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Characterization of graph-based hierarchical watersheds: theory and algorithms

Deise Santana Maia · Jean Cousty · Laurent Najman · Benjamin Perret

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Abstract We propose a characterization of hierarchical watersheds built in the framework of edge-weighted graphs. Based on the notion of binary partition hierarchy by altitude ordering, we provide sufficient and necessary conditions for a hierarchy to be a hierarchical watershed. Those conditions are further used to design an efficient algorithm to recognize hierarchical watersheds. Furthermore, as an immediate application of our theoretical results, we present experiments with the combinations of hierarchical watersheds studied in [15]. Namely, we test if combinations of hierarchical watersheds are hierarchical watersheds themselves.

1 Introduction

Watershed [4, 7] is a well established segmentation technique in the field of mathematical morphology. The idea behind this technique is related to the topographic definition of watersheds: dividing lines between neighboring catchment basins, *i.e.*, regions whose collected water drains to a common point. We say that the point (or region) of lowest altitude of a catchment basin is a (local) minimum of a topographic surface. In the context of digital image processing, gray-level image (gradients) can be treated as topographic surfaces whose altitudes are determined by the pixel gray-levels. The minima of an image are the regions of uniform grey-level surrounded by pixels of strictly greater gray-levels. A watershed segmentation is a partition of the set of pixels of an image into its catchment basins.

Hierarchical watersheds [8, 3, 26, 20] are sequences of nested partitions which correspond to filterings of an initial watershed segmentation [7, 4]. Given an image I, a hierarchical watershed of I can be obtained by iteratively merging neighboring catchment basins of I according to a predefined ordering on the minima of I.

Several well-known image segmentation techniques are modeled in the framework of graphs [5, 30, 12, 11, 13, 6], including (hierarchical) watersheds [19, 7, 8, 10, 25]. In this context, images are often represented as (edge) weighted graphs whose vertices correspond to pixels and whose edge weights convey the dissimilarity between neighboring pixels. Let G be a graph whose edges are weighted by a map w. A minimum of w is a subgraph of G with equal edge weights that is surrounded by edges with strictly greater weights. A hierarchical watershed of (G, w) for a sequence of minima $\mathcal{S}$ of w is constructed by merging the catchment basins of (G, w) following the sequence $\mathcal{S}$.

Hierarchical watersheds can feature several distinct aspects of an image. The minima of a weighted graph are commonly ordered by extinction values based on a regional attribute A, *e.g.* area and volume [33]. We then expect the resulting hierarchical watershed to highlight the regions that stand out with respect to this attribute A. Besides being versatile, hierarchical watersheds can be computed by the efficient algorithm proposed in [10, 25], whose time complexity is the same as minimum spanning tree algorithms. Moreover, as shown in [27], the performance of hierarchical watersheds based on regional attributes is competitive when compared to other hierarchical segmentation methods.

In this study, we tackle the problem of recognizing hierarchical watersheds. More precisely, we aim to solve the following problem:

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(P) given a weighted graph (G, w) and a hierarchy of partitions $\mathcal{H}$, determine if $\mathcal{H}$ is a hierarchical watershed of (G, w).

Problem (P) can be related to the problem studied in [14,2]. In [14,2], the authors search for a minimum set of markers which lead to a given watershed segmentation. In our case, we are interested in ordering a predefined set of markers (the set of all minima of w) that allows to solve the minimum set of markers problem for the series of all watershed segmentations (partitions) of a given hierarchy.

This article is a theoretical and experimental extension of the conference paper [16]. Our main contributions are the following: (1) a characterization of hierarchical watersheds in the framework of weighted graphs (Theorem 4); (2) an efficient algorithm to recognize hierarchical watersheds (Algorithm 1); (3) the proofs of the properties and theorem first established in [16]; and (4) experimental results with the proposed algorithm applied to the combinations of hierarchical watersheds assessed in [15].

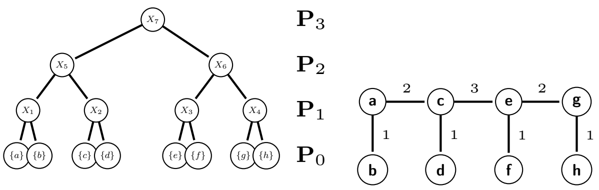
In section 2, we present the basic notions for handling hierarchies with graphs. In section 3, we formally state the problem of recognizing hierarchical watersheds and we present a characterization of hierarchical watersheds to arbitrary graphs. In section 4, we present an efficient algorithm to recognize hierarchical watersheds. In section 5, we introduce the notion of flattened (simplified) hierarchical watersheds and an algorithm to recognize this type of hierarchy. Finally, we present experimental results with the proposed algorithms in section 6.

2 Background notions

In this section, we first introduce hierarchies of partitions. Then, we review the definition of graphs, connected hierarchies and saliency maps. Subsequently, we define hierarchical watersheds.

2.1 Hierarchies of partitions

Let V be a set. A *partition* (of V) is a set $\mathbf{P}$ of non empty disjoint subsets of V whose union is V. Any element of a partition $\mathbf{P}$ is called a *region of* $\mathbf{P}$. Let $\mathbf{P}_1$ and $\mathbf{P}_2$ be two partitions. We say that $\mathbf{P}_1$ is a *refinement* of $\mathbf{P}_2$ if every element of $\mathbf{P}_1$ is included in an element of $\mathbf{P}_2$. A *hierarchy (of partitions)* is a sequence $\mathcal{H} = (\mathbf{P}_0, \ldots, \mathbf{P}_\ell)$ of partitions such that $\mathbf{P}_{i-1}$ is a refinement of $\mathbf{P}_i$, for any i in $\{1, \ldots, \ell\}$ and such that $\mathbf{P}_n = \{V\}$. Let $\mathcal{H} = (\mathbf{P}_0, \ldots, \mathbf{P}_\ell)$ be a hierarchy of partitions. Any region of a partition $\mathbf{P}$ of $\mathcal{H}$ is called a *region of* $\mathcal{H}$. The set of all regions of $\mathcal{H}$ is denoted by $\mathscr{R}(\mathcal{H})$.



(a) (b)

Fig. 1: (a): A representation of a hierarchy of partitions $\mathcal{H} = (\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3)$ on the set $\{a, b, c, d, e, f, g, h\}$. (b): A weighted graph (G, w).

A hierarchy of partitions can be represented as a tree whose nodes correspond to regions, as shown in Figure 1(a). Given a hierarchy $\mathcal{H}$ and two regions X and Y of $\mathcal{H}$, we say that X *is a parent of* Y *(or that* Y *is a child of* X*)* if $Y \subset X$ and X is minimal for this property, *i.e.*, if there is a region Z such that $Y \subseteq Z \subset X$, then we have $Y = Z$. It can be seen that any region $X \neq V$ of $\mathcal{H}$ has exactly one parent. For any region X such that $X \neq V$, we write $parent(X) = Y$ where Y is the unique parent of X. For any region R of $\mathcal{H}$, if R is not the parent of any region of $\mathcal{H}$, we say that R *is a leaf region (of* $\mathcal{H}$*)*. Otherwise, we say that R *is a non-leaf region (of* $\mathcal{H}$*)*. The set of all non-leaf regions of $\mathcal{H}$ is denoted by $\mathscr{R}^*(\mathcal{H})$.

In Figure 1(a), the regions of a hierarchy $\mathcal{H}$ are linked to their parents (and to their children) by straight lines.

2.2 Graphs, connected hierarchies and saliency maps

A *graph* is a pair $G = (V, E)$, where V is a finite set and E is a set of pairs of distinct elements of V, *i.e.*, $E \subseteq \{\{x, y\} \subseteq V \mid x \neq y\}$. Each element of V is called a *vertex (of* G*)*, and each element of E is called an *edge (of* G*)*. To simplify the notations, the set of vertices and edges of a graph G will be also denoted by $V(G)$ and $E(G)$, respectively.

Let $G = (V, E)$ be a graph and let X be a subset of V. A sequence $\pi = (x_0, \ldots, x_n)$ of elements of X is a *path (in* X*) from* x_0 *to* x_n if $\{x_{i-1}, x_i\}$ is an edge of G for any i in $\{1, \ldots, n\}$. Given a path $\pi = (x_0, \ldots, x_n)$ from a vertex x_0 to a vertex x_n in V, for any edge $u = \{x_{i-1}, x_i\}$ for any i in $\{1, \ldots, n\}$, we say that u *is an edge in* π and that u *is in* π. The subset X of V is said to be *connected* if, for any x and y in X, there exists a path from x to y. The subset X is a *connected component* of G if X is connected and if, for any connected subset Y of V, if $X \subseteq Y$, then we have $X = Y$. In the following, we denote by $CC(G)$ the set of all connected components of G. It is well known that this set $CC(G)$ of all connected components of G is a partition of the set V.

Let $G = (V, E)$ be a graph. *A partition of V is connected for G* if each of its regions is connected and *a hierarchy on V is connected (for G)* if every one of its partitions is connected. For example, the hierarchy of Figure 1(a) is connected for the graph of Figure 1(b).

Let G be a graph. If w is a map from the edge set of G to the set $\mathbb{R}$ of real numbers, then the pair (G, w) is called an *(edge) weighted graph.* If (G, w) is a weighted graph, for any edge u of G, the value $w(u)$ is called the *weight of u (for w).*

Important notation: in the sequel of this article, the symbol (G, w) denotes a weighted graph whose vertex set is connected. To shorten the notation, the vertex set of G is denoted by V and its edge set is denoted by E. Without loss of generality, we also assume that the range of w is included in the set $\mathbb{E}$ of all integers from 0 to $|E| - 1$ (otherwise, one could always consider an increasing one-to-one correspondence from the set $\{w(u) \mid u \in E\}$ into the subset $\{0, ..., |\{w(u) \mid u \in E\}| - 1\}$ of $\mathbb{E}$).

Let λ be any element in $\mathbb{R}$. The *λ-level set of (G, w)* is the graph $(V, E_\lambda(G))$ such that $E_\lambda(G) = \{u \in E(G) \mid w(u) \leq \lambda\}$. The sequence

$$\mathcal{QFZ}(w) = (CC(G_{\lambda,w}) \mid \lambda \in \mathbb{E}) \quad (1)$$

where $G_{\lambda,w}$ is the λ-level set of (G, w), is a hierarchy called the *Quasi-Flat Zones (QFZ) hierarchy (of w)* [23, 21, 32, 10].

As established in [9], a connected hierarchy can be equivalently treated by means of a weighted graph through the notion of a saliency map. Given a hierarchy $\mathcal{H} = (\mathbf{P}_0, \ldots, \mathbf{P}_\ell)$ which is connected for G, the *saliency map of $\mathcal{H}$* is the map from E into $\{0, \ldots, \ell\}$, denoted by $\Phi(\mathcal{H})$, such that, for any edge $u = \{x, y\}$ in E, the value $\Phi(\mathcal{H})(u)$ is the lowest value i in $\{0, \ldots, \ell\}$ such that x and y belong to a same region of $\mathbf{P}_i$. It follows that any connected hierarchy has a unique saliency map. Moreover, any hierarchy $\mathcal{H}$ connected for G is precisely the quasi-flat zones hierarchy of its own saliency map: $\mathcal{H} = \mathcal{QFZ}(\Phi(\mathcal{H}))$.

For instance, the map depicted in Figure 1(b) is the saliency map of the hierarchy of Figure 1(a).

Saliency maps are closely related to the notion of ultrametric distances [24, 1]. Let $\mathcal{H} = (\mathbf{P}_1, \ldots, \mathbf{P}_\ell)$ be a hierarchy on V. Let d be a map from $V \times V$ into $\mathbb{R}$ such that, for any pair (x, y) of vertices in $V \times V$, the value $d(x, y)$ is the greatest edge weight λ in a path π from x to y (resp. y to x) in $(G, \Phi(\mathcal{H}))$ such that, for any other path π' from x to y (resp. y to x), the greatest edge weight in π' is greater than or equal to λ. We can affirm that (V, d) is an ultrametric space. Moreover, for any two vertices x and y in V, by the definition of saliency maps and considering its link with QFZ hierarchies, we may say that $d(x, y)$ is the lowest value λ such that x and y belong to a same region of the partition $\mathbf{P}_\lambda$ of $\mathcal{H}$. Furthermore, if G is a complete graph, we can conclude that $(V, \Phi(\mathcal{H}))$ is an ultrametric space.

2.3 Hierarchical minimum spanning forests and watersheds

The watershed segmentation, see *e.g.* [4, 26, 7], derives from the topographic notion of watershed lines and catchment basins. In [7], the authors formalize watersheds in the framework of weighted graphs and show the optimality of watersheds in the sense of minimum spanning forests. In this section, we present hierarchical watersheds following the definition of hierarchies of minimum spanning forests presented in [8, 10].

We say that the graph $G = (V, E)$ *is a forest* if, for any edge u in E, the number of connected components of the graph $(V, E \setminus \{u\})$ is greater than the number of connected components of G. Given another graph G', we say that *G' is a subgraph of G*, denoted by $G' \sqsubseteq G$, if $V(G')$ is a subset of V and $E(G')$ is a subset of E. Let G'' be a subgraph of G and let G' be a subgraph of G''. The graph G'' is a *Minimum Spanning Forest (MSF) of G rooted in G'* if:

1. the graphs G and G'' have the same set of vertices, *i.e.*, $V(G'') = V$; and
2. each connected component of G'' includes exactly one connected component of G'; and
3. the sum of the weight of the edges of G'' is minimal among all subgraphs of G for which the above conditions 1 and 2 hold true.

A MSF of (G, w) rooted in a single vertex of G is a tree (connected forest) called a *Minimum Spanning Tree (MST) of (G, w)*.

Let k be a value in $\mathbb{R}$. A connected subgraph G' of G is a *(regional) minimum* (of w) at level k if:

1. the set of edges $E(G')$ of G' is not empty; and
2. for any edge u in $E(G')$, the weight of u is equal to k; and
3. for any edge $\{x, y\}$ in $E \setminus E(G')$ such that $|\{x, y\} \cap V(G')| \geq 1$, the weight of $\{x, y\}$ is strictly greater than k.

Important notation: in the sequel of this article, we denote by n the number of minima of w. Every sequence of minima of w considered in this article is a sequence of n pairwise distinct minima of w and, therefore, for the sake of simplicity, we use the term *sequence of minima of w* instead of *sequence of n pairwise distinct minima of w*.

Let $\{G_1, \dots, G_\ell\}$ be a set of graphs. We denote by $\sqcup\{G_1, \dots, G_\ell\}$ the graph $(\cup\{V(G_j) \mid j \in \{1, \dots, \ell\}\}, \cup\{E(G_j) \mid j \in \{1, \dots, \ell\}\})$. In the following, we define hierarchical watersheds based on minimum spanning forests following the definition of [8,10].

Definition 1 (hierarchical watershed [8,10])**.** *Let $\mathcal{S} = (M_1, \dots, M_n)$ be a sequence of minima of w. Let $(G_0, \dots, G_{n-1})$ be a sequence of subgraphs of G such that:*

1. *for any i in $\{0, \dots, n-1\}$, the graph G_i is a MSF of G rooted in $\sqcup\{M_j \mid j \in \{i+1, \dots, n\}\}$; and*
2. *for any i in $\{1, \dots, n-1\}$, G_{i-1} is a subgraph of G_i.*

The sequence $\mathcal{T} = (CC(G_0), \dots, CC(G_{n-1}))$ is called a hierarchical watershed of (G, w) for $\mathcal{S}$. *Given a hierarchy $\mathcal{H}$, we say that $\mathcal{H}$* is a hierarchical watershed of (G, w) *if there exists a sequence $\mathcal{S}$ of minima of w such that $\mathcal{H}$ is a hierarchical watershed of (G, w) for $\mathcal{S}$.*

For instance, let (G, w) and $\mathcal{H}$ be respectively the weighted graph and the hierarchy shown in Figure 2(a) and (b). We can see that $\mathcal{H}$ is the hierarchical watershed of (G, w) for the sequence (C, A, B, D) of minima of w.

3 Characterization of hierarchical watersheds

In this section, we solve the following recognition problem:

(P) given a weighted graph (G, w) and a hierarchy of partitions $\mathcal{H}$, determine if $\mathcal{H}$ is a hierarchical watershed of (G, w).

A naive approach to solve Problem (P) is to test if there is a sequence $\mathcal{S}$ of minima of w such that $\mathcal{H}$ is the hierarchical watershed of (G, w) for $\mathcal{S}$. However, there exist $n!$ sequences of minima of w, which leads to an algorithm of factorial time complexity.

To solve Problem (P) more efficiently, we propose in section 3.2 a characterization of hierarchical watersheds (Theorem 4) based on the binary partition hierarchy by altitude ordering (section 3.1) which, as stated in [10], is known to be closely related to hierarchical watersheds. Then, we present a sketch of the proof of Theorem 4 by linking one-side increasing maps to the notion of extinction values as defined in [10]. Based on our proposed characterization of hierarchical watersheds, we design an efficient algorithm (Algorithm 1) to solve Problem (P).

3.1 Binary partition hierarchies by altitude ordering

Binary partition trees [28] are widely used for hierarchical image representation. In this section, we describe the case where regions linked by the lowest edge weights are the first regions to be merged in the hierarchy [10]. This particular case is deeply linked to single-linkage clustering [9].

Let $\prec$ be a total ordering (on E), *i.e.*, $\prec$ is a binary relation that is transitive and trichotomous: for any u and v in E only one of the relations $u \prec v$, $v \prec u$ and $v = u$ holds true. We say that $\prec$ *is an altitude ordering (on E) for w* if, for any u and v in E, if $w(u) < w(v)$, then $u \prec v$. Let $\prec$ be an altitude ordering for w. Let k be any element in $\{1, \dots, |E|\}$. We denote by $u_k^{\prec}$ the k-th element of E with respect to $\prec$. We set $\mathbf{B}_0 = \{\{x\} \mid x \in V\}$. The *$k$-partition of V (by the ordering $\prec$)* is defined by $\mathbf{B}_k = \{\mathbf{B}_{k-1}^y \cup \mathbf{B}_{k-1}^x\} \cup (\mathbf{B}_{k-1} \setminus \{\mathbf{B}_{k-1}^x, \mathbf{B}_{k-1}^y\})$ where $u_k^{\prec} = \{x, y\}$ and $\mathbf{B}_{k-1}^x$ and $\mathbf{B}_{k-1}^y$ are the regions of $\mathbf{B}_{k-1}$ that contain x and y, respectively. The sequence $(\mathbf{B}_i \mid i = 0 \text{ or } \mathbf{B}_i \neq \mathbf{B}_{i-1})$ is a hierarchy on V. This hierarchy $(\mathbf{B}_i \mid i = 0 \text{ or } \mathbf{B}_i \neq \mathbf{B}_{i-1})$, denoted by $\mathcal{B}_{\prec}$, is called the *binary partition hierarchy (by altitude ordering) of (G, w) by $\prec$*.

Let $\mathcal{B}$ be a hierarchy on V. We say that $\mathcal{B}$ *is a binary partition hierarchy (by altitude ordering) of (G, w)* if there is an altitude ordering $\prec$ for w such that $\mathcal{B}$ is the binary partition hierarchy of (G, w) by $\prec$.

Let $\prec$ be an altitude ordering for w. We can associate any non-leaf region X of the binary partition hierarchy $\mathcal{B}_{\prec}$ of (G, w) by $\prec$ to the lowest rank r such that $\mathbf{B}_r$ contains X. This rank is called the *rank of X*. Let X be a non-leaf region of $\mathcal{B}_{\prec}$ and let r be the rank of X. The *building edge of X* is the r-th edge for $\prec$. Given an edge u in E, if u is the building edge of a region of $\mathcal{B}_{\prec}$, we say that *u is a building edge for $\prec$*. Given a building edge u for $\prec$, we denote the region of $\mathcal{B}_{\prec}$ whose building edge is u by R_u. The set of all building edges for $\prec$ is denoted by $E_{\prec}$.

Let (G, w) be the weighted graph illustrated in Figure 2(a) and let $\mathcal{B}$ be the binary partition hierarchy of (G, w) illustrated in Figure 2(c). We can see that $\mathcal{B}$ is the binary partition hierarchy of (G, w) by the altitude ordering $\prec$ such that $\{a, b\} \prec \{c, d\} \prec \{e, f\} \prec \{g, h\} \prec \{a, c\} \prec \{e, g\} \prec \{c, e\}$. The building edge of each non-leaf region R of $\mathcal{B}$ is shown above the node that represents R.

Let $\mathcal{B}$ be a binary partition hierarchy of (G, w) and let X and Y be two distinct regions of $\mathcal{B}$. If the parent of X is equal to the parent of Y, we say that X is a sibling of Y, that Y is a sibling of X and that X and Y are siblings. It can be seen that any region $R \neq V$

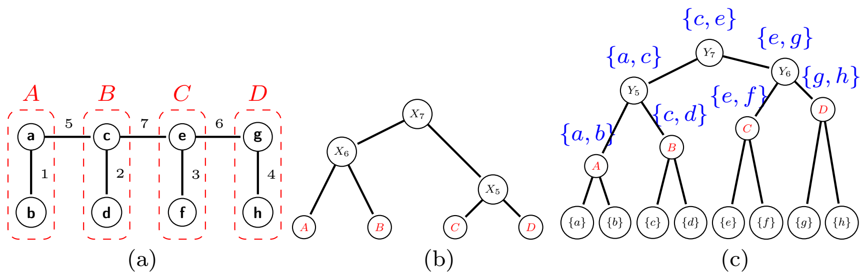


Fig. 2: (a): A weighted graph (G, w) with four minima delimited by the dashed lines. (b): The hierarchical watershed of (G, w) for the sequence (C, A, B, D) of minima of w. (c): The unique binary partition hierarchy $\mathcal{B}$ of (G, w).

of $\mathcal{B}$ has exactly one sibling and we denote this unique sibling of R by $sibling(R)$.

Important remark: by abuse of terminology, when no confusion is possible, if M is a minimum of w, we call the set $V(M)$ of vertices of M as a minimum of w.

As established in [25], given an altitude ordering $\prec$ for w, the minima of w can be extracted from the binary partition hierarchy $\mathcal{B}_\prec$ as well as the watershed-cut edges for $\prec$, whose definition is given bellow.

Definition 2 (watershed-cut edge)**.** *Let $\prec$ be an altitude ordering for w and let u be a building edge for $\prec$. We say that u* is a watershed-cut edge (of (G, w)) for $\prec$ *if each child of the region R_u of $\mathcal{B}_\prec$ includes at least one minimum of w.*

3.2 Characterization of hierarchical watersheds

In [16], the authors propose a characterization of hierarchical watersheds for solving the problem of recognizing hierarchical watersheds in the context of weighted graphs which are trees with pairwise distinct edge weights. In this section, we generalize the characterization of hierarchical watersheds introduced in [16] to arbitrary graphs. To ease the reading of this section, the proofs of the properties and theorems stated here are delayed to the appendix of this article.

Let $\prec$ be an altitude ordering for w and let f be a map from E into $\mathbb{R}$. The *supremum descendant value of R for f and $\prec$* is the supremum edge weight among the building edges of the regions included in R: $\vee\{f(v) \mid v \in E_\prec), R_v \subseteq R\}$, where $\vee\{\} = 0$. We denote by *supremum descendant map for f and $\prec$* the map that assigns every region of $\mathcal{B}_\prec$ into its supremum descendant value for f and $\prec$.

The next definition introduces the notion of one-side increasing map. As established later in Theorem 4, the notion of one-side increasing map is closed linked to the saliency maps of hierarchical watersheds.

Definition 3 (one-side increasing map)**.** *Let $\prec$ be an altitude ordering for w and let f be a map from E into $\mathbb{R}$. Let $\triangledown$ be the supremum descendant map for f and $\prec$. We say that f is* one-side increasing for $\prec$ *if:*

1. *$\{f(u) \mid u \in E_\prec\} = \{0, \ldots, n-1\}$;*
2. *for any edge u in $E_\prec$, the weight $f(u)$ is greater then zero if and only if u is a watershed-cut edge for $\prec$; and*
3. *for any edge u in $E_\prec$, there exists a child R of R_u such that $f(u)$ is greater than or equal to the supremum descendant value $\triangledown(R)$.*

The next theorem, whose proof is given in Appendix F, states that hierarchical watersheds can be characterized as the hierarchies whose saliency maps are one-side increasing maps.

Theorem 4. *Let $\mathcal{H}$ be a hierarchy on V. The hierarchy $\mathcal{H}$ is a hierarchical watershed of (G, w) if and only if there is an altitude ordering $\prec$ for w such that the saliency map $\Phi(\mathcal{H})$ is one-side increasing for $\prec$.*

Let $\mathcal{H}$ be the hierarchy of Figure 3(a), let $\Phi(\mathcal{H})$ be the saliency map of $\mathcal{H}$ shown in Figure 3(b), and let $\mathcal{B}$ be the binary partition hierarchy of (G, w) (Figure 2) shown in Figure 3(b). As the edges of G have pairwise distinct edges for w, we can conclude that $\mathcal{B}$ is the binary partition hierarchy of (G, w) by the unique altitude ordering $\prec$ for w. We can verify that $\Phi(\mathcal{H})$ is one-side increasing for $\prec$. By Theorem 4, we may affirm that $\Phi(\mathcal{H})$ is the saliency map of a hierarchical watershed of (G, w) and that, consequently, the hierarchy $\mathcal{H}$ is a hierarchical watershed of (G, w). On the other, let $\mathcal{H}'$ and $\Phi(\mathcal{H}')$ be the hierarchy and the saliency map of Figure 3(d) and (e), respectively. We can see that $\Phi(\mathcal{H}')$ is not one-side increasing for $\prec$. Indeed, the weight $\Phi(\mathcal{H}')(\{c, e\})$ of the building edge of the region Y_7 of $\mathcal{B}$ is 1, which is lower than both $\vee\{\Phi(\mathcal{H}')(v) \mid R_v \subseteq Y_5\} = 2$ and $\vee\{\Phi(\mathcal{H}')(v) \mid R_v \subseteq Y_6\} = 3$. Hence, the condition 3 of Definition 3 is not satisfied by $\Phi(\mathcal{H}')$. Thus, by Theorem 4, as $\prec$ is the unique altitude ordering for w, we may deduce that $\Phi(\mathcal{H}')$ is not the saliency map of a hierarchical watershed of (G, w) and that $\mathcal{H}'$ is not a hierarchical watershed of (G, w).

Let $\mathcal{H}$ be a hierarchy on V. By Theorem 4, in order to test if $\mathcal{H}$ is a hierarchical watershed of (G, w), we need to test if there is an altitude ordering $\prec$ for w such that the saliency map $\Phi(\mathcal{H})$ is one-side increasing for $\prec$. In the worst case, there exists $|E|!$ possible altitude orderings for w. Hence, the naive approach to verify that $\Phi(\mathcal{H})$ is one-side increasing for an altitude ordering for w has a factorial time complexity, which is the same time complexity as the algorithm to verify that $\mathcal{H}$ is a hierarchical watershed for a sequence of minima of w. Actually, as stated later in Property 5, it is sufficient to test if $\Phi(\mathcal{H})$ is one-side increasing for a single altitude ordering for w, which is the key idea behind our efficient algorithm (Algorithm 1) to recognize hierarchical watersheds.

Let f and g be two maps from E into $\mathbb{R}$. A *lexicographic ordering for* (f, g) is a total ordering $\prec$ on E such that, for any two edges u and v in E, we have $u \prec v$ if $f(u) < f(v)$ or if $f(u) = f(v)$ and $g(u) \leq g(v)$. We can note that any lexicographic ordering for (f, g) is an altitude ordering for f.

Property 5. *Let $\mathcal{H}$ be a hierarchy on V and let $\prec$ be a lexicographic ordering for (w, f). The hierarchy $\mathcal{H}$ is a hierarchical watershed of (G, w) if and only if $\Phi(\mathcal{H})$ is one-side increasing for $\prec$.*

The proof of Property 5 is presented in Appendix B.

In the remaining of this section, we present the building blocks of the proof of Theorem 4. More precisely, we state the link between the notions of one-side increasing map, hierarchical watershed and the method to compute hierarchical watersheds introduced in [10, 25].

Let $\prec$ be an altitude ordering for w and let $\mathcal{S} = (M_1, \ldots, M_n)$ be a sequence of minima of w. Let R be a region of the binary partition hierarchy $\mathcal{B}_\prec$ by $\prec$. Using the terminology of [10], the *extinction value of R (for $\prec$ and $\mathcal{S}$)* is zero if there is no minimum of w included in R and, otherwise, it is the maximum value i in $\{1, \ldots, n\}$ such that the minimum M_i is included in R. Let ϵ be the map from the set of regions of $\mathcal{B}_\prec$ into $\mathbb{R}$ such that, for any region R of $\mathcal{B}_\prec$, the value $\epsilon(R)$ is the extinction value of R. We say that ϵ *is the extinction map for $\prec$ and $\mathcal{S}$* and that ϵ *is an extinction map for $\prec$ (resp. $\mathcal{S}$)*. The following property, whose proof is detailed in Appendix C, characterizes extinction maps.

Property 6. *Let $\prec$ be an altitude ordering for w and let ϵ be a map from the regions of $\mathcal{B}_\prec$ into $\mathbb{R}$. The map ϵ is an* extinction map for $\prec$ *if and only if the following statements hold true:*

1. *$\{\epsilon(R) \mid R$ is a region of $\mathcal{B}_\prec\} = \{0, \ldots, n\}$;*
2. *for any two distinct minima M_1 and M_2 of w, we have $\epsilon(M_1) \neq \epsilon(M_2)$; and*
3. *for any region R of $\mathcal{B}_\prec$, we have that $\epsilon(R)$ is equal to $\vee\{\epsilon(M)$ such that M is a minimum of w included in $R\}$.*

We provide an example of an extinction map in Figure 4. We can see that the map ϵ is the extinction map for the unique altitude ordering for w (Figure 2(a)) and for the sequence $\mathcal{S} = (B, A, D, C)$ of minima of w.

The next property clarifies the relation between hierarchical watersheds and extinction maps. As established in [10], given a sequence $\mathcal{S}$ of minima of w, we can compute the saliency map of a hierarchical watershed for $\mathcal{S}$ by considering any extinction map for $\mathcal{S}$. As the edge weights of w are not necessarily pairwise distinct, given any sequence $\mathcal{S}$ of minima of w, there might be several distinct hierarchical watersheds of (G, w) for $\mathcal{S}$. Let $\mathcal{S}$ be a sequence of minima of w. As established in the following property, we can associated any hierarchical watershed $\mathcal{H}$ of (G, w) for $\mathcal{S}$ with an altitude ordering $\prec$ for w such that, for any building edge u for $\prec$, the weight of u for the saliency map $\Phi(\mathcal{H})$ is obtained from the extinction map for $\prec$ and $\mathcal{S}$.

Property 7. *Let $\mathcal{H}$ be a hierarchy on V. The hierarchy $\mathcal{H}$ is a hierarchical watershed of (G, w) if and only if there exists an altitude ordering $\prec$ for w and an extinction map ϵ for $\prec$ such that*

1. *$(V, E_\prec)$ is a MST of $(G, \Phi(\mathcal{H}))$; and*
2. *for any edge u in $E_\prec$, we have that $\Phi(\mathcal{H})(u)$ is equal to $\min\{\epsilon(R)$ such that R is a child of $R_u\}$.*

The proof of Property 7 is detailed in Appendix A. The intuition of the forward implication of Theorem 4 can be obtained from the definition of hierarchical watersheds (Definition 1) and from Property 7. Let $\mathcal{H}$ be a hierarchical watershed of (G, w). By the definition of hierarchical watersheds, we can infer that $\mathcal{H}$ is a sequence $(\mathbf{P}_0, \ldots, \mathbf{P}_{n-1})$ of n partitions, and that the vertices connected by watershed-cut edges are not in a same region of the partition $\mathbf{P}_0$ of $\mathcal{H}$. Hence, we can infer that the range of $\Phi(\mathcal{H})$ is the set $\{0, \ldots, n-1\}$ and that the watershed-cut edges have non-zero weights for $\Phi(\mathcal{H})$, which correspond partially to the conditions 1 and 2 for $\Phi(\mathcal{H})$ to be one-side increasing for an altitude ordering for w. By the statement 3 of the property on extinction maps (Property 6), we can infer that any extinction map is increasing on the regions of a binary partition hierarchy of (G, w). By Property 7, there exists an altitude ordering $\prec$ for w and an extinction map ϵ for $\prec$ such that, for any edge u in $E_\prec$, we have that $\Phi(\mathcal{H})(u)$ is equal to $\min\{\epsilon(R)$ such that R is a child of $R_u\}$. As ϵ is increasing on the regions of $\mathcal{B}_\prec$,

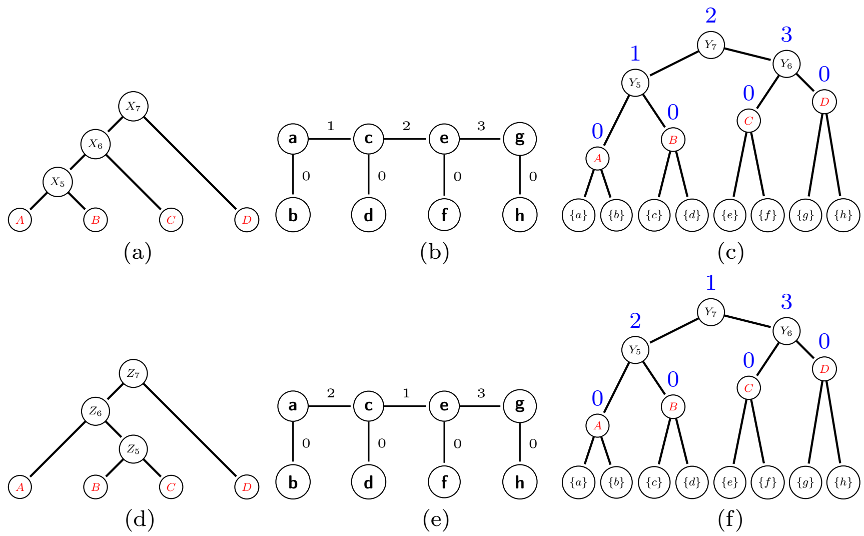


Fig. 3: (a) and (d): The hierarchies $\mathcal{H}$ and $\mathcal{H}'$, respectively. (b) and (e): the weighted graphs $(G, \Phi(\mathcal{H}))$ and $(G, \Phi(\mathcal{H}'))$, respectively. (c) and (f): The maps $\Phi(\mathcal{H})$ and $\Phi(\mathcal{H}')$ represented on the hierarchy $\mathcal{B}$ of Figure 2(c), where, for each edge u, the values $\Phi(\mathcal{H})(u)$ and $\Phi(\mathcal{H}')(u)$ are shown above the region R_u of $\mathcal{B}$.

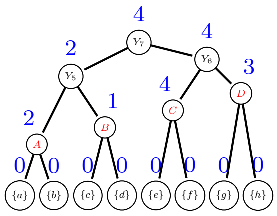


Fig. 4: An extinction map ϵ for the unique altitude ordering of (G, w) of Figure 2(a).

we may say that, for any edge u in $E_\prec$, there is a child R of R_u such that $\Phi(\mathcal{H})(u)$ is greater than the weight of any building edge of the regions included in R, which corresponds to the condition 3 for $\Phi(\mathcal{H})$ to be one-side increasing for $\prec$. The reader can refer to Appendix F for a formal and complete proof of the forward implication of Theorem 4.

In order to present the intuition behind the backward implication of Theorem 4, we introduce the notion of approximated extinction maps. To introduce approximated extinction maps, we first present the auxiliary notions of non-leaf ordering and dominant region.

Definition 8 (non-leaf ordering). *Let $\prec$ be an altitude ordering for w and let f be a map from E into $\mathbb{R}$. Let $\triangledown$ be the supremum descendant map for f and $\prec$. The* non-leaf ordering for f and $\prec$ *is a total ordering $\ll$ on the building edges for $\prec$, such that, for any two building edges u and v for $\prec$, we have $u \ll v$ if $\triangledown(R_u) < \triangledown(R_u)$ or if $\triangledown(R_u) = \triangledown(R_v)$ and $u \prec v$.*

Definition 9 (dominant region). *Let $\prec$ be an altitude ordering for w and let f be a map from E into $\mathbb{R}$. Let $\ll$ be the non-leaf ordering for f and $\prec$. Let R be a non-leaf region of $\mathcal{B}_\prec$ different from V. Let u and v be the building edges of respectively R and the sibling of R. We say that R* is a dominant region for f and $\prec$ *if:*

1. *there is a minimum of w included in R; and*
2. *either:*
   - *$v \ll u$; or*
   - *there is no minimum of w included in the sibling of R.*

For instance, let (G, w) be the weighted graph shown in Figure 2(a), let $\prec$ be the unique altitude ordering for w, let $\mathcal{B}$ be the binary partition hierarchy by $\prec$ of Figure 2(c), and let $\Phi(\mathcal{H})$ be the map of Figure 3(b). Let $\ll$ be the non-leaf ordering for $\Phi(\mathcal{H})$ and $\prec$ such that $\{a,b\} \ll \{c,d\} \ll \{e,f\} \ll \{g,h\} \ll \{a,c\} \ll \{c,e\} \ll \{e,g\}$. The dominant regions of $\mathcal{B}$ for $\Phi(\mathcal{H})$ and $\prec$ are the regions B, D and Y_6.

Definition 10 (approximated extinction map). *Let $\prec$ be an altitude ordering for w and let f be a map from E into $\mathbb{R}$. Let $\triangledown$ be the supremum descendant map for f and $\prec$. The* approximated extinction map for f and $\prec$ *is the map ξ from the set of regions of $\mathcal{B}_\prec$ into $\mathbb{R}$ such that:*

1. *$\xi(R) = \triangledown(R) + 1$ if R is the vertex set V of G; and*
2. *$\xi(R) = \xi(parent(R))$ if R is a dominant region for f and $\prec$; and*
3. *$\xi(R) = f(u)$, where u is the building edge of the parent of R, otherwise.*

The next lemma establishes that the approximated extinction map of any one-side increasing map is indeed an extinction map.

Lemma 11. *Let $\prec$ be an altitude ordering for w and let f be a map from E into $\mathbb{R}$ such that f is one-side increasing for $\prec$. The approximated extinction map for f and $\prec$ is an extinction map for $\prec$.*

For instance, let us consider the weighted graph (G, w) of Figure 2(a) and its unique altitude ordering $\prec$. We can verify that the extinction map ϵ of Figure 4 is precisely the approximated extinction map for $\Phi(\mathcal{H})$ (Figure 3(b)) and $\prec$.

The next lemma is the key result for establishing the backward implication of Theorem 4.

Lemma 12. *Let $\prec$ be an altitude ordering for w and let f be a map from E into $\mathbb{R}$ such that f is one-side increasing for $\prec$. Let ξ be the approximated extinction map for f and $\prec$. Then, for any edge u in $E_\prec$, we have:*

$$f(u) = min\{\xi(R) \text{ such that } R \text{ is a child of } R_u\}.$$

The proof of lemmas 11 and 12 are presented in appendices D and E, respectively. The backward implication of Theorem 4 is a consequence of Lemmas 11 and 12 and the backward implication of Property 7. Let $\mathcal{H}$ be a hierarchy and let $\prec$ be an altitude ordering for w such that $\Phi(\mathcal{H})$ is one-side increasing for $\prec$. Let ξ be the approximated extinction map for $\Phi(\mathcal{H})$ and $\prec$. By Lemma 12, for any edge u in $E_\prec$, we have $\Phi(\mathcal{H})(u) = min\{\xi(R)$ such that R is a child of $R_u\}$. By Lemma 11, the map ξ is an extinction map for $\prec$. Then, by the backward implication of Property 7, we conclude that $\Phi(\mathcal{H})$ is the saliency map of a hierarchical watershed of (G, w) and that $\mathcal{H}$ is a hierarchical watershed of (G, w).

Let (G, w) be the graph of Figure 2(a) and let $\Phi(\mathcal{H})$ be the saliency map of Figure 3(b). Let $\prec$ be the unique altitude ordering of w. As stated previously, $\Phi(\mathcal{H})$ is one-side increasing for $\prec$. To illustrate Lemma 12, we can verify that, for any edge u of G, we have $\Phi(\mathcal{H})(u)$ equal to $\min\{\epsilon(R)$ such that R is a child of $R_u\}$ for any edge u in $E_\prec$ where ϵ (shown in Figure 4) is the approximated extinction map for $\Phi(\mathcal{H})$ and $\prec$.

4 Recognition algorithm for hierarchical watersheds

In this section, we present an efficient algorithm to recognize hierarchical watersheds based on Property 5. Given any hierarchy $\mathcal{H}$ on V, to test if $\mathcal{H}$ is a hierarchical watershed of (G, w), it is sufficient to verify that the saliency map $\Phi(\mathcal{H})(u)$ of $\mathcal{H}$ is one-side increasing for a lexicographic ordering for (w, f).

Algorithm 1 provides a description of our algorithm to recognize hierarchical watersheds. The inputs are a weighted graph $((V, E), w)$ and the saliency map f of a hierarchy $\mathcal{H}$ on V. In this implementation, the edges in E are represented by unique indexes ranging from 0 to $|E| - 1$. The first step (line 1) of Algorithm 1 is to compute a lexicographic ordering $\prec$ for (w, f). Then, the binary partition hierarchy $\mathcal{B}$ by $\prec$ is computed at line 2. Subsequently, the set of minima of w and the watershed-cut edges for $\prec$ are computed at lines 3-9. As established in [25], every minimum of w is a region of $\mathcal{B}$. After computing the set of minima of w, the watershed-cut edges for $\prec$ can be obtained by browsing the hierarchy $\mathcal{B}$ starting from the leaf regions and by iteratively counting the number of minima included in each region of $\mathcal{B}$. At lines 10-16, for each building edge u of $\mathcal{B}$, the value $Max[u]$ is computed, where $Max[u]$ is the maximal value $f(v)$ such that the region R_v is a subset of the region R_u. We can affirm that, for each building edge u of $\mathcal{B}$, the value $Max[u]$ is the supremum descendant value of the region R_u for f and $\prec$. Finally, the last **for** loop (lines $18 - 26$) verifies that the three conditions of Definition 3 for f to be one-side increasing for $\prec$ hold true. If any of those three conditions is not satisfied, then the algorithm halts and returns **false** and, otherwise, it returns **true**.

We will now perform a complexity analysis of Algorithm 1. Given that the lexicographic ordering for (w, f) can be obtained through the merging sort algorithm, the time complexity of this step is $O(|E|log|E|)$. As established in [25], any binary partition hierarchy can be computed in quasi-linear time with respect to $|E|$ provided that the edges in E are already sorted or can be sorted in linear time. More specifically, the time complexity to compute the binary partition hierarchy $\mathcal{B}$ is $O(|E| \times \alpha|V|)$, where α is a slowly growing inverse of the single-valued Ackermann function. Having computed the binary partition hierarchy $\mathcal{B}$, the computation of the minima of w and of the watershed-cut edges for $\prec$ can be performed in linear time with respect to $|V|$ as stated in [25]. At lines $10 - 16$, the array Max can be computed recursively from the leaves to the root in linear time $O(V)$. Finally, each instruction between lines 19 and 26 can be performed in constant time, which implies that the last **for** loop has a linear time complexity with respect to $|V|$. Therefore, the overall time complexity of Algorithm 1 is $O(|E|log|E|)$.

We illustrate Algorithm 1 in Figure 5. The inputs are the weighted graph (G, w) and the saliency map f of Figure 5. We first obtain the lexicographic ordering $\prec$ for (w, f) such that $\{a, b\} \prec \{c, d\} \prec \{e, f\} \prec \{g, h\} \prec \{i, j\} \prec \{a, c\} \prec \{g, i\} \prec \{c, e\} \prec \{d, f\} \prec \{e, g\} \prec \{b, d\} \prec \{f, h\} \prec \{h, j\}$. Then, we obtain the binary partition hierarchy $\mathcal{B}$ by $\prec$, the minima of w (in red) and the four watershed-cut edges of w (underlined). Subsequently, we compute the array Max. For each edge u of G, the value $Max[u]$ is the greatest value in the set $\{f(v) \mid R_v \subseteq R_u\}$. We can verify that the range of f is $\{0, 1, 2, 3, 4\}$ and that, among the building edges for $\prec$, all (and only) the watershed-cut edges for $\prec$ have non-zero weights for f. Therefore, the conditions 1 and 2 of Definition 3 for f to be one-side increasing for $\prec$ hold true. Finally, we test the condition 3 of Definition 3. For each watershed-cut edge u of G, we test if $f(u) \geq Max[v]$ for an edge v such that R_v is a child of R_u. For the building edges of the regions Y_6, Y_7 and Y_8 the condition 3 holds true, but this is not the case for Y_9. Consequently, f is not one-side increasing for $\prec$ and Algorithm 1 returns *false*.

5 Flattened hierarchical watersheds

Let $\mathcal{H}$ be a hierarchy on V. According to Definition 1, in order for $\mathcal{H}$ to be a hierarchical watershed of (G, w), we need $\mathcal{H}$ to have precisely n partitions such that, for any i in $\{1, \ldots, n\}$, the partition $\mathbf{P}_i$ of $\mathcal{H}$ is the result of merging exactly two regions of $\mathbf{P}_{i-1}$. In other words, the hierarchy $\mathcal{H}$ is associated to a total ordering of the minima of w. However, there are cases where hierarchies based on the watershed transform are computed without considering a total ordering on the minima. As an example, we can cite the waterfall algorithm proposed in [3], which can also be formulated in the framework of weighted graphs. At each step of the waterfall algorithm, several catchment basins of the original image can be merged. Therefore, it would be also interesting to characterize any hierarchy which is a "simplified" (flattened) version of a hierarchical watershed, *i.e.*, a hierarchy composed of partitions of a hierarchical watershed.

Definition 13 (flattening of hierarchies [29]). *Let $\mathcal{H}$ and $\mathcal{H}'$ be two hierarchies on V such that any partition of $\mathcal{H}$ is a partition of $\mathcal{H}'$. We say that $\mathcal{H}$ is a* flattening *of $\mathcal{H}'$.*

Let $\mathcal{H}$ and $\mathcal{H}'$ be two hierarchies on V such that $\mathcal{H}$ is a flattening of $\mathcal{H}'$. If $\mathcal{H}'$ is a hierarchical watershed of (G, w), then we say that $\mathcal{H}$ *is a flattened hierarchical watershed of* (G, w). The following property characterizes flattened hierarchical watersheds.

Property 14. *Let $\mathcal{H}$ be a hierarchy on V. The hierarchy $\mathcal{H}$ is a flattened hierarchical watershed of (G, w) if and only if there is an altitude ordering $\prec$ for w such that:*

1. *$(V, E_\prec)$ is a MST of $(G, \Phi(\mathcal{H}))$; and*

Algorithm 1 Recognition of hierarchical watersheds

Data: $((V, E), w)$: a weighted graph
f: the saliency map of a hierarchy $\mathcal{H}$ on V
Result: *true* if $\mathcal{H}$ is a hierarchical watershed of (G, w) and *false* otherwise
// In this algorithm, we consider that the default value of any array position is zero
Compute a lexicographic order $\prec$ for (w, f)
Compute the binary partition hierarchy $\mathcal{B}$ by $\prec$ using the algorithm proposed in [25]
Compute the regions of $\mathcal{B}$ which are minima of w using the algorithm proposed in [25]
// Computation of the array WS of watershed edges for $\prec$ and the number k of watershed-cut edges for $\prec$
Declare WS as an array of $|E|$ integers
$k := 0$
for each building edge u in increasing order for $\prec$ **do**
  if both children of R_u include at least one minimum of w **then**
    $WS[u] := 1$
    $k := k + 1$
// Computation of the array Max such that, for any building edge u for $\prec$, we have $Max[u]$ equal to the supremum descendant value of u for f and $\prec$
Declare Max as an array of $|E|$ real numbers
for each building edge u in increasing order for $\prec$ **do**
  $Max[u] := f[u]$
  for each child X of R_u **do**
    if X is not a leaf node of $\mathcal{B}$ **then**
      $v :=$ the building edge of X
      $Max[u] := max(Max[u], Max[v])$
// Testing of the conditions 1, 2 and 3 of Definition 3 for f to be one-side increasing for $\prec$
Declare *range* as an array of $|E|$ integers
for each building edge u for $\prec$ **do**
  if $f[u] \notin \{0, 1, \ldots, k\}$ **then**
    return *false*
  if $f[u] \neq 0$ and $range[f[u]] \neq 0$ **then**
    return *false*
  $range[f[u]] := 1$
  if $(WS[u] = 0$ and $f[u] \neq 0)$ or $(WS[u] = 1$ and $f[u] = 0)$ **then**
    return *false*
  if both children of R_u are non-leaf regions of $\mathcal{B}$ **then**
    $v_1 :=$ *the building edge of a child of* u
    $v_2 :=$ *the building edge of* $sibling(R_{v_1})$
    if $f[u] < Max[v_1]$ and $f[u] < Max[v_2]$ **then**
      return *false*
return *true*

2. *for any edge u in $E_\prec$, if u is not a watershed-cut edge for $\prec$, then $\Phi(\mathcal{H})(u)$ is zero; and*
3. *for any edge u in $E_\prec$, there exists a child R of R_u such that $f(u)$ is greater than or equal to $\bigtriangledown(R)$, where $\bigtriangledown$ is the supremum descendant map for $\Phi(\mathcal{H})$ and $\prec$.*

Algorithm 2 describes our algorithm to recognize flattened hierarchical watersheds, which is very similar to the algorithm to recognize hierarchical watersheds

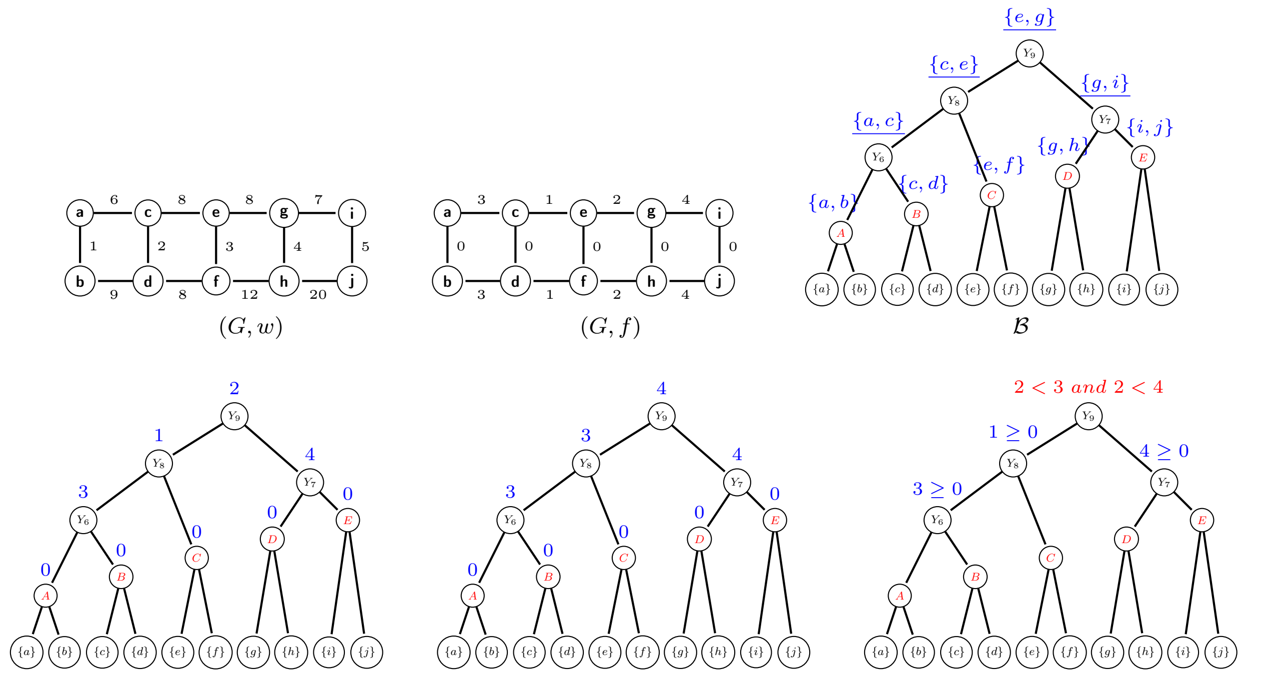


Fig. 5: Toy example of Algorithm 1. Given the weighted graphs (G, w) and (G, f), we test if f is the saliency map of a hierarchical watershed of (G, w). We first compute the lexicographic ordering $\prec$ for (w, f) such that $\{a, b\} \prec \{c, d\} \prec \{e, f\} \prec \{g, h\} \prec \{i, j\} \prec \{a, c\} \prec \{g, i\} \prec \{c, e\} \prec \{d, f\} \prec \{e, g\} \prec \{b, d\} \prec \{f, h\} \prec \{h, j\}$. Then, we obtain the binary partition hierarchy $\mathcal{B}_\prec$ by $\prec$, along with the minima of w (in red) and the watershed-cut edges for $\mathcal{B}$ (underlined). Subsequently, we obtain the array Max as detailed in Algorithm 1. We may conclude that conditions 1 and 2 of Definition 3 hold true for f, but not the condition 3. Hence, f is not the saliency map of a hierarchical watershed of (G, w).

(Algorithm 1). The only difference between algorithms 2 and 1 is that, in Algorithm 2, we do not test if the first condition of Definition 3 holds true, and we test if $(V, E_\prec)$ is a MST of the input map (G, f). The verification that $(V, E_\prec)$ is a MST of (G, f) can be done in time $O(|E|log(|E|))$ by checking if the weight of a MST of (G, f) is equal to the weight of $(V, E_\prec)$.

Algorithm 2 Recognition of flattened hierarchical watersheds

Data: $((V, E), w)$: a weighted graph
f: the saliency map of a hierarchy $\mathcal{H}$ on V
Result: *true* if $\mathcal{H}$ is a flattened hierarchical watershed of (G, w) and *false* otherwise

```
/* Lines 1 − 16 of Algorithm 1 */
// The ordering ≺, the binary partition hierarchy B and the arrays WS and Max are defined as in Algorithm 1
// Testing of the conditions 1, 2 and 3 of Property 14 for f to be a flattened hierarchical watershed of ((V,E),w)
if (V, E≺)) is not a MST of ((V,E), f) then
    return false
for each building edge u for ≺ do
    if (WS[u] = 0 and f[u] ≠ 0) then
        return false
    if both children of R_u are non-leaf regions of B then
        v1 := the building edge of a child of u
        v2 := the building edge of sibling(R_v1)
        if f[u] ≤ Max[v1] and f[u] ≤ Max[v2] then
            return false
return true
```

6 Experimental results

An immediate application of the recognition of (flattened) hierarchical watersheds is on the combinations of hierarchical watersheds assessed in [15]. In [15], the authors showed that combining hierarchies is a good alternative method to outperform individual hierarchical watersheds, which raises the question of whether the resulting combinations are hierarchical watersheds or flattened hierarchical watersheds.

In [15], the authors combine hierarchies through their saliency maps using four functions: infimum, supremum, linear combination and concatenation. As established in the companion paper [29], combinations of hierarchical watersheds with the aforementioned functions are not hierarchical watersheds in general. Indeed, by applying Algorithm 1 to the combinations of hierarchical watersheds assessed in [15], we verified that the first condition of the definition of one-side increasing maps (Definition 3) is not satisfied by any combination. Hence, by Property 5, none of those combinations is a hierarchical watershed. In fact, combining hierarchies often act by simplifying the input hierarchies in the sense that, from a level i to a level $i+1$ of the resulting combination, zero or more than one pair of regions are merged, which suggests that combinations may result in flattened hierarchical watersheds.

The hierarchical watersheds assessed in [15] are based on the following attributes: area [22,33], diagonal of bounding box (DBB) [31], dynamics [20], (topological) height [31], number of descendants, number of minima and volume [33]. In our experiments, we also included a novel attribute based on the number of parent nodes introduced in [27]. We applied Algorithm 2 to combinations of pairs of hierarchical watersheds based on each of the aforementioned attributes using the functions average, supremum and infimum. The experiments were performed on the 200 images of the test set of the Berkeley Segmentation Dataset and Benchmark [18]. The results are shown in Table 1. In each cell of Table 1, we present the number of combinations with respectively average, supremum and infimum (among 200) that are flattened hierarchical watersheds. We can observe that the majority of the combinations with average and supremum are flattened hierarchical watersheds, which is not the case for combinations with infimum.

$\mathcal{H}_1$ | $\mathcal{H}_2$	Area	DBB	Dyn	Height	Desc	Min	Vol	Parent
Area	-	138	125	135	152	59	113	79
	-	200	194	194	200	183	198	182
	-	119	113	124	127	47	60	62
DBB	-	-	127	134	136	62	119	82
	-	-	195	197	200	184	198	183
	-	-	115	122	113	48	102	65
Dyn	-	-	-	117	124	104	126	105
	-	-	-	195	195	189	196	192
	-	-	-	91	111	96	112	100
Height	-	-	-	-	134	108	128	110
	-	-	-	-	195	185	194	186
	-	-	-	-	123	99	106	97
Desc	-	-	-	-	-	63	114	83
	-	-	-	-	-	185	199	180
	-	-	-	-	-	52	98	65
Min	-	-	-	-	-	-	66	171
	-	-	-	-	-	-	179	199
	-	-	-	-	-	-	53	158
Vol	-	-	-	-	-	-	-	80
	-	-	-	-	-	-	-	177
	-	-	-	-	-	-	-	66
Parent	-	-	-	-	-	-	-	-
	-	-	-	-	-	-	-	-
	-	-	-	-	-	-	-	-

Table 1: In each cell, we show the number of combinations of hierarchical watersheds with average (red), supremum (blue) and infimum (black) among 200 that are flattened hierarchical watersheds.

Given any two hierarchical watersheds $\mathcal{H}_1$ and $\mathcal{H}_2$ computed from the same image gradient g and based on distinct attributes, we cannot guarantee that the saliency maps of $\mathcal{H}_1$ and $\mathcal{H}_2$ are one-side increasing for a same altitude ordering for g. However, if that were the case, the resulting combination of $\mathcal{H}_1$ and $\mathcal{H}_2$ with infimum would be a flattened hierarchical watershed, as established in [29]. By applying Algorithm 2 to combinations of saliency maps that are one-side increasing for a same altitude ordering, we observed that all combinations with infimum are flattened hierarchical watersheds as expected. Interestingly, this was also the case for the combinations with average. Regarding the combinations with supremum, among all 5600 combinations, only one combination with volume and diagonal of bounding box, and three combinations with volume and height were not flattened hierarchical watersheds.

Our experimental results suggest that most of the combinations of hierarchical watersheds assessed in [15] are "approximations" of flattened hierarchical watersheds in the sense that, by swapping the weight of a few edges in the combinations of saliency maps, we could obtain a flattened hierarchical watershed. This speculative conclusion may be investigated in future research linking the results established here with the method to convert any hierarchy into a hierarchical watershed introduced in [17].

7 Conclusion

In this article, we aimed at solving the problem of recognition of hierarchical watersheds. We generalized the characterization of hierarchical watersheds proposed in [16] to arbitrary graphs and, based on this characterization, we designed an efficient algorithm to determine if a hierarchy is a hierarchical watershed of any given weighted graph. Knowing that hierarchical watersheds are associated to total orderings on the minima of a graph and that this is not the case for other relevant hierarchical segmentation methods based on the watershed transform, we introduced the notion of flattened hierarchical watersheds, which is a relaxed definition of hierarchical watersheds. Then, we presented experimental results with the combinations of hierarchical watershed assessed in [15]. We concluded that none of those combinations are hierarchical watersheds but most of them are flattened hierarchical watersheds.

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A Proof of Property 7

(**Property 7**). *Let $\mathcal{H}$ be a hierarchy on V. The hierarchy $\mathcal{H}$ is a hierarchical watershed of (G, w) if and only if there exists an altitude ordering $\prec$ for w and an extinction map ϵ for $\prec$ such that*

1. *$(V, E_\prec)$ is a MST of $(G, \Phi(\mathcal{H}))$; and*
2. *for any edge u in $E_\prec$, we have: $\Phi(\mathcal{H})(u) = \min\{\epsilon(R)$ such that R is a child of $R_u\}$.*

To prove Property 7, we first present a result established in [10] and other auxiliary lemmas.

Let $\prec$ be an altitude ordering for w, let $\mathcal{B}_\prec$ be the binary partition hierarchy by $\prec$ and let $\mathcal{S} = (M_1, \ldots, M_n)$ be a sequence of minima of w. Let u be a building edge for $\prec$ and let X be the region of $\mathcal{B}_\prec$ whose building edge is u. The *persistence value of u (for $\prec$ and $\mathcal{S}$)* is the minimum of the extinction values of the children of X. Let ρ be the map from the building edges for $\prec$ into $\mathbb{R}$ such that, for any building edge u for $\prec$, $\rho(u)$ is the persistence value of u. We say that *ρ is the persistence map (for $\prec$ and $\mathcal{S}$)*. We denote by B_i the set of building edges for $\prec$ whose persistence value is lower than or equal to i.

Definition 15. (hierarchy induced by an altitude ordering and a sequence of minima [10]) *Let $\prec$ be an altitude ordering for w, let $\mathcal{S} = (M_1, \ldots, M_n)$ be a sequence of minima of w and let ρ be the persistence map for $\prec$ and $\mathcal{S}$. The sequence of partitions $(CC(V, B_0), \ldots, CC(V, B_{n-1}))$ is a hierarchy called* the hierarchy induced by $\prec$ and $\mathcal{S}$.

Lemma 16 (Property 12 of [10]). *Let $\mathcal{S} = (M_1, \ldots, M_n)$ be a sequence of minima of w and let $\mathcal{H}$ be a hierarchy on V. The hierarchy $\mathcal{H}$ is a hierarchical watershed of (G, w) for $\mathcal{S}$ if and only if there exists an altitude ordering $\prec$ such that $\mathcal{H}$ is the hierarchy induced by $\prec$ and $\mathcal{S}$.*

Lemma 17. *Let $\prec$ be an altitude ordering for w and let ϵ be an extinction map for $\prec$. Let X and Y be two regions of $\mathcal{B}_\prec$. If $X \subseteq Y$, then $\epsilon(X) \leq \epsilon(Y)$.*

Proof *Since $\mathcal{B}_\prec$ is a hierarchy, we can affirm that, for any two regions Y and Z of $\mathcal{B}_\prec$, if $Y \subseteq Z$, then all minima of w included in Y are also included in Z and, therefore, $\epsilon(Y) \leq \epsilon(Z)$.* □

From the results established in [25], we can state the following lemma.

Lemma 18. *Let $\mathcal{B}$ be a binary partition hierarchy of (G, w). Then, any minimum of w is a region of $\mathcal{B}$.*

Lemma 19. *Let $\prec$ be an altitude ordering on the edges of G for w, let $\mathcal{S} = (M_1, \ldots, M_n)$ be a sequence of minima of w and let ρ be the persistence map for $\prec$ and $\mathcal{S}$. The range of ρ is $\{0, \ldots, n-1\}$.*

Proof *Let ϵ be the extinction map for $\prec$ and $\mathcal{S}$. We will prove that (1) for any building edge u for $\prec$, $\rho(u)$ is in $\{0, \ldots, n-1\}$, and that, (2) for any i in $\{0, \ldots, n-1\}$, there is a building edge u for $\prec$ such that $\rho(u) = i$.*

1. *$\{0, \ldots, n-1\} \subseteq range(\rho)$. First, we prove that 0 is in $range(\rho)$. By Property 6, there is a region X of $\mathcal{B}_\prec$ whose extinction value is zero. Therefore, the persistence value of the building edge u of the parent of X is equal to zero: $\rho(u) = 0$. Now, we will prove that any i in $\{1, \ldots, n-1\}$ is in $range(\rho)$. Let i be a value in $\{1, \ldots, n-1\}$. By Lemma 18, the minimum M_i is a region of $\mathcal{B}_\prec$. Then, there is a region of $\mathcal{B}_\prec$ whose extinction value is i. Let X be the largest region of $\mathcal{B}_\prec$ whose extinction value is i. We can say that $X \neq V$ because M_n is included in V and, therefore, $\epsilon(V) = n$. Let Z be the parent of X. We can infer that the extinction value $\epsilon(Z)$ of Z is strictly greater than i. Therefore, there is a minimum M_j with $j > i$ included in the sibling of X. Hence, the extinction value of $sibling(X)$ is also strictly greater than i. Then, the persistence value of the building edge of Z, being the minimum of the extinction value of its children, is i.*
2. *$range(\rho) \subseteq \{0, \ldots, n-1\}$. Let u be an edge in $E_\prec$. By Property 6 (statement 1), and as the persistence value of u is equal to the extinction value of a child of R_u, we have that $\rho(u)$ is in $\{0, \ldots, n\}$. Moreover, the persistence value $\rho(u)$ of u is lower than n because, if the extinction value of one child X of R_u is n, then the minimum M_n is included in X and M_n is not included in $sibling(X)$, which implies that the extinction value of $sibling(X)$ is strictly lower than n. Therefore, since $\rho(u) = min\{\epsilon(X), \epsilon(sibling(X))\}$, the persistence value of u is strictly lower than n. Thus, we have that $range(\rho) \subseteq \{0, \ldots, n-1\}$.* □

Lemma 20. *Let $\prec$ be an altitude ordering for w, let $\mathcal{S} = (M_1, \ldots, M_n)$ be a sequence of minima of w and let ρ be the persistence map for $\prec$ and $\mathcal{S}$. Let $\mathcal{H}$ be the hierarchy induced by $\prec$ and $\mathcal{S}$. For any edge u in $E_\prec$, we have $\Phi(\mathcal{H})(u) = \rho(u)$.*

Proof *By Definition 15, the hierarchy $\mathcal{H}$ is the sequence $(CC(V, B_0), \ldots, CC(V, B_{n-1}))$ such that, for any i in $\{0, \ldots, n-1\}$, B_i is the set of building edges for $\prec$ whose persistence values are lower than or equal to i. Let $u = \{x, y\}$ be a building edge for $\prec$ and let i be the persistence value of u. We can say that x and y are in the same region of $CC(V, B_i)$ but in distinct regions of $CC(V, B_{i-1})$ if $i \neq 0$. Therefore, since $CC(V, B_i)$ is the i-th partition of $\mathcal{H}$, by the definition of saliency maps, we have $\Phi(\mathcal{H})(u) = i$.* □

The following lemma, established in [9], links MSTs and QFZ hierarchies.

Lemma 21 (Theorem 4 of [9]). *A subgraph G' of G is a MST of (G, w) if and only if:*
1. *the QFZ hierarchy of G' and G are the same; and*
2. *the graph G' is minimal for statement 1,* i.e., *for any subgraph G'' of G', if the quasi-flat zone hierarchy of G'' for w is the one of G for w, then we have $G'' = G'$.*

Lemma 22. *Let $\prec$ be an altitude ordering for w and let $\mathcal{S} = (M_1, \ldots, M_n)$ be a sequence of minima of w. Let $\mathcal{H}$ be the hierarchy induced by $\prec$ and $\mathcal{S}$. Then $(V, E_\prec)$ is a MST of $(G, \Phi(\mathcal{H}))$.*

Proof *Let α denote the sum of the weight of the edges in $E_\prec$ in the map $\Phi(\mathcal{H})$: $\alpha = \sum_{e \in E_\prec} \Phi(\mathcal{H})(e)$. Let ρ be the persistence map for $\prec$ and $\mathcal{S}$. By Lemma 20, we can affirm that, for any edge u in $E_\prec$, we have $\Phi(\mathcal{H})(u) = \rho(u)$. Hence, we have $\alpha = \sum_{e \in E_\prec} \rho(e)$. We will first prove that α is precisely $0 + 1 + \cdots + n - 1$. We know that, for any edge u in $E_\prec$:*
1. *if u is a watershed-cut edge for $\prec$, then each child of R_u contains at least one minimum of w. Therefore, the extinction values of both children of R_u is non-zero, and, consequently, the persistence value $\rho(u)$ of u is non-zero.*
2. *otherwise, if u is not a watershed-cut edge for $\prec$, then there exists a child X of R_u such that there is no minimum of w included in X. Therefore, the extinction value of X is zero. Since the extinction value of $sibling(X)$ is at least zero by Lemma 35 (statement 1), the persistence value $\rho(u)$ of u, being the minimum between the extinction values of X and $sibling(X)$, is also zero.*

Hence, since there are $n-1$ watershed-cut edges for $\prec$, and since only the watershed-cut edges for $\prec$ have non-zero persistence values, we can conclude that, for any i in $\{1,\dots,n-1\}$, there is exactly one edge u in $E_\prec$ such that $\rho(u)=i$. Hence, $\alpha=\sum_{e\in E_\prec}\rho(e)=0+1+\cdots+n-1$.

Now, in order to prove that $(V,E_\prec)$ is a MST of $(G,\Phi(\mathcal{H}))$, we will prove that, for any MST G' of $(G,\Phi(\mathcal{H}))$, the sum of the weight of the edges in G' is greater than or equal to α. Let G' be a MST of $(G,\Phi(\mathcal{H}))$. As G' is a MST of $(G,\Phi(\mathcal{H}))$, by the condition 1 of Lemma 21, we have that G and G' have the same quasi-flat zones hierarchies: $\mathcal{QFZ}(G,\Phi(\mathcal{H}))=\mathcal{QFZ}(G',\Phi(\mathcal{H}))$. As $\Phi(\mathcal{H})$ is the saliency map of $\mathcal{H}$, we have that $\mathcal{H}=\mathcal{QFZ}(G,\Phi(\mathcal{H}))$. Therefore, $\mathcal{H}=\mathcal{QFZ}(G',\Phi(\mathcal{H}))$. Let i be a value in $\{1,\dots,n-1\}$. Since $\sum_{e\in E_\prec}\Phi(\mathcal{H})(e)=0+1+\cdots+n-1$, we can say that $\{1,\dots,n-1\}$ is a subset of the range of $\Phi(\mathcal{H})$. Therefore, $\mathcal{H}$ is composed of at least n distinct partitions. Let $\mathcal{H}$ be the sequence $(\mathbf{P}_0,\dots,\mathbf{P}_{n-1},\dots)$. Since the partitions $\mathbf{P}_i$ and $\mathbf{P}_{i-1}$ are distinct, then there exists a region in $\mathbf{P}_i$ which is not in $\mathbf{P}_{i-1}$. Therefore, there is a region X of $\mathbf{P}_i$ which is composed of several regions $\{R_1,R_2,\dots\}$ of $\mathbf{P}_{i-1}$. Then, there are two adjacent vertices x and y such that x and y are in distinct regions in $\{R_1,R_2,\dots\}$. Let x and y be two adjacent vertices such that x and y are in distinct regions in $\{R_1,R_2,\dots\}$. Hence, the lowest j such that x and y belong to the same region of $\mathbf{P}_j$ is i. Thus, there exists an edge $u=\{x,y\}$ in $E_\prec$ such that $\Phi(\mathcal{H})(u)=i$. Hence, the sum of the weight of the edges of G' is at least $1+\cdots+n-1$, which is equal to α. Therefore, the graph $(V,E_\prec)$ is a MST of $(G,\Phi(\mathcal{H}))$. □

Proof (of Property 7) We first prove the forward implication of this property. Let $\mathcal{H}$ be a hierarchical watershed of (G,w). Then there is a sequence $\mathcal{S}$ of minima of w such that $\mathcal{H}$ is the hierarchical watershed of (G,w) for $\mathcal{S}$. Let $\mathcal{S}$ be the sequence of minima of w such that $\mathcal{H}$ is the hierarchical watershed of (G,w) for $\mathcal{S}$. By Lemma 16, there is an altitude ordering $\prec$ such that $\mathcal{H}$ is the hierarchy induced by $\prec$ and $\mathcal{S}$. Let $\prec$ be an altitude ordering such that $\mathcal{H}$ is the hierarchy induced by $\prec$ and $\mathcal{S}$. Then, by Lemma 22, $(V,E_\prec)$ is a MST of $(G,\Phi(\mathcal{H}))$. We will now prove the second statement of Property 7. By Lemma 20, for any edge u in $E_\prec$, $\Phi(\mathcal{H})(u)$ is equal to the persistence value $\rho(u)$ of u for $\prec$ and $\mathcal{S}$. By the definition of persistence values, for edge u in $E_\prec$, the persistence value of u for $\prec$ and $\mathcal{S}$ is the minimum extinction value of the children of R_u. Therefore, we can conclude that, for edge u in $E_\prec$, $\Phi(\mathcal{H})(u)=\min\{\epsilon(R)$ such that R is a child of $R_u\}$, where ϵ is the extinction map for $\prec$ and $\mathcal{S}$. Hence, there exists an extinction map ϵ such that, for edge u in $E_\prec$, $\Phi(\mathcal{H})(u)=\min\{\epsilon(R)$ such that R is a child of $R_u\}$.

We will now prove the backward implication of Property 7. Let $\mathcal{H}$ be a hierarchy on V such that there exists an altitude ordering $\prec$ for w and an extinction map ϵ for $\prec$ such that:

1. $(V,E_\prec)$ is a MST of $(G,\Phi(\mathcal{H}))$; and
2. for any edge u in $E_\prec$, we have: $\Phi(\mathcal{H})(u)=\min\{\epsilon(R)$ such that R is a child of $R_u\}$.

Let G' denote the graph $(V,E_\prec)$. By Lemma 21 (statement 1), as G' is a MST of $(G,\Phi(\mathcal{H}))$, we have that G' and G have the same quasi-flat zones hierarchies (for $\Phi(\mathcal{H})$): $\mathcal{QFZ}(G',\Phi(\mathcal{H}))=\mathcal{QFZ}(G,\Phi(\mathcal{H}))$. Let ρ be the persistence map for $\prec$ and $\mathcal{S}$. By the definition of persistence values, we can affirm that, for any edge u in $E_\prec$, we have $\Phi(\mathcal{H})(u)=\rho(u)$. Hence, we can say that $\mathcal{QFZ}(G',\Phi(\mathcal{H}))=\mathcal{QFZ}(G',\rho))$. Let $\mathcal{H}'$ be the hierarchy induced by $\prec$ and $\mathcal{S}$. By Lemma 22, G' is a MST of $(G,\Phi(\mathcal{H}'))$. Hence, by Lemma 21, G' and G have the same quasi-flat zones hierarchies (for $\Phi(\mathcal{H}')$): $\mathcal{QFZ}(G',\Phi(\mathcal{H}'))=\mathcal{QFZ}(G,\Phi(\mathcal{H}'))$. By Lemma 20, for edge u in $E_\prec$, we have $\Phi(\mathcal{H}')(u)=\rho(u)$, which is equal to $\Phi(\mathcal{H})(u)$ as stated previously. Thus, $\mathcal{QFZ}(G',\Phi(\mathcal{H}'))=\mathcal{QFZ}(G',\Phi(\mathcal{H}))$ and, consequently, $\mathcal{H}$ and $\mathcal{H}'$ are equal. By Lemma 16, $\mathcal{H}'$ is a hierarchical watershed of (G,w). Therefore, $\mathcal{H}$ is a hierarchical watershed of (G,w). □

B Proof of Property 5

(**Property 5**). *Let $\mathcal{H}$ be a hierarchy on V and let $\prec$ be a lexicographic ordering for (w,f). The hierarchy $\mathcal{H}$ is a hierarchical watershed of (G,w) if and only if $\Phi(\mathcal{H})$ is one-side increasing for $\prec$.*

Let $\mathcal{H}$ be a hierarchy on V. By Theorem 4, $\mathcal{H}$ is a hierarchical watershed of (G,w) if and only if there is an altitude ordering for w such that the saliency map $\Phi(\mathcal{H})$ of $\mathcal{H}$ is one-side increasing for $\prec$. In order to prove Property 5, we will prove in the following lemma that, if the saliency map $\Phi(\mathcal{H})$ is one-side increasing for an altitude ordering for w, then $\Phi(\mathcal{H})$ is one-side increasing for any lexicographic ordering for $(w,\Phi(\mathcal{H}))$.

Given a map f from E into $\mathbb{R}$, we say that f *is a saliency map* if there is an hierarchy $\mathcal{H}$ on V such that f is the saliency map of $\mathcal{H}$.

Lemma 23. *Let f be a saliency map and let $\prec_f$ be a lexicographic ordering for (w,f). If there exists an altitude ordering $\prec$ for w such that f is one-side increasing for $\prec$, then f is one-side increasing for $\prec_f$.*

Let $\prec$ be an ordering on E and let $(u_1,\dots,u_{|E|})$ be the sequence of edges in E such that, for any i in $\{1,\dots,|E|-1\}$, we have $u_i\prec u_{i+1}$. This sequence $(u_1,\dots,u_{|E|})$ is called the *sequence (of edges) induced by* $\prec$. In order to prove Lemma B, we first introduce the notion of *critical rank* and the notion of *switch* in the context of lexicographic orderings, and other auxiliary lemmas.

Definition 24 (critical rank). *Let f be a saliency map and let $\prec$ be an altitude ordering for w. Let $(u_1,\dots,u_{|E|})$ be the sequence induced by $\prec$. Let k be a value such that $u_k\prec u_{k+1}$ and such that $w(u_k)=w(u_{k+1})$ and $f(u_k)\geq f(u_{k+1})$. We say that k* is a critical rank for f and $\prec$.

Definition 25 (switch). *Let f be a saliency map and let $\prec$ be an altitude ordering for w. Let $(u_1,\dots,u_{|E|})$ be the sequence induced by $\prec$. Let k be a critical rank for f and $\prec$, and let $\prec_k$ be the ordering such that $(u_1,\dots,u_{k+1},u_k,\dots,u_{|E|})$ is the sequence induced by $\prec_k$. We say that $\prec_k$ is a switch of $\prec$ for f (and k).*

Lemma 26. *Let f be a saliency map, let $\prec$ be an altitude ordering for w and let $\prec'$ be a switch of $\prec$ for f. Then $\prec'$ is an altitude ordering for w.*

Proof Let $\prec'$ be the switch of $\prec$ for a critical rank k for f and $\prec$. Let $(u_1,\dots,u_{|E|})$ be the sequence induced by $\prec$. Then $(u_1,\dots,u_{k+1},u_k,\dots,u_{|E|})$ is the sequence induced by $\prec'$. We may affirm that, for any edge v different from u_{k+1}, if $v\prec u_k$ (resp. $u_k\prec v$) then $v\prec' u_k$ (resp. $u_k\prec' v$). Similarly, for any edge v different from u_k, if $v\prec u_{k+1}$ (resp. $u_{k+1}\prec v$) then $v\prec' u_{k+1}$ ($u_{k+1}\prec' v$). Finally, for any two edges u and v such that $\{u,v\}\cap\{u_k,u_{k+1}\}=\emptyset$, if $u\prec v$ (resp. $v\prec u$), then $u\prec' v$ (resp. $v\prec' u$). Hence, for any two edges u and v such that $w(u)<w(v)$, by the definition

of critical rank, we may say that $\{u, v\} \neq \{u_k, u_{k+1}\}$ *and, consequently, as* $u \prec v$*, then* $u \prec' v$*. Hence,* $\prec'$ *is an altitude ordering for* w*.* □

Lemma 27. *Let* $\prec$ *be an altitude ordering for* w *and let* f *be a saliency map. Let* $\prec'$ *be a lexicographic ordering for* (w, f)*. There exists a sequence* $(\prec_0, \prec_1, \ldots, \prec_\ell)$ *of altitude orderings for* w *such that* $\prec_0$ *is equal to* $\prec$*,* $\prec_\ell$ *is equal to* $\prec'$ *and, for any* i *in* $\{1, \ldots, \ell\}$*,* $\prec_i$ *is a switch of* $\prec_{i-1}$*.*

Proof Let $(u_1, \ldots, u_{|E|})$ *be the sequence induced by* $\prec$ *and let* $(u'_1, \ldots, u'_{|E|})$ *be the sequence induced by* $\prec'$*. Let* k *be the smallest value such that* $u_k \neq u'_k$*. In this case, there is an* $i > k$ *such that* $u'_k = u_i$*. As* $\prec'$ *is a lexicographic ordering for* (w, f)*, for any edge* u_j *such that* $k < j \leq i$*, we have* $f(u_j) \geq f(u_{j-1})$*. Hence, there is a sequence* S *of switches for critical ranks ranging from* $i - 1$ *to* k *such that, in the last ordering* $\prec^*$ *of the sequence* S*, the edge with rank* k *for the ordering* $\prec^*$ *is precisely the edge* u'_k*. Let* $(u^*_1, \ldots, u^*_{|E|})$ *be the sequence induced by* $\prec^*$*. We conclude that, for any* $i \leq k$*, we have* $u^*_k = u'_k$*. Hence, the smallest value* m *such that* $u^*_m \neq u'_m$ *is strictly greater than* k*. By performing this procedure iteratively (like the bubble sort algorithm), the resulting ordering converge to* $\prec'$*.* □

Lemma 28. *Let* $\prec$ *be an altitude ordering for* w *and let* f *be a saliency map such that* f *is one-side increasing for* $\prec$*. Let* v_1 *and* v_2 *be two edges of* E*. If* $f(v_1)$ *is equal to* $f(v_2)$*, then neither* v_1 *nor* v_2 *is a watershed-cut edge for* $\prec$*.*

Proof Since f *is one-side increasing for* $\prec$*, by Definition 3, we have* $\{f(u) \mid u \in E_\prec\} = \{0, \ldots, n-1\}$ *and we have that, for any edge* u *in* $E_\prec$*,* $f(u)$ *is greater than* 0 *if and only if* u *is a watershed-cut edge for* $\prec$*. Since* w *has* n *minima, there are* $n-1$ *watershed-cut edges for* $\prec$*. Hence, the watershed-cut edges for* $\prec$ *have pairwise distinct edge weights ranging from* 1 *to* $n - 1$*. Therefore, neither* v_1 *nor* v_2 *is a watershed-cut edge for* $\prec$*.*

Let $\prec$ be an altitude ordering for w and let f be a saliency map such that f is one-side increasing for $\prec$. By Lemma 26, every switch of $\prec$ is an altitude ordering for w. By Lemma 27, any lexicographic ordering for (w, f) can be obtained by a sequence of switches starting from $\prec$. Hence, to prove Lemma , we can simply prove that f is one-side increasing for any switch of $\prec$. Let $(u_1, \ldots, u_{|E|})$ be the sequence induced by $\prec$. Then $(u_1, \ldots, u_{k+1}, u_k, \ldots, u_{|E|})$ is the sequence induced by $\prec'$. In order to prove that f is one-side increasing for the switch $\prec'$ for k, we should consider the following cases:

1. Neither u_k nor u_{k+1} is a building edge for $\prec$;
2. Both u_k and u_{k+1} are building edges for $\prec$ and $R_{u_k} \cap R_{u_{k+1}} = \emptyset$;
3. Both u_k and u_{k+1} are building edges for $\prec$ and $R_{u_k} \subset R_{u_{k+1}}$;
4. Only u_{k+1} is a building edge for $\prec$; and
5. Only u_k is a building edge for $\prec$.

The following lemmas 30, 31, 32, 33 and **??** prove that, for each of those five cases, the saliency map f is one-side increasing for the switch $\prec'$ for k. Before considering those five cases, we first present the following auxiliary lemma.

Lemma 29. *Let* $\prec$ *be an altitude ordering for* w *and let* f *be a saliency map such that* f *is one-side increasing for* $\prec$*. Let* $\prec'$ *be an altitude ordering for* w *such that the set of building edges for* $\prec'$ *is equal to the set of building edges for* $\prec$ *and such that the set of regions of* $\mathcal{B}_\prec$ *is equal to the set of regions of* $\mathcal{B}_{\prec'}$*. Then* f *is one-side increasing for* $\prec'$*.*

Proof In the definition of one-side increasing maps (Definition 3), the three conditions for f *to be one-side increasing for* $\prec$ *take into consideration only the weight of the building edges for* $\prec$ *and parenthood relationship between the regions of* $\prec$*. Hence, as the set of building edges for* $\prec'$ *is the same set of building edges for* $\prec$ *and as they have the same set of regions, we can conclude that the three conditions of Definition 3 for* f *to the one-side increasing for* $\prec'$ *are satisfied.* □

Lemma 30. *Let* $\prec$ *be an altitude ordering for* w *and let* f *be a saliency map such that* f *is one-side increasing for* $\prec$*. Let* $(u_1, \ldots, u_{|E|})$ *be the sequence induced by* $\prec$*. Let* k *be a critical rank for* f *and* $\prec$ *such that neither* u_k *nor* u_{k+1} *is a building edge for* $\prec$*. Then* f *is one-side increasing for the switch* $\prec'$ *for* k*.*

Proof Let $(\mathbf{B}_0, \mathbf{B}_1, \ldots, \mathbf{B}_{|E|})$ *be the sequence of partitions (of* V*) such that, for any* i *in* $\{1, \ldots, |E|\}$*, the partition* $\mathbf{B}_i$ *is the* i*-partition by the ordering* $\prec$ *(as defined in Section 3.1). Let* $(\mathbf{B}'_0, \mathbf{B}'_1, \ldots, \mathbf{B}'_{|E|})$ *be the sequence of partitions such that, for any* i *in* $\{1, \ldots, |E|\}$*, the partition* $\mathbf{B}'_i$ *is the* i*-partition by the ordering* $\prec'$*.*

By the definition of binary partition hierarchy and, as neither u_k *nor* u_{k+1} *is a building edge for* $\prec$*, we may say that:*

I the partition $\mathbf{B}_k$ *is equal to the partition* $\mathbf{B}_{k-1}$*, and*
II the partition $\mathbf{B}_{k+1}$ *is equal to the partition* $\mathbf{B}_k$*,*
III which implies that $\mathbf{B}_{k-1} = \mathbf{B}_k = \mathbf{B}_{k+1}$*.*

Let $u_k = \{s, r\}$ *and* $u_{k+1} = \{x, y\}$*. By the definition of switch, the sequence* $(u_1, \ldots, u_{k+1}, u_k, \ldots, u_{|E})$ *is the sequence induced by* $\prec'$*. We may infer that, for any* $i < k$*, the* i*-partition by the ordering* $\prec'$ *is equal to the* i*-partition by the ordering* $\prec$*. Hence, as* u_{k+1} *is the edge of rank* k *for* $\prec'$ *and since* $\mathbf{B}'_{k-1} = \mathbf{B}_{k-1}$*, the* k*-partition for the ordering* $\prec'$ *is the partition* $\mathbf{B}'_k = \{\mathbf{B}^y_{k-1} \cup \mathbf{B}^x_{k-1}\} \cup (\mathbf{B}_{k-1} \setminus \{\mathbf{B}^x_{k-1}, \mathbf{B}^y_{k-1}\})$*. By the statement I,* $\mathbf{B}_{k-1} = \mathbf{B}_k$*, which implies that* $\mathbf{B}'_k = \{\mathbf{B}^y_k \cup \mathbf{B}^x_k\} \cup (\mathbf{B}_k \setminus \{\mathbf{B}^x_k, \mathbf{B}^y_k\})$*. Therefore, we have that:*

IV $\mathbf{B}'_k$ *is equal to the partition* $\mathbf{B}_{k+1}$

As $\mathbf{B}_{k+1} = \mathbf{B}_k = \mathbf{B}_{k-1}$ *by statement III, we have that*

V $\mathbf{B}'_k = \mathbf{B}_{k+1} = \mathbf{B}_{k-1} = \mathbf{B}'_{k-1}$

By statement V, as $\mathbf{B}'_k = \mathbf{B}'_{k-1}$*, we conclude that* u_{k+1} *is not a building edge for* $\prec'$*.*

Now, as u_k *is the edge of rank* $k + 1$ *for* $\prec'$*, the* $k + 1$*-partition for the ordering* $\prec'$ *is the partition* $\mathbf{B}'_{k+1} = \{{\mathbf{B}'_k}^s \cup {\mathbf{B}'_k}^r\} \cup (\mathbf{B}'_k \setminus \{{\mathbf{B}'_k}^s, {\mathbf{B}'_k}^r\})$*. By statement V, we have* $\mathbf{B}'_k = \mathbf{B}'_{k-1}$*. Since* $\mathbf{B}'_{k-1} = \mathbf{B}_{k-1}$*, then, by statement III, we have that* $\mathbf{B}'_k = \mathbf{B}_{k-1}$*. Therefore, we conclude that:*

VI $\mathbf{B}'_{k+1} = \{\mathbf{B}^s_{k-1} \cup \mathbf{B}^r_{k-1}\} \cup (\mathbf{B}_{k-1} \setminus \{\mathbf{B}^s_{k-1}, \mathbf{B}^r_{k-1}\})$

By the definition of $\mathbf{B}'_{k+1}$ *in the statement VI, we have:*

VII $\mathbf{B}'_{k+1} = \mathbf{B}_k$

By statement IV, $\mathbf{B}'_k = \mathbf{B}_{k+1}$ *and, by statement III,* $\mathbf{B}_k = \mathbf{B}_{k+1}$*. Hence,* $\mathbf{B}_k = \mathbf{B}'_k$*. Thus, by the statement VII, we conclude that* $\mathbf{B}'_{k+1} = \mathbf{B}'_k$*. Therefore,* u_k *is not a building edge for* $\prec'$*.*

Since the sequences induced by the orderings $\prec$ *and* $\prec'$ *are equal for any* $i > k + 1$*, and since* $\mathbf{B}'_{k+1} = \mathbf{B}'_k = \mathbf{B}_k = \mathbf{B}_{k+1}$*, we may affirm that,* $\mathbf{B}_i = \mathbf{B}'_i$ *for any* $i > k + 1$*. Therefore, the set of building edges for* $\prec$ *is equal to the set of building edges for* $\prec'$*, and the set of partitions and regions of* $\mathcal{B}_\prec$ *is equal to the set of partitions and regions of* $\mathcal{B}_{\prec'}$*. By Lemma 29,* f *is one-side increasing for* $\prec'$*.* □

Lemma 31. *Let $\prec$ be an altitude ordering for w and let f be a saliency map such that f is one-side increasing for $\prec$. Let $(u_1, \ldots, u_{|E|})$ be the sequence induced by $\prec$. Let k be a critical rank for f and $\prec$ such that both u_k and u_{k+1} are building edges for $\prec$ and such that $R_{u_k} \cap R_{u_{k+1}} = \emptyset$. Then f is one-side increasing for the switch $\prec'$ for k.*

Proof Let $(\mathbf{B}_0, \mathbf{B}_1, \ldots, \mathbf{B}_{|E|})$ be the sequence of partitions (of V) such that, for any i in $\{1, \ldots, |E|\}$, the partition $\mathbf{B}_i$ is the i-partition by the ordering $\prec$. Let $(\mathbf{B}'_0, \mathbf{B}'_1, \ldots, \mathbf{B}'_{|E|})$ be the sequence of partitions such that, for any i in $\{1, \ldots, |E|\}$, the partition $\mathbf{B}'_i$ is the i-partition by the ordering $\prec'$. By the definition of switch, the sequence $(u_1, \ldots, u_{k+1}, u_k, \ldots, u_{|E})$ is the sequence induced by $\prec'$. As the sequences induced by $\prec$ and by $\prec'$ are equal for any edge with rank $i < k$, we may affirm that:

I $\mathbf{B}_i = \mathbf{B}'_i$ for any $i < k$

Let $u_k = \{s, r\}$ and $u_{k+1} = \{x, y\}$. As u_k and u_{k+1} are building edges for $\prec$, we have that:

II $\mathbf{B}_k \neq \mathbf{B}_{k-1}$, and
III $\mathbf{B}_{k+1} \neq \mathbf{B}_k$

As u_{k+1} is the edge of rank k for $\prec'$, we have that the k-partition for the ordering $\prec'$ is $\mathbf{B}'_k = \{{\mathbf{B}'_{k-1}}^x \cup {\mathbf{B}'_{k-1}}^y\} \cup (\mathbf{B}'_{k-1} \setminus \{{\mathbf{B}'_{k-1}}^x, {\mathbf{B}'_{k-1}}^y\})$. By the statement I, $\mathbf{B}'_{k-1}$ and $\mathbf{B}_{k-1}$ are equal. Then $\mathbf{B}'_k = \{\mathbf{B}^x_{k-1} \cup \mathbf{B}^y_{k-1}\} \cup (\mathbf{B}_{k-1} \setminus \{\mathbf{B}^x_{k-1}, \mathbf{B}^y_{k-1}\})$.
By definition, we have:

IV $\mathbf{B}_k = \{\mathbf{B}^s_{k-1} \cup \mathbf{B}^r_{k-1}\} \cup (\mathbf{B}_{k-1} \setminus \{\mathbf{B}^s_{k-1}, \mathbf{B}^r_{k-1}\})$, and
V $\mathbf{B}_{k+1} = \{\mathbf{B}^x_k \cup \mathbf{B}^y_k\} \cup (\mathbf{B}_k \setminus \{\mathbf{B}^x_k, \mathbf{B}^y_k\})$

By our hypothesis, $R_{u_k} \cap R_{u_{k+1}} = \emptyset$, which means that the regions R_{u_k} and $R_{u_{k+1}}$ of $\mathcal{B}_\prec$ (whose building edges are respectively u_k and u_{k+1}) have no intersection. As u_k is a building edge for $\prec$, we have $R_{u_k} = \{\mathbf{B}^s_{k-1} \cup \mathbf{B}^r_{k-1}\}$. Similarly, as u_{k+1} is a building edge for $\prec$, we have $R_{u_{k+1}} = \{\mathbf{B}^x_k \cup \mathbf{B}^y_k\}$. Since $R_{u_k} \cap R_{u_{k+1}} = \emptyset$, we have that:

VI neither x nor y is in the region $\mathbf{B}^s_{k-1}$ (resp. $\mathbf{B}^r_{k-1}$), and
VII neither s nor r is in the region $\mathbf{B}^x_k$ (resp. $\mathbf{B}^y_k$)

By VI and VII, we can conclude that $\mathbf{B}^s_{k-1}$, $\mathbf{B}^r_{k-1}$, $\mathbf{B}^x_k$ and $\mathbf{B}^y_k$ are all distinct regions of the partition $\mathbf{B}_{k-1}$. Hence, we have:

VIII $\mathbf{B}^x_k = \mathbf{B}^x_{k-1}$, and
IX $\mathbf{B}^y_k = \mathbf{B}^y_{k-1}$

By definition, as u_{k+1} is the edge of rank k for $\prec'$, we have:

X $\mathbf{B}'_k = \{{\mathbf{B}'_{k-1}}^x \cup {\mathbf{B}'_{k-1}}^y\} \cup (\mathbf{B}'_{k-1} \setminus \{{\mathbf{B}'_{k-1}}^x, {\mathbf{B}'_{k-1}}^y\})$

By I and X, we conclude that:

XI $\mathbf{B}'_k = \{\mathbf{B}^x_{k-1} \cup \mathbf{B}^y_{k-1}\} \cup (\mathbf{B}_{k-1} \setminus \{\mathbf{B}^x_{k-1}, \mathbf{B}^y_{k-1}\})$

By VIII, IX and XI, we conclude:

XII $\mathbf{B}'_k = \{\mathbf{B}^x_k \cup \mathbf{B}^y_k\} \cup (\mathbf{B}_k \setminus \{\mathbf{B}^x_k, \mathbf{B}^y_k\})$

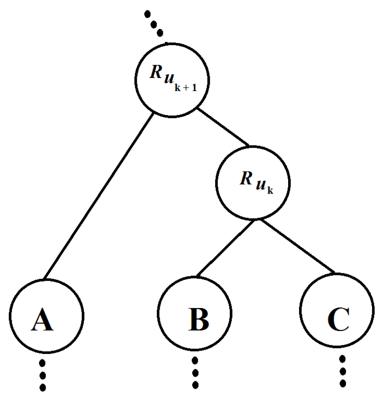
As $\mathbf{B}^x_k$ and $\mathbf{B}^y_k$ are distinct regions, we may say that $\mathbf{B}'_k$ is different from $\mathbf{B}'_{k-1}$. Hence, u_{k+1} is a building edge for $\prec'$.

Now, as u_k is the edge of rank $k+1$ for $\prec'$, we have that the $(k+1)$-partition for the ordering $\prec'$ is $\mathbf{B}'_{k+1} = \{{\mathbf{B}'_k}^s \cup {\mathbf{B}'_k}^r\} \cup (\mathbf{B}'_k \setminus \{{\mathbf{B}'_k}^s, {\mathbf{B}'_k}^r\})$. By statement VII, we have that neither s nor r are in the regions $\mathbf{B}^x_k$ and $\mathbf{B}^y_k$. Hence, by the statement XII, s and r belong to distinct regions of $\mathbf{B}'_k$. Therefore, ${\mathbf{B}'_k}^s \neq {\mathbf{B}'_k}^r$. Consequently, $\mathbf{B}'_{k+1}$ is different from $\mathbf{B}'_k$. Hence, u_k is a building edge for $\prec'$.

Moreover, we conclude that $\mathbf{B}'_{k+1} = \mathbf{B}_{k+1}$ because both partitions result from the union of the four distinct regions of $\mathbf{B}_{k-1}$ containing s, r, x and y. Hence, for any $i > k+1$, as the sequences induced by $\prec$ and $\prec'$ are equal, we can conclude that any partition $\mathbf{B}_i$ is equal to the partition $\mathbf{B}'_i$ for any $i > k+1$. Therefore, by Lemma 29, f is one-side increasing for $\prec'$. □

Lemma 32. *Let $\prec$ be an altitude ordering for w and let f be a saliency map such that f is one-side increasing for $\prec$. Let $(u_1, \ldots, u_{|E|})$ be the sequence induced by $\prec$. Let k be a critical rank for f and $\prec$ such that both u_k and u_{k+1} are building edges for $\prec$ and such that $R_{u_k} \subset R_{u_{k+1}}$. Then f is one-side increasing for the switch $\prec'$ for k.*

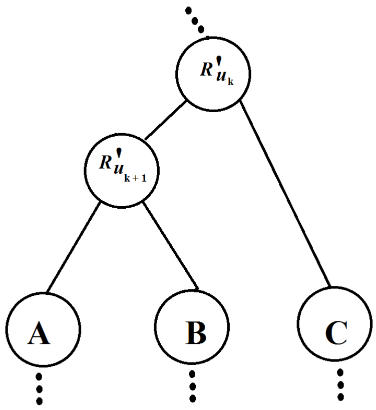
Proof By our hypothesis, the region R_{u_k} of $\mathcal{B}_\prec$ is a subset of the region $R_{u_{k+1}}$ of $\mathcal{B}_\prec$. Let A be the region of $\mathcal{B}_\prec$ such that $R_{u_{k+1}} = R_{u_k} \cup A$. Let B and C be the children of R_{u_k}. This situation is illustrated in the following figure.



$\mathcal{B}_\prec$

Let $u_k = \{s, r\}$ and $u_{k+1} = \{x, y\}$. As u_{k+1} is a building edge for $\prec$, we conclude that x are y belong to two distinct regions in $\{A, B, C\}$. Without loss of generality, let us assume that x belongs to A and that y belongs to B. Let $\mathbf{B}_{k-1}$ be the $(k-1)$-partition for $\prec$. We can say that the regions A, B and C belong to $\mathbf{B}_{k-1}$. Moreover, we know that $\mathbf{B}_{k-1}$ is equal to the $(k-1)$-partition for $\prec'$ because, for any $i < k$, the edge of rank i for $\prec$ is also the edge of rank i for $\prec'$. Since u_{k+1} is the edge of rank k for $\prec'$, we can conclude that the k-partition $\mathbf{B}'_k$ for $\prec'$ is the partition $\{A \cup B\} \cup (\mathbf{B}_{k-1} \setminus \{A, B\})$. As the region $\{A \cup B\}$ is not in the partition $\mathbf{B}'_{k-1}$, we can conclude that $\mathbf{B}'_k$ is different from $\mathbf{B}'_{k-1}$. Hence, u_{k+1} is the building edge of the region $R'_{u_{k+1}} = \{A \cup B\}$ of $\mathcal{B}_{\prec'}$.

Now, without loss of generality, let us assume that s belongs to B and that r belongs to C. By our hypothesis, u_k is the edge of rank $k+1$ for $\prec'$. In the partition $\mathbf{B}'_k$, we know that s and r belong to distinct regions because s is in $\{A \cup B\}$ and r is in C. Hence, the region $\{A \cup B \cup C\}$ is a region of $\mathbf{B}'_{k+1}$ and we have $\mathbf{B}'_{k+1} \neq \mathbf{B}'_k$. Therefore, u_k is a building edge for $\prec'$. This situation is illustrated in the following figure.



$\mathcal{B}_{\prec'}$

We can infer that the $(k+1)$-partition for $\prec'$ is equal to the $(k+1)$-partition for $\prec$. Since the edge of rank i for $\prec$ is also the edge of rank i for $\prec'$, we can conclude that the set of building edges for $\prec$ is equal to the set of building edges for $\prec'$.

Now, we will prove that f is one-side increasing for $\prec'$. To that end, we will demonstrated that the three conditions of the definition of one-side increasing maps (Definition 3) hold true for f.

1. *We first prove that the condition 1 of Definition 3 holds true for f. Since the set $E_\prec$ of building edges for $\prec$ is equal to the set $E_{\prec'}$ of building edges for $\prec'$, we can conclude that $\{f(u) \mid u \in E_{\prec'}\}$ is equal to $\{f(u) \mid u \in E_\prec\} = \{0, \ldots, n-1\}$. Thus, the first condition for f to be one-side increasing for $\prec'$ holds true.*
2. *We now prove that the condition 2 of Definition 3 holds true for f.*
In order to prove this condition, we consider four cases: (2.1) both u_k and u_{k+1} are watershed-cut edges for $\prec$; (2.2) neither u_k nor u_{k+1} is a watershed-cut edge for $\prec$; (2.3) only u_k is a watershed-cut for $\prec$; and (2.4) only u_{k+1} is a watershed-cut for $\prec$.

(2.1) *If both u_k and u_{k+1} are watershed-cut edges for $\prec$, then there is at least one minimum of w included in each of the regions A, B and C. Since A and B are the children of $R'_{u_{k+1}}$, we may say that u_{k+1} is a watershed-cut edge for $\prec'$. Since $\{A \cup B\}$ and C are the children of R'_{u_k} and since there is at least one minimum included in each of the children of R'_{u_k}, we may say that u_k is a watershed-cut edge for $\prec'$. Hence, both u_k and u_{k+1} are watershed-cut edges for $\prec'$.*

(2.2) *If neither u_k nor u_{k+1} is a watershed-cut edge for $\prec$, then there are at least two regions among A, B and C that do not include any minimum of w. Hence, there is at least one child of each of the regions R'_{u_k} and $R'_{u_{k+1}}$ that do not include any minimum of w. Hence, neither u_k nor u_{k+1} is a watershed-cut edge for $\prec'$.*

(2.3) *If u_k is a watershed-cut edge for $\prec$ and if u_{k+1} is not watershed-cut edge for $\prec$, then there is at least one minimum included in each of the regions B and C and there is no minimum included in A. Hence, as A is a child of the region $R'_{u_{k+1}}$ of $\mathcal{B}_{\prec'}$ and as there is no minimum of w included in A, u_{k+1} is not a watershed-cut edge for $\prec'$. Since there is at least one minimum included in each of the regions B and C, and since B and C are included in distinct children of the region R'_{u_k}, we can conclude that u_k is a watershed-cut edge for $\prec'$.*

(2.4) *If u_{k+1} is a watershed-cut edge for $\prec$ and if u_k is not watershed-cut edge for $\prec$. As k is a critical rank for f and $\prec$, we have that $f(u_k) \geq f(u_{k+1})$. However, by the definition of one-side increasing maps (Definition 3), we have $f(u_{k+1}) > 0$ and $f(u_k) = 0$, which contradicts our hypothesis. Therefore, the case where u_{k+1} is a watershed-cut edge for $\prec$ and if u_k is not watershed-cut edge for $\prec$ does not happen.*

Therefore, we can conclude that the set of watershed-cut edges for $\prec$ is equal to the set of watershed-cut edges for $\prec'$. Then, the second condition for f to be one-side increasing for $\prec'$ holds true.

3. *We finally prove that the condition 3 of Definition 3 holds true for f. As k is a critical rank for f and $\prec$, we have that $f(u_k) \geq f(u_{k+1})$. We will consider two cases: (3.1) $f(u_k) = f(u_{k+1})$; and (3.2) $f(u_k) > f(u_{k+1})$.*

(3.1) *If $f(u_k) = f(u_{k+1})$, by Lemma 28, neither u_k nor u_{k+1} is a watershed-cut edge for $\prec$. Since neither u_k nor u_{k+1} is a watershed-cut edge for $\prec$, as proven in the case (2.2), neither u_k nor u_{k+1} is a watershed-cut edge for $\prec'$. Hence, there is at least one child of the region R'_{u_k} (resp. $R'_{u_{k+1}}$) that does not include any minimum of w. Let Z be the child of R'_{u_k} (resp. $R'_{u_{k+1}}$) that does not include any minimum of w. We can infer that there is no watershed-cut edge v for $\prec'$ such that $R_v \subseteq Z$. Then, for any edge v such that $R_v \subseteq Z$, we have $f(v) = 0$. Since $f(u_k) = 0$ (resp. $f(u_{k+1}) = 0$), we can affirm that there is a child Z of R'_{u_k} (resp. $R'_{u_{k+1}}$) such that $f(u_k) \geq \{f(v) \mid R_v \subseteq Z\}$ (resp. $f(u_{k+1}) \geq \{f(v) \mid R_v \subseteq Z\}$).*

(3.2) *Let us assume that $f(u_k) > f(u_{k+1})$. Since f is one-side increasing for $\prec$, by Definition 3 (statement 3), we conclude that, for any edge v such that v is the building edge of a region included in A, we have $f(u_{k+1}) \geq f(v)$. In the hierarchy $\mathcal{B}_{\prec'}$, the region $R'_{u_{k+1}}$ is the parent of A, so the statement 3 of Definition 3 holds true for $R'_{u_{k+1}}$.*
We will now prove that the statement 3 of Definition 3 holds true for R'_{u_k}. By Definition 3, we know that there is a child Z of R_{u_k} such that for any edge v such that v is the building edge of a region included in Z, we have $f(u_k) \geq f(v)$. Let us first assume that $Z = C$. Since C is also a child of the region R'_{u_k} of $\mathcal{B}_{\prec'}$, the statement 3 of Definition 3 holds true for R'_{u_k}. Now, let us assume that $Z = B$. We will prove that, for the building edge v of any region included in $\{A \cup B \cup R_{u_{k+1}}\}$, we have $f(u_k) \geq f(v)$. By our assumption $f(u_k) > f(u_{k+1})$. Moreover, for any edge v such that v is the building edge of a region included in A, we have $f(u_{k+1}) \geq f(v)$. Therefore, for the building edge v of any region included in $\{A \cup B \cup R_{u_{k+1}}\}$, we have $f(u_k) \geq f(v)$. □

Lemma 33. *Let $\prec$ be an altitude ordering for w and let f be a saliency map such that f is one-side increasing for $\prec$. Let $(u_1, \ldots, u_{|E|})$ be the sequence induced by $\prec$. Let k be a critical rank for f and $\prec$ such that u_{k+1} is a building edge for $\prec$ and such that u_k is not a building edge for $\prec$. Then f is one-side increasing for the switch $\prec'$ for k.*

Proof *Let $(\mathbf{B}_0, \mathbf{B}_1, \ldots, \mathbf{B}_{|E|})$ be the sequence of partitions (of V) such that, for any i in $\{1, \ldots, |E|\}$, the partition $\mathbf{B}_i$ is the i-partition by the ordering $\prec$ (as defined in Section*

3.1). Let $(\mathbf{B}'_0, \mathbf{B}'_1, \ldots, \mathbf{B}'_{|E|})$ be the sequence of partitions such that, for any i in $\{1, \ldots, |E|\}$, the partition $\mathbf{B}'_i$ is the i-partition by the ordering $\prec'$. As the sequences induced by $\prec$ and by $\prec'$ are equal for any edge with rank $i < k$, we may affirm that:

I. *$\mathbf{B}_i = \mathbf{B}'_i$ for any $i < k$*

By the definition of binary partition hierarchy and since u_k is not a building edge for $\prec$, we may say that:

II. *the partition $\mathbf{B}_k$ is equal to the partition $\mathbf{B}_{k-1}$.*

Let $u_k = \{s, r\}$ and $u_{k+1} = \{x, y\}$. Since $\mathbf{B}_k = \mathbf{B}_{k-1}$ and since $\mathbf{B}_k = \{\mathbf{B}^s_{k-1} \cup \mathbf{B}^r_{k-1}\} \cup (\mathbf{B}_{k-1} \setminus \{\mathbf{B}^s_{k-1}, \mathbf{B}^r_{k-1}\})$, we conclude that the regions $\mathbf{B}^s_{k-1}$ and $\mathbf{B}^r_{k-1}$ of the partition $\mathbf{B}_{k-1}$ are equal: $\mathbf{B}^s_{k-1} = \mathbf{B}^r_{k-1}$. By the statement I, we may say that the regions ${\mathbf{B}'_{k-1}}^s$ and ${\mathbf{B}'_{k-1}}^r$ of the partition $\mathbf{B}'_{k-1}$ are equal as well. Hence:

III. *the partition $\mathbf{B}'_k$ is equal to the partition and $\mathbf{B}'_{k-1}$*

Therefore, u_k is not a building edge for $\prec'$.
Since u_{k+1} is a building edge for $\prec$, we have that:

IV. *the partition $\mathbf{B}_{k+1}$ is different from the partition $\mathbf{B}_k$.*

By the statement IV, we conclude that the regions $\mathbf{B}^x_k$ and $\mathbf{B}^y_k$ of the partition $\mathbf{B}_k$ are distinct. By the statement III, we have that $\mathbf{B}'_k = \mathbf{B}'_{k-1}$. Then, by statement I, we have $\mathbf{B}'_k = \mathbf{B}_{k-1}$. Hence, by statement II, we have $\mathbf{B}'_k = \mathbf{B}_k$. Therefore, the regions $\mathbf{B}^x_k$ and $\mathbf{B}^y_k$ also belong to the partition $\mathbf{B}'_k$. Consequently, since x and y are in distinct regions in the partition $\mathbf{B}'_k$, we conclude that u_{k+1} is a building edge for $\prec'$. Therefore, the set $E_\prec$ of building edges for $\prec$ is equal to the set $E_{\prec'}$ of building edges for $\prec'$.

Moreover, we conclude that $\mathbf{B}'_{k+1} = \mathbf{B}_{k+1}$ because both partitions result from the union of the two distinct regions of $\mathbf{B}_{k-1}$ containing x and y. Hence, for any $i > k+1$, as the edge of rank i for $\prec$ is also the edge of rank i for $\prec'$, we can conclude that any partition $\mathbf{B}_i$ is equal to the partition $\mathbf{B}'_i$. Hence, $\mathcal{B}_\prec$ and $\mathcal{B}_\prec$ have the same set of regions.

Since $E_\prec = E_{\prec'}$ and since $\mathcal{B}_\prec$ and $\mathcal{B}_\prec$ have the same set of regions, by Lemma 29, f is one-side increasing for $\prec'$. □

Lemma 34. *Let $\prec$ be an altitude ordering for w and let f be a saliency map such that f is one-side increasing for $\prec$. Let $(u_1, \ldots, u_{|E|})$ be the sequence induced by $\prec$. Let k be a critical rank for f and $\prec$ such that u_k is a building edge for $\prec$ and such that u_{k+1} is not a building edge for $\prec$. Then f is one-side increasing for the switch $\prec'$ for k.*

Proof Let $(\mathbf{B}_0, \mathbf{B}_1, \ldots, \mathbf{B}_{|E|})$ be the sequence of partitions (of V) such that, for any i in $\{1, \ldots, |E|\}$, the partition $\mathbf{B}_i$ is the i-partition by the ordering $\prec$. Let $(\mathbf{B}'_0, \mathbf{B}'_1, \ldots, \mathbf{B}'_{|E|})$ be the sequence of partitions such that, for any i in $\{1, \ldots, |E|\}$, the partition $\mathbf{B}'_i$ is the i-partition by the ordering $\prec'$. As the sequences induced by $\prec$ and by $\prec'$ are equal for any edge with rank $i < k$, we may affirm that:

I. *$\mathbf{B}_i = \mathbf{B}'_i$ for any $i < k$*

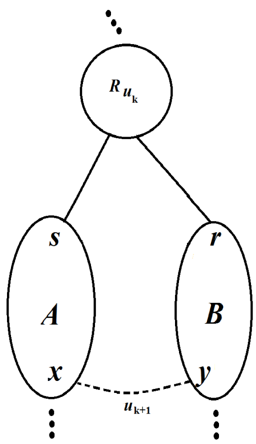
Since u_k is a building edge for $\prec$, we have that:

II. *$\mathbf{B}_k$ is different from $\mathbf{B}_{k-1}$*

Let $u_k = \{s, r\}$ and $u_{k+1} = \{x, y\}$. Since $\mathbf{B}_k \neq \mathbf{B}_{k-1}$, we conclude that s and r are in distinct regions of $\mathbf{B}_{k-1}$. As u_{k+1} is not a building edge for $\prec$, we consider two cases: (1) x and y belong to a unique region of $\mathbf{B}_{k-1}$; and (2) x and y belong to two distinct regions of $\mathbf{B}_{k-1}$.

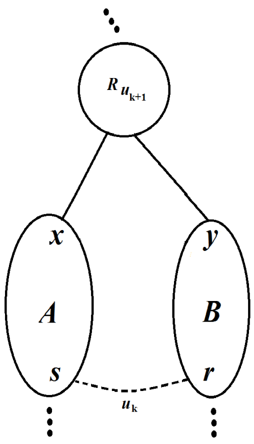
(1) *Let us consider that x and y belong to a unique region of $\mathbf{B}_{k-1}$. By the statement I, we have $\mathbf{B}'_{k-1} = \mathbf{B}_{k-1}$. Hence, x and y belong to a unique region of $\mathbf{B}'_{k-1}$ and, therefore, u_{k+1} is not a building edge for $\prec'$. We will now prove that u_k is a building edge for $\prec'$. Since u_k is a building edge for $\prec$, we have that s and r belong to two distinct regions of the partition $\mathbf{B}_{k-1}$. Since u_{k+1} is not a building edge for $\prec'$, we have $\mathbf{B}'_k = \mathbf{B}'_{k-1}$. Then, by the statement I, we have $\mathbf{B}'_k = \mathbf{B}'_{k-1} = \mathbf{B}_{k-1}$. Therefore, s and r belong to two distinct regions of the partition $\mathbf{B}'_k$. Hence, u_k is a building edge for $\prec'$.*
Therefore, the set $E_\prec$ of building edges for $\prec$ is equal to the set $E_{\prec'}$ of building edges for $\prec'$.
Moreover, we conclude that $\mathbf{B}'_{k+1} = \mathbf{B}_{k+1}$ because both partitions result from the union of the two distinct regions of $\mathbf{B}_{k-1}$ containing s and r. Hence, for any $i > k+1$, as the edge of rank i for $\prec$ is also the edge of rank i for $\prec'$, we can conclude that any partition $\mathbf{B}_i$ is equal to the partition $\mathbf{B}'_i$. Thus, $\mathcal{B}_\prec$ and $\mathcal{B}_\prec$ have the same set of regions.
Since $E_\prec = E_{\prec'}$ and since $\mathcal{B}_\prec$ and $\mathcal{B}_\prec$ have the same set of regions, by Lemma 29, f is one-side increasing for $\prec'$.

(2) *We now consider that x and y belong to two distinct regions of $\mathbf{B}_{k-1}$. Let A and B be the regions of $\mathbf{B}_{k-1}$ such that $s \in A$ and $r \in B$. Since x and y belong to two distinct regions of $\mathbf{B}_{k-1}$ and since $\mathbf{B}_k = \{A \cup B\} \cup (\mathbf{B}_{k-1} \setminus \{A, B\})$, we conclude that either x or y is in A, and that either s or r is in B. Without loss of generality, let us assume that $x \in A$ and $y \in B$. This situation is illustrated in the following figure.*



$\mathcal{B}_\prec$

Since u_{k+1} is the edge of rank k for the ordering $\prec'$, we can say that the k-partition $\mathbf{B}'_k$ by the ordering $\prec'$ is $\{A \cup B\} \cup (\mathbf{B}'_{k-1} \setminus \{A, B\})$ because A and B are the regions of $\mathbf{B}'_{k-1}$ that contain respectively x and y. As the region $\{A \cup B\}$ do not belong to the partition $\mathbf{B}'_{k-1}$, we have that u_{k+1} is the building edge of the region $\{A \cup B\}$. Hence, u_{k+1} is a building edge for $\prec'$.
Since u_k is the edge of rank $k+1$ for the ordering $\prec'$, we may conclude that $\mathbf{B}'_{k+1} = \mathbf{B}'_k$ because the s and r belong to the same region $\{A \cup B\}$ of $\mathbf{B}'_k$. Therefore, u_k is not a building edge for $\prec'$. This situation is illustrated in the following image.



$\mathcal{B}_{\prec'}$

We conclude that $\mathcal{B}_\prec$ and $\mathcal{B}_{\prec'}$ have the same set of regions but not the same set of building edges: $E_{\prec'} = E_\prec \setminus \{u_k\} \cup \{u_{k+1}\}$. Hence, the only difference between the hierarchies $\mathcal{B}_\prec$ and $\mathcal{B}_{\prec'}$ is the building edge of the region $\{A \cup B\}$. Therefore, we may say that, if the weight of the building edge of $\{A \cup B\}$ for $\prec$ is equal to the weight of the building edge of $\{A \cup B\}$ for $\prec'$, then f is also one-side increasing for $\prec'$. To that end, we will prove that $f(u_k) = f(u_{k+1})$.
By Lemma 22, as f is one-side increasing for $\prec$, we have that:

III. $(V, E_\prec)$ is a MST of (G, f)

By the statement III and by Lemma 21, we conclude that:

IV. the hierarchy $\mathcal{QFZ}(G, f)$ is equal to the hierarchy $\mathcal{QFZ}((V, E_\prec), f)$

Statement IV implies that f is the saliency map of the hierarchy $\mathcal{QFZ}((V, E_\prec), f)$. Hence, for any edge $u = \{a, b\}$ in E, $f(u)$ is the maximum weight in the unique path between a and b in $((V, E_\prec), f)$. We can affirm that:

V. the unique path between x and y in $((V, E_\prec), f)$ is a path that includes the edge u_k

By the statement V and by the definition of saliency maps, we have $f(u_{k+1}) \geq f(u_k)$. Since k is a critical rank for f and $\prec$, we have $f(u_{k+1}) \leq f(u_k)$. Therefore, we have $f(u_k) = f(u_{k+1})$, which completes the proof that f is one-side increasing for $\prec'$. □

C Proof of Property 6

(Property 6). *Let $\prec$ be an altitude ordering for w and let ϵ be a map from the regions of $\mathcal{B}_\prec$ into $\mathbb{R}$. The map ϵ is an* extinction map for $\prec$ *if and only if the following statements hold true:*

- *$\{\epsilon(R) \mid R$ is a region of $\mathcal{B}_\prec\} = \{0, \ldots, n\}$;*
- *for any two distinct minima M_1 and M_2 of w, we have $\epsilon(M_1) \neq \epsilon(M_2)$; and*
- *for any region R of $\mathcal{B}_\prec$, we have that $\epsilon(R)$ is equal to $\vee\{\epsilon(M)$ such that M is a minimum of w included in $R\}$, where $\vee\{\} = 0$.*

We prove the forward and backward implications of Property 6 in Lemma 35 and Lemma 36, respectively.

Lemma 35. *Let $\prec$ be an altitude ordering for w and let ϵ be a map from the regions of $\mathcal{B}_\prec$ into $\mathbb{R}$. If the map ϵ is an* extinction map for $\prec$, *then the following statements hold true:*

1. *$\{\epsilon(R) \mid R$ is a region of $\mathcal{B}_\prec\} = \{0, \ldots, n\}$;*
2. *for any two distinct minima M_1 and M_2 of w, we have $\epsilon(M_1) \neq \epsilon(M_2)$; and*
3. *for any region R of $\mathcal{B}_\prec$, we have that $\epsilon(R)$ is equal to $\vee\{\epsilon(M)$ such that M is a minimum of w included in $R\}$, where $\vee\{\} = 0$.*

Proof Let ϵ be an extinction map for $\prec$. Then, by the definition of extinction maps, there is a sequence $\mathcal{S} = (M_1, \ldots, M_n)$ of minima of w such that ϵ is the extinction map for $\prec$ and $\mathcal{S}$. We will prove that the statements 1, 2 and 3 hold true for ϵ.

To prove that the statement 1 holds true, we will first prove that $\{\epsilon(R) \mid R$ is a region of $\mathcal{B}_\prec\} \subseteq \{0, \ldots, n\}$. Since w has n minima, the extinction value of any region of $\mathcal{B}_\prec$ which includes a minimum of w is in the set $\{1, \ldots, n\}$. On the other hand, for any region R of $\mathcal{B}_\prec$ which do not include any minimum of w, we have that $\epsilon(R) = 0$. Hence, $\{\epsilon(R) \mid R$ is a region of $\mathcal{B}_\prec\} \subseteq \{0, \ldots, n\}$. We will now prove that $\{0, \ldots, n\} \subseteq \{\epsilon(R) \mid R$ is a region of $\mathcal{B}_\prec\}$. As $\mathcal{B}_\prec$ has at least one leaf-region composed of a single vertex of G, we can affirm that there is at least one region of $\mathcal{B}_\prec$ which do not include any minimum of w and whose extinction value for $\prec$ and $\mathcal{S}$ is zero. Then, 0 is in $\{\epsilon(R) \mid R$ is a region of $\mathcal{B}_\prec\}$. Now, let i be a value in $\{1, \ldots, n\}$. For the minimum M_i, we may affirm that M_i is the unique minimum of w included in M_i and, therefore, $\epsilon(M_i) = i$. Hence, i is in $\{\epsilon(R) \mid R$ is a region of $\mathcal{B}_\prec\}$. We may conclude that, for any i in $\{0, \ldots, n\}$, i is in $\{\epsilon(R) \mid R$ is a region of $\mathcal{B}_\prec\}$. Therefore, the range of ϵ is $\{0, \ldots, n\}$, which corresponds to the statement 1 of Lemma 35.

By the definition of extinction maps, for any minimum M_i, for i in $\{1, \ldots, n\}$, we have $\epsilon(M_i) = i$ because M_i is the only minimum of w included in M_i. Therefore, for any two distinct minima M_i and M_j, for i, j in $\{1, \ldots, n\}$, we have $\epsilon(M_i) = i$ and $\epsilon(M_j) = j$ and, consequently, $\epsilon(M_i)$ is different from $\epsilon(M_j)$. Hence, the statement 2 of Lemma 35 holds true for ϵ.

The statement 3 of Lemma 35 is precisely the definition of extinction values: for any region R of $\mathcal{B}_\prec$, the extinction value of R is zero if there is no minimum of w included in R and, otherwise, it is the maximal i (which is equal to $\epsilon(M_i)$) such that M_i is included in R. □

Lemma 36. *Let $\prec$ be an altitude ordering for w and let ϵ be a map from the regions of $\mathcal{B}_\prec$ into $\mathbb{R}$ such that:*

1. *$\{\epsilon(R) \mid R$ is a region of $\mathcal{B}_\prec\} = \{0, \ldots, n\}$;*
2. *for any two distinct minima M_1 and M_2 of w, we have $\epsilon(M_1) \neq \epsilon(M_2)$; and*
3. *for any region R of $\mathcal{B}_\prec$, we have that $\epsilon(R)$ is equal to $\vee\{\epsilon(M)$ such that M is a minimum of w included in $R\}$, where $\vee\{\} = 0$.*

Then the map ϵ is an extinction map for $\prec$.

Proof To prove that ϵ is an extinction map for $\prec$, we will show that there exists a sequence $S = (M_1, \ldots, M_n)$ of minima of w such that, for any region R of $\mathcal{B}_\prec$, the value $\epsilon(R)$ is the extinction value of R for $\prec$ and $\mathcal{S}$.

Let $\mathcal{S} = (M_1, \ldots, M_n)$ be a sequence of minima of w ordered in non-decreasing order for ϵ, i.e., *for any two distinct minima M_i and M_j, with i, j in $\{1, \ldots, n\}$, if $\epsilon(M_i) < \epsilon(M_j)$ then $i < j$.*

By the hypothesis 2, this sequence $\mathcal{S}$ is unique. By the hypothesis 3, for any region R of $\mathcal{B}$ such that there is no minimum of w included in R, $\epsilon(R) = \vee\{\} = 0$, so $\epsilon(R)$ is the extinction value of R for $\prec$ and $\mathcal{S}$.

Since w has n minima, for any minimum M of w, the value $\epsilon(M)$ is in $\{1, \ldots, n\}$. Otherwise, by contradiction, let us assume that there exists a minimum M' of w such that $\epsilon(M') = 0$. Then, there is a value i in $\{1, \ldots, n\}$ such that, for any minimum M'' of w, the value $\epsilon(M'')$ is different from i. Consequently, by the hypothesis 3, the range of ϵ would be $\{0, \ldots, n\} \setminus \{i\}$, which contradicts the hypothesis 1. Therefore, for any minimum M_i of w, for i in $\{1, \ldots, n\}$, as our assumption that $\epsilon(M_i) < \epsilon(M_j)$ implies that $i < j$, we have that $\epsilon(M_i) = i$. Thus, $\epsilon(M_i)$ is the extinction value of M_i for $\prec$ and S.

It follows that, by the hypothesis 3, for any region R of $\mathcal{B}_\prec$ such that there is a minimum of w included in R, the value $\epsilon(R)$ is the maximum value i (which is equal to $\epsilon(M_i)$) in $\{1, \ldots, n\}$ such that M_i is included in R.

Thus, for any region R of $\mathcal{B}_\prec$, the value $\epsilon(R)$ is the extinction value of R for $\prec$ and $\mathcal{S}$. Therefore, the map ϵ is an extinction map for $\prec$. □

D Proof of Lemma 11

(Lemma 11). *Let $\prec$ be an altitude ordering for w, let f be a map from E into $\mathbb{R}$ such that f is one-side increasing for $\prec$, and let ξ be the approximated extinction map for f and $\prec$. The map ξ is an extinction map for $\prec$.*

In order to prove Lemma 11, we prove in Lemmas 38, 39 and 43 that the three conditions of Property 6 for ξ to be an extinction map are satisfied. We first establish the following auxiliary lemma.

Lemma 37. *Let $\prec$ be an altitude ordering for w and let f be a map from E into $\mathbb{R}$ such that f is one-side increasing for $\prec$. Then, the two following statements hold true:*

1. *the set $\{f(e) \mid e \text{ is a watershed} - \text{cut edge for } \prec\}$ is equal to $\{1, \ldots, n-1\}$; and*
2. *for any two distinct watershed-cut edges u and v for $\mathcal{B}$, we have $f(u) \neq f(v)$.*

Proof *By the Definition 3 (statement 1), we have $\{f(u) \mid u \in E_\prec\} = \{0, \ldots, n-1\}$ and, by Definition 3 (statement 2), only the weight of the watershed-cut edges for $\prec$ are strictly greater than zero. Then, $\{f(e) \mid e \text{ is a watershed} - \text{cut edge for } \prec\} = \{1, \ldots, n-1\}$. Hence, for any i in $\{1, \ldots, n-1\}$, there is a watershed-cut edge e for $\prec$ such that $f(e) = i$. Moreover, as there are $n-1$ watershed-cut edges for $\prec$, for any two distinct watershed-cut edges u and v for $\prec$, we have $f(u) \neq f(v)$.* □

Lemma 38. *Let $\prec$ be an altitude ordering for w, let f be a map from E into $\mathbb{R}$ such that f is one-side increasing for $\prec$, and let ξ be the approximated extinction map for f and $\prec$. The range of ξ is $\{0, \ldots, n\}$.*

Proof We will prove that: (1) for any i in $\{0, \ldots, n\}$, there is a region R of $\mathcal{B}_\prec$ such that $\xi(R) = i$; and (2) for any region R of $\mathcal{B}_\prec$, we have $\xi(R)$ in $\{0, \ldots, n\}$.

(1) We first prove statement (1). We start by proving that there is a region R of $\mathcal{B}_\prec$ such that $\xi(R) = n$. Let R be the set V of vertices of G. Then, by Definition 10 (statement 1), we have $\xi(R) = \triangledown(R) + 1$, where $\triangledown$ is the supremum descendant map for f and $\prec$. By Definition 3 (statement 1), we have $\{f(u) \mid u \in E_\prec\} = \{0, \ldots, n-1\}$. As $\triangledown(V) = \vee\{f(u) \mid R_u \subseteq V\} = \vee\{0, \ldots, n-1\} = n-1$, we have that $\xi(R) = n - 1 + 1 = n$.
We will now show that there is a region R of $\mathcal{B}_\prec$ such that $\xi(R) = 0$. Let R be a region of $\mathcal{B}_\prec$ such that there is no minimum of w included in R. Then R is not a minimum of w and, consequently, the building edge of the parent of R is not a watershed-cut edge for $\prec$. Let u be building edge of the parent of R. Since there is no minimum of w included in R, by Definition 9, R is not a dominant region for f and $\prec$. By the statement 3 of the definition of approximated extinction maps (Definition 10), we have $\xi(R) = f(u)$. Since f is a one-side increasing map and since u is not a watershed-cut edge for $\prec$, we have $f(u) = 0$. Therefore, we have $\xi(R) = f(u) = 0$.
Finally, we will prove that, for any i in $\{1, \ldots, n-1\}$, there is a region R of $\mathcal{B}_\prec$ such that $\xi(R) = i$. By Lemma 37, we can say that, for any i in $\{1, \ldots, n-1\}$, there is a watershed-cut u edge for $\prec$ such that $f(u) = i$. Let u be a watershed-cut edge for $\prec$ and let X and Y be the children of R_u. Since u is a watershed-cut edge for $\prec$, both X and Y contain at least a minimum of w and, then, neither X nor Y are leaf regions of $\mathcal{B}_\prec$. Let $\ll$ be the non-leaf ordering for f and $\prec$. Since $\ll$ is a total ordering, we have either $X \ll Y$ or $Y \ll X$. Then, exactly one child of R_u is a dominant region for f and $\prec$. Let Y (resp. X) be the child of R_u which is not a dominant region for f and $\prec$. By Definition 10 (statement 3), we have $\xi(Y) = f(u)$ (resp. $\xi(X) = f(u)$). Therefore, for any i in $\{1, \ldots, n-1\}$, there is a watershed-cut edge u for $\prec$ such that $f(u) = i$ and such that there is a child Z of R_u such that $\xi(Z) = i$.

(2) We will now prove the statement (2). Let R be a region of $\mathcal{B}_\prec$. If $R = V$, then $\xi(R) = n$, as established in the proof of statement (1). Otherwise, let v be the building edge of the parent of R. By Definition 10, the value $\xi_f(R)$ is either $f(v)$ or $\xi(parent(R))$. Hence, either $\xi_f(R)$ is equal to $f(v)$ for a building edge v for $\prec$, or $\xi_f(R)$ is equal to $\xi(V) = n$. It is enough to prove that n and $f(v)$ are in $\{0, \ldots, n\}$. As f is one-side increasing for $\prec$, by Definition 3 (statement 1), we have $\{f(u) \mid u \in E_\prec\} = \{0, \ldots, n-1\}$. Since v is a building edge for $\prec$, we may say that $f(v)$ is in $\{0, \ldots, n-1\}$. □

Lemma 39. *Let $\prec$ be an altitude ordering for w and let f be a map from E into $\mathbb{R}$ such that f is one-side increasing for $\prec$. Let ξ be the approximated extinction map for f and $\prec$. For any two minima M_1 and M_2 of w, if $\xi(M_1) = \xi(M_2)$, then $M_1 = M_2$.*

To prove Lemma 39, we first present the Lemmas 40, 41 and 42. In the following, for any non-leaf region X of a binary partition hierarchy $\mathcal{B}$ of (G, w), we denote by u_X the building edge of X.

Lemma 40. *Let $\prec$ be an altitude ordering for w and let f be a map from E into $\mathbb{R}$ such that f is one-side increasing for $\prec$. Let ξ be the approximated extinction map for f and $\prec$. For any region X of $\mathcal{B}_\prec$ such that there is a minimum M of w such that $M \subset X$, there is a child Y of X such that:*

1. *$\xi(Y) = \xi(X)$;*
2. *$\xi(sibling(Y)) = f(u_X)$; and*
3. *there is a minimum of w included in Y.*

Proof *Let X be a region such that there is a minimum M of w such that $M \subset X$. Then, there is a child Z of X such that there is a minimum M such that $M \subseteq Z$. Let Z be a child X such that there is a minimum M such that $M \subseteq Z$. We consider two cases: (1) $sibling(Z)$ is a leaf-region of $\mathcal{B}_\prec$; and (2) $sibling(Z)$ is a non-leaf region of $\mathcal{B}_\prec$.*

(1) If $sibling(Z)$ is a leaf-region of $\mathcal{B}_\prec$, then, by Definition 9, Z is a dominant region for f and $\prec$ and $sibling(Z)$ is not a dominant region for f and $\prec$. Hence, by Definition 10, $\xi(Z) = \xi(X)$ and $\xi(sibling(Z)) = f(u_X)$.

(2) Let us now assume that $sibling(Z)$ is a non-leaf region of $\mathcal{B}_\prec$. Since X is not a minimum of w and since there is a minimum of w included in Z, we can conclude that there is a minimum of w included in $sibling(Z)$ as well. Let $\ll$ be the non-leaf ordering for f and $\prec$. As the non-leaf ordering $\ll$ is a total ordering on the non-leaf regions of $\mathcal{B}_\prec$, we have either $Z \ll sibling(Z)$ or $sibling(Z) \ll Z$. Then, by the definition of dominant regions (Definition 9), we have that either Z or $sibling(Z)$ is a dominant region for f and $\prec$. Let us assume that Z is a dominant region for f and $\prec$. Then, by Definition 10, we have $\xi(Z) = \xi(X)$ and $\xi(sibling(Z)) = f(u_X)$. Otherwise, if $sibling(Z)$ is a dominant region for f and $\prec$, we have $\xi(sibling(Z)) = \xi(X)$ and $\xi(Z) = f(u_X)$. Since both Z and $sibling(Z)$ include at least one minimum of w, we may say that there is a child Y of X for which the hypothesis 1, 2 and 3 hold true. □

Lemma 41. *Let $\prec$ be an altitude ordering for w and let f be a map from E into $\mathbb{R}$ such that f is one-side increasing for $\prec$. Let ξ be the approximated extinction map for f and $\prec$. Let u be a watershed-cut edge for $\prec$. Then, there is a minimum M of w such that $\xi(M) = f(u)$.*

Proof *As u is a watershed-cut edge for $\prec$, each child of R_u includes at least one minimum of w. Then, there is a minimum M of w such that $M \subset R_u$. By Lemma 40, there is a child Y_1 of R_u such that $\xi(Y_1) = f(u)$. If Y_1 is a minimum of w, then the property holds true. Otherwise, if Y_1 is not a minimum of w, it means that there is a minimum M of w such that $M \subset Y_1$. By Lemma 40, there is a child Y_2 of Y_1 such that $\xi(Y_2) = \xi(Y_1) = f(u)$ and such that there is a minimum of w included in Y_2. Again, if Y_2 is a minimum of w, then the property holds true. Otherwise, we can apply this same reasoning indefinitely. We can define a sequence $(Y_1, \ldots, Y_p)$ of regions of $\mathcal{B}_\prec$ where Y_p is a minimum of w and such that $\xi(Y_p) = \cdots = \xi(Y_1) = f(u)$ and $Y_i \subset Y_{i-1}$ for any i in $\{2, \ldots, p\}$. Therefore, there is a minimum Y_p included in R_u such that $\xi(Y_p) = f(u)$.* □

Lemma 42. *Let $\prec$ be an altitude ordering for w and let f be a map from E into $\mathbb{R}$ such that f is one-side increasing for $\prec$. Let ξ be the approximated extinction map for f and $\prec$. Let X be a region of $\mathcal{B}_\prec$ such that X contains at least one minimum of w. There exists a minimum $M \subseteq X$ such that $\xi(M) = \xi(X)$.*

Proof *If X is a minimum of w, then it is trivial. Otherwise, by Lemma 40, there is a child Y_1 of X such that $\xi(Y_1) = \xi(X)$ and such that there is a minimum of w included in Y_1. If Y_1 is a minimum of w, then the property holds true. Otherwise, by Lemma 40, there is a child Y_2 of Y_1 such that $\xi(Y_2) = \xi(Y_1) = \xi(X)$ and such that there is a minimum of w included in Y_2. Again, if Y_2 is a minimum of w, then the property holds true. Otherwise, we can apply this same reasoning indefinitely. We can define a sequence $(Y_1, \ldots, Y_p)$ of regions of $\mathcal{B}_\prec$ where Y_p is a minimum of w and such that $\xi(Y_p) = \cdots = \xi(Y_1) = \xi(X)$ and $Y_i \subset Y_{i-1}$ for any i in $\{2, \ldots, p\}$. Therefore, there is a minimum Y_p included in X such that $\xi(Y_p) = \xi(Y)$.* □

Proof *(of Lemma 39)*

In order to prove that

(1) for any two minima M_1 and M_2 of w, if $\xi(M_1) = \xi(M_2)$, then $M_1 = M_2$,

we will prove that

(2) for any two minima M_1 and M_2 of w, we have $\xi(M_1) \neq \xi(M_2)$.

As w has n minima, it suffices to prove that, for any i in $\{1, \ldots, n\}$, there is a minimum M of w such that $\xi(M) = i$.

By Lemma 41, for any watershed-cut edge u for $\mathcal{B}_\prec$, there is a minimum M such that $\xi(M) = f(u)$. By Lemma 37, for any i in $\{1, \ldots, n-1\}$, there is a watershed-cut edge such that $f(u) = i$. Then, for any i in $\{1, \ldots, n-1\}$, there is a minimum M of w such that $\xi(M) = i$.

Since, f is one-side increasing for $\prec$, we have $\vee\{f(v) \mid R_v \in V\} = \{0, \ldots, n-1\}$. Then, we can conclude that $\xi(V) = \vee\{f(v) \mid R_v \in V\} + 1 = (n-1) + 1 = n$. By Lemma 42, there is a minimum M of w such that $\xi(M) = \xi(V) = n$.

Therefore, for any i in $\{1, \ldots, n\}$, there is a minimum M of w such that $\xi(M) = i$. Since w has n minima, it implies that the values $\xi(M_1)$ and $\xi(M_2)$ are distinct for any pair (M_1, M_2) of distinct minima of w. Hence, for any two minima M_1 and M_2 of w, if $\xi(M_1) = \xi(M_2)$, then $M_1 = M_2$. □

Lemma 43. *Let $\prec$ be an altitude ordering for w and let f be a map from E into $\mathbb{R}$ such that f is one-side increasing for $\prec$. Let ξ be the approximated extinction map for f and $\prec$. For any region R of $\mathcal{B}_\prec$, we have $\xi_f(R) = \vee\{\xi_f(M)$ such that M is a minimum of w included in $R\}$.*

To prove Lemma 43, we introduce Lemma 44.

Lemma 44. *Let $\prec$ be an altitude ordering for w and let f be a map from E into $\mathbb{R}$ such that f is one-side increasing for $\prec$. Let ξ be the approximated extinction map for f and $\prec$. Let $\bigtriangledown$ be the supremum descendant map for f and $\prec$. Let X be a region of $\mathcal{B}_\prec$. Then $\xi(X)$ is greater than or equal to the supremum descendant value $\bigtriangledown(X)$ of X.*

Proof *We consider the following cases: (1) $X = V$, (2) $X \neq V$ and X is not a dominant region for f and $\prec$; and (3) X is a dominant region for f and $\prec$. Let $\ll$ be the non-leaf ordering for f and $\prec$.*

1. *If $X = V$, then $\xi(X) = \xi(V) = \bigtriangledown(V) + 1$ (first case of Definition 10). Then, $\xi(X)$ is clearly than $\bigtriangledown(X)$.*
2. *If $X \neq V$ and if X is not a dominant region for f and $\prec$, then $\xi(X) = f(u)$ (third case of Definition 10), where u is the building edge of the parent of X. By the definition of dominant regions, we consider two cases: (a) there is no minimum M of w such that $M \subseteq X$; or (b) $X \ll sibling(X)$.*
   - *(a) If there is no minimum M of w such that $M \subseteq X$, then there is no descendant of X whose building edge is a watershed-cut edge for $\prec$. Hence, for any edge v such that $R_v \subseteq X$, u is not a watershed-cut edge for $\prec$ and, since f is one-side increasing for $\prec$, we have $f(v) = 0$ Definition 3 (statement 2). Therefore, $\bigtriangledown(X) = 0$. By Definition 3 (statement 1), we have $\{f(v) \mid v \in E_\prec\} = \{0, \ldots, n-1\}$. Hence, $\xi(X)$, being equal to $f(u)$, is greater than or equal to $\bigtriangledown(X) = 0$.*
   - *(b) If $X \ll sibling(X)$, then, by the definition of non-leaf ordering, we have:*
     - *i. either $\bigtriangledown(X) < \bigtriangledown(sibling(X))$; or*
     - *ii. $\bigtriangledown(X) = \bigtriangledown(sibling(X))$ and $u_X \prec u_{sibling(X)}$.*

Thus, we have $\bigtriangledown(X) \leq \bigtriangledown(sibling(X))$. Since f is one-side increasing for $\prec$, by the statement 3 of Definition 3, there is a child Y of $parent(X)$ such that $f(u) \geq \vee\{f(v) \mid R_v \subseteq Y\}$. Hence, there is a child Y of $parent(X)$ such that $f(u) \geq \bigtriangledown(Y)$. Then, we have $f(u) \geq \bigtriangledown(X)$ or $f(u) \geq \bigtriangledown(sibling(X))$. In the case where $f(u) \geq \bigtriangledown(sibling(X))$, this also implies that $f(u) \geq \bigtriangledown(X)$ because $\bigtriangledown(X) \leq \bigtriangledown(sibling(X))$. Therefore, $\xi(X)$, being equal to $f(u)$, is greater than or equal to $\bigtriangledown(X)$.

3. *If X is a dominant region for f and $\prec$, then $\xi(X) = \xi(parent(X))$ (second case of Definition 10). We will prove by induction that this lemma holds true for any dominant region for f and $\prec$. In the base step, we consider that $parent(X)$ is V. In the inductive step, we show that, if the property holds true for $parent(X)$, then it also holds true for X. Please note that, if $parent(X)$ is not a dominant region for f and $\prec$, the property holds for $parent(X)$ as proven in the previous case.*
   (a) *Base step: if $parent(X)$ is V, then $\xi(X) = \xi(V) = \bigtriangledown(V) + 1$ (first case of Definition 10). We can see that $\bigtriangledown(V) \geq \bigtriangledown(X)$ because, for any edge u such that $R_u \subseteq X$, we also have $R_u \subseteq V$. Then, $\xi(X)$, being equal to $\bigtriangledown(V) + 1$, is greater than $\bigtriangledown(X)$.*
   (b) *Inductive step: let us assume that $\xi(parent(X)) \geq \bigtriangledown(parent(X))$. Since $\xi(X) = \xi(parent(X))$, we have $\xi(X) \geq \bigtriangledown(parent(X))$. We can affirm that, for any edge v in $E_\prec$ such that $R_v \subseteq X$, we also have $R_v \subseteq parent(X)$. Hence, $\bigtriangledown(parent(X)) \geq \bigtriangledown(X)$. Therefore, $\xi(X)$, being equal to $\xi(parent(X))$, is greater than or equal to $\bigtriangledown(X)$.* □

Proof (of Lemma 43) We will prove that, for any region X of $\mathcal{B}_\prec$, we have $\xi(X) = \vee\{\xi_f(M)$ such that M is a minimum of w included in $X\}$. Let X be a region of $\mathcal{B}_\prec$. We consider two cases: (1) there is a minimum of w included in X; and (2) there is no minimum of w included in X.

(1) If there is no minimum of w included in X, then X is not a dominant region for f and $\prec$. Then $\xi(X) = f(u)$ (third condition of Definition 10), where u is the building edge of $parent(X)$. The edge u is not a watershed-cut edge for $\prec$ because the child X of R_u does not include any minimum of w. Hence, since f is one-side increasing for $\prec$, by the statement 2 of Definition 3, we have $f(u) = 0$. Therefore, $\xi(X)$, being equal to $f(u)$, is also equal to $\vee\{\xi(M)$ such that M is a minimum of w included in $R\} = \vee\{\} = 0$.

(2) Let us assume that there is at least one minimum of w included in X. If X is a minimum of w, then $\xi(X) = \vee\{\xi_f(M)$ such that M is a minimum of w included in $X\} = \vee\{\xi_f(X)\}$.
In order to prove the case where X is not a minimum of w, we will first demonstrate that $\xi(X) \geq \vee\{\xi(Y) \mid Y \subseteq X\}$. To prove that $\xi(X) \geq \vee\{\xi(Y) \mid Y \subseteq X\}$, it is enough to demonstrate that, for any region Z of $\mathcal{B}_\prec$, we have $\xi(Z) \geq \vee\{\xi(Y) \mid Y \textit{ is a child of } Z\}$. Let Z be a region of $\mathcal{B}_\prec$. If Z is a leaf region of $\mathcal{B}_\prec$, then $\xi(Z) \geq \vee\{\xi(Y) \mid Y \textit{ is a child of } Z\} = \vee\{\} = 0$ because, by Lemma 38, $\xi(Z)$ is in $\{0, \ldots, n\}$. Let us now assume that Z is not a leaf region of $\mathcal{B}_\prec$ and let Y be a child of Z. If Y is a dominant region for f and $\prec$, then $\xi(Y) = \xi(Z)$ and, consequently, $\xi(Z) \geq \xi(Y)$. Otherwise, if Y is not a dominant region for f and $\prec$, then $\xi(Y) = f(v)$, where v is the building edge of Z. By Lemma 44, $\xi(Z) \geq \bigtriangledown(Z)$ and, consequently, $\xi(Z) \geq f(u)$. Hence, $\xi(Z) \geq \xi(Y)$.
We can now prove that $\xi(X) = \vee\{\xi_f(M)$ such that M is a minimum of w included in $X\}$ in the case where X is not a minimum of w. By Lemma 42, there is a minimum M of w such that $M \subset X$ and such that $\xi(M) = \xi(X)$. Let M be the minimum of w such that $\xi(M) = \xi(X)$. Since $\xi(X) \geq \vee\{\xi(Y) \mid Y \subseteq X\}$, we can say that $\xi(X) = \vee\{\xi_f(M')$ such that M' is a minimum of w included in $X\}$. □

E Proof of Lemma 12

(Lemma 12). *Let $\prec$ be an altitude ordering for w and let f be a map from E into $\mathbb{R}$ such that f is one-side increasing for $\mathcal{B}_\prec$. Then, for any u in $E_\prec$, we have:*

$$f(u) = min\{\xi(R) \textit{ such that } R \textit{ is a child of } R_u\}.$$

Proof Let u be an edge in $E_\prec$. By the definition of dominant regions, we have that at most one child of R_u is a dominant region for f and $\prec$. Therefore, there is a child of R_u which is not a dominant region for f and $\prec$. Let X be the child of R_u which is not a dominant region for f and $\prec$. Then, $\xi(X) = f(u)$ (by the third condition of Definition 10). If $sibling(X)$ is not a dominant region for f and $\prec$, then $\xi(sibling(X)) = f(u)$ as well and, consequently, $f(u) = min\{\xi(R)$ such that R is a child of $R_u\} = min\{f(u), f(u)\}$. Otherwise, let us assume that $sibling(X)$ is a dominant region for f and $\prec$. Then, $\xi(sibling(X)) = \xi(R_u)$. By Lemma 44, we can infer that $\xi(R_u) \geq f(u)$. Therefore, $min\{\xi(Y)$ such that Y is a child of $R_u\} = min\{\xi_f(X), \xi(sibling(X))\} = min\{f(u), \xi(R_u)\} = f(u)$. □

F Proof of Theorem 4

(Theorem 4). *Let $\mathcal{H}$ be a hierarchy on V. The hierarchy $\mathcal{H}$ is a hierarchical watershed of (G, w) if and only if there is an altitude ordering $\prec$ for w such that $\Phi(\mathcal{H})$ is one-side increasing for $\prec$.*

Proof We prove the forward and backward implications of Theorem 4 in Lemma 45 and Lemma 46, respectively.

Lemma 45. *Let $\mathcal{H}$ be a hierarchy on V. If the hierarchy $\mathcal{H}$ is a hierarchical watershed of (G, w), then there exists an altitude ordering $\prec$ for w such that $\Phi(\mathcal{H})$ is one-side increasing for $\prec$.*

Proof By Lemma 16, there is a sequence of minima $\mathcal{S}$ of w such that $\mathcal{H}$ is the hierarchy induced by $\prec$ and $\mathcal{S}$. In order to prove that $\Phi(\mathcal{H})$ is one-side increasing for $\prec$, by Definition 3, we will prove that the following three statements hold true:

1. *$\{\Phi(\mathcal{H})(e) \mid e \in E_\prec\} = \{0, \ldots, n-1\}$;*
2. *for any edge u in $E_\prec$, $\Phi(\mathcal{H})(u) > 0$ if and only if u is a watershed-cut edge for $\prec$; and*
3. *for any edge u in $E_\prec$, there exists a child R of R_u such that $\Phi(\mathcal{H})(u) \geq \vee\{\Phi(\mathcal{H})(v)$ such that R_v is included in $R\}$, where $\vee\{\} = 0$.*

In the sequel of this proof, let ρ and ϵ be respectively the persistence map and the extinction map for $\prec$ and $\mathcal{S}$.

1. *By Lemma 20, we have $\{\Phi(\mathcal{H})(e) \mid e \in E_\prec\} = \{\rho(e) \mid e \in E_\prec\}$. Then, as Lemma 19 states that the range of ρ is $\{0, \ldots, n-1\}$, we can conclude that $\{\Phi(\mathcal{H})(e) \mid e \in E_\prec\}$ is the set $\{0, \ldots, n-1\}$.*

2. *Let u be a building edge for $\prec$. Given the following propositions:*
   (a) u is a watershed-cut edge
   (b) $\Phi(\mathcal{H})(u) > 0$
   we will prove that (a) implies (b), and that not (b) implies not (a).
   If u is a watershed-cut edge for $\prec$, then both children of R_u contain at least one minimum of w. Therefore, the extinction value of both children of R_u is non-zero and, consequently, the persistence value $\rho(u)$ of u is non-zero. Moreover, by Lemma 20, in this case we have $\Phi(\mathcal{H})(e) = \rho(e)$ for any building edge e for $\prec$. Thus, $\Phi(\mathcal{H})(u)$ is non-zero.
   On the other hand, if u is not a watershed-cut edge for $\prec$, then there is a child X of R_u which does not contain any minimum of w. Therefore, the extinction value of X is equal to 0: $\epsilon(X) = 0$. Since, by definition $\rho(u) = min\{\epsilon(X), \epsilon(sibling(X))\}$ and the minimal extinction value is zero, we can say that $\rho(u) = 0$. Again, by Lemma 20, in this case we have $\Phi(\mathcal{H})(e) = \rho(e)$ for any building edge e for $\prec$ and thus, $\Phi(\mathcal{H})(u)$ is equal to 0.
3. *Let u be a building edge for $\prec$. The persistence value of u is the extinction value of a child X of R_u. Let X be a child of R_u such that $\rho(u)$, the persistence value of u, is equal to $\epsilon(X)$, the extinction value of X. By Lemma 17, for any region Y of $\mathcal{B}_\prec$ such that $Y \subseteq X$, we have $\epsilon(Y) \leq \epsilon(X)$ and, as $X \subseteq R_u$, $\epsilon(Y) \leq \epsilon(R_u)$. Let v be the building edge of a region $Z \subseteq X$. Then, we can say that the extinction value of both children of Z is less than or equal to the extinction value $\epsilon(X)$. Hence, $\rho(v) \leq \epsilon(X)$ and, then, $\rho(v) \leq \rho(u)$. By Lemma 20, we can conclude that $\Phi(\mathcal{H})(v) \leq \Phi(\mathcal{H})(u)$. Hence, $\Phi(\mathcal{H})(u) \geq \vee\{\Phi(\mathcal{H})(v)$ such that R_v is included in $X\}$.* □

Lemma 46. *Let $\mathcal{H}$ be a hierarchy on V and let $\prec$ be an altitude ordering such that $\Phi(\mathcal{H})$ is one-side increasing for $\prec$. Then the hierarchy $\mathcal{H}$ is a hierarchical watershed of (G, w).*

Proof *Let ξ be the approximated extinction map for $\Phi(\mathcal{H})$ and $\prec$. By Lemma 12, for any edge in $E_\prec$, we have $\Phi(\mathcal{H})(u) = min\{\xi(R)$ such that R is a child of $R_u\}$. By Lemma 11, the map ξ is an extinction map for $\prec$. Then, by the backward implication of Property 7, the hierarchy $\mathcal{H}$ is a hierarchical watershed of (G, w).* □

G Proof of Property 14

(Property 14). *Let $\mathcal{H}$ be a hierarchy on V. The hierarchy $\mathcal{H}$ is a flattened hierarchical watershed of (G, w) if and only if there is an altitude ordering $\prec$ for w such that:*

1. *$(V, E_\prec)$ is a MST of $(G, \Phi(\mathcal{H}))$; and*
2. *for any edge u in $E_\prec$, if u is not a watershed-cut edge for $\prec$, then $\Phi(\mathcal{H})(u) = 0$; and*
3. *for any edge u in $E_\prec$, there exists a child R of R_u such that $\Phi(\mathcal{H})(u) \geq \vee\{\Phi(\mathcal{H})(v)$ such that R_v is included in $R\}$, where $\vee\{\} = 0$.*

To prove Property 14, we establish the following lemma.

Lemma 47. *Let $\prec$ be an altitude ordering for w and let $\mathcal{H}$ be a hierarchy on V such that $\Phi(\mathcal{H})$ is one-side increasing for $\prec$. Then $(V, E_\prec)$ is a MST of $(G, \Phi(\mathcal{H}))$.*

Proof *Let α denote the sum of the weight of the edges in $E_\prec$ in the map $\Phi(\mathcal{H})$: $\alpha = \sum_{e \in E_\prec} \Phi(\mathcal{H})(e)$. As $\Phi(\mathcal{H})$ is one-side increasing for $\prec$, by the condition 1 of Definition 3, we can affirm that $\alpha = 0 + 1 + \cdots + n - 1$. In order to prove that $(V, E_\prec)$ is a MST of $(G, \Phi(\mathcal{H}))$, we will prove that, for any MST G' of $(G, \Phi(\mathcal{H}))$, the sum of the weight of the edges in G' is greater than or equal to α. Let G' be a MST of $(G, \Phi(\mathcal{H}))$. As G' is a MST of $(G, \Phi(\mathcal{H}))$, by the condition 1 of Lemma 21, we have that G and G' have the same quasi-flat zones hierarchy: $\mathcal{QFZ}(G, \Phi(\mathcal{H})) = \mathcal{QFZ}(G', \Phi(\mathcal{H}))$. As $\Phi(\mathcal{H})$ is the saliency map of $\mathcal{H}$, we have that $\mathcal{H} = \mathcal{QFZ}(G, \Phi(\mathcal{H}))$. Therefore, $\mathcal{H} = \mathcal{QFZ}(G', \Phi(\mathcal{H}))$. Let i be a value in $\{1, \ldots, n-1\}$. By the condition 1 of Definition 3, we can say that $\{1, \ldots, n-1\}$ is a subset of the range of $\Phi(\mathcal{H})$. Therefore, $\mathcal{H}$ is composed of at least n distinct partitions. Let $\mathcal{H}$ be the sequence $(\mathbf{P}_0, \ldots, \mathbf{P}_{n-1}, \ldots)$. Since the partitions $\mathbf{P}_i$ and $\mathbf{P}_{i-1}$ are distinct, then there exists a region in $\mathbf{P}_i$ which is not in $\mathbf{P}_{i-1}$. Therefore, there is a region X of $\mathbf{P}_i$ which is composed of a several regions $\{R_1, R_2, \ldots\}$ of $\mathbf{P}_{i-1}$. Then, there are two adjacent vertices x and y such that x and y are in distinct regions in $\{R_1, R_2, \ldots\}$. Let x and y be two adjacent vertices such that x and y are in distinct regions in $\{R_1, R_2, \ldots\}$. Hence, the lowest j such that x and y belong to the same region of $\mathbf{P}_j$ is i. Thus, there exists an edge $u = \{x, y\}$ in $E_\prec$ such that $\Phi(\mathcal{H})(u) = i$. Hence, the sum of the weight of the edges of G' is at least $1 + \cdots + n - 1$, which is equal to α. Therefore, the graph $(V, E_\prec)$ is a MST of $(G, \Phi(\mathcal{H}))$.* □

The reader can observe that the statement 3 of the above property is precisely the statement 3 of the definition of one-side increasing maps (Definition 3), and that the statement 2 is an implication of the statement 2 of Definition 3. The statement 1 of the above property corresponds to a property of one-side increasing maps established in Lemma 47.

In order to prove Property 14, we establish some auxiliary lemmas on MSTs and saliency maps.

In the following, we state a well-known property of spanning trees in Lemma 48.

Let x and y be two vertices in V and let $\pi = (x_0, \ldots, x_p)$ be a path from x to y. For any edge $u = \{x_{i-1}, x_i\}$ for i in $\{1, \ldots, p\}$, we say that u *is in* π or that π *includes* u.

Lemma 48. *Let G' be a spanning tree of a weighted graph (G, f). Let $u = \{x, y\}$ be an edge in $E \setminus E(G')$ and let π be the path from x to y (resp. y to x) in G'. The graph G' is a MST of (G, f) if and only if $f(u) \geq f(v)$ for any edge v in π.*

The following lemma characterizes MSTs of saliency maps.

Lemma 49. *Let f be the saliency map of a hierarchy on V and let G' be a spanning tree of (G, f). Let $u = \{x, y\}$ be an edge in $E \setminus E(G')$ and let π be the path from x to y (resp. y to x) in G'. Let v be an edge of greatest weight in π. The graph G' is a MST of (G, f) if and only if $f(u) = f(v)$.*

Proof *We will first prove the forward implication of this lemma. Let G' be a MST of $(G, \Phi(\mathcal{H}))$. Then, by Lemma 48, for any edge e in the path π, we have $\Phi(\mathcal{H})(e) \leq \Phi(\mathcal{H}(u)$. Hence, $\Phi(\mathcal{H})(v) \leq \Phi(\mathcal{H}(u)$. Let us assume that $\Phi(\mathcal{H})(v) < \Phi(\mathcal{H})(u)$. Then, given $\lambda = \Phi(\mathcal{H})(v)$, in the λ-level set of $(G, \Phi(\mathcal{H}))$, the vertices x and y are connected, which implies that, by the definition of saliency maps, $\Phi(\mathcal{H}(u)$ is less or equal to $\Phi(\mathcal{H})(v)$, which contradicts our assumption. Hence, $\Phi(\mathcal{H})(v) = \Phi(\mathcal{H}(u)$.*

Now, let us assume that $\Phi(\mathcal{H})(u)$ is equal to the greatest weight among the edges in π. Then, for any edge e in the path π, we have $\Phi(\mathcal{H})(e) \leq \Phi(\mathcal{H}(u)$. Then, by Lemma 48, G' is a MST of $(G, \Phi(\mathcal{H}))$. □

Lemma 50. *Let $\mathcal{H}'$ be a hierarchy on V and let $\mathcal{H}$ be a flattening of $\mathcal{H}'$. Let u and v be two distinct edges in E such that $\Phi(\mathcal{H})(u) < \Phi(\mathcal{H})(v)$. Then $\Phi(\mathcal{H}')(u) < \Phi(\mathcal{H}')(v)$.*

Proof *Let $u = \{x_1, y_1\}$ and $v = \{x_2, y_2\}$. As $\Phi(\mathcal{H})(u) < \Phi(\mathcal{H})(v)$, there is a partition $\mathbf{P}$ of $\mathcal{H}$ such that x_1 and y_1 belong to the same region of $\mathbf{P}$ and we such that x_2 and y_2 do not belong to the same region of $\mathbf{P}$. As $\mathbf{P}$ is a partition of $\mathcal{H}'$, there is a partition in $\mathcal{H}'$ such that x_1 and y_1 belong to the same region of this partition but x_2 and y_2 do not. Then, $\Phi(\mathcal{H}')(u) < \Phi(\mathcal{H}')(v)$.* □

Lemma 51. *Let $\mathcal{H}'$ be a hierarchy on V and let $\mathcal{H}$ be a flattening of $\mathcal{H}'$. Let u and v be two distinct edges in E such that $\Phi(\mathcal{H}')(u) \leq \Phi(\mathcal{H}')(v)$. Then $\Phi(\mathcal{H})(u) \leq \Phi(\mathcal{H})(v)$.*

Proof *Let $u = \{x_1, y_1\}$ and $v = \{x_2, y_2\}$. As $\Phi(\mathcal{H}')(u) \leq \Phi(\mathcal{H}')(v)$, then for any partition $\mathbf{P}$ of $\mathcal{H}'$, if x_2 and y_2 are in the same region of $\mathbf{P}$, then x_1 and y_1 are in the same region of $\mathbf{P}$ as well. As any partition of $\mathcal{H}$ is also a partition of $\mathcal{H}'$, we may say that for any partition $\mathbf{P}$ of $\mathcal{H}$, if x_2 and y_2 are in the same region of $\mathbf{P}$, then x_1 and y_1 are in the same region of $\mathbf{P}$. Hence, $\Phi(\mathcal{H})(u) \leq \Phi(\mathcal{H})(v)$.* □

The forward and backward implications of Property 14 are proven in Lemmas 52 and 53, respectively.

Lemma 52. *Let $\mathcal{H}$ be a flattened hierarchical watershed of (G, w). Then, there is an altitude ordering $\prec$ for w such that:*

1. *$(V, E_\prec)$ is a MST of $(G, \Phi(\mathcal{H}))$; and*
2. *for any building edge u for $\prec$, if u is not a watershed-cut edge for $\prec$, then $\Phi(\mathcal{H})(u) = 0$; and*
3. *for any building edge u for $\prec$, there exists a child R of R_u such that $\Phi(\mathcal{H})(u) \geq \vee\{\Phi(\mathcal{H})(v)$ such that R_v is included in $R\}$, where $\vee\{\} = 0$.*

Proof *As $\mathcal{H}$ is a flattened hierarchical watershed of (G, w), by Definition 13, there is a hierarchical watershed $\mathcal{H}_w$ of (G, w) such that $\mathcal{H}$ is a flattening of $\mathcal{H}_w$. By Theorem 4, there is an altitude ordering $\prec$ for w such that $\Phi(\mathcal{H}_w)$ is one-side increasing for $\prec$. Let $\prec$ be the altitude ordering for w such that $\Phi(\mathcal{H}_w)$ is one-side increasing for $\prec$. By Lemma 22, $(V, E_\prec)$ is a MST of $(G, \Phi(\mathcal{H}_w))$. Let G' denote the graph $(V, E_\prec)$. By Lemma 21, $\mathcal{H}_w$ is the hierarchy $\mathcal{QFZ}(G', \Phi(\mathcal{H}_w))$. Then, any partition of $\mathcal{H}$ is a partition of $\mathcal{QFZ}(G', \Phi(\mathcal{H}_w))$. By the definition of saliency maps, we can affirm that any partition of $\mathcal{QFZ}(G, \Phi(\mathcal{H}))$ is a partition of $\mathcal{QFZ}(G', \Phi(\mathcal{H}_w))$.*

In the following, we will prove that the three statements hold true for $\prec$.

1. *We will first prove that G' is a MST of $(G, \Phi(\mathcal{H}))$. By contradiction, let us assume that G' is not a MST of $(G, \Phi(\mathcal{H}))$. Then, by Lemma 49, there is an edge $u = \{x, y\}$ such that u is in $E \setminus E(G')$ and such that $\Phi(\mathcal{H})(u)$ is different from the greatest weight among the edges in the path π from x to y in $(G', \Phi(\mathcal{H}))$. Let v be an edge of greatest weight in π. As $\mathcal{H}$ is equal to $\mathcal{QFZ}(G, \Phi(\mathcal{H}))$, we may affirm that $\Phi(\mathcal{H})(u)$ is lower than $\Phi(\mathcal{H})(v)$ because, otherwise, the vertices x and y would be connected in the λ-level set of $(G, \Phi(\mathcal{H}))$ for a λ lower than $\Phi(\mathcal{H})(u)$, which contradicts the fact that $\Phi(\mathcal{H})$ is a saliency map. Hence, we have $\Phi(\mathcal{H})(u) < \Phi(\mathcal{H})(v)$. Then, by Lemma 51, as $\mathcal{H}$ is a flattening of $\mathcal{H}_w$, we may conclude that $\Phi(\mathcal{H}_w)(u) < \Phi(\mathcal{H}_w)(v)$. Hence, the weight $\Phi(\mathcal{H}_w)(u)$ is different from the greatest weight among the edges in the path π. Therefore, by Lemma 49, G' is not a MST of $(G, \Phi(\mathcal{H}_w))$, which contradicts our assumption. Hence, we may conclude that G' is a MST of $(G, \Phi(\mathcal{H}))$.*
2. *We will now prove the second condition for $\mathcal{H}$ to be a flattened hierarchical watershed of (G, w). As $\mathcal{H}_w$ is one-side increasing for $\prec$, by the second condition of Definition 3, for any watershed-cut edge $u = \{x, y\}$ for $\prec$, we have $\Phi(\mathcal{H}_w)(u) = 0$. Then, for any partition $\mathbf{P}$ of $\mathcal{H}_w$, x and y belong to the same region of $\mathbf{P}$. Therefore, as any partition of $\mathcal{H}$ is a partition of $\mathcal{H}_w$, we can say that, for any partition $\mathbf{P}$ of $\mathcal{H}$, x and y belong to the same region of $\mathbf{P}$. Hence, the lowest λ such that x and y are the same partition $\mathbf{P}_\lambda$ of $\mathcal{H}$ is zero. Hence, $\Phi(\mathcal{H})(u) = 0$.*
3. *We will now prove the third condition for $\mathcal{H}$ to be a flattened hierarchical watershed of (G, w). By the third statement of Definition 3, we have that, for any edge u in $E_\prec$, there exists a child R of R_u such that $\Phi(\mathcal{H}_w)(u) \geq \vee\{\Phi(\mathcal{H}_w)(v) \mid R_v \subseteq R\}$. Let u be an edge in $E_\prec$ and let R be the child of R_u such that $\Phi(\mathcal{H}_w)(u) \geq \vee\{\Phi(\mathcal{H}_w)(v) \mid R_v \subseteq R\}$. Let v be an edge in $E_\prec$ such that $R_v \subseteq R$. Then, $\Phi(\mathcal{H}_w)(u) \geq \Phi(\mathcal{H}_w)(v)$. Hence, by Lemma 51, $\Phi(\mathcal{H})(u) \geq \Phi(\mathcal{H})(v)$. Therefore, we may conclude that $\Phi(\mathcal{H})(u) \geq \vee\{\Phi(\mathcal{H})(v) \mid R_v \subseteq R\}$.* □

The following lemma corresponds to the backward implication of Property 14.

Lemma 53. *Let $\mathcal{H}$ be a hierarchy on V and let $\prec$ be an altitude ordering for w such that:*

1. *$(V, E_\prec)$ is a MST of $(G, \Phi(\mathcal{H}))$; and*
2. *for any edge u in $E_\prec$, if u is not a watershed-cut edge for $\prec$, then $\Phi(\mathcal{H})(u) = 0$; and*
3. *for any edge u in $E_\prec$, there exists a child R of R_u such that $\Phi(\mathcal{H})(u) \geq \vee\{\Phi(\mathcal{H})(v)$ such that R_v is included in $R\}$, where $\vee\{\} = 0$.*

Then $\mathcal{H}$ is a flattened hierarchical watershed of (G, w).

In order to prove Lemma 53, we first state two auxiliary lemmas. From Property 10 of [10], we can deduce the following lemma linking binary partition hierarchies and MSTs.

Lemma 54. *Let $\mathcal{B}$ be a binary partition hierarchy of (G, w). The graph $(V, E_\prec)$ is a MST of (G, w).*

By Property 12 of [10] linking hierarchical watersheds and hierarchies induced by an altitude ordering and a sequence of minima, and by Lemma 21, we infer the following lemma.

Lemma 55. *Let G' be a MST of (G, w) and let $\mathcal{H}$ be a hierarchical watershed of (G', w). Then $\mathcal{H}$ is also a hierarchical watershed of (G, w).*

Proof (of Lemma 53) Let $\mathcal{H}$ be a hierarchy on V such that there is an altitude ordering $\prec$ for w such that:

1. $(V, E_\prec)$ is a MST of $(G, \Phi(\mathcal{H}))$; and
2. for edge u in $E_\prec$, if u is not a watershed-cut edge for $\prec$, then $\Phi(\mathcal{H})(u) = 0$; and
3. for edge u in $E_\prec$, there exists a child R of R_u such that $\Phi(\mathcal{H})(u) \geq \vee\{\Phi(\mathcal{H})(v)$ such that R_v is included in $R\}$, where $\vee\{\} = 0$.

We will prove that $\mathcal{H}$ is a flattened hierarchical watershed of (G, w). To this end, we will prove that there is a hierarchical watershed $\mathcal{H}_w$ of (G, w) such that any partition of $\mathcal{H}$ is also a partition of $\mathcal{H}_w$. Let G' denote the graph $(V, E_\prec)$. By Lemma

54, G' is a MST of (G, w). Moreover, by Lemma 55, given a hierarchical watershed $\mathcal{H}_w$ of a MST of (G, w), we can say that $\mathcal{H}_w$ is also a hierarchical watershed of (G, w). Hence, we can simply prove that there is a hierarchical watershed $\mathcal{H}_w$ of (G', w) such that any partition of $\mathcal{H}$ is also a partition of $\mathcal{H}_w$.

To define the hierarchy $\mathcal{H}_w$, we first define a map f from $E_\prec$ into $\mathbb{R}$ such that f is one-side increasing for $\prec$. Since G' is a tree, by the definition of saliency maps, we can say that f is the saliency map of the hierarchy $\mathcal{QFZ}(G', f)$. By Theorem 4, as f is one-side increasing for $\prec$, we may say that $\mathcal{QFZ}(G', f)$ is a hierarchical watershed of (G', w).

In the map f, the edges which are not watershed-cut edges for $\prec$ are assigned to zero, and the watershed-cut edges for $\prec$ are ranked according to their weights in w and in $\Phi(\mathcal{H})$. Let $\prec_2$ be a total ordering on the set $\{u$ is a watershed-cut edge for $\prec\}$ such that, for any two watershed-cut edges u and v for $\prec$, we have $u \prec_2 v$ if and only if $\Phi(\mathcal{H})(u) < \Phi(\mathcal{H})(v)$ or if $\Phi(\mathcal{H})(u) = \Phi(\mathcal{H})(v)$ and $u \prec v$. The map f is defined as follows:

$$f(u) = \begin{cases} 0 & if\ u\ is\ not\ a\ watershed-cut \\ & edge\ for\ \prec \\ rank\ of\ u\ for\ \prec_2 & otherwise \end{cases} \tag{2}$$

We first demonstrate that f is one-side increasing for $\prec$.

1. By the definition of f, as there are $n-1$ watershed-cut edges for $\prec$, we can say that, for any i in $\{1, \dots, n-1\}$, there is a watershed-cut edge u for $\prec$ such that the rank of u for $\prec_2$ is i and, consequently, $f(u) = i$. On the other hand, as w has at least one minimum, there is at least one edge e in $E_\prec$ such that e is not a watershed-cut edge for $\prec$ and such that $f(e) = 0$. Hence, we have $\{f(e) \mid u \in E_\prec\} = \{0, \dots, n-1\}$. Therefore, the statement 1 of Definition 3 holds true for f.
2. For any edge u, by the definition of f, $f(u)$ is non-zero if and only if u is not a watershed-cut edge for $\prec$, so the statement 2 of Definition 3 holds true for f.
3. Let u be a building edge for $\prec$. If u is not a watershed-cut edge for $\prec$, then there is a child X of R_u such that there is no minimum of w included in X. Hence, none of the building edges of the descendants of X is a watershed-cut edge for $\prec$. By the definition of f, we have $f(u) = 0$ and, for any edge v such that $R_v \subseteq X$, we have $f(v) = 0$. Hence, $f(u) \geq \vee\{f(v)$ such that R_v is included in $X\}$. Otherwise, let us assume that u is a watershed-cut edge for $\prec$. Then there is at least one minimum of w included in each child of R_u. By the hypothesis 3, there is a child X of R_u such that $\Phi(\mathcal{H})(u) \geq \vee\{\Phi(\mathcal{H})(v)$ such that R_v is included in $X\}$. Let X be the child of R_u such that $\Phi(\mathcal{H})(u) \geq \vee\{\Phi(\mathcal{H})(v)$ such that R_v is included in $X\}$. Let e be a building edge for $\prec$ such that $R_e \subseteq X$. If e is not a watershed-cut edge for $\prec$, then $f(e) = 0$ and, consequently, $f(u) > f(e)$. Otherwise, if e is a watershed-cut edge for $\prec$, then we have $\Phi(\mathcal{H})(u) \geq \Phi(\mathcal{H})(e)$ and $e \prec u$, which implies that $e \prec_2 u$. Consequently, by the definition of f, we have $f(u) > f(e)$. Therefore, $f(u) \geq \vee\{f(v)$ such that R_v is included in $X\}$. Then, the third condition of Definition 3 holds true for f.

Hence, f is one-side increasing for $\prec$ and, as stated previously, $\mathcal{QFZ}(G', f)$ is a hierarchical watershed of (G', w) (resp. (G, w)). Now, we only need to prove that any partition of $\mathcal{H}$ is a partition of $\mathcal{QFZ}(G', f)$. By the hypothesis 1, G' is a MST of $(G, \Phi(\mathcal{H}))$. Then, by Lemma 21, we can say that $\mathcal{H}$ is the QFZ hierarchy of $(G', \Phi(\mathcal{H}))$. We will prove that any partition of $\mathcal{QFZ}(G', \Phi(\mathcal{H}))$ is also a partition of $\mathcal{QFZ}(G', f)$.

Let the range of $\Phi(\mathcal{H})$ be the set $\{0, \dots, \ell\}$: $\{\Phi(\mathcal{H})(u) \mid u \in E_\prec\} = \{0, \dots, \ell\}$. Let λ be a value in $\{0, \dots, \ell\}$. Let $G'_{\lambda, \Phi(\mathcal{H})}$ be the λ-level set of $(G', \Phi(\mathcal{H}))$. Let α be the greatest value in $\{f(u) \mid u \in E(G'_{\lambda, \Phi(\mathcal{H})})\}$. We will prove that the α-level set of (G', f) is equal to the λ-level set of $(G', \Phi(\mathcal{H}))$. Since α is the greatest value in the set $\{f(u) \mid u \in E(G'_{\lambda, \Phi(\mathcal{H})})\}$, we can see that any edge v in the λ-level set of $(G', \Phi(\mathcal{H}))$ also belongs to the α-level set of (G', f). Now, we also need to prove that there is no edge u in the α-level set of (G', f) such that u is not in the λ-level set of $(G', \Phi(\mathcal{H}))$.

Let u be an edge which is not in the λ-level set of $(G', \Phi(\mathcal{H}))$. Then, $\Phi(\mathcal{H})(u) > \lambda$ and, for any edge v in the λ-level set of $(G', \Phi(\mathcal{H}))$, we have $\Phi(\mathcal{H})(u) > \Phi(\mathcal{H})(v)$. Since the minimum value of λ is zero, we can say that $\Phi(\mathcal{H})(u) > 0$ and, by the hypothesis 2, u is a watershed-cut edge for $\prec$. Let v be an edge in the λ-level set of $(G', \Phi(\mathcal{H}))$. Since $\Phi(\mathcal{H})(u) > \Phi(\mathcal{H})(v)$, if v is a watershed-cut edge for $\prec$, then $v \prec_2 u$ and $f(u) > f(v)$. Otherwise, if v is not a watershed-cut edge for $\prec$, by the definition of f, we have $f(v) = 0$ and $f(u) > f(v)$. Thus, for any edge v in the λ-level set of $(G', \Phi(\mathcal{H}))$, we have $f(u) > f(v)$ and, therefore, $f(u) > \alpha$. Then, u is not in the α-level set of (G', f).

Therefore, we can conclude that the α-level set of (G', f) is equal to the λ-level set of $(G', \Phi(\mathcal{H}))$. As the partitions of $\mathcal{H}$ are given by the set of connected components of the level sets of $(G', \Phi(\mathcal{H}))$, we can affirm that any partition of $\mathcal{H}$ is also a partition of $\mathcal{QFZ}(G', f)$. Therefore, there is a hierarchical watershed $\mathcal{H}_w = \mathcal{QFZ}(G', f)$ of (G', w) (resp. (G, w)) such that any partition of $\mathcal{H}$ is also a partition of $\mathcal{H}_w$. Then, $\mathcal{H}$ is a flattened hierarchical watershed of (G', w) (resp. (G, w)). $\square$