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NEW PANCAKE SERIES FOR π

YANNICK SAOUTER

Introduction. In [1], Dalzell proved that $\pi = \frac{22}{7} - \int_0^1 \frac{t^4(1-t)^4}{1+t^2} dt$. He then used this equation to derive a new series converging to π . In [2], Backhouse studied the general case of integrals of the form $\int_0^1 \frac{t^m(1-t)^n}{1+t^2} dt$ and derived conditions on m and n so that they could be used to evaluate π . As a sequel, he derived accurate rational approximations of π . This work was extended in [3] where new rational approximations of π are obtained. Some related integrals of the forms $\int_0^1 \frac{t^m(1-t)^n}{1+t^2} P(t) dt$ and $\int_0^1 \frac{t^m(1-t)^n}{\sqrt{1-t^2}} P(t) dt$ with $P(t)$ being of polynomial form are also investigated. In [4], the author gives more new approximations and new series for the case $m = n = 4k$. In [5], new series for π are obtained with the integral $\int_0^\alpha \frac{t^{12m}(\alpha-t)^{12m}}{1+t^2} dt$ where $\alpha = 2 - \sqrt{3}$. The general problem of improving the convergence speed of the arctan series by transformation of the argument has also been considered in [6, 7]. In this work, the author considers an alternative form for the denominators in integrals. As a result, new series are obtained for multiples of π by some algebraic numbers.

The alternative integral. In the following θ will be a real number with $0 < \theta < \pi$ and we define $I_\theta = \int_0^1 \frac{dt}{t^2 - 2t \cos \theta + 1}$. We have $t^2 - 2t \cos \theta + 1 = (t - e^{i\theta})(t - e^{-i\theta})$ and thus I_θ is a proper well-defined integral. The computation of I_θ is a classical exercise of integration. We have

$$\begin{aligned} I_\theta &= \int_0^1 \frac{dt}{(t - \cos \theta)^2 + \sin^2 \theta} = \frac{1}{\sin^2 \theta} \int_0^1 \frac{dt}{1 + \left(\frac{t - \cos \theta}{\sin \theta}\right)^2} \\ &= \frac{1}{\sin \theta} \int_{-\frac{\cos \theta}{\sin \theta}}^{\frac{1 - \cos \theta}{\sin \theta}} \frac{du}{1 + u^2} \end{aligned}$$

Moreover:

$$\begin{aligned} \frac{1 - \cos \theta}{\sin \theta} &= \frac{1 - \cos^2(\theta/2) + \sin^2(\theta/2)}{2 \sin(\theta/2) \cos(\theta/2)} = \frac{\sin(\theta/2)}{\cos(\theta/2)} = \tan(\theta/2) \\ -\frac{\cos \theta}{\sin \theta} &= -\frac{1}{\tan(\theta)} = -\tan\left(\frac{\pi}{2} - \theta\right) = \tan\left(\theta - \frac{\pi}{2}\right) \end{aligned}$$

Therefore:

$$I_\theta = \frac{1}{\sin \theta} \left[\arctan u \right]_{\tan(\theta - \frac{\pi}{2})}^{\tan(\theta/2)} = \frac{\pi - \theta}{2 \sin \theta}$$

This integral can alternatively be computed by first decomposing the integrand as:

$$\frac{1}{t^2 - 2t \cos \theta + 1} = \frac{1}{2i \sin \theta} \left[\frac{1}{t - e^{i\theta}} - \frac{1}{t - e^{-i\theta}} \right]$$

and then integrating with the function $\log$ extended to the complex domain.

Factorizing pancake functions. We let $P_\theta(x) = (x - e^{i\theta})(x - e^{-i\theta})$. Let now $A(x) = x^m(1 - x)^n$ be a function with m, n integers such that $m, n \geq 1$. Such functions have been called pancake functions because of the shape of their graph. Following the steps of previous works, we are then looking for a factorization $A(x) = P_\theta(x)Q(x) + R$ where $Q(x)$ is a polynomial with real coefficients and R is a real constant. In the following, such a decomposition will be called an acceptable factorization. We then prove the following theorem.

Theorem 1. *Acceptable factorizations exist only for θ being a rational multiple of π . Let then $\theta = r\pi/s$ with r and s positive integers with $0 < r < s$ and $(r, s) = 1$. Then an acceptable factorization exists according to the following criteria:*

- *For any r and s values, the pair (m, n) with n even is a solution if and only if $m + n/2 = 0[\text{mod } s]$,*
- *If r and s are odd, the pair (m, n) with n odd is also a solution if and only if $2m + n = 0[\text{mod } s]$.*

If an acceptable factorization exists then necessarily $A(e^{i\theta}) = A(e^{-i\theta}) = R$. Since $A(x)$ is a polynomial with real coefficients, we have $\bar{A}(x) = A(\bar{x})$ for any $x \in \mathbb{C}$. Thus we have $R = \bar{R}$ and R is a real number. Reciprocally if $A(e^{i\theta}) = R$ with R real, then by conjugacy $A(e^{-i\theta}) = R$. Then the polynomial $A(x) - R$ has $x - e^{i\theta}$ and $x - e^{-i\theta}$ as factors. Since $0 < \theta < \pi$, both factors are distinct and $A(x) - R$ admit $P_\theta(x)$ as a factor. We also have $A(e^{i\theta}) = e^{im\theta}(1 - e^{i\theta})^n = e^{i(m+n/2)\theta}(-2i \sin(\theta/2))^n$. Therefore $\text{Arg}(e^{i\theta}) = ((m + n/2)\theta - n\pi/2)[\text{mod } 2\pi]$ and $A(e^{i\theta})$ is real if and only if:

$$(1) \quad (m + n/2)\theta = n\pi/2[\text{mod } \pi]$$

It is clear that if the latter equation has solutions then θ is necessarily a rational multiple of π . Therefore let r and s integers such that $\theta = r\pi/s$, $0 < r < s$ and $(r, s) = 1$. Equation (1) becomes then successively:

$$(2) \quad \begin{aligned} (m + n/2)r\pi &= ns\pi/2[\text{mod } s\pi] \\ (2m + n)r\pi &= ns\pi[\text{mod } 2s\pi] \\ (2m + n)r &= ns[\text{mod } 2s] \end{aligned}$$

We suppose first that n is even, so that $n = 2n'$ with n' an integer. Then (2) successively becomes:

$$(3) \quad \begin{aligned} (2m + 2n')r &= 0[\text{mod } 2s] \\ (m + n')r &= 0[\text{mod } s] \\ m + n' &= 0[\text{mod } s] \end{aligned}$$

since $(r, s) = 1$. We suppose now that n is odd. Then (2) becomes:

$$(4) \quad (2m + n)r = s[\text{mod } 2s]$$

Modulo 2, we obtain then $r = s[\text{mod } 2]$. Therefore, since $(r, s) = 1$, we have necessarily $r = s = 1[\text{mod } 2]$ and both r and s are odd. Modulo s , (4) gives

$$(5) \quad \begin{aligned} (2m + n)r &= 0[\text{mod } s] \\ 2m + n &= 0[\text{mod } s] \end{aligned}$$

Reciprocally if $2m + n = 0[\text{mod } s]$ and $r = s = 1[\text{mod } 2]$ then (4) is satisfied.

If $\theta = \pi/2$, using Theorem 1, the criterion derived in [2] is recovered. Thus the latter theorem is a generalization of this work. Another remark that can be made is that, since acceptable factorizations exist only for values of θ being rational multiple of π , in these cases, corresponding values of I_θ are products of π by some real algebraic numbers.

Series development. We suppose now that this is the case and this factorization will be used to develop I_θ in series. For $x \in (0, 1)$, we have $\frac{1}{P_\theta(x)} = \frac{Q(x)}{A(x)-R}$, so that:

$$(6) \quad \begin{aligned} I_\theta &= - \int_0^1 \frac{Q(x)}{R - A(x)} \\ &= - \frac{1}{R} \int_0^1 \frac{Q(x)}{1 - \frac{A(x)}{R}} \end{aligned}$$

For any real number t such that $t \neq 1$ and any integer $K \geq 0$, we have

$$\sum_{k=0}^K t^k = \frac{1 - t^{K+1}}{1 - t}$$

We suppose then that $|A(x)/R| \leq M < 1$ for all x in $(0, 1)$. Therefore we have

$$(7) \quad RI_\theta + \sum_{k=0}^K \left[\int_0^1 \left(\frac{A(x)}{R} \right)^k Q(x) dx \right] = - \int_0^1 \frac{(A(x)/R)^{K+1} Q(x)}{1 - \frac{A(x)}{R}} dx$$

However, by the mean value theorem, we have

$$(8) \quad \begin{aligned} \left| \int_0^1 \frac{(A(x)/R)^{K+1} Q(x)}{1 - \frac{A(x)}{R}} \right| &\leq \sup_{x \in (0,1)} \left| \frac{(A(x)/R)^{K+1} Q(x)}{1 - \frac{A(x)}{R}} \right| \\ &\leq \frac{M^{K+1}}{1 - M} \sup_{x \in (0,1)} |Q(x)| \end{aligned}$$

Now it is clear that the right-hand side of (8) tends to 0 when K goes to infinity. Therefore, passing to the limit, we obtain

$$(9) \quad I_\theta = - \sum_{k=0}^{\infty} \frac{1}{R^{k+1}} \int_0^1 Q(x) (A(x))^k dx$$

The same result can be obtained by using the uniform convergence of the series $\sum_{k=0}^{\infty} t^k$ to the function $\frac{1}{1-t}$ in $(0, M)$. The commutation of sum and integral is then justified and the result follows. It can also be remarked that functions $(A(x))^k$ are pancake functions and since $Q(x)$ is a polynomial, integrands of (9) are sums of pancake functions. At this point, it is useful to recall definitions and well-known properties of the Beta and Gamma Eulerian functions.

Definition 2. The function $\Gamma(x)$ is defined for $x > 0$ by $\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt$. If $n \in \mathbb{N}$, we have $\Gamma(n+1) = n!$. The function $B(x, y)$ is defined for $x, y > 0$ by $B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt$ and we have $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.

Therefore the integral I_θ is the sum of beta integrals and can be evaluated by the use of Definition 2. More precisely, if we set $Q(x) = \sum_{i=0}^{\deg(Q)} q_i x^i$, we have, if

convergence is assumed:

$$\begin{aligned}
(10) \quad I_\theta &= - \sum_{k=0}^{\infty} \frac{1}{R^{k+1}} \int_0^1 \left[\sum_{i=0}^{\deg(Q)} q_i x^i \right] x^{mk} (1-x)^{nk} dx \\
&= - \sum_{k=0}^{\infty} \frac{1}{R^{k+1}} \left[\sum_{i=0}^{\deg(Q)} q_i \int_0^1 x^{mk+i} (1-x)^{nk} dx \right] \\
&= - \sum_{k=0}^{\infty} \frac{1}{R^{k+1}} \left[\sum_{i=0}^{\deg(Q)} q_i B(mk+i+1, nk+1) \right] \\
&= - \sum_{k=0}^{\infty} \frac{1}{R^{k+1}} \left[\sum_{i=0}^{\deg(Q)} q_i \frac{\Gamma(mk+i+1)\Gamma(nk+1)}{\Gamma((m+n)k+i+2)} \right] \\
&= - \sum_{k=0}^{\infty} \frac{1}{R^{k+1}} \left[\sum_{i=0}^{\deg(Q)} q_i \frac{(mk+i)!(nk)!}{((m+n)k+i+1)!} \right]
\end{aligned}$$

Using Stirling's formula, i.e. $n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$, it is then possible to find the order of convergence of the final right-hand side of (10). We have, for k large:

$$\begin{aligned}
(11) \quad &\frac{(mk+i)!(nk)!}{((m+n)k+i+1)!} \sim \\
&\left(\frac{mk+i}{e}\right)^{mk+i} \left(\frac{nk}{e}\right)^{nk} \left(\frac{e}{(m+n)k+i+1}\right)^{(m+n)k+i+1} \times \\
&\sqrt{\frac{2\pi(mk+i)(nk)}{(m+n)k+i+1}} \\
&\sim e \frac{(mk+i)^{mk+i} (nk)^{nk}}{((m+n)k+i+1)^{(m+n)k+i+1}} \sqrt{\frac{2\pi mnk}{m+n}} \\
&= e \frac{(mk)^{mk+i} (nk)^{nk}}{((m+n)k)^{(m+n)k+i+1}} \frac{(1+\frac{i}{mk})^{mk+i}}{(1+\frac{i+1}{(m+n)k})^{(m+n)k+i+1}} \sqrt{\frac{2\pi mnk}{m+n}} \\
&\sim \frac{e}{k} \frac{(m)^{mk+i} (n)^{nk}}{((m+n))^{(m+n)k+i+1}} \frac{e^i}{e^{i+1}} \sqrt{\frac{2\pi mnk}{m+n}} \\
&\sim \frac{1}{\sqrt{k}} \left(\frac{m^m n^n}{(m+n)^{m+n}}\right)^k \frac{m^i}{(m+n)^{i+1}} \sqrt{\frac{2\pi mn}{m+n}}
\end{aligned}$$

Therefore the final right-hand side of (10) is convergent if and only if $\frac{m^m n^n}{(m+n)^{m+n}} < |R|$. In order to justify the passage to the limit in (10), we need to check that $|A(x)| < |R|$ on the entire interval $(0, 1)$. Therefore, we are now looking for an upper bound for function $A(x)$. Since the function $\log$ is concave on the positive

real axis, for $x \in (0, 1)$, we have, successively:

$$\begin{aligned} \frac{m \log(nx) + n \log(m - mx)}{m + n} &\leq \log\left(\frac{mn}{m + n}\right) \\ (nx)^m (m - mx)^n &\leq \left(\frac{mn}{m + n}\right)^{m+n} \\ A(x) = \frac{(nx)^m (m - mx)^n}{n^m m^n} &\leq \frac{m^m n^n}{(m + n)^{m+n}} = A\left(\frac{m}{m + n}\right) \end{aligned}$$

Therefore we obtain the same condition as in the previous paragraph. Moreover, since the bound is tight, the condition cannot be relaxed.

With a little work, it is also possible to find an upper bound independent of m and n . For $m, n \geq 1$, we have

$$\left(\frac{m}{n + m}\right)^m \left(\frac{n}{n + m}\right)^n \leq \frac{mn}{(m + n)^2} = \frac{1}{2 + \frac{m}{n} + \frac{n}{m}} \leq \frac{1}{4}$$

However, we have $R = A(e^{i\theta})$ and from preceding computations, we obtain $|R| = (2 \sin(\theta/2))^n$. Therefore if we have

$$2^{n+2} \sin^n(\theta/2) > 1$$

the convergence criterion is met and the series development of (9) is valid. Alternatively, the following sensitive criterion can also be used:

$$2^n \sin^n(\theta/2) > \frac{m^m n^n}{(n + m)^{m+n}}$$

Examples. In this section, we detail computations for some values of θ in both cases of Theorem 1 if possible. The case $\theta = \pi/2$ is the classical case already exposed in [2] and is thus not addressed.

• $\theta = \pi/3$. We have then $r = 1$, $s = 3$, $I_{\pi/3} = \frac{2\pi}{3\sqrt{3}}$ and $P_{\pi/3}(x) = x^2 - x + 1$. From what precedes, acceptable pairs are of the form $(m, 2n)$ with $m+n = 0 \pmod{3}$ and of the form (m, n) with n odd and $2m + n = 0 \pmod{3}$. The simplest case for odd n is obtained with the pair $(m, n) = (1, 1)$. We have then $A(x) = x(1 - x)$, $R = 1$ and $Q(x) = -1$. We have $2^{n+2} \sin^n(\theta/2) = 4$ and the convergence criterion is met. From (10), we have then:

$$\frac{2\pi}{3\sqrt{3}} = \sum_{k=0}^{\infty} \frac{(k!)^2}{(2k+1)!}$$

From (11), the convergence of the series is of order $\mathcal{O}(k^{-1/2} \cdot 4^{-k})$. This series was already obtained by Euler [8] using his transformation of the arctan summation series. For even n , the simplest case is obtained by the pair $(m, n) = (2, 2)$. We have then $A(x) = (x(1 - x))^2$, $R = 1$ and $Q(x) = x^2 - x - 1$. Since $2^{n+2} \sin^n(\theta/2) = 4$, the series development is valid and (10) gives

$$\begin{aligned} \frac{2\pi}{3\sqrt{3}} &= \sum_{k=0}^{\infty} \left[\frac{(2k)!(2k)!}{(4k+1)!} + \frac{(2k+1)!(2k)!}{(4k+2)!} - \frac{(2k+2)!(2k)!}{(4k+3)!} \right] \\ &= \sum_{k=0}^{\infty} \frac{(2k)!^2}{(4k+3)!} (20k^2 + 24k + 7) \end{aligned}$$

The general term of this series is then of order $\mathcal{O}(k^{-1/2} \cdot 16^{-k})$.

• $\theta = \pi/4$. We have then $r = 1$, $s = 4$, $I_{\pi/4} = \frac{3\sqrt{2}\pi}{8}$ and $P_{\pi/4}(x) = x^2 - \sqrt{2}x + 1$. From Theorem 1, acceptable factorizations exist only for even values of n and the condition is then $m + n/2 = 0[\mod 4]$. The simplest case is then $(m, n) = (3, 2)$ for which $R = 2 - \sqrt{2}$. We have $2^{n+2} \sin^n(\theta/2) \sim 2.34 > 1$, so that the convergence criterion is met. After computations the following series is obtained from (10).

$$\frac{3\sqrt{2}\pi}{8} = \sum_{k=0}^{\infty} \frac{(3k)!(2k)!}{(5k+4)!} \frac{1}{(2-\sqrt{2})^{k+1}} \times \\ [(163 - 20\sqrt{2})k^3 + (298 - 46\sqrt{2})k^2 + (177 - 34\sqrt{2})k + 34 - 8\sqrt{2}]$$

Using (11), it is found that the general term of this series is of order $\mathcal{O}(k^{-1/2} \cdot (\frac{108}{3125(2-\sqrt{2})})^k)$. The second simplest case is $(m, n) = (2, 4)$ for which we have $R = -6 + 4\sqrt{2}$. We have then $2^{n+2} \sin^n(\theta/2) \sim 1.37 > 1$ and the convergence criterion is met. We obtain

$$\frac{3\sqrt{2}\pi}{8} = \sum_{k=0}^{\infty} \frac{(2k+1)!(4k)!}{(6k+5)!} \frac{1}{(-6+4\sqrt{2})^{k+1}} \times \\ [(-2576 + 1560\sqrt{2})k^3 + (-4648 + 2800\sqrt{2})k^2 + (-2592 + 1550\sqrt{2})k \\ - 424 + 250\sqrt{2}]$$

The general term of this series is $\mathcal{O}(k^{-1/2} \cdot (\frac{16}{729(6-4\sqrt{2})})^k)$.

• $\theta = \pi/6$. In this case, we have then $r = 1$, $s = 6$, $I_{\pi/6} = \frac{5\pi}{6}$ and $P_{\pi/6}(x) = x^2 - \sqrt{3}x + 1$. Again acceptable factorizations only exist for even values of n . The two simplest cases are $(m, n) = (5, 2)$ and $(m, n) = (4, 4)$. In the first case, we have $2^{n+2} \sin^n(\theta/2) \sim 1.07$ and the convergence criterion is met. In the second case, we have $2^{n+2} \sin^n(\theta/2) \sim 0.28$. However, we also have $2^n \sin^n(\theta/2) \sim 0.07$ and $m^m n^n / (m+n)^{m+n} = 1/256 \sim 0.004$ and thus the sensitive convergence criterion is met. We obtain then the following two series.

$$\frac{5\pi}{6} = \sum_{k=0}^{\infty} \frac{(2k)!(5k)!}{(7k+6)!} \frac{1}{(2-\sqrt{3})^{k+1}} \times \\ [(19149 - 2072\sqrt{3})k^5 + (55007 - 7068\sqrt{3})k^4 + (61627 - 9464\sqrt{3})k^3 \\ + (33685 - 6240\sqrt{3})k^2 + (9008 - 2036\sqrt{3})k + 948 - 264\sqrt{3}]$$

$$\frac{5\pi}{6} = \sum_{k=0}^{\infty} \frac{(4k)!(4k+3)!}{(8k+7)!} \frac{1}{(-7+4\sqrt{3})^{k+1}} \times \\ [(-13888 + 7552\sqrt{3})k^3 + (-26544 + 14448\sqrt{3})k^2 + (-16184 + 8828\sqrt{3})k \\ - 3144 + 1722\sqrt{3}]$$

General terms of these series are respectively $\mathcal{O}(k^{-1/2} \cdot (\frac{12500}{823543(2-\sqrt{3})})^k)$ and $\mathcal{O}(k^{-1/2} \cdot (\frac{1}{256(7-4\sqrt{3})})^k)$.

• $\theta = \pi/15$. In this case, we have $r = 1$ and $s = 15$. From Theorem (1), acceptable factorizations exist for (m, n) such that n is odd and $2m + n = 0[\mod 15]$. A possible case is $(m, n) = (5, 5)$. However, we have then $2^n \sin^n(\theta/2) \sim 0.0004$ while $m^m n^n / (m+n)^{m+n} = 1/1024 \sim 0.00097$. Therefore, in this case, the

sensitive convergence criterion fails. Equation (6) still holds but the right-hand side of (7) is then diverging, so that (10) is no more justified. Therefore the existence of an acceptable factorization is not a sufficient condition to obtain convergent series. However, in this case, other choices of parameters, like $(m, n) = (7, 1)$, give proper convergent series.

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