is trivial when \( t = 0 \) by definition of \( u_0 \). When \( t > 0 \), we observe that for all \( u \in \mathcal{F}(\mathbb{R}) \) such that \( u > 0 \), the function \( e^{-\frac{t}{2}(\tanh t)x^2} u > 0 \) is also positive, and on the other hand, we notice by using the explicit formula for the Fourier transform of Gaussian functions that
\[ e^{\frac{t}{2} \sinh(2t)\partial_x^2} u = \left( \frac{2\pi}{\sinh(2t)} \right)^n \exp \left( -\frac{x^2}{2 \sinh(2t)} \right) * u > 0, \]
where \( * \) denotes the convolution product. This proves that (6.14) holds. Now, let us consider the function \( \varphi \) defined for all \( t \geq 0 \) by
\[ \varphi(t) = \varepsilon_t e^{-\frac{t}{2}(\tanh t)x^2} e^{\frac{t}{2} \sinh(2t)\partial_x^2} e^{-\frac{t}{2}(\tanh t)x^2} u_0 \in \mathcal{F}(\mathbb{R}^+) \).
The rest of the proof consists in checking that \( \varphi(t) > 0 \) for all \( t \geq 0 \). This property combined with (6.14) will prove that \( \varepsilon_t > 0 \) for all \( t \geq 0 \). Since \( \varepsilon_t \in \{-1, 1\} \), it will then follow that \( \varepsilon_t = 1 \) for all
We first deduce from [20] (Theorem 4.2) that the function \( t \geq 0 \mapsto e^{-t(x^2 - \delta^2)}u_0 \in \mathcal{F}(\mathbb{R}^n) \) is continuous which implies from (6.13) and (6.15) the continuity of the function \( \varphi \) from \([0, +\infty)\) to \( \mathcal{F}(\mathbb{R}) \). As a consequence of (6.14) and (6.15), the Schwartz function \( \varphi(t) \) is not the zero function for all \( t \geq 0 \). Let \( x \in \mathbb{R} \). The previous discussion implies that the function \( t \geq 0 \mapsto \varphi(t)(x) \in \mathbb{R}^n \) is continuous and does not vanish. Moreover, it follows from (6.13) and (6.15) that \( \varphi(0)(x) = u_0(x) > 0 \). We deduce that \( \varphi(t)(x) > 0 \) for all \( t \geq 0 \). As a consequence, \( \varphi(t) > 0 \) for all \( t \geq 0 \). This proves that (6.7) holds. The injectivity of the operators \( e^{-tJH} \) is then a straightforward consequence of (6.7) since the operators \( e^{-\frac{t}{2}(\tanh t)x^2} \) and \( e^{\frac{t}{2}\sinh(2t)\delta^2} \) are themselves injective. This ends the proof of Proposition 6.8.

Notice that the injectivity property of the evolution operators \( e^{-tJH} \) can also be readily proved by using the Hermite basis of \( L^2(\mathbb{R}^n) \) and a direct calculus.

**Corollary 6.9.** Let \( q : \mathbb{R}^{2n} \to \mathbb{R} \) be a non-negative quadratic form \( q \geq 0 \). Then, for all \( t \geq 0 \), the evolution operator \( e^{-tq^w} \) generated by the accretive quadratic operator \( q^w(x, D_x) \) is injective.

**Proof.** We deduce from [19] (Theorem 21.5.3) that there exists a real linear symplectic transformation \( \chi : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \) such that for all \((x, \xi) \in \mathbb{R}^{2n},

\[
(q \circ \chi)(x, \xi) = \sum_{j=1}^k \lambda_j (\xi_j^2 + x_j^2) + \sum_{j=k+1}^{k+l} x_j^2,
\]

with \( k, l \geq 0 \) and \( \lambda_j > 0 \) for all \( 1 \leq j \leq k \). By the symplectic invariance of the Weyl quantization, [19] (Theorem 28.5.9), we can find a metaplectic operator \( T \) satisfying

\[
q^w(x, D_x) = T^{-1} \left( \sum_{j=1}^k \lambda_j (D_{\xi_j}^2 + x_j^2) + \sum_{j=k+1}^{k+l} x_j^2 \right) T.
\]

Let \( t \geq 0 \). It follows from (6.17) that the evolution operator \( e^{-tq^w} \) writes as

\[
e^{-tq^w} = T^{-1} \left( \prod_{j=1}^k e^{-t\lambda_j (D_{\xi_j}^2 + x_j^2)} \right) \left( \sum_{j=k+1}^{k+l} e^{-tx_j^2} \right) T.
\]

We deduce from (6.18) and Proposition 6.8 that the operator \( e^{-tq^w} \) is the composition of injective operators, so is itself injective. This ends the proof of Corollary 6.9.

### 6.5. Spectrum localization.

The following result provides a localization for the spectrum of matrices of the form \( JA \), with \( J \) the symplectic matrix defined in (1.9) and \( A \) a Hermitian positive semidefinite matrix.

**Lemma 6.10.** Let \( A \in \mathcal{H}_c(\mathbb{C}) \) be a Hermitian positive semidefinite matrix and \( J \in Sp_{2n}(\mathbb{R}) \) be the symplectic matrix given by (1.9). Then, the spectrum of the matrix \( JA \) is purely imaginary, that is \( \sigma(JA) \subset i\mathbb{R} \).

**Proof.** We first assume that the matrix \( A \) is Hermitian positive definite. Under this assumption, we observe that \( \sqrt{A}(JA)(\sqrt{A})^{-1} = \sqrt{A}J\sqrt{A} \). The matrix \( JA \) is therefore conjugated to a skew-Hermitian matrix and its spectrum is then purely imaginary. When \( A \) is only Hermitian positive semidefinite, we can consider \( (A_p)_p \) a sequence of Hermitian positive definite matrices that converges to \( A \). Since the eigenvalues of a complex matrix are continuous with respect to this matrix according to [22] (Theorem II.5.1), and that \( \sigma(JA_p) \subset i\mathbb{R} \) from the beginning of the proof, we deduce that the eigenvalues of the matrix \( JA \) are purely imaginary. This ends the proof of Lemma 6.10.

### 6.6. Taylor expansion in a non-commutative setting.

In the next lemma, we prove a composition result of Taylor expansions for functions taking values in non-commutative rings. It will be useful in the end of Subsection 6.7. Notice that we consider holomorphic functions in a neighborhood of 0, but the proof works the same near any point of \( \mathbb{C} \). Let us recall that \( \mathcal{C}(X, Y) \) denotes the ring of non-commutative polynomials in \( X \) and \( Y \), and that for all non-negative integer \( k \geq 0 \), we consider \( \mathcal{C}_{k,0}(X, Y) \) the finite-dimensional subspace of \( \mathcal{C}(X, Y) \) of non-commutative polynomials of degree smaller than or equal to \( k \) vanishing in \((0, 0)\). In the following, given \( \rho > 0 \), the notation \( \mathbb{D}(0, \rho) \) denotes the open disk in \( \mathbb{C} \) centered in 0 of radius \( \rho \), and \( B((0, 0), \rho) \) stands for the open ball in \( M_{2n}(\mathbb{R}) \times M_{2n}(\mathbb{R}) \) centered in \((0, 0)\) of radius with respect to the norm \( \| \cdot \|_\infty \) defined in the notations in p.11.
Lemma 6.11. Let $f : \mathbb{D}(0, \rho) \rightarrow \mathbb{C}$ be an analytic function, with $\rho > 0$. We consider $P \in \mathbb{C}_{k,0}(X,Y)$, with $k \geq 0$ a non-negative integer, and $R: B((0,0), \rho) \rightarrow M_{2\alpha}(\mathbb{C})$ a function satisfying that there exists a positive constant $C > 0$ such that for all $(M, N) \in B((0,0), \rho)$ we have

\begin{equation}
\|R(M,N)\| \leq C \| (M,N) \|^{k+1}.
\end{equation}

Then, there exists $\rho' \in (0, \rho)$, depending continuously on $P$ and $C$, such that the function

\[ f \circ (P + R) : (M, N) \mapsto \sum_{j=0}^{+\infty} \frac{f^{(j)}(0)}{j!} (P(M, N) + R(M,N))^j, \]

is well defined on $B((0,0), \rho')$. Furthermore, there exists a continuous map $\Psi : \mathbb{C}_{k,0}(X,Y) \rightarrow \mathbb{C}_{k,0}(X,Y)$ and a function $R' : B((0,0), \rho') \rightarrow M_{2\alpha}(\mathbb{C})$ such that for all $(M, N) \in B((0,0), \rho')$,

\[ f(P(M, N) + R(M,N)) = f(0)I_{2n} + \Psi(P)(M,N) + R'(M,N), \]

with

\[ \|R'(M,N)\| \leq \Gamma_{C,P} \|(M,N)\|^{k+1}, \]

$\Gamma_{C,P} > 0$ denoting a positive constant which depends continuously on $C$ and $P$.

Proof. Since the functions $P$ and $R$ tend to $(0,0)$ as $(M,N)$ goes to $(0,0)$, if $\rho' \in (0, \rho)$ is chosen sufficiently small, then for $(M,N) \in B((0,0), \rho')$, we have $\|P(M,N)\| < \rho/4$ and $\|R(M,N)\| < \rho/4$. Consequently, the function $f \circ (P + R)$ is well defined on $B((0,0), \rho')$. Let $(M,N) \in B((0,0), \rho')$. Realizing a Taylor expansion of the function $f$ (considered as a map on $M_{2\alpha}(\mathbb{C})$) in $P(M,N)$, we get that

\[ f \circ (P + R)(M,N) = f \circ P(M,N) + \int_{0}^{1} df(P(M,N) + \alpha R(M,N))(R(M,N)) \, d\alpha, \]

where $df$ denotes the differential of the function $f$. The second term in the right-hand side of the above equality is a remainder term. Indeed, since $\|P(M,N) + \alpha R(M,N)\| < \rho/2$ for all $0 \leq \alpha \leq 1$ with our choice of $\rho' \in (0, \rho)$, we deduce from (6.19) that this term satisfies

\[ \left| \int_{0}^{1} df(P(M,N) + \alpha R(M,N))(R(M,N)) \, d\alpha \right| \leq C \left( \sup_{\|L\| < \rho/2} \|df(L)\| \right) \|(M,N)\|^{k+1}. \]

Consequently, we focus on the term $f \circ P(M,N)$. Since the function $f$ is analytic on $\mathbb{D}(0, \rho)$, we can consider $(a_j)_{j \geq 1} \in \mathbb{C}^\infty$ the coefficients of the Taylor expansion of $f$ and write

\[ \forall z \in \mathbb{D}(0,\rho), \quad f(z) = \sum_{j=0}^{+\infty} a_j z^j. \]

Naturally, $f \circ P(M,N)$ can be decomposed as

\[ f \circ P(M,N) = f(0)I_{2n} + Q(P(M,N)) + P(M,N)^{k+1} \sum_{j=0}^{+\infty} a_{j+k+1} P(M,N)^j, \]

where $Q \in \mathbb{C}_k[X]$ is a polynomial of degree smaller than or equal to $k$ vanishing in 0 and depending only on $f$, given by $Q(X) = \sum_{j=1}^{k} a_j X^j$. The third term in the right-hand side of the above equality is also a remainder term. Indeed, since the polynomial $P$ vanishes in $(0,0)$, there exists a positive constant $M_P > 0$ depending continuously (and only) on $P$ such that

\[ \|P(M,N)\| \leq M_P \|(M,N)\|_{\infty}. \]

With the previous choice of $\rho' \in (0, \rho)$, $\|P(M,N)\| < \rho/4$, we obtain that

\[ \left\| P(M,N)^{k+1} \sum_{j=0}^{+\infty} a_{j+k+1} P(M,N)^j \right\| \leq M_P^{k+1} \|(M,N)\|^{k+1} \sum_{j=0}^{+\infty} |a_{j+k+1}| \left( \frac{\rho}{4} \right)^j. \]

Notice that the sum in the right-hand side is finite since the function $f$ is analytic on $\mathbb{D}(0, \rho)$. Finally, we just have to observe that $Q \circ P \in \mathbb{C}_{k^2,0}(X,Y)$ is a non-commutative polynomial vanishing in $(0,0)$ and depending continuously on $P$. The sum of its terms of degree smaller than or equal to $k$ defines $\Psi(P)$ and its higher order terms are remainder terms bounded by $\|(M,N)\|^{k+1}$, up to a constant also depending continuously on $P$. This ends the proof of Lemma 6.11. \[ \square \]
6.7. A perturbation result. To end this Appendix, we give the proof of a quite technical lemma which is instrumental in Section 2 and Section 3. Let \( q : \mathbb{R}^{2n} \to \mathbb{C} \) be a complex-valued quadratic form with a non-negative real-part \( \text{Re} q \geq 0 \). We consider \( Q \in \mathbb{S}_{2n}(\mathbb{C}) \) the matrix of \( q \) in the canonical basis of \( \mathbb{R}^{2n} \). \( F \) the Hamilton map of \( q \) and \( S \) its singular value. Let \( 0 \leq k_0 \leq 2n - 1 \) be the smallest integer such that \( 1.13 \) holds. Moreover, we consider the time-dependent quadratic form \( \kappa_t : \mathbb{C}^{2n} \to \mathbb{R} \) defined in accordance with the convention \( 1.42 \) for all \( t \geq 0 \) and \( X \in \mathbb{C}^{2n} \) by

\[
\kappa_t(X) = \sum_{k=0}^{k_0} t^{2k} \text{Re} q((\text{Im} F)^k X) = \sum_{k=0}^{k_0} t^{2k} |\sqrt{\text{Re} Q}(\text{Im} F)^k X|^2.
\]

The following lemma investigates the perturbations of the quadratic form \( \kappa_t \):

Lemma 6.12. Let \( (G_\alpha)_{0 \leq \alpha \leq 1} \) be a family of functions \( G_\alpha : B((0, 0), \rho) \to M_{2n}(\mathbb{C}) \), with \( \rho > 0 \), satisfying on the one hand that there exist a family \( (P_\alpha)_{0 \leq \alpha \leq 1} \) of non-commutative polynomials \( P_\alpha \in \mathbb{C}_{k_0,0}(X, Y) \) depending continuously on the parameter \( 0 \leq \alpha \leq 1 \), a family \( (R_\alpha)_{0 \leq \alpha \leq 1} \) of functions \( R_\alpha : B((0, 0), \rho) \to M_{2n}(\mathbb{C}) \) and a positive constant \( C > 0 \) such that for all \( 0 \leq \alpha \leq 1 \) and \( (M, N) \in B((0, 0), \rho) \),

\[
G_\alpha(M, N) = I_{2n} + P_\alpha(M, N) + R_\alpha(M, N),
\]

with

\[
\|R_\alpha(M, N)\| \leq C\|\alpha(M, N)\|^{k_0 + 1},
\]

and on the other hand that for all \( 0 \leq \alpha \leq 1 \) and \( t \geq 0 \) such that \( (t \text{Re} F, t \text{Im} F) \in B((0, 0), \rho) \),

\[
G_\alpha(t \text{Re} F, t \text{Im} F)(S + iS) \subseteq S + iS.
\]

Then, there exist some positive constants \( c > 0 \) and \( 0 < T \leq 1 \) such that for all \( 0 \leq t \leq T \), \( 0 \leq \alpha \leq 1 \) and \( X \in \mathbb{C}^{2n} \),

\[
\kappa_t(G_\alpha(t \text{Re} F, t \text{Im} F)X) \geq c\kappa_t(X).
\]

Proof. By definition \( 6.20 \) of the time-dependent quadratic form \( \kappa_t \), the estimate we want to prove writes for all \( 0 \leq \alpha \leq 1 \), \( 0 \leq t \ll 1 \) small enough and \( X \in \mathbb{C}^{2n} \) as

\[
\sum_{k=0}^{k_0} t^{2k} |\sqrt{\text{Re} Q}(\text{Im} F)^k X|^2 \geq c \sum_{k=0}^{k_0} t^{2k} |\sqrt{\text{Re} Q}(\text{Im} F)^k X|^2.
\]

By using the two classical inequalities that hold for all \( m \geq 1 \) and \( a_1, \ldots, a_m \geq 0 \),

\[
\sqrt{a_1 + \ldots + a_m} \leq \sqrt{a_1} + \ldots + \sqrt{a_m},
\]

and

\[
(a_1 + \ldots + a_m)^2 \leq 2^{m-1}(a_1^2 + \ldots + a_m^2),
\]

we notice that in order to prove the estimate \( 6.24 \), it is in fact sufficient to establish that for all \( 0 \leq \alpha \leq 1 \), \( 0 \leq t \ll 1 \) small enough and \( X \in \mathbb{C}^{2n} \),

\[
\sum_{k=0}^{k_0} t^k |\sqrt{\text{Re} Q}(\text{Im} F)^k X|^2 \geq c \sum_{k=0}^{k_0} t^k |\sqrt{\text{Re} Q}(\text{Im} F)^k X|^2.
\]

Indeed, we deduce from \( 6.25 \) and \( 6.20 \) that when \( 6.27 \) holds, we have that for all \( 0 \leq \alpha \leq 1 \), \( 0 \leq t \ll 1 \) small enough and \( X \in \mathbb{C}^{2n} \),

\[
\sum_{k=0}^{k_0} t^{2k} |\sqrt{\text{Re} Q}(\text{Im} F)^k X|^2 \geq \frac{1}{2^{k_0}} \left( \sum_{k=0}^{k_0} t^k |\sqrt{\text{Re} Q}(\text{Im} F)^k X|^2 \right)^2 \geq \frac{c^2}{2^{k_0}} \left( \sum_{k=0}^{k_0} t^k |\sqrt{\text{Re} Q}(\text{Im} F)^k X|^2 \right)^2
\]

\[
= \frac{c^2}{2^{k_0}} \left( \sum_{k=0}^{k_0} t^{2k} |\sqrt{\text{Re} Q}(\text{Im} F)^k X|^2 \right)^2 \geq \frac{c^2}{2^{k_0}} \sum_{k=0}^{k_0} t^{2k} |\sqrt{\text{Re} Q}(\text{Im} F)^k X|^2,
\]

which is the required estimate. We therefore focus on proving the estimate \( 6.27 \). First of all, let us write the functions \( G_\alpha \) under a more manageable form. Since the non-commutative polynomials \( P_\alpha \in \mathbb{C}_{k_0}(X, Y) \) have a degree smaller than or equal to \( k_0 \), vanish on \( (0, 0) \) and depend continuously
on the parameter $0 \leq \alpha \leq 1$, there exist some continuous functions $\sigma_{j,m} : [0, 1] \to \mathbb{C}$, with 
$1 \leq j \leq k_0$ and $m \in \{0, 1\}^j$, such that for all $0 \leq \alpha \leq 1$,
\begin{equation}
(6.29) \quad P_{\alpha}(X,Y) = \sum_{j=1}^{k_0} \sum_{m \in \{0,1\}^j} \sigma_{j,m}(\alpha) X^{m_1} Y^{1-m_1} \cdots X^{m_j} Y^{1-m_j}.
\end{equation}

With an abuse of notation, we denote the above non-commutative product by
\[ X^{1-m_1} Y^{1-m_1} \cdots X^{m_j} Y^{1-m_j} = \prod_{\ell=1}^{j} X^{m_{\ell}} Y^{1-m_{\ell}}. \]

We deduce from (6.21) and (6.29) that for all $0 \leq \alpha \leq 1$, $0 \leq k \leq k_0$ and $(M,N) \in B((0,0), \rho)$,
\begin{equation}
(6.30) \quad G_{\alpha}(M,N) = I_{2n} + \sum_{j=1}^{k_0} \sum_{m \in \{0,1\}^j} \sigma_{j,m}(\alpha) \prod_{\ell=1}^{j} M^{m_{\ell}} N^{1-m_{\ell}} + R_{\alpha,k}(M,N),
\end{equation}
where the remainder terms are given by
\begin{equation}
(6.31) \quad R_{\alpha,k}(M,N) = \sum_{j=k+1}^{k_0} \sum_{m \in \{0,1\}^j} \sigma_{j,m}(\alpha) \prod_{\ell=1}^{j} M^{m_{\ell}} N^{1-m_{\ell}} + R_{\alpha,k}(M,N).
\end{equation}

Since the functions $\sigma_{j,m}$ are continuous on $[0,1]$, we deduce from (6.22) and (6.31) that there exists a positive constant $C_0 > 0$ such that for all $0 \leq \alpha \leq 1$, $0 \leq k \leq k_0$ and $(M,N) \in B((0,0), \rho)$,
\begin{equation}
(6.32) \quad \|R_{\alpha,k}(M,N)\| \leq C_0 \|(M,N)\|^{k+1}.
\end{equation}

We can now tackle the proof of the estimate (6.27). We begin by studying the matrices
\[ t^k(\text{Im } F)^k G_{\alpha}(t \text{Re } F, t \text{Im } F). \]
Let $T_0 > 0$ be such that $(t \text{Re } F, t \text{Im } F) \in B((0,0), \rho)$ for all $0 \leq t \leq T_0$. It follows from (6.30) that for all $0 \leq \alpha \leq 1$, $0 \leq k \leq k_0$ and $0 \leq t \leq T_0$,
\begin{equation}
(6.33) \quad t^k(\text{Im } F)^k G_{\alpha}(t \text{Re } F, t \text{Im } F) = t^k(\text{Im } F)^k + \sum_{j=1}^{k_0-k} \sum_{m \in \{0,1\}^j} \sigma_{j,m}(\alpha) t^{k+j} (\text{Im } F)^k \prod_{\ell=1}^{j} (\text{Re } F)^{m_{\ell}} (\text{Im } F)^{1-m_{\ell}} + R_{\alpha,k_0-k}(t \text{Re } F, t \text{Im } F).
\end{equation}

We deduce that for all $0 \leq \alpha \leq 1$, $0 \leq k \leq k_0$ and $0 \leq t \leq T_0$,
\begin{equation}
(6.34) \quad \sum_{m \in \{0,1\}^j} \sigma_{j,m}(\alpha) t^{k+j} (\text{Im } F)^k \prod_{\ell=1}^{j} (\text{Re } F)^{m_{\ell}} (\text{Im } F)^{1-m_{\ell}} = \sigma_{j,0}(\alpha) t^{k+j} (\text{Im } F)^{k+j} + \sum_{m \in \{0,1\}^j \setminus \{0\}} \sigma_{j,m}(\alpha) t^{k+j} (\text{Im } F)^k \prod_{\ell=1}^{j} (\text{Re } F)^{m_{\ell}} (\text{Im } F)^{1-m_{\ell}}.
\end{equation}
For all $m \in \{0,1\}^j \setminus \{0\}$, we can write
\begin{equation}
(6.35) \quad \prod_{\ell=1}^{j} (\text{Re } F)^{m_{\ell}} (\text{Im } F)^{1-m_{\ell}} = A_m(\text{Re } F)(\text{Im } F)^{n_m},
\end{equation}
where $n_m$ is a non-negative integer satisfying $0 \leq n_m \leq j-1$ and $A_m \in M_{2n}(\mathbb{R})$ is a real matrix product of $j-1-n_m$ matrices belonging to $\{\text{Re } F, \text{Im } F\}$. It follows from (6.31) and (6.35) that for all $0 \leq \alpha \leq 1$, $0 \leq k \leq k_0$ and $0 \leq t \leq T_0$,
\begin{equation}
(6.36) \quad \sum_{j=1}^{k_0-k} \sum_{m \in \{0,1\}^j} \sigma_{j,m}(\alpha) t^{k+j} (\text{Im } F)^k \prod_{\ell=1}^{j} (\text{Re } F)^{m_{\ell}} (\text{Im } F)^{1-m_{\ell}} = \sum_{j=1}^{k_0-k} \sum_{m \in \{0,1\}^j \setminus \{0\}} \sigma_{j,m}(\alpha) t^{k+j} (\text{Im } F)^k A_m(\text{Re } F)(\text{Im } F)^{n_m}.
\end{equation}
Moreover, the second term in the right-hand side of the above equality can be written as

\begin{equation}
(6.37) \quad \sum_{j=1}^{k_0-k} \sum_{m \in \{0,1\} \setminus \{0\}} \sigma_{j,m}(\alpha)t^{k+j}(\text{Im } F)^kA_m(\text{Re } F)(\text{Im } F)^{\eta_m} = \sum_{j=1}^{k_0-k-1} \sum_{m \in \{0,1\} \setminus \{0\}} \sigma_{j,m}(\alpha)t^{k+j}(\text{Im } F)^kA_m(\text{Re } F)(\text{Im } F)^{\eta_m} + \sum_{j=1}^{k_0-k-1} \sum_{m \in \{0,1\} \setminus \{0\}} \sigma_{j,m}(\alpha)t^{k+j}(\text{Im } F)^kA_m(\text{Re } F)(\text{Im } F)^{\eta_m} \equiv \sum_{j=1}^{k_0-k-1} \sum_{m \in \{0,1\} \setminus \{0\}} \sigma_{j,m}(\alpha)t^{k+j}(\text{Im } F)^kA_m \in M_{2n}(\mathbb{C}),
\end{equation}

where we set

\begin{equation}
(6.38) \quad B_{\alpha,p,k}(t) = \sum_{j=1}^{k_0-k} \sum_{m \in \{0,1\} \setminus \{0\}} \sigma_{j,m}(\alpha)t^{k+j}(\text{Im } F)^kA_m \in M_{2n}(\mathbb{C}).
\end{equation}

We deduce from (6.33), (6.36) and (6.37) that for all $0 \leq \alpha \leq 1$, $0 \leq k \leq k_0$ and $0 \leq t \leq T_0$,

\begin{equation}
(6.39) \quad t^k(\text{Im } F)^kG_\alpha(t \text{ Re } F, t \text{ Im } F) = t^k(\text{Im } F)^k + \sum_{j=1}^{k_0-k-1} \sigma_{j,0}(\alpha)t^{k+j}(\text{Im } F)^k + \sum_{j=1}^{k_0-k-1} t^{p+1}B_{\alpha,p,k}(t)(\text{Re } F)(\text{Im } F)^p + t^k(\text{Im } F)^kR_{\alpha,k_0-k}(t \text{ Re } F, t \text{ Im } F).
\end{equation}

The triangle inequality therefore implies that for all $0 \leq \alpha \leq 1$, $0 \leq k \leq k_0$, $0 \leq t \leq T_0$ and $X \in \mathbb{C}^{2n}$,

\begin{equation}
(6.40) \quad t^k|\sqrt{\text{Re } Q(\text{Im } F)^kG_\alpha(t \text{ Re } F, t \text{ Im } F)}X| \geq |t^k\sqrt{\text{Re } Q(\text{Im } F)^k}X + \sum_{j=1}^{k_0-k} t^{k+j}\sigma_{j,0}(\alpha)\sqrt{\text{Re } Q(\text{Im } F)^kX}| - \sum_{j=1}^{k_0-k-1} t^{p+1}\sqrt{\text{Re } QB_{\alpha,p,k}(t)\text{Re } F)(\text{Im } F)^pX} - t^k|\sqrt{\text{Re } Q(\text{Im } F)^kR_{\alpha,k_0-k}(t \text{ Re } F, t \text{ Im } F)}X|.
\end{equation}

Our aim is now to control the two first terms appearing in the right-hand side of the above estimate. To that end, we begin by noticing that since $(\sigma_{j,m})_{1 \leq j \leq k_0, m \in \{0,1\}}$ is a finite family of continuous functions defined on $[0,1]$, and by definition (6.38) of the terms $B_{\alpha,p,k}(t)$, there exists a positive constant $c_0 > 0$ such that for all $0 \leq \alpha \leq 1$, $0 \leq k \leq k_0$, $1 \leq j \leq k-k_0$ and $m \in \{0,1\}$,

\begin{equation}
(6.41) \quad \left|\sigma_{j,m}(\alpha)\right| + \|\sqrt{\text{Re } QB_{\alpha,p,k}(t)\text{Re } F)(\text{Im } F)^pX} \right| \leq c_0.
\end{equation}

Then, the first term can be controlled in the following way: from (6.41) and Lemma 6.14 we have that for all $0 \leq k \leq k_0-1$ and $\eta_k \in \mathbb{R}_+^{k_0-k}$, there exists a positive constant $\gamma_{\eta_k} > 0$, such that for all $0 \leq \alpha \leq 1$, $0 \leq t \leq T_0$ and $X \in \mathbb{C}^{2n}$,

\begin{equation}
(6.42) \quad t^k\sqrt{\text{Re } Q(\text{Im } F)^k}X + \sum_{j=1}^{k_0-k} t^{k+j}\sigma_{j,0}(\alpha)\sqrt{\text{Re } Q(\text{Im } F)^kX} \geq \gamma_{\eta_k}t^k\sqrt{\text{Re } Q(\text{Im } F)^k}X - c_0\sum_{j=1}^{k_0-k} (\eta_j)t^{k+j}\sqrt{\text{Re } Q(\text{Im } F)^k}X.
\end{equation}

Notice that when $k = k_0$, the sum appearing in the left-hand side of the estimate (6.42) is reduced to zero, which motivates to set $\gamma_{\eta_{k_0}} = 1$. By using that $F = JQ$ and (6.41), we derive the following
Therefore, there exist some positive constants $c_0 \leq c$ for all $\eta$ where the functions $\gamma_0$ for all $\eta,0 \leq t \leq T_0$ and $X \in \mathbb{C}^{2n}$,

$$
(6.43) \quad \sum_{p=0}^{k_0-1} t^{p+1} |\sqrt{Re Q}B_{\alpha,p,k}(t)(Re F)(Im F)^p X| \\
\quad \leq \sum_{p=0}^{k_0-1} t^{p+1} |\sqrt{Re Q}B_{\alpha,p,k}(t)J(\sqrt{Re Q}(Im F)^p X) | \leq c_0 \sum_{p=0}^{k_0} t^{p+1} |\sqrt{Re Q}(Im F)^p X|.
$$

We deduce from (6.40), (6.42) and (6.43) that for all $0 \leq \alpha \leq 1, 0 \leq t \leq T_0$ and $X \in \mathbb{C}^{2n}$,

$$
p_{\alpha,t}(X) \geq \sum_{k=0}^{k_0} \gamma_{\alpha} k^k |\sqrt{Re Q}(Im F)^k X| - c_0 \sum_{k=0}^{k_0-1} \sum_{j=1}^{k_0-k} (\eta_j) t^{k+j} |\sqrt{Re Q}(Im F)^{k+j} X| \\
\quad - c_0(k_0 + 1) \sum_{p=0}^{k_0} t^{p+1} |\sqrt{Re Q}(Im F)^p X| - \sum_{k=0}^{k_0} t^k |\sqrt{Re Q}(Im F)^k R_{\alpha,k_0-k}(t Re F, t Im F) X|,
$$

where the functions $p_{\alpha,t}$ are the ones appearing in the left-hand side of the estimate (6.27), defined for all $0 \leq \alpha \leq 1, 0 \leq t \leq T_0$ and $X \in \mathbb{C}^{2n}$ by

$$
p_{\alpha,t}(X) = \sum_{k=0}^{k_0} t^k |\sqrt{Re Q}(Im F)^k G_{\alpha}(t Re F, t Im F) X|.
$$

We make the change of indexes $j' = k$ and $k' = j + k$ in the following sum

$$
\sum_{k=0}^{k_0-1} \sum_{j=1}^{k_0-k} (\eta_j) t^{k+j} |\sqrt{Re Q}(Im F)^{k+j} X| = \sum_{k=1}^{k_0} \left( \sum_{j=0}^{k-1} (\eta_j) t^{k-j} \right) t^k |\sqrt{Re Q}(Im F)^k X|.
$$

Considering the quantity

$$
(6.45) \quad \varepsilon_{\eta,k,t} = \gamma_{\eta} - c_0 \sum_{j=0}^{k-1} (\eta_j) t^{-j} - c_0(k_0 + 1)t,
$$

and the remainder term

$$
(6.46) \quad \Sigma_{\alpha,t}(X) = \sum_{k=0}^{k_0} t^k |\sqrt{Re Q}(Im F)^k R_{\alpha,k_0-k}(t Re F, t Im F) X|,
$$

we deduce that for all $0 \leq \alpha \leq 1, 0 \leq t \leq T_0$ and $X \in \mathbb{C}^{2n}$, $p_{\alpha,t}(X)$ satisfies the estimate

$$
(6.47) \quad p_{\alpha,t}(X) \geq (\gamma_{\eta} - c_0(k_0 + 1)t) |\sqrt{Re Q}X| + \sum_{k=1}^{k_0} \varepsilon_{\eta,k,t} t^k |\sqrt{Re Q}(Im F)^k X| - \Sigma_{\alpha,t}(X).
$$

Now, we determine the $\eta_k \in (\mathbb{R})^{k_0-k}$. We would like to have $c_0(\eta_j) t^{-j} = \gamma_{\eta}$. Therefore, we define for all $0 \leq k \leq k_0 - 1$ and $1 \leq j \leq k_0 - k$,

$$
(6.48) \quad (\eta_k)_j = \gamma_{\eta_k} t^{-j} = c_0(k_0 + 1)t.
$$

This construction seems implicit but, in fact, it is not. Indeed, to define $\eta_k$, we just need to know $\gamma_{\eta_k}$ for the indexes $k + 1 \leq \ell \leq k_0$ and since $\gamma_{\eta_{k_0}} = 1$, we can proceed by induction. With this construction (6.48) of $\eta_k$, we have that for all $1 \leq k \leq k_0$ and $0 \leq t \leq T_0$,

$$
(6.49) \quad \varepsilon_{\eta,k,t} = \gamma_{\eta_k} t^{-j} - c_0(k_0 + 1)t.
$$

We deduce from this construction and (6.47) that for all $0 \leq \alpha \leq 1, 0 \leq t \leq T_0$ and $X \in \mathbb{C}^{2n}$,

$$
p_{\alpha,t}(X) \geq \sum_{k=0}^{k_0} \left( \gamma_{\eta_k} t^{-j} - c_0(k_0 + 1)t \right) t^k |\sqrt{Re Q}(Im F)^k X| - \Sigma_{\alpha,t}(X).
$$

Therefore, there exist some positive constants $c_1 > 0$ and $0 < T_1 < T_0$ such that for all $0 \leq \alpha \leq 1, 0 \leq t \leq T_1$ and $X \in \mathbb{C}^{2n}$,

$$
(6.49) \quad p_{\alpha,t}(X) \geq c_1 \sum_{k=0}^{k_0} t^k |\sqrt{Re Q}(Im F)^k X| - \Sigma_{\alpha,t}(X).
$$
Now, we prove that the reminder term $\Sigma_{\alpha,t}$ can be controlled by $\sum_{k=0}^{k_0} t^k |\sqrt{\text{Re} Q(\text{Im} F)^k} X|$. To that end, we begin by observing from (6.32) and (6.40) that $0 \leq \alpha \leq 1$, $0 \leq t \leq T_1$ and $X \in \mathbb{C}^{2n}$,

$$\Sigma_{\alpha,t}(X) \leq C_0 \sum_{k=0}^{k_0} t^k \left| \sqrt{\text{Re} Q(\text{Im} F)^k} X \right|^2.$$

(6.50)

Moreover, it follows from the assumption (6.23) that for all $0 \leq \alpha \leq 1$, $0 \leq t \leq \min(1,T_1)$ and $X \in (S + iS)^{\perp}$,

$$\Sigma_{\alpha,t}(X) \leq t^{k_0+1} \left( \sum_{k=0}^{k_0} \left| \sqrt{\text{Re} Q(\text{Im} F)^k} X \right|^2 \right)^{\frac{1}{2}}.$$

Then, the inequality (6.23), the estimate (6.50) and Lemma 6.13 imply that there exists a positive constant $c_2 > 0$ such that for all $0 \leq \alpha \leq 1$, $0 \leq t \leq \min(1,T_1)$ and $X \in (S + iS)^{\perp}$,

$$\Sigma_{\alpha,t}(X) \leq c_2 t^{k_0+1} \left( \sum_{k=0}^{k_0} \left| \sqrt{\text{Re} Q(\text{Im} F)^k} X \right|^2 \right)^{\frac{1}{2}}.$$

where the orthogonality is taken with respect to the Hermitian structure of $\mathbb{C}^{2n}$. This estimate combined with (6.49) shows the existence of positive constants $c_3 > 0$ and $0 < T_2 < T_1$ such that for all $0 \leq \alpha \leq 1$, $0 \leq t \leq T_2$ and $X \in (S + iS)^{\perp}$,

$$\Sigma_{\alpha,t}(X) \leq c_3 \left( \sum_{k=0}^{k_0} t^k \left| \sqrt{\text{Re} Q(\text{Im} F)^k} X \right|^2 \right)^{\frac{1}{2}}.$$

Now, it only remains to check that the estimate (6.51) can be extended to all $X \in \mathbb{C}^{2n}$. To that end, we notice that for all $0 \leq k \leq k_0$, $X \in S + iS$ and $Y \in \mathbb{C}^{2n}$,

$$\sqrt{\text{Re} Q(\text{Im} F)^k}(X + Y) = \sqrt{\text{Re} Q(\text{Im} F)^k}Y,$$

since $\sqrt{\text{Re} Q(\text{Im} F)}(S + iS) = \{0\}$ by definition (4.13) of the singular space $S$. This implies that for all $0 \leq t \leq T_2$ and $X \in \mathbb{C}^{2n}$ written $X = X_{S+iS} + X_{(S+iS)^{\perp}}$, with $X_{S+iS} \in S + iS$ and $X_{(S+iS)^{\perp}} \in (S+iS)^{\perp}$ according to the decomposition $\mathbb{C}^{2n} = (S+iS) \oplus (S+iS)^{\perp}$, the orthogonality being taken with respect to the Hermitian structure of $\mathbb{C}^{2n}$, we have

$$\sum_{k=0}^{k_0} t^k \left| \sqrt{\text{Re} Q(\text{Im} F)^k} X \right|^2 = \sum_{k=0}^{k_0} t^k \left| \sqrt{\text{Re} Q(\text{Im} F)^k} X_{(S+iS)^{\perp}} \right|^2.$$

Moreover, it follows from the assumption (6.23) that for all $0 \leq \alpha \leq 1$, $0 \leq t \leq T_2$ and $X \in S + iS$,

$$G_{\alpha}(t \text{Re} F, t \text{Im} F)X \in S + iS.$$

We deduce from (6.45), (6.52) and (6.54) that for all $0 \leq \alpha \leq 1$, $0 \leq t \leq T_2$ and $X \in \mathbb{C}^{2n}$,

$$p_{\alpha,t}(X) = p_{\alpha,t}(X_{(S+iS)^{\perp}}).$$

As a consequence of (6.53) and (6.55), the estimate (6.51) can be extended to all $0 \leq \alpha \leq 1$, $0 \leq t \leq T_2$ and $X \in \mathbb{C}^{2n}$. This ends the proof of Lemma 6.12 \qed

The two following lemmas are used to prove Lemma 6.12.

**Lemma 6.13.** There exists a positive constant $c > 0$ such that for all $0 \leq t \leq 1$ and $X \in (S + iS)^{\perp}$,

$$\kappa_t(X) \geq ct^{2k_0} |X|^2,$$

where the orthogonality is taken with respect to the Hermitian structure of $\mathbb{C}^{2n}$.

**Proof.** We begin by observing that for all $0 \leq t \leq 1$ and $X \in \mathbb{C}^{2n}$,

$$\kappa_t(X) \geq t^{2k_0} \sum_{k=0}^{k_0} \left| \sqrt{\text{Re} Q(\text{Im} F)^k} X \right|^2.$$

(6.56)

It follows from (1.24), (4.17) and (4.8) that there exists a positive constant $c > 0$ such that for all $X \in S^{\perp}$,

$$\kappa_t(X) \geq t^{2k_0} \sum_{k=0}^{k_0} \left| \sqrt{\text{Re} Q(\text{Im} F)^k} X \right|^2 \geq ct^{2k_0} |X|^2,$$

since $V_{k_0}^{\perp} = S^{\perp}$. Moreover, if $X \in (S + iS)^{\perp}$, then $\text{Re} X, \text{Im} X \in S^{\perp}$ and since $\kappa_t$ is a non-negative quadratic form, we deduce that

$$ct^{2k_0} |X|^2 = c_t^{2k_0} \left| \text{Re} X \right|^2 + c_t^{2k_0} \left| \text{Im} X \right|^2 \leq \kappa_t(\text{Re} X) + \kappa_t(\text{Im} X) = \kappa_t(X).$$

This ends the proof of Lemma 6.13 \qed
Lemma 6.14. Let $m \in \mathbb{N}^*$ and $\eta \in (\mathbb{R}^*)^m$. Then, we have that for all $x, y_1, \ldots, y_m \in \mathbb{C}^m$,
\[
    |x + \sum_{j=1}^{m} y_j| \geq \frac{|x|}{1 + \eta_{\min}} - \sum_{j=1}^{m} \eta_j |y_j|,
\]
with $\eta_{\min} = \min_{1 \leq j \leq m} \eta_j$.

Proof. Let $x, y_1, \ldots, y_m \in \mathbb{C}^m$. We consider $\alpha = \frac{1}{1 + \eta_{\min}}$ and distinguish two cases:

1. On the one hand, if $\alpha|x| \geq \sum_{j=1}^{m} |y_j|$, we have that
\[
    |x + \sum_{j=1}^{m} y_j| + \sum_{j=1}^{m} \eta_j |y_j| \geq |x + \sum_{j=1}^{m} y_j| \geq |x| - \sum_{j=1}^{m} |y_j| \geq |x|(1 - \alpha) = \frac{|x|}{1 + \eta_{\min}}.
\]

2. On the other hand, when $\alpha|x| \leq \sum_{j=1}^{m} |y_j|$, it follows that
\[
    |x + \sum_{j=1}^{m} y_j| + \sum_{j=1}^{m} \eta_j |y_j| \geq \sum_{j=1}^{m} \eta_j |y_j| \geq \alpha \eta_{\min} |x| = \frac{|x|}{1 + \eta_{\min}}.
\]
This ends the proof of Lemma 6.14.

To end this subsection, let us detail why Lemma 6.12 can be applied to the functions $G$ and $G_\alpha$ respectively defined in (3.14) and (4.22).

Lemma 6.15. The function $G$ defined in (3.14) satisfy the assumptions of Lemma 6.12.

Proof. Let us recall that the function $G$ is given by
\[
    G(M, N) = \left( \frac{\sqrt{e^{-2i(M+N)}e^{-2i(M-iN)} + I_{2n}}}{2} \right)^{-1}.
\]
The matrix exponential being defined as the sum of an absolutely convergent series, the product of the two exponentials is given by the following Cauchy product for all $(M, N) \in M_{2n}(\mathbb{R}) \times M_{2n}(\mathbb{R}),$
\[
    e^{-2i(M+iN)}e^{-2i(M-iN)} = \sum_{j=0}^{+\infty} \frac{(-2i)^j}{j!} \sum_{\ell=0}^{j} \binom{j}{\ell} (M + iN)^\ell (M - iN)^{j-\ell}.
\]
Let us consider the non-commutative polynomial $P$ defined by
\[
    P(X, Y) = \sum_{j=1}^{k_n} \frac{(-2i)^j}{j!} \sum_{\ell=0}^{j} \binom{j}{\ell} (X + iY)^\ell (X - iY)^{j-\ell} \in \mathbb{C}_{k_n,0}(X, Y).
\]
We also consider the function $R : (M, N) \in M_{2n}(\mathbb{R}) \times M_{2n}(\mathbb{R}) \to M_{2n}(\mathbb{C})$ defined for all $(M, N) \in M_{2n}(\mathbb{R}) \times M_{2n}(\mathbb{R})$ by
\[
    R(M, N) = \sum_{j=k_n+1}^{+\infty} \frac{(-2i)^j}{j!} \sum_{\ell=0}^{j} \binom{j}{\ell} (M + iN)^\ell (M - iN)^{j-\ell}.
\]
With these notations, the product of exponentials takes the following form for all $(M, N) \in M_{2n}(\mathbb{R}) \times M_{2n}(\mathbb{R}),$
\[
    e^{-2i(M+iN)}e^{-2i(M-iN)} = I_{2n} + P(M, N) + R(M, N).
\]
Notice that the term $R(M, N)$ is a remainder since for all $\rho > 0$ there exists a positive constant $c > 0$ such that for all $(M, N) \in B((0, 0), \rho),$
\[
    ||R(M, N)|| \leq c ||(M, N)||^{k_n+1}.
\]
Now applying Lemma 6.11 with $\rho = 1$ (it could be chosen arbitrarily) and the analytic function
\[
    f : z \in \mathbb{D}(1, 1) \mapsto ((\sqrt{z} + 1)/2)^{-1},
\]
we deduce that there exists $\rho' \in (0, 1)$ such that the function $G$ is well defined on $B((0, 0), \rho')$ and satisfies the assumptions (6.21) and (6.22) of Lemma 6.12 on $B((0, 0), \rho')$ (with no dependence with respect to the parameter $0 \leq \alpha \leq 1$ here).

Always in order to apply Lemma 6.12 to the function $G$, it remains to check that for all $t \geq 0$ such that $(t \Re F, t \Im F) \in B((0, 0), \rho'),$
\[
    G(t \Re F, t \Im F)(S+iS) \subset S+iS.
\]
Notice that $G(t \text{Re } F, t \text{Im } F) = \Phi_t$ from the definitions \((3.3)\) and \((6.57)\) of the matrices $\Phi_t$ and of the function $G$ respectively. The inclusion we aim at proving is therefore equivalent to the following one for all $t \geq 0$ such that $(t \text{Re } F, t \text{Im } F) \in B((0,0), \rho')$,

$$\Phi_t(S + iS) \subset S + iS. \tag{6.61}$$

Since the matrix function $((\sqrt{\gamma} + I_{2n})/2)^{-1}$ is analytic on $B(I_{2n}, 1)$ (from the analyticity of the function \((6.60)\) on $D(1,1)$), there exists a sequence of complex numbers $(\sigma_j)_{j \geq 1}$ such that

$$\forall A \in B(I_{2n}, 1), \quad \left( \frac{\sqrt{\gamma} + I_{2n}}{2} \right)^{-1} = I_{2n} + \sum_{j=1}^{+\infty} \sigma_j (A - I_{2n})^j.$$

It follows that the matrix $\Phi_t$ is the sum of the following series for all $t \geq 0$ such that $(t \text{Re } F, t \text{Im } F) \in B((0,0), \rho')$,

$$\Phi_t = I_{2n} + \sum_{j=1}^{+\infty} \sigma_j (e^{-2itF} e^{-2it\Phi} - I_{2n})^j. \tag{6.62}$$

Since $(\text{Re } F)S = \{0\}$ and $(\text{Im } F)S \subset S$ from \((1.11)\), the two inclusions $FS \subset S + iS$ and $\overline{F}S \subset S + iS$ hold. They imply in particular that $e^{-2itF} S \subset S + iS$ and $e^{-2it\Phi} S \subset S + iS$ for all $t \geq 0$. The inclusion \((6.61)\) is then a consequence of this observation and \((3.62)\).

**Lemma 6.16.** The family of functions $(G_\alpha)_{0 \leq \alpha \leq 1}$ defined in \((1.22)\) satisfies the assumptions of Lemma \ref{lemma6.13}.

**Proof.** We recall that the matrix functions $G_\alpha$ are defined for all $0 \leq \alpha \leq 1$ by

$$G_\alpha(M, N) = \exp \left( -\frac{\alpha}{2} \log \left( e^{-2i(M+iN)} e^{2i(M-iN)} \right) \right). \tag{6.63}$$

Similarly to the previous study of the function $G$ in the proof of Lemma \ref{lemma6.15} we deduce that there exists $\rho > 0$ and $C > 0$ such that the function

$$(M, N) \mapsto \log \left( e^{-2i(M+iN)} e^{2i(M-iN)} \right)$$

is well defined on $B((0,0), \rho)$ and can be written as

$$\forall (M, N) \in B((0,0), \rho), \quad \log \left( e^{-2i(M+iN)} e^{2i(M-iN)} \right) = P(M, N) + R(M, N),$$

where $P \in C_{b_0,0}(X, Y)$ and $R$ is a remainder term

$$\forall (M, N) \in B((0,0), \rho), \quad \|R(M, N)\| \leq C\|\!(M, N)\!\|_\infty^{b_0+1}. \tag{6.64}$$

Now, observing that the set $\{-(\alpha/2)P : 0 \leq \alpha \leq 1\}$ is bounded, we deduce from Lemma \ref{lemma6.11} applied with $f = \exp$ that there exists $\rho' \in (0, \rho)$ and $C' > 0$ (independent of $\alpha$) such that for all $0 \leq \alpha \leq 1$, the function $G_\alpha$ is well defined on $B((0,0), \rho')$ and there exists $R_\alpha : B((0,0), \rho') \to M_{2n}(\mathbb{C})$ satisfying

$$\forall (M, N) \in B((0,0), \rho'), \quad \|R_\alpha(M, N)\| \leq C\|\!(M, N)\!\|_\infty^{b_0+1}, \tag{6.65}$$

such that

$$\forall (M, N) \in B((0,0), \rho'), \quad G_\alpha(M, N) = I_{2n} + \sum_{j=0}^{+\infty} \sigma_{\alpha,j}(A - I_{2n})^j.$$
However, we have already noticed that the vector space $S + iS$ is stable by the matrices $e^{-2itF}$ and $e^{-2itP}$. The inclusion (6.63) is therefore a consequence of this observation and (6.65).

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