# Stability and asymptotic properties of a linearized hydrodynamic medium model for dispersive media in nanophotonics

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**Abstract**

We analyze the stability of a linearized hydrodynamical model describing the response of nanometric dispersive metallic materials illuminated by optical light waves that is the situation occurring in nanoplasmonics. This model corresponds to the coupling between the Maxwell system and a PDE describing the evolution of the polarization current of the electrons in the metal. We show the well posedness of the system, polynomial stability and optimal energy decay rate. We also investigate the numerical stability for a discontinuous Galerkin type approximation and several explicit time integration schemes.

**AMS (MOS) subject classification** 35Q61, 93D20, 35B35, 65M12
**Key Words** Maxwell's equations, dispersive media, stability

## 1 Introduction

Nanophotonics is the field that manages to exploit the interaction of light with nanometer scaled structures. With, nowadays, the ability of designing nanometer scaled devices, came the exponential growth of potential applications of nanophotonics. Subwavelength imaging is one of the famous example see *e.g.* [24, 9] and references therein. Most of very interesting features in nanophotonics come from the possibility to enhance fields leading to the creation of very good absorbers or emitters (see one example in *e.g.* [9, 33, 23]). All these reasons make nanophotonics a very active field of research. Nanoplasmonics, one of the major subfield of Nanophotonics is of particular interest. It is based on the exploitation of plasmons (see [21] for a physical insight). These occur when the light interact with nanoscaled *metals*. Modelling is at the heart of the understanding of nanoplasmonics. It relies on the description of the reaction of the electrons of the metal to an applied external electric field. Popular classical models rely on a mechanical description of the movement of the electrons. These descriptions lead to the famous Drude and Drude-Lorentz models that are describing the electric dispersive nature of metals at optical frequencies. Indeed, electrons exhibit a delay in response to the applied electric field and a polarization that characterizes a dispersive media. These models give very good results when the size of the device is not smaller than $\simeq 15nm$.

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Below this threshold, the repulsive interaction between electrons in the metal do play an important role. Models that take into account of these effects are called "non-local" in the sense that the reaction of the electron not only depends on the applied electric field at its precise position but also on the field around it. To model these effects, one can describe the metal as a fluid of electron and makes use of a hydrodynamical description (see [4]). This is the point of view that we adopt in this paper and we focus on the linear response of such systems. The equations that we consider come from a linearization of the non linear hydrodynamical model around a static equilibrium. We refer the reader to [4] for details. The resulting system of equations is a linear hyperbolic system of PDE's that encompass the "non-local" character of the response through a linearized quantum pressure term. These equations write formally as:

$$\left\{\begin{array}{l} \varepsilon_0\varepsilon_L\partial_t E - \operatorname{curl} H = -J, \\ \mu\partial_t H + \operatorname{curl} E = 0, \\ J_{tt} + \gamma\partial_t J - \beta^2\nabla(\operatorname{div} J) = \varepsilon_0\omega_p^2\partial_t E, \end{array}\right. \tag{1.1}$$

The system of PDE consists in a linear coupling between Maxwell's equations (with $(E, H)$ the electromagnetic field) with a PDE that describes the evolution of the polarization current $J$. Classically, $\varepsilon_0$, the vacuum permeability, $\varepsilon_L$, the relative permeability of the media and $\mu$, its permittivity, are physical constants. Furthermore, $\omega_p^2$ is the plasma frequency and $\beta$ is the so-called "nonlocal" parameter. One should notice that if $\beta = 0$, the system reduces to Drude dispersive model. The system (1.1) has been first investigated numerically in [13] (with Nédélec elements), and later in [30] (with a Discontinuous Galerkin framework) with an emphasis on computational aspects; the benefit for nanoplasmonics has been shown. However, no theoretical study of the continuous model was provided in the latter. In [15], well posedness has been investigated for (1.1), with zero normal trace for current $J$, using variational techniques without considering charge conservation. Let us also mention a similar study of existence and uniqueness that can be found in [8] (in german) together with a numerical approximation based on a splitting scheme. In this work, we first investigate the question of well posedness for several types of boundary conditions with the point of view of semigroup theory and including charge conservation, inherent to this system. Stability is an important feature with regards to the complete understanding of the phenomenon and has also an impact on the development of adapted numerical frameworks. We thus also propose to investigate polynomial stability and optimal polynomial decay. This has been studied in details for all classical dispersive media in [26] but not for the more involved system (1.1) for which we propose to extend the latter results. We are also concerned with the behavior of numerical schemes with respect to (polynomial) stability. In [15], the authors also proposed a conforming space discretization framework with a leap-frog time integration strategy and provide some numerical analysis of it and academic convergence test cases. Here, we adopt a different point of view and propose to push the numerical analysis further. We especially focus on discrete stability and discrete energy decay. We use the Discontinuous Galerkin discretization framework of [30, 31, 29] combined with several explicit time integration schemes (from Leap-frog to explicit Runge-Kutta schemes). We concentrate on establishing, using energy techniques precise stability results, with CFL condition explicit in the physical parameters and polynomial orders. Furthermore, we prove that the charge constraint inherent to (1.1), is weakly preserved at the discrete level. Last we provide some 2D numerical tests that study the precise type of discrete energy decay.

The paper is organized as follows: in section 2 we present the different notations and the model. The well-posedness of the problem is then proved in section 3 by using semi-group theory. Section

4 is devoted to the polynomial decay of the energy. In section 5, we look at the optimality of the polynomial decay. Finally, in section 6, we investigate the numerical approximation and provide some numerical stability results.

# 2 Well-posedness of the systems

## 2.1 Notations

Let $\Omega$ be a open bounded simply connected Lipschitz domain of $\mathbb{R}^2$ or $\mathbb{R}^3$. We will denote by $\Gamma$ its boundary. The $L^2(\Omega)$-inner product (resp. norm) will be denoted by $\langle\cdot,\cdot\rangle$ (resp. $\|\cdot\|$). The usual norm and semi-norm of $H^s(\Omega)$ $(s \geq 0)$ are denoted by $\|\cdot\|_{s,\Omega}$ and $|\cdot|_{s,\Omega}$, respectively. For $s = 0$ we drop the index $s$.

For further uses, let us introduce the following spaces:

$$H_0^1(\Omega) := \{u \in H^1(\Omega) | u = 0 \text{ on } \partial\Omega\},$$

that is a Hilbert space for the inner product

$$\int_\Omega \nabla u \cdot \nabla v \, dx, \forall u, v \in H_0^1(\Omega).$$

Set

$$\begin{aligned} H_0(\mathrm{div};\Omega) &= \{\chi \in L^2(\Omega)^3 | \operatorname{div}\chi \in L^2(\Omega) \text{ and } \chi \cdot n = 0 \text{ on } \Gamma\}, \\ \mathcal{K}(\Omega) &= \{\chi \in L^2(\Omega)^3 | \mathrm{div}\chi = 0\}, \\ \hat{\mathcal{K}}(\Omega) &= \{\chi \in \mathcal{K}(\Omega) | \chi \cdot n = 0 \text{ on } \Gamma\} = \mathcal{K}(\Omega) \cap H_0(\mathrm{div};\Omega). \end{aligned}$$

Similarly, we recall that

$$\begin{aligned} H(\mathrm{curl};\Omega) &= \{\chi \in L^2(\Omega)^3 | \operatorname{curl}\chi \in L^2(\Omega)^3\}, \\ H_0(\mathrm{curl};\Omega) &= \{\chi \in L^2(\Omega)^3 | \operatorname{curl}\chi \in L^2(\Omega)^3 \text{ and } \chi \times n = 0 \text{ on } \Gamma\}. \end{aligned}$$

Recall also the spaces

$$\begin{aligned} X_T(\Omega) := H(\mathrm{curl};\Omega) \cap H_0(\mathrm{div},\Omega) &= \{\chi \in H_0(\mathrm{div},\Omega) | \operatorname{curl}\chi \in L^2(\Omega)^3\}, \\ X_N(\Omega) := H(\mathrm{div},\Omega) \cap H_0(\mathrm{curl};\Omega) &= \{\chi \in H(\mathrm{div},\Omega) | \operatorname{curl}\chi \in L^2(\Omega)^3 \text{ and } \chi \times n = 0 \text{ on } \Gamma\}, \end{aligned}$$

both are Hilbert space with the norm

$$\|\chi\|^2_{X(\Omega)} = \int_\Omega (|\operatorname{curl}\chi|^2 + |\operatorname{div}\chi|^2)\, dx.$$

Recall that the next Green's formula holds (see Lemma 3.1 of [25] or Lemma 2.5 of [11, p. 91]):

$$\int_\Omega (\operatorname{curl} E \cdot E' + E \cdot \operatorname{curl} E')\, dx = 0, \forall E \in H_0(\mathrm{curl},\Omega), E' \in H(\mathrm{curl},\Omega). \tag{2.1}$$

We also denote $\mathcal{O} := \Omega \times ]0, +\infty[$ and $\Sigma := \Gamma \times ]0, +\infty[$.

## 2.2 Mixed first order form of the model

This model, based on a linearization of a hydrodynamical model that describes the metal as an electron gas [4], reads:

$$\begin{cases} \varepsilon_0\varepsilon_L\partial_t E - \operatorname{curl} H = -J \text{ in } \mathcal{O}, \\ \mu\partial_t H + \operatorname{curl} E = 0 \text{ in } \mathcal{O}, \\ J_{tt} + \gamma\partial_t J - \beta^2\nabla(\operatorname{div} J) = \varepsilon_0\omega_p^2\partial_t E \text{ in } \mathcal{O}, \end{cases} \tag{2.2}$$

where $E$ (resp. $H$) is the electric (resp. magnetic) field and $J$ is the polarization current. The parameters $\beta$ (driving the "non locality" in space), $\omega_p$ (the plasma frequency), $\gamma, \varepsilon_0, \varepsilon_L$ are physical quantities that can be assumed to be positive and constants. For shortness, we set $\varepsilon = \varepsilon_0\varepsilon_L$. As usual $\partial_t E = \frac{\partial E}{\partial t}$ is the partial derivative of $E$ with respect to the time $t$. In this setting and for further use, it is natural to rewrite this system in a mixed form as a first order system of PDEs:

$$\begin{cases} \varepsilon_0\varepsilon_L\partial_t E - \operatorname{curl} H = -J \text{ in } \mathcal{O}, \\ \mu\partial_t H + \operatorname{curl} E = 0 \text{ in } \mathcal{O}, \\ \partial_t J - \beta^2\nabla Q = \varepsilon_0\omega_p^2 E - \gamma J \text{ in } \mathcal{O}, \\ \partial_t Q - \operatorname{div} J = 0 \text{ in } \mathcal{O}. \end{cases} \tag{2.3}$$

**Remark 2.1** *Here the new unknown $Q$ plays the role of a charge.*

This system has to be completed with initial conditions:

$$E(.,0) = E_0(.), H(.,0) = H_0(.), J(.,0) = J_0(.), Q(.,0) = Q_0(.) \text{ in } \Omega, \tag{2.4}$$

in suitable spaces that will be specified later, and with boundary conditions. Later on, we will focus on several type of boundary conditions. Either the electric boundary conditions

$$E \times n = 0, H \cdot n = 0, \operatorname{div} J = 0, Q = 0, \tag{2.5}$$

or the magnetic boundary conditions

$$E \cdot n = 0, H \times n = 0, J \cdot n = 0, \nabla Q \cdot n = 0. \tag{2.6}$$

Here and below $n$ denotes the unit outer normal vector on the considered boundary.

We will detail in each dedicated section, the type of setting (in terms of hypotheses on the boundary) that will be used.

## 2.3 The system with electric or magnetic boundary conditions

### 2.3.1 The case of electric boundary conditions

In this section, we begin by the study the following system with electric boundary conditions.

$$\begin{cases} \varepsilon_0\varepsilon_L\partial_t E - \operatorname{curl} H = -J \text{ in } \mathcal{O}, \\ \mu\partial_t H + \operatorname{curl} E = 0 \text{ in } \mathcal{O}, \\ \partial_t J - \beta^2\nabla Q = \varepsilon_0\omega_p^2 E - \gamma J \text{ in } \mathcal{O}, \\ \partial_t Q - \operatorname{div} J = 0 \text{ in } \mathcal{O}. \\ E \times n = 0, H \cdot n = 0, \operatorname{div} J = 0, Q = 0 \text{ on } \Sigma, \\ E(.,0) = E_0(.), H(.,0) = H_0(.), J(.,0) = J_0(.), Q(.,0) = Q_0(.) \text{ in } \Omega, \end{cases} \tag{2.7}$$

The existence of a solution to (2.11) will be obtained by using semigroup theory in the appropriate Hilbert setting that we describe below (see for instance [16, 25, 26]).

Introduce the Hilbert space

$$\mathcal{H} = \{(F,G,R,S)^\top \in H(\mathrm{div},\Omega) \times \hat{\mathcal{K}}(\Omega) \times L^2(\Omega)^3 \times L^2(\Omega), \mathrm{div}(\varepsilon F) = -S \text{ on } \Omega\},$$

with the inner product

$$((F,G,R,S)^\top,(F',G',R',S')^\top)_{\mathcal{H}} := \int_\Omega (\varepsilon_0\varepsilon_L F\cdot\bar{F}' + \mu G\cdot\bar{G}' + \frac{1}{\varepsilon_0\omega_p^2}R\cdot\bar{R}' + \frac{\beta^2}{\varepsilon_0\omega_p^2}S\cdot\bar{S}')\,dx, \tag{2.8}$$

The space $\mathcal{H}$ is indeed a Hilbert space for the associated norm thanks to the divergence conditions.

Note that the equations imply a divergence free constraint on $H$ and a divergence constraint on $\varepsilon E$ based on the original problem (2.3). Indeed the first and second equations in (2.3) formally yields respectively

$$(\mathrm{div}(\varepsilon E) + Q)_t = (\mathrm{div}\, H)_t = 0 \text{ in } \mathcal{O}.$$

Therefore

$$(\mathrm{div}(\varepsilon E) + Q)(x,t) = (\mathrm{div}(\varepsilon E) + Q)(x,0) \text{ and } \mathrm{div}\, H(x,t) = \mathrm{div}\, H(x,0),\ \forall x\in\Omega, t>0,$$

and if we assume the divergence free properties at $t=0$, they will remain valid for $t>0$.

We define the unbounded operator $\mathcal{A}$ as follows:

$$\mathcal{D}(\mathcal{A}) := \Big\{(F,G,R,S)^\top \in \mathcal{H} |\, \mathrm{curl}\, G \in L^2(\Omega)^3, R\in H(\mathrm{div},\Omega),\ S\in H_0^1(\Omega) \text{ and } F\in X_N(\Omega)\Big\}, \tag{2.9}$$

and for all $U=(E,H,J,Q)^\top \in \mathcal{D}(\mathcal{A})$, $\mathcal{A}U$ is given by

$$\mathcal{A}U = \begin{pmatrix} \varepsilon_0^{-1}\varepsilon_L^{-1}(\mathrm{curl}\, H - J) \\ -\mu^{-1}\,\mathrm{curl}\, E \\ \beta^2\nabla Q + \varepsilon_0\omega_p^2 E - \gamma J \\ \mathrm{div}\, J \end{pmatrix}. \tag{2.10}$$

The model (2.7) can then be rewritten as follows

$$\begin{cases} \partial_t U = \mathcal{A}U, \\ U(0) = U_0, \end{cases} \tag{2.11}$$

where $U$ is the vectorial unknown

$$U = \begin{pmatrix} E \\ H \\ J \\ Q \end{pmatrix}, \tag{2.12}$$

where $E,H,J,Q \in L^2(\Omega)^3$ and for smooth enough $E,H,J$ and $Q$,

**Theorem 2.2** *The operator $\mathcal{A}$ defined by (2.10) with domain (2.9) generates a $C_0$-semigroup of contractions $(T(t))_{t\geq 0}$ on $\mathcal{H}$. Therefore for all $U_0\in\mathcal{H}$, the problem (2.11) has a weak solution $U\in C([0,\infty),H)$ given by $U=TU_0$.*
*If moreover $U_0\in D(\mathcal{A}^k)$, with $k\in\mathbb{N}^*$, the problem (2.11) has a strong solution $U\in C([0,\infty),D(\mathcal{A}^k))\cap C^1([0,\infty),D(\mathcal{A}^{k-1}))$.*

**Proof.** It suffices to show that $\mathcal{A}$ is a maximal dissipative operator (see [16, 25]), then by Lumer-Phillips' theorem it generates a $C_0$-semigroup of contractions $(T(t))_{t\geq 0}$ on $\mathcal{H}$.

Let us first show the dissipativity. For $U = (E, H, J, Q)^\top \in \mathcal{D}(\mathcal{A})$, we have

$$(\mathcal{A}U, U)_{\mathcal{H}} = \int_\Omega \left( (\operatorname{curl} H - J) \cdot \bar{E} - \operatorname{curl} E \cdot \bar{H} + \frac{1}{\varepsilon_0 \omega_p^2} (\beta^2 \nabla Q + \varepsilon_0 \omega_p^2 E - \gamma J) \cdot \bar{J} + \frac{\beta^2}{\varepsilon_0 \omega_p^2} \operatorname{div} J \bar{Q} \right) dx.$$

Hence by Green's formula (2.1), we find that

$$(\mathcal{A}U, U)_{\mathcal{H}} = \int_\Omega (H \cdot \operatorname{curl} \bar{E} - \operatorname{curl} E \cdot \bar{H} + \frac{\beta^2}{\varepsilon_0 \omega_p^2} (\operatorname{div} J \cdot \bar{Q} - \operatorname{div} \bar{J} \cdot Q) + \frac{1}{\varepsilon_0 \omega_p^2} (E \cdot \bar{J} - J \cdot \bar{E}) - \frac{1}{\varepsilon_0 \omega_p^2} \gamma |J|^2) \, dx.$$

Taking the real part of this identity, we obtain

$$\Re(\mathcal{A}U, U)_{\mathcal{H}} = -\frac{\gamma}{\varepsilon_0 \omega_p^2} \int_\Omega |J|^2 \, dx.$$

This shows that $\mathcal{A}$ is dissipative.

Let us go on with the maximality. Let $\lambda > 0$ be fixed. For $(F, G, R, S)^\top \in \mathcal{H}$, we look for $U = (E, H, J, Q)^\top \in \mathcal{D}(\mathcal{A})$ such that

$$(\lambda I - \mathcal{A}) U = (F, G, R, S)^\top. \tag{2.13}$$

According to (2.10) this is equivalent to

$$\varepsilon \lambda E - \operatorname{curl} H + J = \varepsilon F, \tag{2.14}$$

$$\mu \lambda H + \operatorname{curl} E = \mu G, \tag{2.15}$$

$$\lambda J - \beta^2 \nabla Q - \varepsilon_0 \omega_p^2 E + \gamma J = R, \tag{2.16}$$

$$\lambda Q - \operatorname{div} J = S. \tag{2.17}$$

Assume for the moment that $U$ exists. Then the first and second equation allow to eliminate $J$ and $H$ since they are equivalent to

$$H = -\frac{1}{\mu \lambda} \operatorname{curl} E + \frac{1}{\lambda} G, \tag{2.18}$$

$$J = -\varepsilon \lambda E + \operatorname{curl} H + \varepsilon F. \tag{2.19}$$

Thus

$$J = -\varepsilon \lambda E - \frac{1}{\mu \lambda} \operatorname{curl} \operatorname{curl} E + \frac{1}{\lambda} \operatorname{curl} G + \varepsilon F \tag{2.20}$$

Furthermore the last equation gives

$$Q = \frac{1}{\lambda} \operatorname{div} J + \frac{1}{\lambda} S, \tag{2.21}$$

so that we recover the constraint since $(E, H, J, Q)$ and $(F, G, R, S)$ belong to $\mathcal{H}$:

$$Q = -\varepsilon \operatorname{div} E + \frac{\varepsilon}{\lambda} \operatorname{div} F + \frac{1}{\lambda} S \tag{2.22}$$

i.e.

$$Q = -\varepsilon \operatorname{div} E \tag{2.23}$$

Replacing in the third equation, we get

$$-\left(\varepsilon\lambda(\lambda+\gamma)+\varepsilon_0\omega_p^2\right)E - \left(\frac{\lambda+\gamma}{\mu\lambda}\right)\operatorname{curl}\operatorname{curl}E + \varepsilon\beta^2\nabla\operatorname{div}E = R - \frac{\lambda+\gamma}{\lambda}\operatorname{curl}G - \varepsilon(\lambda+\gamma)F \tag{2.24}$$

that corresponds to a problem with only $E$ as unknown.

We now consider the following variational problem: Find $E \in X_N(\Omega)$ such that

$$a_\lambda(E,E') = F_\lambda(E'), \forall E' \in X_N(\Omega), \tag{2.25}$$

where

$$a_\lambda(T,T') = \int_\Omega \left((\varepsilon\lambda(\lambda+\gamma)+\varepsilon_0\omega_p^2)T\cdot\bar{T}' + \frac{\lambda+\gamma}{\mu\lambda}\operatorname{curl}T\cdot\operatorname{curl}\bar{T}' + \varepsilon\beta^2\operatorname{div}T\operatorname{div}\bar{T}'\right)dx, \tag{2.26}$$

and

$$F_\lambda(T') = \int_\Omega \left(-R\cdot\bar{T}' + \frac{\lambda+\gamma}{\lambda}G\cdot\operatorname{curl}\bar{T}' + \varepsilon(\lambda+\gamma)F\cdot\bar{T}'\right)dx, \tag{2.27}$$

for all $T, T' \in X_N(\Omega)$. Let us prove that this problem is well posed. As for $\lambda > 0$, $a_\lambda$ is clearly a sesquilinear, continuous and coercive form on $X_N(\Omega)$ and $F_\lambda$ is a conjugate linear and continuous form on $X_N(\Omega)$, by Lax-Milgram lemma, problem (2.25) has a unique solution $E \in X_N(\Omega)$.

We would like to come back to problem (2.13), with $E$ in hand. We thus define $H$ by (2.18); $H \in L^2(\Omega)$. Let us prove a regularity result on $H$. We first notice that (2.25) is equivalent to

$$\begin{aligned}\int_\Omega \left((\varepsilon\lambda(\lambda+\gamma)+\varepsilon_0\omega_p^2)E\cdot\bar{E}' - (\lambda+\gamma)H\cdot\operatorname{curl}\bar{E}' + \varepsilon\beta^2\operatorname{div}E\operatorname{div}\bar{E}'\right)dx \\ = \int_\Omega \left(-R\cdot\bar{E}' + \varepsilon(\lambda+\gamma)F\cdot\bar{E}'\right)dx, \forall E' \in X_N(\Omega).\end{aligned} \tag{2.28}$$

In a first step we show that this identity implies that $\operatorname{div}E$ belongs to $H^1(\Omega)$. For that purpose, we use the same argument as in the proof of Theorem 1.1 in [10]. As test function we take $E' = \nabla\varphi$, with $\varphi \in D(\Delta^{Dir}) := \{\psi \in H^1(\Omega) | \Delta\psi \in L^2(\Omega) \text{ and } \psi = 0 \text{ on } \Gamma\}$. Then by integration by parts in (2.28), we get

$$\int_\Omega \operatorname{div}E\left[-(\varepsilon\lambda(\lambda+\gamma)+\varepsilon_0\omega_p^2)\bar{\varphi} + \varepsilon\beta^2\Delta\bar{\varphi}\right]dx = \int_\Omega (-R+\varepsilon(\lambda+\gamma)F)\cdot\nabla\bar{\varphi}\,dx, \forall\varphi \in D(\Delta^{Dir}).$$

On the other hand, thanks to Lax Milgram lemma again, there exists a unique solution $q \in H_0^1(\Omega)$ to

$$\int_\Omega (\varepsilon\lambda(\lambda+\gamma)+\varepsilon_0\omega_p^2)q\bar{\varphi} + \varepsilon\beta^2\nabla q\cdot\nabla\bar{\varphi})\,dx = \int_\Omega (R-\varepsilon(\lambda+\gamma)F)\cdot\nabla\bar{\varphi}\,dx, \forall\varphi \in H_0^1(\Omega).$$

Restricting test-functions to $D(\Delta^{Dir})$, we get

$$\int_\Omega q(\varepsilon\lambda(\lambda+\gamma)+\varepsilon_0\omega_p^2)\bar\varphi-\varepsilon\beta^2\Delta\bar\varphi)\,dx=\int_\Omega (R-\varepsilon(\lambda+\gamma)F)\cdot\nabla\bar\varphi\,dx, \forall\varphi\in D(\Delta^{Dir}).$$

This implies that $q-\operatorname{div}E$ is orthogonal to the range of $(\varepsilon\lambda(\lambda+\gamma)+\varepsilon_0\omega_p^2)Id-\beta^2\Delta$, since in that case this range is the full $L^2(\Omega)$, we conclude that $\operatorname{div}E=q$, so that $\operatorname{div}E\in H_0^1(\Omega)$.

Now we come back to (2.28) and take test functions $E'\in\mathcal{D}(\Omega)^3$ to get

$$(\varepsilon\lambda(\lambda+\gamma)+\varepsilon_0\omega_p^2)E-(\lambda+\gamma)\operatorname{curl}H-\varepsilon\beta^2\nabla\operatorname{div}E=-R+(\lambda+\gamma)\varepsilon F \text{ in } \mathcal{D}'(\Omega)^3. \tag{2.29}$$

As $\operatorname{div}E\in H^1(\Omega)$, this identity guarantees that $H$ belongs to $H(\operatorname{curl};\Omega)$ and since we have (2.15), $H\in\mathcal{K}(\Omega)$. We can now define $J$ by (2.19). We obtain $J\in(L^2(\Omega))^2$. Furthermore since $\operatorname{div}E\in H^1(\Omega)$ and $\operatorname{div}F\in L^2$ (since $(F,G,R,S)\in\mathcal{H}$), one obtains $\operatorname{div}J\in L^2(\Omega)$. Thus $Q:=\dfrac{1}{\lambda}\operatorname{div}J+\dfrac{1}{\lambda}S$ is well defined. It remains to prove that $Q\in H_0^1(\Omega)$ and thus we will have $\operatorname{div}J=0$ on $\Gamma$. Let us consider (2.28) and use the expression of $J$:

$$\int_\Omega\Big((\lambda+\gamma)(-J+\operatorname{curl}H+\varepsilon F)\cdot\bar E'-(\lambda+\gamma)H\cdot\operatorname{curl}\bar E'+\frac{\beta^2}{\lambda}\operatorname{div}(-J+\varepsilon F)\operatorname{div}\bar E'\Big)\,dx$$
$$+\int_\Omega\varepsilon_0\omega_p^2E\cdot\bar E'\,dx=\int_\Omega\big(-R\cdot\bar E'+\varepsilon(\lambda+\gamma)F\cdot\bar E'\big)\,dx, \forall E'\in X_N(\Omega).$$

This gives

$$\int_\Omega\Big(-(\lambda+\gamma)J\cdot\bar E'-\frac{\beta^2}{\lambda}\operatorname{div}J\operatorname{div}\bar E'+\frac{\varepsilon\beta^2}{\lambda}\operatorname{div}F\operatorname{div}\bar E'\Big)\,dx+\int_\Omega\varepsilon_0\omega_p^2E\cdot\bar E'\,dx$$
$$=-\int_\Omega R\cdot\bar E'\,dx, \forall E'\in X_N(\Omega).$$

But since $\dfrac{1}{\lambda}\operatorname{div}J=Q-\dfrac{1}{\lambda}S$,

$$\int_\Omega\Big(-(\lambda+\gamma)J\cdot\bar E'-\beta^2(Q-\frac{1}{\lambda}S)\operatorname{div}\bar E'+\frac{\varepsilon\beta^2}{\lambda}\operatorname{div}F\operatorname{div}\bar E'\Big)\,dx+\int_\Omega\varepsilon_0\omega_p^2E\cdot\bar E'\,dx$$
$$=-\int_\Omega R\cdot\bar E'\,dx, \forall E'\in X_N(\Omega).$$

Using the divergence constraint in the space $\mathcal{H}$, we get

$$\int_\Omega\Big(-(\lambda+\gamma)J\cdot\bar E'-\beta^2Q\operatorname{div}\bar E'\Big)\,dx+\int_\Omega\varepsilon_0\omega_p^2E\cdot\bar E'\,dx$$
$$=-\int_\Omega R\cdot\bar E'\,dx, \forall E'\in X_N(\Omega). \tag{2.30}$$

Thus in the sense of distributions

$$-(\lambda+\gamma)J+\beta^2\nabla Q+\varepsilon_0\omega_p^2E=-R.$$

This shows that $Q\in H^1(\Omega)$ and as a result we also show that $Q\in H_0^1(\Omega)$ (by integration by parts in (2.30)). The constraint is recovered from (2.21) and the definition of $J$. The surjectivity of $\lambda I-\mathcal{A}$ is proved. ■

We continue by the study of the kernel of $\mathcal{A}$.

**Lemma 2.3** *One has*

$$\ker \mathcal{A} := \{0\}.$$

**Proof.** $U = (E, H, P, Q)^\top \in \mathcal{D}(\mathcal{A})$ belongs to $\ker \mathcal{A}$ if and only if

$$\operatorname{curl} H - J = 0, \tag{2.31}$$

$$\operatorname{curl} E = 0, \tag{2.32}$$

$$\beta^2 \nabla Q + \varepsilon_0 \omega_p^2 E - \gamma J = 0, \tag{2.33}$$

$$\operatorname{div} J = 0. \tag{2.34}$$

Taking into account (2.33), (2.34) implies that

$$\int_\Omega (\beta^2 \nabla Q \cdot \nabla Q + \varepsilon_0 \omega_p^2 E \cdot \nabla Q - \gamma J \cdot \nabla Q)\, dx = 0.$$

Integrating by parts, and reminding that $\varepsilon \operatorname{div} E = -Q$, we get

$$\int_\Omega (\beta^2 |\nabla Q|^2 + \frac{\varepsilon_0}{\varepsilon} \omega_p^2 |Q|^2)\, dx = 0,$$

consequently $Q = 0$ and therefore $\operatorname{div} E = 0$. Since $\operatorname{curl} E = 0$ and $E \in X_N(\Omega)$, we deduce that $E = 0$ (recalling that $\Omega$ is supposed to be simply connected and Proposition 3.14 of [1]). This then gives that $J = 0$.

For $H$, we notice that (2.31) implies that $H$ is curl free. As it is already in $\mathcal{K}(\Omega)$, we deduce that $H = 0$ as for $E$. ■

We define the energy of (2.11) in $\mathcal{H}$ by

$$\mathcal{E} = \frac{1}{2} \int_\Omega (\varepsilon |E|^2 + \mu |H|^2 + \frac{1}{\varepsilon_0 \omega_p^2} |J|^2 + \frac{\beta^2}{\varepsilon_0 \omega_p^2} |Q|^2)\, dx, \text{ on } ]0, +\infty[. \tag{2.35}$$

From the above computations (dissipativeness of $\mathcal{A}$), we deduce that

**Proposition 2.4** *The solution $(E, H, J, Q)$ of (2.11) with initial datum in $\mathcal{D}(\mathcal{A})$ satisfies*

$$\frac{d}{dt} \mathcal{E} = -\frac{\gamma}{\varepsilon_0 \omega_p^2} \int_\Omega |J|^2\, dx \text{ on } ]0, +\infty[.$$

*Therefore the energy is non increasing.*

### 2.3.2 The case of magnetic boundary conditions

In a similar manner, we can prove an existence and uniqueness result for the operator with magnetic boundary conditions. Since the proof are quite similar, we choose not to reproduce it here in details.

The model (2.37) can be rewritten in the form (2.11) with $\mathcal{A}$ defined by (2.10). The first difference is the Hilbert space $\mathcal{H}$ defined here by

$$\mathcal{H} = \{(E, H, P, Q)^\top \in H(\operatorname{div}, \Omega) \times \hat{J}(\Omega) \times H(\operatorname{div}; \Omega) \times L^2(\Omega)^3 | \operatorname{div}(\varepsilon E + P) = 0 \text{ in } \Omega\},$$

but equipped with the same inner product (2.8). The second difference is the domain of the operator $\mathcal{A}$:

$$\mathcal{D}(\mathcal{A}) := \Big\{(E,H,P,Q)^\top \in \mathcal{H} | E \in X_N(\Omega), H \in X_T(\Omega), \operatorname{div} E, \operatorname{div} P \in H_0^1(\Omega), \text{ and } Q \in H(\operatorname{div},\Omega)\Big\}. \tag{2.36}$$

$$\begin{cases} \varepsilon_0\varepsilon_L\partial_t E - \operatorname{curl} H = -J \text{ in } \mathcal{O}, \\ \mu\partial_t H + \operatorname{curl} E = 0 \text{ in } \mathcal{O}, \\ \partial_t J - \beta^2\nabla Q = \varepsilon_0\omega_p^2 E - \gamma J \text{ in } \mathcal{O}, \\ \partial_t Q - \operatorname{div} J = 0 \text{ in } \mathcal{O}. \\ E\cdot n = 0, H\times n = 0, J\cdot n = 0, \nabla Q\cdot n = 0 \text{ on } \Sigma, \\ E(.,0) = E_0(.), H(.,0) = H_0(.), J(.,0) = J_0(.), Q(.,0) = Q_0(.) \text{ in } \Omega, \end{cases} \tag{2.37}$$

**Theorem 2.5** *The operator $\mathcal{A}$ defined by (2.10) with domain (2.36) generates a $C_0$-semigroup $(T(t))_{t\geq 0}$ on $\mathcal{H}$.*

# 3 Stability results

Our stability results are based on a frequency domain approach. Recall that the polynomial decay of the energy can be obtained by using the next result stated in Theorem 2.4 of [5] (see also [2, 3, 20] for weaker variants and [27, 14] for exponential decay):

**Lemma 3.1** *A $C_0$ semigroup $e^{t\mathcal{L}}$ of contractions on a Hilbert space satisfies*

$$||e^{t\mathcal{L}}U_0|| \leq C\, t^{-\frac{1}{l}}||U_0||_{\mathcal{D}(\mathcal{L})}, \quad \forall U_0 \in \mathcal{D}(\mathcal{L}), \quad \forall t > 1,$$

*as well as*

$$||e^{t\mathcal{L}}U_0|| \leq C\, t^{-1}||U_0||_{\mathcal{D}(\mathcal{L}^l)}, \quad \forall U_0 \in \mathcal{D}(\mathcal{L}^l), \quad \forall t > 1,$$

*for some constant $C > 0$ and for some positive integer $l$ if*

$$\rho(\mathcal{L}) \supset i\mathbb{R}, \tag{3.1}$$

*and*

$$\limsup_{|\xi|\to\infty} \frac{1}{\xi^l}\,\|(i\xi - \mathcal{L})^{-1}\| < \infty, \tag{3.2}$$

*hold.*

## 3.1 Electric boundary conditions

In order to check the assumptions of Lemma 3.1 for $\mathcal{A}$, we first analyze the assumption (3.1).

**Lemma 3.2** *We have*

$$0 \in \rho(\mathcal{A}) := \{\lambda \in \mathbb{C} | \lambda Id - \mathcal{A} \text{ is densely defined and has a continuous inverse}\}.$$

**Proof.** Let $(F,G,R,S)^\top \in \mathcal{H}$, we look for $U=(E,H,J,Q)^\top \in \mathcal{D}(\mathcal{A})$ such that

$$\mathcal{A}U = (F,G,R,S)^\top. \tag{3.3}$$

According to (2.10) this is equivalent to

$$\operatorname{curl} H - J = \varepsilon F, \tag{3.4}$$

$$-\operatorname{curl} E = \mu G, \tag{3.5}$$

$$\beta^2 \nabla Q + \varepsilon_0 \omega_p^2 E - \gamma J = R, \tag{3.6}$$

$$\operatorname{div} J = S. \tag{3.7}$$

Suppose for a moment that such a $U=(E,H,J,Q)^\top \in \mathcal{D}(\mathcal{A})$ exists. One has by (3.6),

$$\beta^2 \int_\Omega \nabla Q \cdot \nabla \bar{\psi} dx + \varepsilon_0 \omega_p^2 \int_\Omega E \cdot \nabla \bar{\psi} dx - \gamma \int_\Omega J \cdot \nabla \bar{\psi} dx = \int_\Omega R \cdot \nabla \bar{\psi} dx, \text{ for all } \psi \in H_0^1(\Omega). \tag{3.8}$$

This gives

$$\beta^2 \int_\Omega \nabla Q \cdot \nabla \bar{\psi} dx - \varepsilon_0 \omega_p^2 \int_\Omega \operatorname{div} E \cdot \bar{\psi} dx + \gamma \int_\Omega \operatorname{div} J \cdot \bar{\psi} dx = \int_\Omega R \cdot \nabla \bar{\psi} dx, \text{ for all } \psi \in H_0^1(\Omega). \tag{3.9}$$

Since $\varepsilon \operatorname{div} E = -Q$ and (3.7), we find

$$\beta^2 \int_\Omega \nabla Q \cdot \nabla \bar{\psi} dx + \frac{\varepsilon_0}{\varepsilon} \omega_p^2 \int_\Omega Q \cdot \bar{\psi} dx = -\gamma \int_\Omega S \cdot \bar{\psi} dx + \int_\Omega R \cdot \nabla \bar{\psi} dx, \text{ for all } \psi \in H_0^1(\Omega). \tag{3.10}$$

We now go back to the problem (3.3). Let us introduce the sesquilinear continuous coercive form $\tilde{a}$ on $H_0^1(\Omega)$ as:

$$\forall (\varphi,\psi) \in H_0^1(\Omega),\ \tilde{a}(\varphi,\psi) = \beta^2 \int_\Omega \nabla \varphi \cdot \nabla \bar{\psi} dx + \frac{\varepsilon_0}{\varepsilon} \omega_p^2 \int_\Omega \varphi \cdot \bar{\psi} dx \tag{3.11}$$

and the conjugate linear form $\tilde{F}$:

$$\forall \psi \in H_0^1(\Omega), \tilde{F}(\psi) = -\gamma \int_\Omega S \cdot \bar{\psi} dx + \int_\Omega R \cdot \nabla \bar{\psi} dx. \tag{3.12}$$

Thanks to Lax Milgram theorem, there exists $Q \in H_0^1(\Omega)$ such that

$$\tilde{a}(Q,\psi) = F(\psi), \forall \psi \in H_0^1(\Omega). \tag{3.13}$$

Then, let us denote by $\varphi \in H_0^1(\Omega)$ the unique solution to the following variational problem

$$\int_\Omega \nabla \varphi \cdot \nabla \bar{\varphi}' dx = \int_\Omega \frac{Q}{\varepsilon} \bar{\varphi}',\ \forall \varphi' \in H_0^1(\Omega). \tag{3.14}$$

Furthermore, we introduce the following variational problem: *Find* $\xi \in X_T(\Omega) \cap \mathcal{K}(\Omega)$ *such that*,

$$\int_\Omega \operatorname{curl} \xi \cdot \operatorname{curl} \bar{\varphi}' dx = -\int_\Omega \mu G \bar{\varphi}' dx,\ \forall \varphi' \in X_T(\Omega) \cap \mathcal{K}(\Omega). \tag{3.15}$$

This variational problem has a unique solution thanks to Lax Milgram lemma applied on $X_T(\Omega) \cap \mathcal{K}(\Omega)$ embedded with the $\|\cdot\|_{X(\Omega)} (= \|\operatorname{curl} \cdot\|_{L^2(\Omega)})$ norm. Let us denote $\xi \in X_T(\Omega) \cap \mathcal{K}(\Omega)$ the unique solution of this problem.

We then define $E := \operatorname{curl} \xi + \nabla \varphi$. We thus have

$$\int_\Omega E \cdot \nabla \bar{\varphi}' dx = \int_\Omega \frac{Q}{\varepsilon} \bar{\varphi}', \, \forall \varphi' \in H_0^1(\Omega). \tag{3.16}$$

Thus

$$\operatorname{div} E = -\frac{Q}{\varepsilon} \tag{3.17}$$

in $L^2(\Omega)$ and $E \in H(\operatorname{div}, \Omega)$.

Furthermore,

$$\int_\Omega E \cdot \operatorname{curl} \bar{\varphi}' dx = -\mu \int_\Omega G \bar{\varphi}', \, \forall \varphi' \in X_T(\Omega) \cap \mathcal{K}(\Omega). \tag{3.18}$$

Since any $\psi' \in X_T(\Omega)$ can be written as

$$\psi' = \nabla \chi' + \varphi_0',$$

with $\chi' \in D(\Delta^{Neu})$ solution of

$$\Delta \chi' = \operatorname{div} \psi', \text{ in } \Omega,$$

and then $\varphi_0' \in X_T(\Omega) \cap \mathcal{K}(\Omega)$, we deduce that

$$\int_\Omega E \cdot \operatorname{curl} \bar{\varphi}' dx = -\mu \int_\Omega G \bar{\varphi}', \, \forall \varphi' \in X_T(\Omega). \tag{3.19}$$

This yields $\operatorname{curl} E = -\mu G$ in $L^2(\Omega)$ and $E \times n = 0$.
Let us define $J = -\dfrac{R}{\gamma} + \dfrac{\beta^2}{\gamma} \nabla Q + \varepsilon_0 \omega_p^2 \dfrac{E}{\gamma}$ in $(L^2(\Omega))^3$. Using (3.13), we deduce that $\forall \psi \in H_0^1(\Omega)$,

$$\int_\Omega J \cdot \nabla \bar{\psi} = -\frac{\varepsilon_0}{\varepsilon \gamma} \omega_p^2 \int_\Omega Q \cdot \bar{\psi} dx - \int_\Omega S \cdot \bar{\psi} dx + \frac{\varepsilon_0 \omega_p^2}{\gamma} \int_\Omega E \cdot \nabla \bar{\psi} dx \tag{3.20}$$

Since we have (3.17), we deduce that

$$\operatorname{div} J = S, \tag{3.21}$$

which gives that $J \in H(\operatorname{div}, \Omega)$.

Finally, the first equation allows to find $H$. Indeed as $H$ has also to be in $\hat{\mathcal{K}}(\Omega)$, we look for $H$ in the form $H = \operatorname{curl} \chi$ with $\chi \in X_N(\Omega) \cap \mathcal{K}(\Omega)$, the unique solution of

$$\int_\Omega \operatorname{curl} \chi \cdot \operatorname{curl} \bar{\psi} \, dx = \int_\Omega (\varepsilon F + J) \cdot \bar{\psi} \, dx, \forall \psi \in X_N(\Omega) \cap \mathcal{K}(\Omega).$$

As $\varepsilon F + J$ is divergence free, this problem implies that

$$\int_\Omega \operatorname{curl} \chi \cdot \operatorname{curl} \bar{\psi} \, dx = \int_\Omega (\varepsilon F - J) \cdot \bar{\psi} \, dx, \forall \psi \in X_N(\Omega), \tag{3.22}$$

because any $\psi \in X_N(\Omega)$ can be written as

$$\psi = \nabla\varphi + \psi_0,$$

with $\varphi \in D(\Delta^{Dir})$ solution of

$$\Delta\varphi = \operatorname{div}\psi \text{ in } \Omega,$$

and then $\psi_0 \in X_N(\Omega) \cap \mathcal{K}(\Omega)$. Problem (3.22) then yields that $H = \operatorname{curl}\psi$ satisfies (3.4). The continuity of the inverse of $\mathcal{A}$ is easily shown by basic estimations coming from the definition of each fields. The proof is thus complete. ■

**Lemma 3.3** *We have*

$$i\mathbb{R} \subset \rho(\mathcal{A}).$$

**Proof.** As the previous lemma has shown that $0 \in \rho(\mathcal{A})$, it remains to show that

$$i\omega \in \rho(\mathcal{A}), \forall\omega \in \mathbb{R} \setminus \{0\}.$$

This means that for $\omega \in \mathbb{R}$, $\omega \neq 0$ and an arbitrary $W = (F, G, R, S)^\top \in \mathcal{H}$, we look for $U = (E, H, J, Q)^\top \in \mathcal{D}(\mathcal{A})$ such that

$$(i\omega - \mathcal{A})U = W, \tag{3.23}$$

that means solution of (2.13) with $\lambda = i\omega$. Hence the arguments of Theorem 2.2 lead first to the problem (2.25) with $\lambda = i\omega$ (with $a_\lambda$ and $F_\lambda$ defined respectively by (2.26) and (2.27)). This problem is equivalent to

$$e^{i\theta} a_{i\omega}(E, E') = e^{i\theta} F_{i\omega}(E'), \forall E' \in X_N(\Omega), \tag{3.24}$$

forall $\theta \in \mathbb{R}$. Hence we look for one $\theta$ such that $e^{i\theta} a_{i\omega}$ is coercive on $X_N(\Omega)$, i.e., such that

$$\Re(e^{i\theta} a_{i\omega}(E, E)) \gtrsim \|E\|^2_{X_N(\Omega)}, \forall E' \in X_N(\Omega).$$

Simple calculations show that this property holds if

$$\begin{aligned} \cos\theta &> 0, \\ \frac{\gamma}{\omega}\tan\theta + 1 &> 0, \\ -\omega\gamma\varepsilon\tan\theta + \varepsilon_0\omega_p^2 - \varepsilon\omega^2 &> 0. \end{aligned}$$

For $\omega > 0$, these conditions are equivalent to

$$\cos\theta > 0, -\frac{\omega}{\gamma} < \tan\theta < \frac{\varepsilon_0\omega_p^2 - \varepsilon\omega^2}{\omega\gamma\varepsilon},$$

and therefore it suffices to choose

$$\theta \in (\theta_0, \theta_1),$$

with $\theta_0 = -\arctan(\frac{\omega}{\gamma})$ and $\theta_1 = \arctan(\frac{\varepsilon_0\omega_p^2 - \varepsilon\omega^2}{\omega\gamma\varepsilon})$.

On the contrary for $\omega < 0$ these conditions are equivalent to

$$\cos\theta > 0, \frac{\varepsilon_0\omega_p^2 - \varepsilon\omega^2}{\omega\gamma\varepsilon} < \tan\theta < -\frac{\omega}{\gamma}$$

and therefore it suffices to chose

$$\theta \in (\theta_1, \theta_0).$$

With this choice, problem (3.24) has a unique solution $E \in X_N(\Omega)$ and the arguments of the proof of Theorem 2.2 yield $U = (E, H, P, Q)^\top \in \mathcal{D}(\mathcal{A})$ solution of (3.23). The fact that $U$ belongs to $\mathcal{H}$ comes from the property $W \in \mathcal{H}$. ■

Now we need to analyze the behaviour of the resolvent on the imaginary axis.

**Lemma 3.4** *The resolvent of the operator of $\mathcal{A}$ satisfies condition (3.2) with $l = 2$, i.e.*

$$\limsup_{|\xi|\to\infty} \frac{1}{\xi^2} \|(i\xi - \mathcal{A})^{-1}\| < \infty. \tag{3.25}$$

**Proof.** We use a contradiction argument, i.e., we suppose that (3.2) is false with $l = 2$. Then there exist a sequence of real numbers $\xi_n \to +\infty$ and a sequence of vectors $Z_n = (E_n, H_n, J_n, Q_n)^\top$ in $\mathcal{D}(\mathcal{A})$ with $\|Z_n\|_{\mathcal{H}} = 1$ such that

$$\xi_n^2 \|(i\xi_n - \mathcal{A})Z_n\|_{\mathcal{H}} \to 0 \text{ as } n \to \infty. \tag{3.26}$$

By (2.10), this is equivalent to

$$\xi_n^2 \|i\varepsilon\xi_n E_n - \operatorname{curl} H_n + J_n\|_\Omega \to 0, \tag{3.27}$$

$$\xi_n^2 \|i\mu\xi_n H_n + \operatorname{curl} E_n\|_\Omega \to 0, \tag{3.28}$$

$$\xi_n^2 \|i\xi_n J_n - \beta^2\nabla Q_n - \varepsilon_0\omega_p^2 E_n + \gamma J_n\|_\Omega \to 0, \tag{3.29}$$

$$\xi_n^2 \|i\xi_n Q_n - \operatorname{div} J_n\|_\Omega \to 0, \tag{3.30}$$

as $n \to +\infty$.

We now notice that

$$\Re\left((i\xi_n - \mathcal{A})Z_n, Z_n\right)_{\mathcal{H}} \le \|(i\xi_n - \mathcal{A})Z_n\|_{\mathcal{H}} \|Z_n\|_{\mathcal{H}} = \|(i\xi_n - \mathcal{A})Z_n\|_{\mathcal{H}} \tag{3.31}$$

and that, by dissipativity of $\mathcal{A}$:

$$\Re\left((i\xi_n - \mathcal{A})Z_n, Z_n\right)_{\mathcal{H}} = \Re\left(i\xi_n\|Z_n\|^2 - \left(\mathcal{A}Z_n, \bar{Z}_n\right)_{\mathcal{H}}\right) = \frac{\gamma}{\varepsilon_0\omega_p^2}\|J_n\|^2. \tag{3.32}$$

From (3.26) we get

$$\xi_n^2 \int_\Omega |J_n|^2\, dx \to 0, \text{ as } n \to +\infty.$$

This means that

$$\xi_n J_n \to 0, \text{ in } L^2(\Omega)^3, \text{ as } n \to +\infty. \tag{3.33}$$

This property and (3.29) imply that

$$\|\beta^2\nabla Q_n + \varepsilon_0\omega_p^2 E_n\| \to 0, \tag{3.34}$$

One has that $(E_n)$ is bounded in $(L^2(\Omega))^3$, so (3.34) implies that $\nabla Q_n$ is bounded in $(L^2(\Omega))^3$. Moreover,

$$|\int_\Omega \left(\beta^2 \nabla Q_n + \varepsilon_0 \omega_p^2 E_n\right) \cdot \nabla Q_n| \leq \|\beta^2 \nabla Q_n + \varepsilon_0 \omega_p^2 E_n\| \|\nabla Q_n\|. \tag{3.35}$$

This implies that

$$\left[\beta^2 \|\nabla Q_n\|^2 + \varepsilon_0 \omega_p^2 \|Q_n\|^2\right] \to 0, \tag{3.36}$$

where we used that $\varepsilon \operatorname{div} E_n = -Q_n$. We thus deduce that

$$Q_n \to 0, \tag{3.37}$$

in $H_0^1(\Omega)$. As a consequence,

$$E_n \to 0, \tag{3.38}$$

in $L^2(\Omega)$. Using that $\varepsilon \operatorname{div} E_n = -Q_n$, we have that $\operatorname{div} E_n \to 0$ in $L^2(\Omega)$.

From (3.27) and the above results:

$$\xi_n^{-1} \operatorname{curl} H_n \to 0, \tag{3.39}$$

in $(L^2(\Omega))^3$.

Since

$$\int_\Omega \operatorname{curl} E_n \cdot \bar{H}_n \, dx = \int_\Omega E_n \cdot \operatorname{curl} \bar{H}_n \, dx,$$

we get

$$\xi_n^{-1} \int_\Omega \operatorname{curl} E_n \cdot \bar{H}_n \, dx = o(1). \tag{3.40}$$

Now by (3.28) and the fact that $\|H_n\| = 0(1)$, we have

$$\xi_n^{-1} \int_\Omega (i\mu\xi_n H_n + \operatorname{curl} E_n) \cdot \bar{H}_n \, dx = o(1),$$

and by (3.40) we get

$$H_n \to 0, \text{ in } L^2(\Omega)^3. \tag{3.41}$$

In conclusion, we have shown that

$$Z_n \to 0, \text{ in } \mathcal{H},$$

which contradicts $\|Z_n\|_{\mathcal{H}} = 1$. ■

The previous Lemmas allow to check the hypotheses of Lemma 3.1 and then lead to the next stability results.

**Theorem 3.5** *Problem (2.11) is polynomially stable in $\mathcal{H}$, more precisely there exists a positive constant $C$ such that*

$$\mathcal{E}(t) \leq C\, t^{-1} ||U_0||^2_{\mathcal{D}(\mathcal{A})}, \forall t > 0, \tag{3.42}$$

*for all $U_0 \in \mathcal{D}(\mathcal{A})$.*

### 3.2 Magnetic boundary conditions

Comparing subsections 2.3 and 2.3.2, we see that it mainly suffices to exchange the role of $X_T(\Omega)$ and $X_N(\Omega)$, of $\Delta^{Neu}$ and $\Delta^{Dir}$, of $H_0(\mathrm{div};\Omega)$ and $H(\mathrm{div};\Omega)$ etc... Hence the arguments of the previous subsection can be adapted to prove that Lemmas 3.2, 3.3 and 3.4 hold. By Lemma 3.1, Theorem 3.5 is valid for system (2.37).

## 4 Optimal energy decay rate

### 4.1 A general result

The optimality of the decay is based on the next general principle, see [26, Le 5.1] or [19, 34].

**Lemma 4.1** *Consider a $C^0$-semigroup $T(t)$ acting on a complex Hilbert space $\mathcal{H}$ with infinitesimal generator $\mathcal{A}$. Assume that the two points below hold.*

*(i) For all $k \in \mathbb{N}^*$, we assume given a family of eigenvalues $\lambda_k$ of $\mathcal{A}$ of the form $\lambda_k = -\sigma_k + i\tau_k$ (repeated according to their multiplicities) with $\sigma_k, \tau_k \in \mathbb{R}$ and $\frac{c_1}{k^\delta} < \sigma_k < \frac{c_2}{k^\delta}$, where $0 < c_1 < c_2$ and $\delta > 0$ are independent of $k$.*

*(ii) The eigenvectors $\phi_k, k \geq 1$ associated with the eigenvalues $\lambda_k$ are orthonormal, in the sense that*

$$(\phi_k, \phi_{k'})_{\mathcal{H}} = \delta_{k,k'}, \forall (k,k') \in (\mathbb{N}^*)^2.$$

*Let $u_0 \in \mathcal{H}$ be such that*

$$u_0 = \sum_{k\geq 1} a_k \phi_k, \text{ with } |a_k| = \frac{1}{k^q} \text{ and } q > \frac{1}{2}. \tag{4.1}$$

*Then there exists a constant $c > 0$ depending on $u_0$ such that*

$$\|T(t)u_0\|_{\mathcal{H}} \geq \frac{c}{t^{(q-1/2)/\delta}}, \forall t > 1.$$

### 4.2 Electric boundary conditions

Recall [22] that the operator $A_N$ defined by

$$D(A_N) = \{E \in H(\mathrm{curl},\Omega) | \operatorname{div} E = 0 \text{ in } \Omega, \operatorname{curl}\operatorname{curl} E \in L^2(\Omega)^3 \text{ and } E\times n = 0, \operatorname{curl} E\cdot n = 0 \text{ on } \Gamma\},$$

and

$$A_N E = \operatorname{curl}\operatorname{curl} E, \forall E \in D(A_N),$$

is a positive selfadjoint operator in $L^2(\Omega)^3$ with a compact resolvent. Let us denote by $\{\lambda_{N,k}^2\}_{k\in\mathbb{N}^*}$ the eigenvalues of its discrete spectrum repeated according to their multiplicity. It consists of an increasing sequence that tends to $+\infty$ as $k \to +\infty$.

If $U_0 \in \mathcal{D}(\mathcal{A})$, we define the optimal rational decay rate $\omega(U_0)$ by

$$\omega(U_0) = \sup\{\alpha \in \mathbb{R} : \exists c > 0, \mathcal{E}(t) = \frac{1}{2}\|U(t)\|_{\mathcal{H}}^2 \leq \frac{c}{t^\alpha}, \forall t \geq 0, \text{ with } U \text{ the solution of (2.11)}\}. \tag{4.2}$$

**Lemma 4.2** *For system (2.7), there exists $k_0$ large enough such that $\mathcal{A}$ has eigenvalues $\lambda_k^{\pm}$, for all $k \geq k_0$ satisfying*

$$\lambda_k^{\pm} = \pm i(\varepsilon\mu)^{-1/2}\lambda_{N,k} \pm i\sqrt{\frac{\mu}{\varepsilon}}\frac{\varepsilon_0\omega_p^2}{2\lambda_{N,k}} - \frac{\gamma\varepsilon_0\omega_p^2\mu}{2\lambda_{N,k}^2} + o\left(\frac{1}{\lambda_{N,k}^2}\right), \forall k \geq k_0. \tag{4.3}$$

*Its associated eigenvector $U_k^{\pm}$ is in the form*

$$U_k^{\pm} = c_k^{\pm}\begin{pmatrix} \varphi_{N,k} \\ -\dfrac{1}{\lambda_k^{\pm}\mu}\operatorname{curl}\varphi_{N,k} \\ -\varepsilon\lambda_k^{\pm}\varphi_{N,k} - \dfrac{1}{\lambda_k^{\pm}\mu}\lambda_{N,k}^2\varphi_{N,k} \\ 0 \end{pmatrix}, \tag{4.4}$$

*where $\varphi_{N,k}$ is the eigenvector of the Maxwell operator $A_N$ associated with the eigenvalue $\lambda_{N,k}^2$ and $c_k^{\pm} \neq 0$ is a normalization factor chosen such that*

$$\|U_k^{\pm}\|_{\mathcal{H}} = 1.$$

**Proof.** From the definition of $\mathcal{A}$, if $U = (E, H, J, Q)^{\top} \in D(\mathcal{A})$ is an eigenvector of the operator $\mathcal{A}$ of eigenvalue $\lambda \in \mathbb{C} \setminus \{0\}$, it satisfies

$$\mathcal{A}U = \lambda U,$$

i.e.

$$\begin{cases} \varepsilon\lambda E - \operatorname{curl} H + J = 0 \\ \lambda\mu H = -\operatorname{curl} E, \\ \lambda J - \beta^2\nabla Q - \varepsilon_0\omega_p^2 E + \gamma J = 0, \\ \lambda Q = \operatorname{div} J, \end{cases} \tag{4.5}$$

From the second equation of (4.5), we deduce that $\operatorname{curl}\operatorname{curl} E \in L^2(\Omega)$. Furthermore, we thus have

$$\begin{cases} H = -\dfrac{1}{\lambda\mu}\operatorname{curl} E \\ J = -\varepsilon\lambda E - \dfrac{1}{\lambda\mu}\operatorname{curl}(\operatorname{curl}(E)) \\ Q = \dfrac{1}{\lambda}\operatorname{div} J \\ -\left(\varepsilon\lambda(\lambda+\gamma) + \varepsilon_0\omega_p^2\right)E - \dfrac{\lambda+\gamma}{\mu\lambda}\operatorname{curl}(\operatorname{curl}(E)) + \varepsilon\beta^2\nabla\operatorname{div} E = 0 \end{cases} \tag{4.6}$$

We try to take advantage of the spectral properties of $A_N$. Let us study the equation for $k \in \mathbb{N}^*$:

$$(\lambda+\gamma)\lambda^2\varepsilon\mu + \mu\varepsilon_0\omega_p^2\lambda = -(\lambda+\gamma)\lambda_{N,k}^2. \tag{4.7}$$

(4.7) is equivalent to

$$p_k(\lambda) = 0, \tag{4.8}$$

with $p_k$ the polynomial given by $p_k(\lambda) := (\lambda+\gamma)\left(\lambda^2\varepsilon\mu + \lambda_{N,k}^2\right) + \mu\varepsilon_0\omega_p^2\lambda$.
For each $k \in \mathbb{N}^*$, there exists three complex roots that are different from $-\gamma$. One has $p_k(0) = \gamma\lambda_{N,k}^2$ and $p_k(-\gamma) = -\mu\varepsilon_0\omega_p^2\gamma < 0$, so that there exists one real root $-\gamma < r_k < 0$.
We have that $r_k + \gamma = -\dfrac{\mu\varepsilon_0\omega_p^2 r_k}{r_k^2\varepsilon\mu + \lambda_{N,k}^2}$. Since $-\gamma < r_k < 0$, we deduce that

$$0 < r_k + \gamma < \frac{\mu\varepsilon_0\omega_p^2\gamma}{\lambda_{N,k}^2}. \tag{4.9}$$

So that $r_k \to -\gamma$ as $k \to +\infty$. Moreover there exists $k_0 \in \mathbb{N}^*$ such that for $k \geq k_0$, $p_k$ is strictly increasing. Therefore for $k \geq k_0$, the two other roots are complex conjugates. Let us then denote by $\lambda_k^\pm$ these two complex eigenvalues and $U_k^\pm$ the vector

$$U_k^\pm = \begin{pmatrix} \varphi_{N,k} \\ -\dfrac{1}{\lambda_k^\pm\mu}\operatorname{curl}\varphi_{N,k} \\ -\varepsilon\lambda_k^\pm\varphi_{N,k} - \dfrac{1}{\lambda_k^\pm\mu}\lambda_{N,k}^2\varphi_{N,k} \\ 0 \end{pmatrix}. \tag{4.10}$$

For $k \geq k_0$, $(\lambda_k^\pm, U_k^\pm)$ are eigenvalue-eigenvector pairs since, by construction, each verify (4.6) and thus (4.5). We have that $\varphi_{N,k} \in \mathcal{D}(A_N)$ and we deduce that $U_k^\pm \in \mathcal{D}(\mathcal{A})$. Let us now study the asymptotic of these eigenvalues. Introduce, for the clarity of the reading $\kappa := \varepsilon\mu$ and $\delta := \mu\varepsilon_0\omega_p^2$. $p_k$ thus rewrite

$$p_k(\lambda) = (\lambda+\gamma)\left(\kappa\lambda^2 + \lambda_{N,k}^2\right) + \delta\lambda. \tag{4.11}$$

For $k \geq k_0$, we write $\lambda_k^\pm = \alpha_k \pm i\zeta_k$, with $\zeta_k > 0$. We have the two following equations corresponding to the real and imaginary part of the equation $p_k(\lambda) = 0$, where for the sake of clarity we dropped the superscript $\pm$ and consider for the moment the case of $\lambda^+$ (the case of $\lambda^-$ would be treated similarly),

$$(\alpha_k+\gamma)\left((\alpha_k^2 - \zeta_k^2)\kappa + \lambda_{N,k}^2\right) - 2\alpha_k\zeta_k^2\kappa + \delta\alpha_k = 0, \tag{4.12}$$

$$\zeta_k\left(2(\alpha_k+\gamma)\alpha_k\kappa + (\alpha_k^2 - \zeta_k^2)\kappa + \lambda_{N,k}^2 + \delta\right) = 0. \tag{4.13}$$

Since $\zeta_k \neq 0$, we obtain

$$3\alpha_k^2\kappa + 2\gamma\alpha_k\kappa - \zeta_k^2\kappa + \lambda_{N,k}^2 + \delta = 0. \tag{4.14}$$

This equation has a real solution $\alpha_k$ if and only of its discriminant is non negative, this yealds:

$$\gamma^2\kappa^2 - 3\kappa(\lambda_{N,k}^2 + \delta - \zeta_k^2\kappa) \geq 0. \tag{4.15}$$

In other words, $\zeta_k^2 \geq \dfrac{-\gamma^2\kappa^2 + 3\kappa\lambda_{N,k}^2 + 3\kappa\delta}{3\kappa^2}$. Thus

$$\zeta_k^2 \to +\infty, \text{ as } k \to +\infty.$$

Let us then study more precisely $\alpha_k$. (4.13) gives

$$(\alpha_k^2 - \zeta_k^2)\kappa + \lambda_{N,k}^2 = -\delta - 2(\alpha_k + \gamma)\alpha_k\kappa. \tag{4.16}$$

Plugging this expression in (4.12), we find

$$-2\alpha_k\kappa\left((\alpha_k+\gamma)^2 + \zeta_k^2\right) = \delta\gamma. \tag{4.17}$$

In other words, $\alpha_k$ is a root of $q_k$ where

$$q_k(\alpha) := 2\alpha^3 + 4\alpha^2\gamma + 2\alpha(\gamma^2 + \zeta_k^2) + \frac{\delta\gamma}{\kappa}. \tag{4.18}$$

We have that $q_k(0) = \dfrac{\delta\gamma}{\kappa} > 0$. Also

$$q_k\left(-\frac{\delta\gamma}{2\kappa\zeta_k^2}\right) = -\frac{\delta\gamma^3}{\kappa\zeta_k^2}\left[\frac{\delta^2}{4\kappa^2\zeta_k^4} - \frac{\delta}{\kappa\zeta_k^2} + 1\right], \tag{4.19}$$

so that for $k$ large enough such that $\dfrac{\delta}{\gamma\kappa\zeta_k^2} < \dfrac{1}{2}$, $q_k\left(-\dfrac{\delta\gamma}{2\kappa\zeta_k^2}\right) < 0$. Furthermore, for $k$ large enough, $q_k'(\alpha) > 0$ and $q_k$ is strictly increasing. Thus $\alpha_k$ is unique and $-\dfrac{\delta\gamma}{2\kappa\zeta_k^2} < \alpha_k < 0$. This gives $\alpha_k < 0$ and $\alpha_k \to 0$ as $k \to +\infty$.

We can use more sophisticated $q_k$ to find an asymptotic expansion of $\alpha_k$. Denote $\eta := \dfrac{\delta\gamma}{\kappa}$ and fix $\xi > 0$ such that $\xi > \dfrac{\eta\gamma^2}{2}$. We denote by $\psi_k$ the real quantity $\psi_k := -\dfrac{\eta}{2\zeta_k^2} + \dfrac{\xi}{\zeta_k^4}$. Some easy manipulations gives, if $\chi := 2\xi - \gamma^2\eta$,

$$q_k(\psi_k) = \frac{\chi}{\zeta_k^2}\left[1 + \varphi_k\right], \tag{4.20}$$

where $\varphi_k := \left(\eta^2\gamma + 2\xi\gamma^2\right)\dfrac{1}{\chi\zeta_k^2} + \left(-\dfrac{\eta^3}{4} - 4\eta\xi\gamma\right)\dfrac{1}{\chi\zeta_k^4} + \left(\dfrac{3\eta^2\xi}{2} + 4\xi^2\gamma\right)\dfrac{1}{\chi\zeta_k^6} - \dfrac{3\eta\xi^2}{\chi\zeta_k^8} + \dfrac{2\xi^3}{\chi\zeta_k^{10}}$.

Since $\zeta_k \to +\infty$ as $k \to +\infty$, we deduce that for $k$ sufficiently large: $|\varphi_k| < \dfrac{1}{2}$, so that since the choice of $\xi$ gives $\chi > 0$, we find $q_k(\psi_k) > 0$. Thus

$$-\frac{\eta}{2\zeta_k^2} < \alpha_k < -\frac{\eta}{2\zeta_k^2} + \frac{\xi}{\zeta_k^4}. \tag{4.21}$$

We conclude that

$$\alpha_k = -\frac{\eta}{2\zeta_k^2} + O\left(\frac{1}{\zeta_k^4}\right). \tag{4.22}$$

Reusing (4.12), we first deduce that

$$\zeta_k \sim \frac{\lambda_{N,k}}{\sqrt{\kappa}}. \tag{4.23}$$

Then using (4.22) in (4.12), we obtain the following asymptotic expansion

$$\zeta_k = \frac{\lambda_{N,k}}{\sqrt{\kappa}} + \frac{\delta}{2\sqrt{\kappa}\lambda_{N,k}} + \mu_k, \tag{4.24}$$

with $\mu_k = O\left(\frac{1}{\lambda_{N,k}^3}\right)$. We thus conclude that

$$\lambda_k^+ = -\frac{\mu\varepsilon_0\omega_p^2\gamma}{2\lambda_{N,k}^2} + i\frac{\lambda_{N,k}}{\sqrt{\varepsilon\mu}} + i\sqrt{\frac{\mu}{\varepsilon}}\frac{\varepsilon_0\omega_p^2}{2\lambda_{N,k}} + O\left(\frac{1}{\lambda_{N,k}^3}\right). \tag{4.25}$$

■

Due to this lemma, we can prove the optimal energy decay rate for our system (2.7).

**Theorem 4.3** *For system (2.7), we have*

$$\inf_{u_0\in\mathcal{D}(\mathcal{A})} \omega(u_0) = 1. \tag{4.26}$$

**Proof.** The proof is the same as the one of Theorem 5.5 of [26] since the eigenvectors $U_k^+$ are orthonormal in $\mathcal{H}$ and the asymptotic behavior of the $\lambda_k^+$ is the same as the one from Lemma 5.4 of [26]. ■

## 4.3 Magnetic boundary conditions

Here we need the operator $A_T$ defined by

$$D(A_T) = \{E \in H(\mathrm{curl},\Omega) |\, \mathrm{div}\, E = 0 \text{ in } \Omega, \mathrm{curl}\,\mathrm{curl}\, E \in L^2(\Omega)^3 \text{ and } E\cdot n = 0 \text{ on } \Gamma\},$$

and

$$A_T E = \mathrm{curl}\,\mathrm{curl}\, E, \forall E \in D(A_N),$$

that is a positive selfadjoint operator in $L^2(\Omega)^3$ with a compact resolvent [22]. It is well known that $A_T$ has the same discrete spectrum than $A_T$, that we previously denote by $\{\lambda_{T,k}^2\}_{k\in\mathbb{N}^*}$, and that $\varphi_{T,k}$ is an associated eigenvector corresponding to $A_T$ if and only if $\mathrm{curl}\,\varphi_{T,k}$ is an associated eigenvector corresponding to $A_T$.

Clearly we can prove the

**Lemma 4.4** *For system (2.37), there exists $k_0$ large enough such that $\mathcal{A}$ has eigenvalues $\lambda_k^\pm$, for all $k \geq k_0$ satisfying*

$$\lambda_k^\pm = \pm i(\varepsilon\mu)^{-1/2}\lambda_{T,k} \pm i\sqrt{\frac{\mu}{\varepsilon}}\frac{\varepsilon_0\omega_p^2}{2\lambda_{T,k}} - \frac{\gamma\varepsilon_0\omega_p^2\mu}{2\lambda_{N,k}^2} + o\left(\frac{1}{\lambda_{T,k}^2}\right), \forall k \geq k_0. \tag{4.27}$$

*Its associated eigenvector $U_k^\pm$ is in the form*

$$U_k^\pm = c_k^\pm \begin{pmatrix} \varphi_{T,k} \\ -\frac{1}{\lambda_k^\pm\mu}\mathrm{curl}\,\varphi_{T,k} \\ -\varepsilon\lambda_k^\pm\varphi_{T,k} - \frac{1}{\lambda_k^\pm\mu}\lambda_{T,k}^2\varphi_{T,k} \\ 0 \end{pmatrix}, \tag{4.28}$$

*where $\varphi_{T,k}$ is the eigenvector of the Maxwell operator $A_T$ associated with the eigenvalue $\lambda^2_{T,k}$ and $c_k^{\pm} \neq 0$ is a normalization factor chosen such that*

$$\|U_k^{\pm}\|_{\mathcal{H}} = 1.$$

This Lemma directly leads to the optimality of the decay rate.

**Theorem 4.5** *The optimal rational decay rate (4.26) holds for system (2.37).*

# 5 A high order Discontinuous Galerkin numerical framework

In this section we consider the discretization of the linearized Hydrodynamic dispersive model (2.7), with a space discretization based on a Discontinuous Galerkin (DG) method.

Initially proposed by Reed and Hill [28] in the context of neutron transport problems, DG methods have become very popular and have been applied to a vast field of computational physics and engineering. DG methods have already been successfully used in the context of nanophotonics, see e.g. [7] and [30, 17, 29] (in the context of the study of (2.7)). In a more academic context, one can cite [32], [18]. Indeed, one can clearly benefit from the flexibility of DG methods to deal with complex and heterogeneous structures such as the one encountered in nanophotonics. The cost of the added unknowns resulting from the broken continuity at the interface is reduced by an appropriate parallel computing environment.

In the following, we first detail the scheme that will be used and propose a unified framework allowing to deal with several schemes at the same time. We fist recall the semi-discrete stability estimates presented in [29]. We moreover add a constraint weak preservation result. Then, we establish fully discrete stability estimates using energy techniques and keep track of the physical parameters and polynomial order in the constants. Our results extend the preliminary results obtained in [29] in this direction. We furthermore provide explicit CFL condition with respect to physical parameters and polynomial order. The generality of the framework will open the route to a more thorough stability analysis as a discrete analogue of the first part of this work. This will be part of a future work.

## 5.1 The semi-discrete setting

The classical Discontinuous Galerkin approximation relies on the choice of a non conforming space to approximate the unknown leading to a local weak formulation on each element of the mesh. The communication at the interfaces of cells is recovered via the definition of numerical fluxes (in the same spirit as finite volumes approximations).

We introduce a tetrahedral mesh of the domain $\Omega$ (that we will assume for simplicity to be convex polyhedral in this section) : $\overline{\Omega} = \bigcup_{i \in \mathcal{N}_\Omega} \overline{\Omega}_i$, $\mathcal{N}_\Omega$ being the set of indices of the mesh elements. We furthermore suppose that the mesh is quasi-uniform with quasi-uniformity constant $\eta > 0$. We will denote the mesh size by $h > 0$. Furthermore, for all $i \in \mathcal{N}_\Omega$, $\mathcal{N}_{\Omega_i}$ will denote the set of indices of the neighboring elements of $\Omega_i$ (having a face in common) and $\mathcal{F}_{iq} = \overline{\Omega}_i \cap \overline{\Omega}_q$, $\forall q \in \mathcal{N}_{\Omega_i}$, the internal faces. We also denote by $\mathcal{F}$ the set of all faces of the mesh. We define the finite dimensional non-conforming approximation space as

$$\mathbb{V}_h^p := \left\{ v \in L^2(\Omega), v|_{\Omega_i} \in \mathbb{P}_p(\Omega_i), \forall i \in \mathcal{N}_\Omega \right\}, \tag{5.1}$$

where $\mathbb{P}_p(\Omega_i)$ is the space of polynomials of maximum degree $p \in \mathbb{N}$ on $\Omega_i$. We also denote $\mathbb{W}_h^p = (\mathbb{V}_h^p)^{10}$.

For $\boldsymbol{\vartheta} \in \mathbb{V}_h^p$, and $i \in \mathcal{N}_\Omega$, we denote by $\boldsymbol{\vartheta}_i$ the restriction of $V$ on $\Omega_i$.

The semi-discrete DG formulation write as follows: find $(E_h, H_h, J_h, Q_h) \in \mathbb{W}_h^p$, such that for all $i \in \mathcal{N}_\Omega$, $\forall(\varphi_h, \psi_h, \xi_h, \zeta_h)$ in $\mathbb{W}_h^p$

$$\begin{aligned}
\int_{\Omega_i} \mu_0 (H_h)_t \cdot \psi_h dx &= -\int_{\Omega_i} E_h \cdot \operatorname{curl} \psi_h dx - \int_{\partial\Omega_i} (\boldsymbol{n} \times E_h^*) \cdot \psi_h ds, \\
\int_{\Omega_i} \epsilon_0 \epsilon_\infty (E_h)_t \cdot \varphi_h dx &= \int_{\Omega_i} (H_h \cdot \operatorname{curl} \varphi_h - J_h \cdot \varphi_h) dx + \int_{\partial\Omega_i} (\boldsymbol{n} \times H_h^*) \cdot \varphi_h ds - \int_{\Omega_i} J_h \cdot \varphi_h dx, \\
\int_{\Omega_i} (J_h)_t \cdot \xi_h dx &= -\int_{\Omega_i} \beta^2 Q_h \operatorname{div} \xi_h dx + \int_{\partial\Omega_i} \beta^2 Q_h^* \xi_h \cdot \boldsymbol{n} ds + \int_{\Omega_i} (\epsilon_0 \omega_p^2 E_h \cdot \xi_h - \gamma J_h \cdot \xi_h) dx, \\
\int_{\Omega_i} (Q_h)_t \zeta_h dx &= -\int_{\Omega_i} J_h \cdot \nabla \zeta_h dx + \int_{\partial\Omega_i} J_h^* \cdot \boldsymbol{n} \zeta_h ds.
\end{aligned}$$

The $^*$ quantities refer to the flux at the interface that one has to define. Several choices are available for these fluxes that will affect the different properties of the scheme such as e.g. dispersion or dissipation. We will work with two basic fluxes, namely the centered and upwind ones, that can be put in the following abstract form. Let $i \in \mathcal{N}_\Omega$ and $l \in \mathcal{N}_{\Omega_i}$, then on $\mathcal{F}_{il}$, we set

$$E_h^* = \frac{1}{2}\left(\{E_h\}_{il} + \alpha Z \mathbf{n} \times [\![H_h]\!]_{il}\right), H_h^* = \frac{1}{2}\left(\{H_h\}_{il} - \alpha Y \mathbf{n} \times [\![E_h]\!]_{il}\right), \tag{5.2}$$

with $Y = \sqrt{\frac{\varepsilon}{\mu}}$ and $Z = \sqrt{\frac{\mu}{\varepsilon}}$,

$$\begin{aligned}
Q_h^* &= \frac{1}{2}\left(\{Q_h\}_{il} - \alpha \frac{1}{\beta} \mathbf{n} \cdot [\![J_h]\!]_{il}\right), \\
\mathbf{n} \cdot J_h^* &= \frac{1}{2}\left(\mathbf{n} \cdot \{J_h\}_{il} - \alpha\beta [\![Q_h]\!]_{il}\right),
\end{aligned} \tag{5.3}$$

with $\alpha \in \{0, 1\}$, $\mathbf{n}$ the outward normal to the considered face and for all $\boldsymbol{\vartheta} \in \mathbb{V}_h^p(\Omega)$, $\{\boldsymbol{\vartheta}\}_{il} = \boldsymbol{\vartheta}_i + \boldsymbol{\vartheta}_l$, $[\![\boldsymbol{\vartheta}]\!]_{il} = \boldsymbol{\vartheta}_i - \boldsymbol{\vartheta}_l$, $\forall(i, l) \in \mathcal{N}_\Omega \times \mathcal{N}_{\Omega_i}$. The case $\alpha = 0$ is referred to as centered flux, while the case $\alpha = 1$ is referred to as upwind flux.

To ease the reading, we introduce several discrete forms $a_h, b_{h,\alpha}, k_h^1, k_h^2, c_{\alpha,h}$ from $\mathbb{W}_h^p \times \mathbb{W}_h^p$ to

$\mathbb{R}$ as

$$
(5.4)\qquad \begin{cases}
a_h(\boldsymbol{\vartheta},\boldsymbol{\vartheta}') = -\left(E_h, \operatorname{curl}\psi_h\right)_h + \left(H_h, \operatorname{curl}\varphi_h\right)_h \\
\qquad\qquad - \dfrac{\beta^2}{\varepsilon_0\omega_p^2}\left(Q_h, \operatorname{div}\xi_h\right)_h - \dfrac{\beta^2}{\varepsilon_0\omega_p^2}\left(J_h, \nabla\zeta_h\right)_h, \\
b_{h,\alpha}(\boldsymbol{\vartheta},\boldsymbol{\vartheta}') = -\langle(\boldsymbol{n}\times\{E_h\}), [\![\psi_h]\!]\rangle_h - \alpha Z\langle n\times[\![H_h]\!], n\times[\![\psi_h]\!]\rangle_h \\
\qquad\qquad + \langle(\boldsymbol{n}\times\{H_h\}), [\![\varphi_h]\!]\rangle_h - \alpha Y\langle n\times[\![E]\!], n\times[\![\varphi_h]\!]\rangle_h \\
\qquad\qquad + \dfrac{\beta^2}{\varepsilon_0\omega_p^2}\langle\{Q_h\}, [\![\xi_h]\!]\cdot\boldsymbol{n}\rangle_h - \alpha\dfrac{\beta}{\varepsilon_0\omega_p^2}\langle[\![J_h]\!]\cdot n, [\![\xi_h]\!]\cdot n\rangle_h \\
\qquad\qquad + \dfrac{\beta^2}{\varepsilon_0\omega_p^2}\langle\{J_h\}\cdot\boldsymbol{n}, [\![\zeta_h]\!]\rangle_h - \alpha\dfrac{\beta^3}{\varepsilon_0\omega_p^2}\langle[\![Q_h]\!], [\![\zeta_h]\!]\rangle_h, \\
k_h^1(\boldsymbol{\vartheta},\boldsymbol{\vartheta}') = -\left(J_h, \varphi_h\right)_h + \left(E_h, \xi_h\right)_h, \\
k_h^2(\boldsymbol{\vartheta},\boldsymbol{\vartheta}') = -\dfrac{\gamma}{\epsilon_0\omega_p^2}\left(J_h, \xi_h\right)_h, \forall(\boldsymbol{\vartheta},\boldsymbol{\vartheta}') \in \mathbb{W}_h^p\times\mathbb{W}_h^p,
\end{cases}
$$

and finally $c_{\alpha,h} = a_h + b_{\alpha,h} + k_h^1 + k_h^2$. Here curl, div and $\nabla$ have to be understood as respectively piecewise curl, divergence and gradient operator (on each $\Omega_i$, $i\in\mathcal{N}_\Omega$). Furthermore, for all $(\boldsymbol{\vartheta},\boldsymbol{\vartheta}')\in\boldsymbol{\vartheta}_h^p\times\boldsymbol{\vartheta}_h^p$,

$$
\left(\boldsymbol{\vartheta},\boldsymbol{\vartheta}'\right)_h = \sum_{i\in\mathcal{N}_\Omega}(\boldsymbol{\vartheta}_i,\boldsymbol{\vartheta}_i')_{L^2(\Omega_i)},
$$

$$
\langle\boldsymbol{\vartheta},\boldsymbol{\vartheta}'\rangle_{\mathcal{F}} = \sum_{F\in\mathcal{F}}(\boldsymbol{\vartheta}_i,\boldsymbol{\vartheta}_i')_{L^2(F)},
$$

with the associated respective norms $\|\cdot\|_h$, $\|\cdot\|_{\mathcal{F}}$.

If there is no ambiguity, we will denote $|||\cdot|||_{\mathcal{H}}$, the norm of linear and bilinear forms on either $(\mathcal{L}(\mathcal{H},\mathbb{C}),\|\cdot\|_{\mathcal{H}})$ and $(\mathcal{B}(\mathcal{H}\times\mathcal{H},\mathbb{C}),\|\cdot\|_{\mathcal{H}})$.
Finally, $|\cdot|_S$ is defined for $\boldsymbol{\vartheta}\in\mathbb{W}_h^p$,

$$
|\boldsymbol{\vartheta}|_S^2 := \sum_{j=1}^{10}\delta_j\|[\![\boldsymbol{\vartheta}_j]\!]\|_{\mathcal{F}}^2
$$

with for $j\in\{1,2,3\}$, $\delta_j = c\varepsilon$, for $j\in\{4,5,6\}$, $\delta_j = c\mu$, for $j\in\{7,8,9\}$, $\delta_j = \frac{\beta}{\varepsilon_0\omega_p^2}$, and $\delta_{10} = \frac{\beta^3}{\varepsilon_0\omega_p^2}$, with $c = \dfrac{1}{\sqrt{\varepsilon\mu}}$ .

Thus, the global semi-discrete weak formulation can be written as follows.
Find $\boldsymbol{\vartheta}_h\in\mathbb{W}_h^p$ such that $\forall\boldsymbol{\vartheta}_h'\in\mathbb{W}_h^p$,

$$
(5.5)\qquad \left(\frac{\partial\boldsymbol{\vartheta}_h}{\partial t},\boldsymbol{\vartheta}_h'\right)_{\mathcal{H}} = c_{\alpha,h}(\boldsymbol{\vartheta}_h,\boldsymbol{\vartheta}_h').
$$

One can easily prove that there exists a unique solution in $\mathcal{C}^1(0,T,\mathbb{W}_h^p)$ with initial conditions $\boldsymbol{\vartheta}_h^0 = \pi_h(\boldsymbol{\vartheta}^0)$, where $\pi_h$ is the corresponding $L^2$ orthogonal projector on $\mathbb{W}_h^p$.

Inverse inequalities and quasi-uniformity of the mesh (with related parameter $\eta$) give

**Proposition 5.1** *There exits $C > 0$ such that for all $h > 0$, and for all $\boldsymbol{\vartheta} \in \mathbb{V}_h^p$,*

$$\begin{aligned}
\|\operatorname{curl}(\boldsymbol{\vartheta})\|_h &\leq C\eta p^2 h^{-1}\|\boldsymbol{\vartheta}\|_h, \\
\|\nabla\boldsymbol{\vartheta}\|_h &\leq C\eta p^2 h^{-1}\|\boldsymbol{\vartheta}\|_h, \\
\|\operatorname{div}\boldsymbol{\vartheta}\|_h &\leq C\eta p^2 h^{-1}\|\boldsymbol{\vartheta}\|_h, \\
\|[\![\boldsymbol{\vartheta}]\!]\|_{\mathcal{F}} &\leq C\eta p h^{-1/2}\|\boldsymbol{\vartheta}\|_h, \\
\|\{\boldsymbol{\vartheta}\}\|_{\mathcal{F}} &\leq C\eta p h^{-1/2}\|\boldsymbol{\vartheta}\|_h.
\end{aligned}$$

In the following, we give some continuity estimates on these bilinear forms that will help us later to complete the stability study.

**Proposition 5.2** *Let $\alpha \in [0,1]$. There exists $C_\alpha > 0$ such that*

$$|||a_h + b_{h,\alpha}|||_{\mathcal{H}} \leq C_\alpha p^2 h^{-1}\eta, \tag{5.6}$$

$$|||k_h^1|||_{\mathcal{H}} \leq \frac{\omega_p}{\sqrt{\varepsilon_\infty}},$$

$$|||k_h^2|||_{\mathcal{H}} \leq \gamma.$$

*Similarly*

$$|||b_{\alpha,h}|||_{\mathcal{H}} \leq C_\alpha \eta p^2 h^{-1},$$

*and finally $\forall(\boldsymbol{\varsigma}, \boldsymbol{\xi}) \in \mathbb{W}_h^p \times \mathbb{W}_h^p$,*

$$|b_{\alpha,h}(\boldsymbol{\varsigma}, \boldsymbol{\xi})| \leq C_\alpha \eta p h^{-\frac{1}{2}}\|\boldsymbol{\varsigma}\||\boldsymbol{\xi}|_S.$$

One has the following result:

**Proposition 5.3** *[29] Let $\alpha \in [0,1]$. For all $\boldsymbol{\vartheta} \in \mathbb{W}_h^p$, it holds*

$$a_h(\boldsymbol{\vartheta}, \boldsymbol{\vartheta}) + b_{\alpha,h}(\boldsymbol{\vartheta}, \boldsymbol{\vartheta}) = -\alpha|\boldsymbol{\vartheta}|_S^2.$$

Furthermore, for all $\boldsymbol{\vartheta} \in \mathbb{W}_h^p$, we easily see that

$$k_h^1(\boldsymbol{\vartheta}, \boldsymbol{\vartheta}) = 0, \tag{5.7}$$

$$k_h^2(\boldsymbol{\vartheta}, \boldsymbol{\vartheta}) = -\frac{\gamma}{\varepsilon_0 \omega_p^2}\sum_{j=7}^{9}\|(\boldsymbol{\vartheta})_j\|^2. \tag{5.8}$$

In the following we will need the canonical projectors $\mathfrak{p} : \mathbb{W}_h^p \to \mathbb{W}_h^p : \boldsymbol{\vartheta} = (F, G, R, S) \to (F, 0, 0, S)$, $\mathfrak{q} : \mathbb{W}_h^p \to \mathbb{W}_h^p : \boldsymbol{\vartheta} = (F, G, R, S) \to (0, G, R, 0)$, $\mathfrak{p}_M : \mathbb{W}_h^p \to \mathbb{W}_h^p : \boldsymbol{\vartheta} = (F, G, R, S) \to (F, G, 0, 0)$, $\mathfrak{p}_H : \mathbb{W}_h^p \to \mathbb{W}_h^p : \boldsymbol{\vartheta} = (F, G, R, S) \to (0, 0, R, S)$. One immediately sees that $\mathfrak{q} = id - \mathfrak{p}$.

**Proposition 5.4** *Let $\mathfrak{s} \in \{\mathfrak{p}, \mathfrak{q}, \mathfrak{p}_M, \mathfrak{p}_H\}$ and $d \in \{a_h, b_{0,h}, k_h^1\}$, for all $(\boldsymbol{\vartheta}, \boldsymbol{\vartheta}') \in \mathbb{W}_h^p \times \mathbb{W}_h^p$, we have*

$$\begin{aligned}
(\boldsymbol{\vartheta}, \mathfrak{s}(\boldsymbol{\vartheta})))_{\mathcal{H}} &= (\mathfrak{s}(\boldsymbol{\vartheta}), \mathfrak{s}(\boldsymbol{\vartheta})))_{\mathcal{H}}, \\
d(\boldsymbol{\vartheta}, \mathfrak{p}(\boldsymbol{\vartheta})) &= d(\mathfrak{q}(\boldsymbol{\vartheta}), \mathfrak{p}(\boldsymbol{\vartheta})), \\
d(\boldsymbol{\vartheta}, \mathfrak{q}(\boldsymbol{\vartheta})) &= d(\mathfrak{p}(\boldsymbol{\vartheta}), \mathfrak{q}(\boldsymbol{\vartheta})), \\
d(\mathfrak{q}(\boldsymbol{\vartheta}), \mathfrak{p}(\boldsymbol{\vartheta}')) + d(\mathfrak{p}(\boldsymbol{\vartheta}'), \mathfrak{q}(\boldsymbol{\vartheta})) &= d(\mathfrak{p}(\boldsymbol{\vartheta}') + \mathfrak{q}(\boldsymbol{\vartheta}), \mathfrak{p}(\boldsymbol{\vartheta}') + \mathfrak{q}(\boldsymbol{\vartheta})).
\end{aligned}$$

*As a consequence, for all $\boldsymbol{\vartheta} \in \mathbb{W}_h^p$, we have*

$$d(\mathfrak{q}(\boldsymbol{\vartheta}), \mathfrak{p}(\boldsymbol{\vartheta})) + d(\mathfrak{p}(\boldsymbol{\vartheta}), \mathfrak{q}(\boldsymbol{\vartheta})) = d(\boldsymbol{\vartheta}, \boldsymbol{\vartheta}).$$

*Furthermore, for all $(\boldsymbol{\vartheta}, \boldsymbol{\vartheta}') \in \mathbb{W}_h^p \times \mathbb{W}_h^p$, we have*

$$\begin{aligned} k_h^2(\boldsymbol{\vartheta}', \mathfrak{p}(\boldsymbol{\vartheta})) &= 0, \\ k_h^2(\boldsymbol{\vartheta}', \mathfrak{q}(\boldsymbol{\vartheta})) &= k_h^2(\mathfrak{q}(\boldsymbol{\vartheta}'), \mathfrak{q}(\boldsymbol{\vartheta})). \end{aligned}$$

We do not detail the proof since it is straightforward.

We also have the following estimate.

**Proposition 5.5** *Let $d \in \{a_h, b_{0,h}\}$, for all $\boldsymbol{\vartheta} \in \mathbb{W}_h^p$, we have*

$$|d(\boldsymbol{\vartheta}, \mathfrak{p}(\boldsymbol{\vartheta}))| \leq Cc\eta p^2 h^{-1} \|\mathfrak{p}_M(\boldsymbol{\vartheta})\|_{\mathcal{H}}^2 + C\beta\eta p^2 h^{-1} \|\mathfrak{p}_H(\boldsymbol{\vartheta})\|_{\mathcal{H}}^2,$$

*with $c = \dfrac{1}{\sqrt{\varepsilon\mu}}$ and $C$ a generic positive constant. More generally, for all $(\boldsymbol{\vartheta}, \boldsymbol{\vartheta}') \in \mathbb{W}_h^p \times \mathbb{W}_h^p$,*

$$|a_h(\boldsymbol{\vartheta}, \boldsymbol{\vartheta}')| \leq Cc\eta p^2 h^{-1} \|\mathfrak{p}_M(\boldsymbol{\vartheta})\|_{\mathcal{H}} \|\mathfrak{p}_M(\boldsymbol{\vartheta}')\|_{\mathcal{H}} + C\beta\eta p^2 h^{-1} \|\mathfrak{p}_H(\boldsymbol{\vartheta})\|_{\mathcal{H}} \|\mathfrak{p}_H(\boldsymbol{\vartheta}')\|_{\mathcal{H}},$$

*and for all $\alpha \in [0, 1]$,*

$$|b_{\alpha,h}(\boldsymbol{\vartheta}, \boldsymbol{\vartheta}')| \leq Cc\eta p^2 h^{-1} \|\mathfrak{p}_M(\boldsymbol{\vartheta})\boldsymbol{\vartheta}\|_{\mathcal{H}} \|\mathfrak{p}_M(\boldsymbol{\vartheta}')\|_{\mathcal{H}} + C\beta\eta p^2 h^{-1} \|\mathfrak{p}_H(\boldsymbol{\vartheta})\|_{\mathcal{H}} \|\mathfrak{p}_H(\boldsymbol{\vartheta}')\|_{\mathcal{H}} + C\eta\alpha p h^{-\frac{1}{2}} |\boldsymbol{\vartheta}|_S \|\boldsymbol{\vartheta}'\|_{\mathcal{H}},$$

*and*

$$|k_h^1(\boldsymbol{\vartheta}, \boldsymbol{\vartheta}')| \leq \frac{\omega_p}{\sqrt{\varepsilon_\infty}} \left( \|\mathfrak{p}_M(\boldsymbol{\vartheta})\|_{\mathcal{H}} \|\mathfrak{p}_M(\boldsymbol{\vartheta}')\|_{\mathcal{H}} + \|\mathfrak{p}_H(\boldsymbol{\vartheta})\|_{\mathcal{H}} \|\mathfrak{p}_H(\boldsymbol{\vartheta}')\|_{\mathcal{H}} \right).$$

**Proof.** We only detail how to obtain the first inequality since the other inequalities are obtained similarly.

For all $\boldsymbol{\vartheta} \in \mathbb{W}_h^p$,

$$|a_h(\boldsymbol{\vartheta}, \mathfrak{p}(\boldsymbol{\vartheta}))| \leq c\sqrt{\varepsilon} \|F\|_h \sqrt{\mu} \|\operatorname{curl}(G)\|_h + \beta \frac{\beta}{\sqrt{\varepsilon_0}\omega_p} \|R\|_h \frac{1}{\sqrt{\varepsilon_0}\omega_p} \|\nabla S\|_h.$$

Using Proposition 5.1, we find that

$$\begin{aligned} |a_h(\boldsymbol{\vartheta}, \mathfrak{p}(\boldsymbol{\vartheta}))| &\leq Cc\eta p^2 h^{-1} \sqrt{\varepsilon} \|F\|_h \sqrt{\mu} \|G\|_h + C\beta\eta p^2 h^{-1} \tfrac{\beta}{\sqrt{\varepsilon_0}\omega_p} \|R\|_h \tfrac{1}{\sqrt{\varepsilon_0}\omega_p} \|S\|_h \\ &\leq Cc\eta p^2 h^{-1} \|\mathfrak{p}_M(\boldsymbol{\vartheta})\|_{\mathcal{H}}^2 + C\beta\eta p^2 h^{-1} \|\mathfrak{p}_H(\boldsymbol{\vartheta})\|_{\mathcal{H}}^2. \end{aligned}$$

■

Combining all the previous propositions, we easily obtain the following result.

**Proposition 5.6** *One has for all $(\boldsymbol{\vartheta}, \boldsymbol{\vartheta}') \in \mathbb{W}_h^p \times \mathbb{W}_h^p$,*

$$\begin{aligned} |c_{h,\alpha}(\boldsymbol{\vartheta}, \boldsymbol{\vartheta}')| \leq & \left( 2Cc\eta p^2 h^{-1} + \frac{\omega_p}{\sqrt{\varepsilon_\infty}} \right) \|\mathfrak{p}_M(\boldsymbol{\vartheta})\|_{\mathcal{H}} \|\mathfrak{p}_M(\boldsymbol{\vartheta}')\|_{\mathcal{H}} + \\ & \left( 2C\beta\eta p^2 h^{-1} + \frac{\omega_p}{\sqrt{\varepsilon_\infty}} + \gamma \right) \|\mathfrak{p}_H(\boldsymbol{\vartheta})\|_{\mathcal{H}} \|\mathfrak{p}_H(\boldsymbol{\vartheta}')\|_{\mathcal{H}} + C\eta\alpha p h^{-\frac{1}{2}} |\boldsymbol{\vartheta}|_S \|\boldsymbol{\vartheta}'\|_{\mathcal{H}}. \end{aligned}$$

All these estimates will serve in proving the stability of the fully discrete schemes that we will consider.

First, we focus on the semi-discrete stability. To this end, we define the energy of the semi-discrete problem by

$$\mathcal{E}_h = \frac{1}{2}(\boldsymbol{\vartheta}_h, \boldsymbol{\vartheta}_h)_{\mathcal{H}}, \text{ on } ]0,T[. \tag{5.9}$$

One has

**Proposition 5.7** *[29] For $\alpha \in \{0,1\}$,*

$$\mathcal{E}_h(t) = \mathcal{E}_h(0) - \frac{\gamma}{\varepsilon_0 \omega_p^2}\|J_h\|^2 - \alpha|\boldsymbol{\vartheta}_h|_S^2,$$

*with $\boldsymbol{\vartheta}_h = (E_h, H_h, J_h, Q_h)$.*

**Proof.** This result easily follows from Proposition 5.3, (5.7), (5.8) and the regularity (in time) on the solution. ■

**Remark 5.8** *The previous Proposition means that we are using a semi-discretization that converges and that adds (if $\alpha \neq 0$) numerical dissipation to the system, (i.e., the term $|\boldsymbol{\vartheta}_h|_S^2$). The dissipation term coming from the continuous setting, i.e., the term $-\dfrac{\gamma}{\varepsilon_0\omega_p^2}\|J_h\|^2$ is itself unchanged.*

As mentioned in [29], a direct combination of the arguments used in [17] allows to conclude to the convergence of the semi-discrete schemes, with classical orders (i.e. $p$ if $\alpha = 0$ and $p + \frac{1}{2}$, if $\alpha = 1$). We will not reproduce the proof here.

Last, we can prove that the constraint is preserved at the semi-discrete level.

**Proposition 5.9** *Let $\mathbb{Y}_h^p \subset H_0^1(\Omega)$ be the space of piecewise continuous polynomials of degree $p$ with zero trace on the boundary. If $\boldsymbol{\vartheta_h} = (E_h, H_h, J_h, Q_h) \in \mathbb{W}_h^p$ is the solution of (5.5), and if at the initial time,*

$$-\langle \varepsilon E_h(0,\cdot), \nabla p_h\rangle + (Q_h(0,\cdot), p_h)_{\mathcal{H}}\rangle = 0, \forall p_h \in \mathbb{Y}_h^p,$$

*then for all $t \in [0,T]$,*

$$-\langle \varepsilon E_h(t,\cdot), \nabla p_h\rangle + \langle Q_h(t,\cdot), p_h\rangle = 0, \forall p_h \in \mathbb{Y}_h^p,$$

*i.e. one has a weak (and discrete) preservation of the constraint* $\operatorname{div}(\varepsilon E) + Q = 0$.

**Proof.** Let $p_h \in \mathbb{Y}_h^p$. Due to the continuity of $p_h$, $\nabla p_h$ has no tangential jump at the element interfaces and has zero tangential trace at the boundary of the domain. Now, we consider the weak formulation (5.5) and choose $\boldsymbol{\vartheta}_h' = (-\frac{\beta^2}{\varepsilon_0\omega_p^2}\nabla p_h, 0, 0, p_h)$, with $p_h \in \mathbb{Y}_h^p$. One thus has, using the tangential continuity of $\nabla p_h$ and $p_h$ at interfaces and the zero boundary condition,

$$\begin{aligned} -\varepsilon\frac{\beta^2}{\varepsilon_0\omega_p^2}\langle \partial_t E_h, \nabla p_h\rangle + \frac{\beta^2}{\varepsilon_0\omega_p^2}\langle \partial_t Q_h, p_h\rangle &= -\frac{\beta^2}{\varepsilon_0\omega_p^2}\langle J_h, \nabla p_h\rangle + \frac{\beta^2}{\varepsilon_0\omega_p^2}\langle J_h, \nabla p_h\rangle \\ &= 0. \end{aligned}$$

This shows that $-\langle E_h, \nabla p_h\rangle + \langle Q_h, p_h\rangle$ is constant in time. Thus, if it is zero at the initial time, it will remain zero at all positive time. ■

## 5.2 Time discretization

We will now focus on the time integration scheme. We will discretize in time by using three time integration schemes: a Leap-frog scheme of order 2 (LF2) and two explicit Runge-Kutta schemes (RK2 of order 2 and RK4 of order 4). We will review the stability properties of these scheme in our precise context. In [29], the stability of the LF2 (with $\alpha = 0$) and RK4 (with $\alpha = 1$) schemes were quickly sketched. Proving the stability of these schemes relies on a generalization of the arguments used in [17], where the focus was put on RK4 schemes. Here, we choose to go more into details, especially by detailing the stability proofs for LF2 and RK2 and giving explicitly the stability constant in terms of the physical parameters.

In this prospect, we introduce a uniform subdivision of the time interval $[0, T]$, with $(t_n)_{n\in[\![0,N]\!]}$, $N \in \mathbb{N}^*$ with time step $\Delta t = \dfrac{T}{N}$.

### 5.2.1 The Leap-Frog scheme of order 2 (LF2)

The LF2 scheme shall preserve the dissipative properties of the semi-discrete scheme. It writes as follows: For $n \in [\![0, N]\!]$, find $\boldsymbol{\vartheta}_h^n = (E_h^n, H_h^{n+\frac{1}{2}}, J_h^{n+\frac{1}{2}}, Q_h^n) \in \mathbb{W}_h^p$ such that for all $i \in \mathcal{N}_\Omega$ and all $(\varphi_h, \psi_h, \xi_h, \zeta_h) \in \mathbb{W}_h^p$,

$$\left(\frac{\boldsymbol{\vartheta}_h^{n+1} - \boldsymbol{\vartheta}_h^n}{\Delta t}, \boldsymbol{\vartheta}_h'\right)_{\mathcal{H}} = a_h(\tilde{\boldsymbol{\vartheta}}_h^n, \boldsymbol{\vartheta}_h') + b_{\alpha,h}(\tilde{\boldsymbol{\vartheta}}_h^n, \boldsymbol{\vartheta}_h') + k_h^1(\tilde{\boldsymbol{\vartheta}}_h^n, \boldsymbol{\vartheta}_h') + \frac{1}{2}k_h^2(\boldsymbol{\vartheta}_h^n + \boldsymbol{\vartheta}_h^{n+1}, \boldsymbol{\vartheta}_h'). \tag{5.10}$$

with $\tilde{\boldsymbol{\vartheta}}_h^n = (E_h^{n+1}, H_h^{n+\frac{1}{2}}, J_h^{n+\frac{1}{2}}, Q_h^{n+1})$.

**Remark 5.10** *If $\alpha = 0$, then the scheme can be easily written in an explicit form. However, if $\alpha \neq 0$, the upwind part of the flux is implicit. Doing so, we loose the flexibility of the locality of DG method combined with a Leap-frog type approximation. we will therefore only concentrate on the case of Leap-frog scheme with centered fluxes (i.e. $\alpha = 0$).*

We focus on energy techniques to prove stability. In [12], the stability of a centered DG scheme with LF2 time integration for Maxwell's equation with absorbing boundary conditions is studied. Following modified energy technique used in the latter, we could investigate the stability of the upwind scheme ($\alpha = 1$) combined to the LF2 time discretization. However, we will not include this case in the following proofs, since we will not use LF2 scheme with upwind fluxes (see previous remark).

First, we point out some straightforward properties that will be used in the sequel. One has

$$\mathfrak{p}(\tilde{\boldsymbol{\vartheta}}_h^n) = \mathfrak{p}(\boldsymbol{\vartheta}_h^{n+1}), \tag{5.11}$$

$$\mathfrak{q}(\tilde{\boldsymbol{\vartheta}}_h^n) = \mathfrak{q}(\boldsymbol{\vartheta}_h^n). \tag{5.12}$$

We then define the fully discrete energy as:

$$\mathcal{E}_h^{n+\frac{1}{2}} := \frac{1}{2}\left(\boldsymbol{\vartheta}_h^n, \tilde{\boldsymbol{\vartheta}_h}^n\right)_{\mathcal{H}}. \tag{5.13}$$

We remark that this energy can be rewritten using the projectors $\mathfrak{p}$ and $\mathfrak{q}$ as:

$$\mathcal{E}_h^{n+\frac{1}{2}} := \frac{1}{2}\left(\mathfrak{q}(\boldsymbol{\vartheta}_h^n), \mathfrak{q}(\boldsymbol{\vartheta}_h^n)\right)_{\mathcal{H}} + \frac{1}{2}\left(\mathfrak{p}(\boldsymbol{\vartheta}_h^{n+1}), \mathfrak{p}(\boldsymbol{\vartheta}_h^n)\right)_{\mathcal{H}}. \tag{5.14}$$

This energy is not necessarily positive, but one has the

**Proposition 5.11** *Let $\alpha = 0$. If $\Delta t\left(Cc\eta p^2 h^{-1} + \frac{\omega_p}{\sqrt{\varepsilon_\infty}}\right) < 1$ and $\Delta t\left(C\beta\eta p^2 h^{-1} + \frac{\omega_p}{\sqrt{\varepsilon_\infty}}\right) < 1$, then the energy is positive definite.*

**Proof.** Let $i \in \mathcal{N}_\Omega$. One has

$$\begin{aligned}\mathcal{E}_h^{n+\frac{1}{2}} &= \frac{1}{2}\left(\boldsymbol{\vartheta}_h^n, \tilde{\boldsymbol{\vartheta}_h}^n - \boldsymbol{\vartheta}_h^n + \boldsymbol{\vartheta}_h^n\right)_{\mathcal{H}} \\ &= \frac{1}{2}\left(\boldsymbol{\vartheta}_h^n, \boldsymbol{\vartheta}_h^n\right)_{\mathcal{H}} + \frac{1}{2}\left(\boldsymbol{\vartheta}_h^n, \tilde{\boldsymbol{\vartheta}_h}^n - \boldsymbol{\vartheta}_h^n\right)_{\mathcal{H}}.\end{aligned}$$

Using that $\tilde{\boldsymbol{\vartheta}_h}^n - \boldsymbol{\vartheta}_h^n = (E_h^{n+1} - E_h^n, 0, 0, Q_h^{n+1} - Q_h^n)$, we easily see that for all $\boldsymbol{\vartheta}_h' = (F', G', R', S') \in \mathbb{W}_h^p$, we have

$$\left(\tilde{\boldsymbol{\vartheta}_h}^n - \boldsymbol{\vartheta}_h^n, \boldsymbol{\vartheta}_h'\right)_{\mathcal{H}} = \left(\boldsymbol{\vartheta}_h^{n+1} - \boldsymbol{\vartheta}_h^n, \mathfrak{p}(\boldsymbol{\vartheta}_h')\right)_{\mathcal{H}}, \tag{5.15}$$

with $\mathfrak{p}(\boldsymbol{\vartheta}_h') = (F', 0, 0, S'))$. Then the scheme (5.10) gives:

$$\begin{aligned}\mathcal{E}_h^{n+\frac{1}{2}} &= \frac{1}{2}\left(\boldsymbol{\vartheta}_h^n, \boldsymbol{\vartheta}_h^n\right)_{\mathcal{H}} + \frac{\Delta t}{2}\left[a_h(\tilde{\boldsymbol{\vartheta}}_h^n, \mathfrak{p}(\boldsymbol{\vartheta}_h^n)) + b_{0,h}(\tilde{\boldsymbol{\vartheta}}_h^n, \mathfrak{p}(\boldsymbol{\vartheta}_h^n)) + k_h^1(\tilde{\boldsymbol{\vartheta}}_h^n, \mathfrak{p}(\boldsymbol{\vartheta}_h^n)) + \frac{1}{2}k_h^2(\boldsymbol{\vartheta}_h^n + \boldsymbol{\vartheta}_h^{n+1}, \mathfrak{p}(\boldsymbol{\vartheta}_h^n))\right] \\ &= \frac{1}{2}\left(\boldsymbol{\vartheta}_h^n, \boldsymbol{\vartheta}_h^n\right)_{\mathcal{H}} + \frac{\Delta t}{2}\left[a_h(\mathfrak{q}(\boldsymbol{\vartheta}_h^n), \mathfrak{p}(\boldsymbol{\vartheta}_h^n)) + b_{0,h}(\mathfrak{q}(\boldsymbol{\vartheta}_h^n), \mathfrak{p}(\boldsymbol{\vartheta}_h^n)) + k_h^1(\tilde{\boldsymbol{\vartheta}}_h^n, \mathfrak{p}(\boldsymbol{\vartheta}_h^n))\right],\end{aligned}$$

since $\frac{1}{2}k_h^2(\boldsymbol{\vartheta}_h^n + \boldsymbol{\vartheta}_h^{n+1}, \mathfrak{p}(\boldsymbol{\vartheta}_h^n)) = 0$.

We furthermore have

$$k_h^1(\tilde{\boldsymbol{\vartheta}}_h^n, \mathfrak{p}(\boldsymbol{\vartheta}_h^n)) = -(J_h^{n+1/2}, E_h^n)_h,$$

so that

$$|k_h^1(\tilde{\boldsymbol{\vartheta}}_h^n, \mathfrak{p}(\boldsymbol{\vartheta}_h^n))| \leq \frac{\omega_p}{\sqrt{\varepsilon_\infty}}\left(\frac{1}{\varepsilon_0\omega_p^2}\|J_h^{n+\frac{1}{2}}\|^2 + \varepsilon\|E_h^n\|^2\right).$$

Combining this last estimate with estimates of Proposition 5.5, we finally obtain:

$$\begin{aligned}\mathcal{E}_h^{n+\frac{1}{2}} \geq\ & \frac{1}{2}\left(\boldsymbol{\vartheta}_h^n, \boldsymbol{\vartheta}_h^n\right)_{\mathcal{H}} - \frac{C}{2}\eta\Delta t p^2 h^{-1}\left[c\|\mathfrak{p}_M(V)\|_{\mathcal{H}}^2 + C\beta\|\mathfrak{p}_H(V)\|_{\mathcal{H}}^2\right] \\ & -\frac{\Delta t}{2}\frac{\omega_p}{\sqrt{\varepsilon_\infty}}\left(\frac{1}{\varepsilon_0\omega_p^2}\|J_h^{n+\frac{1}{2}}\|^2 + \varepsilon\|E_h^n\|^2\right).\end{aligned}$$

Thus

$$\begin{aligned}\mathcal{E}_h^{n+\frac{1}{2}} \geq\ & \frac{1}{2}\Big[(1 - Cc\eta\Delta t p^2 h^{-1})\mu\|H_h^{n+1/2}\|^2 + (1 - Cc\eta\Delta t p^2 h^{-1} - \frac{\omega_p}{\sqrt{\varepsilon_\infty}}\Delta t)\varepsilon\|E_h^n\|^2 \\ & +(1 - C\beta\eta\Delta t p^2 h^{-1} - \frac{\omega_p}{\sqrt{\varepsilon_\infty}}\Delta t)\frac{1}{\varepsilon\omega_p^2}\|J_h^{n+1/2}\|^2 + (1 - C\beta\eta\Delta t p^2 h^{-1})\frac{\beta^2}{\varepsilon\omega_p^2}\|Q_h^n\|^2\Big],\end{aligned}$$

which gives the result. ■

**Remark 5.12** *If $\omega_p$ and $\beta$ are zero, we recover Maxwell's equations and the classical CFL condition. If $\beta = 0$, we recover the so-called Drude model. From the estimate, we see that one has also to refine the time-step accordingly to the plasma frequency $\omega_p$, which is physically coherent. Finally, if all parameters are non-zero, since, physically, the speed of the hydrodynamic wave ($\beta$ here) is always less that the speed of light ($c$ here), the most constrained CFL condition remains the one associated to Maxwell's equations alone (i.e., $\Delta t \left( Cc\eta p^2 h^{-1} + \frac{\omega_p}{\sqrt{\varepsilon_\infty}} \right) < 1$).*

**Proposition 5.13** *One has the following energy principle,*

$$\mathcal{E}_h^{n+\frac{1}{2}} - \mathcal{E}_h^{n-\frac{1}{2}} = -\frac{\gamma}{\varepsilon_0 \omega_p^2} \left\| \frac{J_h^{n-1/2} + J_h^{n+1/2}}{2} \right\|^2 . \tag{5.16}$$

**Proof.** Using the scheme at different times and with different test functions, one obtains

$$\begin{aligned}
\left( \frac{\boldsymbol{\vartheta}_h^n - \boldsymbol{\vartheta}_h^{n-1}}{\Delta t}, \mathfrak{q}(\frac{\boldsymbol{\vartheta}_h^n + \boldsymbol{\vartheta}_h^{n-1}}{2}) + \frac{1}{2}\mathfrak{p}(\boldsymbol{\vartheta}_h^n) \right)_{\mathcal{H}} &= a_h(\tilde{\boldsymbol{\vartheta}}_h^{n-1}, \mathfrak{q}(\frac{\boldsymbol{\vartheta}_h^n + \boldsymbol{\vartheta}_h^{n-1}}{2}) + \frac{1}{2}\mathfrak{p}(\boldsymbol{\vartheta}_h^n)) \\
&\quad + b_{0,h}(\tilde{\boldsymbol{\vartheta}}_h^{n-1}, \mathfrak{q}(\frac{\boldsymbol{\vartheta}_h^n + \boldsymbol{\vartheta}_h^{n-1}}{2}) + \frac{1}{2}\mathfrak{p}(\boldsymbol{\vartheta}_h^n)) \\
&\quad + k_h^1(\tilde{\boldsymbol{\vartheta}}_h^{n-1}, \mathfrak{q}(\frac{\boldsymbol{\vartheta}_h^n + \boldsymbol{\vartheta}_h^{n-1}}{2}) + \frac{1}{2}\mathfrak{p}(\boldsymbol{\vartheta}_h^n)) \\
&\quad + \frac{1}{2} k_h^2(\boldsymbol{\vartheta}_h^{n-1} + \boldsymbol{\vartheta}_h^n, \mathfrak{q}(\frac{\boldsymbol{\vartheta}_h^n + \boldsymbol{\vartheta}_h^{n-1}}{2}) + \frac{1}{2}\mathfrak{p}(\boldsymbol{\vartheta}_h^n)),
\end{aligned}$$

and

$$\begin{aligned}
\left( \frac{\boldsymbol{\vartheta}_h^{n+1} - \boldsymbol{\vartheta}_h^n}{\Delta t}, \frac{1}{2}\mathfrak{p}(\boldsymbol{\vartheta}_h^n) \right)_{\mathcal{H}} &= a_h(\tilde{\boldsymbol{\vartheta}}_h^n, \frac{1}{2}\mathfrak{p}(\boldsymbol{\vartheta}_h^n)) + b_{0,h}(\tilde{\boldsymbol{\vartheta}}_h^n, \frac{1}{2}\mathfrak{p}(\boldsymbol{\vartheta}_h^n)) + k_h^1(\tilde{\boldsymbol{\vartheta}}_h^n, \frac{1}{2}\mathfrak{p}(\boldsymbol{\vartheta}_h^n)) \\
&\quad + \frac{1}{2} k_h^2(\boldsymbol{\vartheta}_h^n + \boldsymbol{\vartheta}_h^{n+1}, \frac{1}{2}\mathfrak{p}(\boldsymbol{\vartheta}_h^n)).
\end{aligned}$$

Summing the two equations, we obtain for the left hand side

$$\begin{aligned}
\left( \frac{\boldsymbol{\vartheta}_h^n - \boldsymbol{\vartheta}_h^{n-1}}{\Delta t}, \mathfrak{q}(\frac{\boldsymbol{\vartheta}_h^n + \boldsymbol{\vartheta}_h^{n-1}}{2}) + \frac{\mathfrak{p}(\boldsymbol{\vartheta}_h^n)}{2} \right)_{\mathcal{H}} + \left( \frac{\boldsymbol{\vartheta}_h^{n+1} - \boldsymbol{\vartheta}_h^n}{\Delta t}, \frac{\mathfrak{p}(\boldsymbol{\vartheta}_h^n)}{2} \right)_{\mathcal{H}} &= \left( \mathfrak{q}(\frac{\boldsymbol{\vartheta}_h^n - \boldsymbol{\vartheta}_h^{n-1}}{\Delta t}), \mathfrak{q}(\frac{\boldsymbol{\vartheta}_h^n + \boldsymbol{\vartheta}_h^{n-1}}{2}) \right)_{\mathcal{H}} \\
&\quad + \left( \frac{\mathfrak{p}(\boldsymbol{\vartheta}_h^{n+1}) - \mathfrak{p}(\boldsymbol{\vartheta}_h^{n-1})}{\Delta t}, \mathfrak{p}(\boldsymbol{\vartheta}_h^n) \right)_{\mathcal{H}}.
\end{aligned}$$

Using (5.14), we find

$$\left( \frac{\boldsymbol{\vartheta}_h^n - \boldsymbol{\vartheta}_h^{n-1}}{\Delta t}, \mathfrak{q}(\frac{\boldsymbol{\vartheta}_h^n + \boldsymbol{\vartheta}_h^{n-1}}{2}) + \frac{1}{2}\mathfrak{p}(\boldsymbol{\vartheta}_h^n) \right)_{\mathcal{H}} + \left( \frac{\boldsymbol{\vartheta}_h^{n+1} - \boldsymbol{\vartheta}_h^n}{\Delta t}, \frac{1}{2}\mathfrak{p}(\boldsymbol{\vartheta}_h^n) \right)_{\mathcal{H}} = \mathcal{E}_h^{n+1/2} - \mathcal{E}_h^{n-1/2} \tag{5.17}$$

For the right hand side, let us group similar terms. Let $d \in \{a_h, b_{0,h}\}$, we have

$$d(\tilde{\boldsymbol{\vartheta}}_h^n, \frac{1}{2}\mathfrak{p}(\boldsymbol{\vartheta}_h^n)) + d(\tilde{\boldsymbol{\vartheta}}_h^{n-1}, \mathfrak{q}(\frac{\boldsymbol{\vartheta}_h^n + \boldsymbol{\vartheta}_h^{n-1}}{2}) + \frac{1}{2}\mathfrak{p}(\boldsymbol{\vartheta}_h^n)) \quad = \quad d(\tilde{\boldsymbol{\vartheta}}_h^n + \tilde{\boldsymbol{\vartheta}}_h^{n-1}, \frac{1}{2}\mathfrak{p}(\boldsymbol{\vartheta}_h^n)) + d(\tilde{\boldsymbol{\vartheta}}_h^{n-1}, \mathfrak{q}(\frac{\boldsymbol{\vartheta}_h^n + \tilde{\boldsymbol{\vartheta}}_h^{n-1}}{2}))$$

Furthermore, using Proposition 5.5, (5.11) and (5.12),

$$d(\tilde{\boldsymbol{\vartheta}}_h^n + \tilde{\boldsymbol{\vartheta}}_h^{n-1}, \frac{1}{2}\mathfrak{p}(\boldsymbol{\vartheta}_h^n)) = d(\mathfrak{q}(\tfrac{\tilde{\boldsymbol{\vartheta}}_h^n + \tilde{\boldsymbol{\vartheta}}_h^{n-1}}{2}), \mathfrak{p}(\boldsymbol{\vartheta}_h^n)) \tag{5.18}$$

$$= d(\mathfrak{q}(\tfrac{\boldsymbol{\vartheta}_h^n + \boldsymbol{\vartheta}_h^{n-1}}{2}), \mathfrak{p}(\boldsymbol{\vartheta}_h^n)) \tag{5.19}$$

Similarly,

$$d(\tilde{\boldsymbol{\vartheta}}_h^{n-1}, \mathfrak{q}(\frac{\boldsymbol{\vartheta}_h^n + \tilde{\boldsymbol{\vartheta}}_h^{n-1}}{2})) = d(\mathfrak{p}(\tilde{\boldsymbol{\vartheta}}_h^{n-1}), \mathfrak{q}(\tfrac{\boldsymbol{\vartheta}_h^n + \tilde{\boldsymbol{\vartheta}}_h^{n-1}}{2})) \tag{5.20}$$

$$= d(\mathfrak{p}(\boldsymbol{\vartheta}_h^n), \mathfrak{q}(\tfrac{\boldsymbol{\vartheta}_h^n + \tilde{\boldsymbol{\vartheta}}_h^{n-1}}{2})) \tag{5.21}$$

$$= d(\mathfrak{p}(\boldsymbol{\vartheta}_h^n), \mathfrak{q}(\tfrac{\boldsymbol{\vartheta}_h^n + \boldsymbol{\vartheta}_h^{n-1}}{2})) \tag{5.22}$$

Finally, from Proposition 5.4 and Proposition 5.3, one finds

$$\begin{aligned} d(\tilde{\boldsymbol{\vartheta}}_h^n, \frac{1}{2}\mathfrak{p}(\boldsymbol{\vartheta}_h^n)) + d(\tilde{\boldsymbol{\vartheta}}_h^{n-1}, \mathfrak{q}(\frac{\boldsymbol{\vartheta}_h^n + \boldsymbol{\vartheta}_h^{n-1}}{2}) + \frac{1}{2}\mathfrak{p}(\boldsymbol{\vartheta}_h^n)) &= d(\mathfrak{q}(\tfrac{\boldsymbol{\vartheta}_h^n + \boldsymbol{\vartheta}_h^{n-1}}{2}), \mathfrak{p}(\boldsymbol{\vartheta}_h^n)) + d(\mathfrak{p}(\boldsymbol{\vartheta}_h^n), \mathfrak{q}(\tfrac{\boldsymbol{\vartheta}_h^n + \boldsymbol{\vartheta}_h^{n-1}}{2})) \\ &= d(\mathfrak{p}(\boldsymbol{\vartheta}_h^n) + \mathfrak{q}(\tfrac{\boldsymbol{\vartheta}_h^n + \boldsymbol{\vartheta}_h^{n-1}}{2}), \mathfrak{p}(\boldsymbol{\vartheta}_h^n) + \mathfrak{q}(\tfrac{\boldsymbol{\vartheta}_h^n + \boldsymbol{\vartheta}_h^{n-1}}{2})) \\ &= 0. \end{aligned}$$

Moreover, by using Propositions 5.4, (5.7) and (5.8), one gets

$$\begin{aligned} k_h^1(\tilde{\boldsymbol{\vartheta}}_h^{n-1}, \mathfrak{q}(\frac{\boldsymbol{\vartheta}_h^n + \boldsymbol{\vartheta}_h^{n-1}}{2}) + \frac{\mathfrak{p}(\boldsymbol{\vartheta}_h^n)}{2}) \quad + \quad k_h^1(\tilde{\boldsymbol{\vartheta}}_h^n, \frac{\mathfrak{p}(\boldsymbol{\vartheta}_h^n))}{2} &= k_h^1(\tilde{\boldsymbol{\vartheta}}_h^{n-1}, \mathfrak{q}(\frac{\boldsymbol{\vartheta}_h^n + \boldsymbol{\vartheta}_h^{n-1}}{2})) \\ &+ k_h^1(\tilde{\boldsymbol{\vartheta}}_h^{n-1} + \tilde{\boldsymbol{\vartheta}}_h^n, \frac{1}{2}\mathfrak{p}(\boldsymbol{\vartheta}_h^n)) \\ &= k_h^1(\mathfrak{p}(\tilde{\boldsymbol{\vartheta}}_h^{n-1}), \mathfrak{q}(\frac{\boldsymbol{\vartheta}_h^n + \boldsymbol{\vartheta}_h^{n-1}}{2})) \\ &+ k_h^1(\mathfrak{q}(\frac{\tilde{\boldsymbol{\vartheta}}_h^{n-1} + \tilde{\boldsymbol{\vartheta}}_h^n}{2}), \mathfrak{p}(\boldsymbol{\vartheta}_h^n)) \\ &= k_h^1(\mathfrak{p}(\boldsymbol{\vartheta}_h^n), \mathfrak{q}(\frac{\boldsymbol{\vartheta}_h^n + \boldsymbol{\vartheta}_h^{n-1}}{2})) \\ &+ k_h^1(\mathfrak{q}(\frac{\boldsymbol{\vartheta}_h^{n-1} + \boldsymbol{\vartheta}_h^n}{2}), \mathfrak{p}(\boldsymbol{\vartheta}_h^n)) \\ &= k_h^1(\mathfrak{p}(\boldsymbol{\vartheta}_h^n) + \mathfrak{q}(\frac{\boldsymbol{\vartheta}_h^n + \boldsymbol{\vartheta}_h^{n-1}}{2}), \mathfrak{p}(\boldsymbol{\vartheta}_h^n) + \mathfrak{q}(\frac{\boldsymbol{\vartheta}_h^n + \boldsymbol{\vartheta}_h^{n-1}}{2})) \\ &= 0. \end{aligned}$$

and

$$\frac{1}{2}k_h^2(\boldsymbol{\vartheta}_h^{n-1}+\boldsymbol{\vartheta}_h^n,\mathfrak{q}(\frac{\boldsymbol{\vartheta}_h^n+\boldsymbol{\vartheta}_h^{n-1}}{2}) \qquad +\frac{1}{2}\mathfrak{p}(\boldsymbol{\vartheta}_h^n))+\frac{1}{2}k_h^2(\boldsymbol{\vartheta}_h^n+\boldsymbol{\vartheta}_h^{n+1},\frac{1}{2}\mathfrak{p}(\boldsymbol{\vartheta}_h^n))$$
$$= k_h^2(\frac{\boldsymbol{\vartheta}_h^{n-1}+\boldsymbol{\vartheta}_h^n}{2},\mathfrak{q}(\frac{\boldsymbol{\vartheta}_h^n+\boldsymbol{\vartheta}_h^{n-1}}{2}))$$
$$= k_h^2(\mathfrak{q}(\frac{\boldsymbol{\vartheta}_h^{n-1}+\boldsymbol{\vartheta}_h^n}{2}),\mathfrak{q}(\frac{\boldsymbol{\vartheta}_h^n+\boldsymbol{\vartheta}_h^{n-1}}{2}))$$
$$= -\frac{\gamma}{\varepsilon_0\omega_p^2}\left\|\frac{J_h^{n-1/2}+J_h^{n+1/2}}{2}\right\|^2.$$

Combining all these equalities, we find the result. ■

Finally, we establish the fully discrete weak constraint preservation property.

**Proposition 5.14** *If for $n\in\{0,..,N\}$, $\boldsymbol{\vartheta_h^n}=(E_h^n,H_h^{n+1/2},J_h^{n+1/2},Q_h^n)\in\mathbb{W}_h^p$ is the solution of* (5.10)*, and if at the initial time,*

$$-\langle\varepsilon E_h^0,\nabla p_h\rangle+\langle Q_h^0,p_h\rangle=0,\forall p_h\in\mathbb{Y}_h^p,$$

*then for all $n\in\{0,\cdots,N\}$,*

$$-\langle\varepsilon E_h^n,\nabla p_h\rangle+\langle Q_h^n,p_h\rangle=0,\forall p_h\in\mathbb{Y}_h^p,$$

*i.e. one has a weak (and discrete) preservation of the constraint* $\mathrm{div}(\varepsilon E)+Q=0$.

**Proof.** The strategy is analogous to the semi-discrete case. Let $\mathbb{Y}_h^p\subset H_0^1(\Omega)$ being the space of piecewise continuous polynomial of degree $p$ with zero trace on the boundary. Let $p_h\in\mathbb{Y}_h^p$. Due to the continuity of $p_h$, $\nabla p_h$ has no tangential jump at the element interfaces and has zero tangential trace at the boundary of the domain. Now, we consider the weak formulation (5.10) and choose $\boldsymbol{\vartheta}_h'=(-\frac{\beta^2}{\varepsilon_0\omega_p^2}\nabla p_h,0,0,p_h)$, with $p_h\in\mathbb{Y}_h^p$. One thus has, $\forall p_h\in\mathbb{Y}_h^p$, using the tangential continuity of $\nabla p_h$ and $p_h$ at interfaces and the zero boundary condition,

$$-\varepsilon\frac{\beta^2}{\varepsilon_0\omega_p^2}\langle\frac{E_h^{n+1}-E_h^n}{\Delta t},\nabla p_h\rangle \quad + \quad \frac{\beta^2}{\varepsilon_0\omega_p^2}\langle\frac{Q_h^{n+1}-Q_h^n}{\Delta t},p_h\rangle$$
$$=-\frac{\beta^2}{\varepsilon_0\omega_p^2}\langle J_h^{n+1/2},\nabla p_h\rangle+\frac{\beta^2}{\varepsilon_0\omega_p^2}\langle J_h^{n+1/2},\nabla p_h\rangle=0.$$

This gives the result using the hypothesis on the initial conditions. ■

### 5.2.2 Explicit Runge Kutta schemes of order 2 and 4.

As mentioned above, the use of upwind fluxes in the case of a Leap-frog discretization is ruining all the advantages and flexibilities of the approach.

The use of upwind fluxes is more appropriate to explicit Runge-Kutta discretization. We focus on explicit Runge Kutta scheme of order 2 (RK2) and explicit Runge-Kutta scheme of order 4 (RK4). We investigate stability results in this context. Mimicking the strategy of [6] and [17], one can establish stability results for both RK2 and RK4 [1]. The situation and properties of the

[1] stability for RK4 was briefly envisaged in [29], without detailing the computations

discrete operators are more general than [6] and [17]. We here choose to present the details of the computations for RK2 to emphasize the energy technique and, in particular, the resulting CFL condition (explicit in physical parameters and polynomial order).

Explicit RK2 schemes can be easily re-written in our context. For all $n \in \{1, \dots, N\}$, find $\boldsymbol{\vartheta}_h^n \in \mathbb{W}_h^p$ with $(L_1^n, \boldsymbol{\vartheta}_h^{n+1/2}, L_2^n) \in \mathbb{W}_h^p \times \mathbb{W}_h^p \times \mathbb{W}_h^p$ defined as follows: for all $\boldsymbol{\vartheta}_h' \in \mathbb{W}_h^p$,

$$\begin{aligned}
\left(L_1^n, \boldsymbol{\vartheta}_h'\right)_{\mathcal{H}} &= c_{\alpha,h}(\boldsymbol{\vartheta}_h^n, \boldsymbol{\vartheta}_h'), \\
\left(\boldsymbol{\vartheta}_h^{n+1/2}, \boldsymbol{\vartheta}_h'\right)_{\mathcal{H}} &= \left(\boldsymbol{\vartheta}_h^n, \boldsymbol{\vartheta}_h'\right)_{\mathcal{H}} + \Delta t \left(L_1^n, \boldsymbol{\vartheta}_h'\right)_{\mathcal{H}}, \\
\left(L_2^n, \boldsymbol{\vartheta}_h'\right)_{\mathcal{H}} &= c_{\alpha,h}(\boldsymbol{\vartheta}_h^{n+1/2}, \boldsymbol{\vartheta}_h'),
\end{aligned}$$

and then for all $\boldsymbol{\vartheta}_h' \in \mathbb{W}_h^p$,

$$\left(\boldsymbol{\vartheta}_h^{n+1}, \boldsymbol{\vartheta}_h'\right)_{\mathcal{H}} = \left(\boldsymbol{\vartheta}_h^n, \boldsymbol{\vartheta}_h'\right)_{\mathcal{H}} + \frac{\Delta t}{2}\left(\left(L_1^n, \boldsymbol{\vartheta}_h'\right)_{\mathcal{H}} + \left(L_2^n, \boldsymbol{\vartheta}_h'\right)_{\mathcal{H}}\right). \tag{5.23}$$

In other words, for all $n \in \{1, \dots, N\}$, find $\boldsymbol{\vartheta}_h^n \in \mathbb{W}_h^p$ with $\boldsymbol{\vartheta}_h^{n+1/2} \in \mathbb{W}_h^p$ defined as follows: for all $\boldsymbol{\vartheta}_h' \in \mathbb{W}_h^p$,

$$\left(\boldsymbol{\vartheta}_h^{n+1/2}, \boldsymbol{\vartheta}_h'\right)_{\mathcal{H}} = \left(\boldsymbol{\vartheta}_h^n, \boldsymbol{\vartheta}_h'\right)_{\mathcal{H}} + \Delta t \left(c_{\alpha,h}(\boldsymbol{\vartheta}_h^n, \boldsymbol{\vartheta}_h')\right), \tag{5.24}$$

$$\left(\boldsymbol{\vartheta}_h^{n+1}, \boldsymbol{\vartheta}_h'\right)_{\mathcal{H}} = \frac{1}{2}\left(\boldsymbol{\vartheta}_h^n + \boldsymbol{\vartheta}_h^{n+1/2}, \boldsymbol{\vartheta}_h'\right)_{\mathcal{H}} + \frac{\Delta t}{2}\left(c_{\alpha,h}(\boldsymbol{\vartheta}_h^{n+1/2}, \boldsymbol{\vartheta}_h')\right).$$

In the case of RK schemes, we simply define the fully discrete energy as

$$\mathcal{E}_h^n := \frac{1}{2}\left(\boldsymbol{\vartheta}_h^n, \boldsymbol{\vartheta}_h^n\right)_{\mathcal{H}}. \tag{5.25}$$

The following results give a stability result under a CFL condition.

**Proposition 5.15** *The scheme is stable under a 4/3-CFL condition given as $\upsilon_3 < 0$, $\upsilon_4 < 0$, $\upsilon_5 < 0$, $\upsilon_1 < 1$ and $\upsilon_2 < 1$, with*

$$\upsilon_1 := 4\Delta t^3 \left(2C c\eta p^2 h^{-1} + \frac{\omega_p}{\sqrt{\varepsilon_\infty}}\right)^4,$$

$$\upsilon_2 := 4\Delta t^3 \left(2C\beta\eta p^2 h^{-1} + \frac{\omega_p}{\sqrt{\varepsilon_\infty}} + \gamma\right)^4,$$

$$\upsilon_3 := \alpha\left(4C^2\Delta t^2\eta^2\alpha p^2 h^{-1}\left(2Cc\eta p^2 h^{-1} + \frac{\omega_p}{\sqrt{\varepsilon_\infty}}\right)^2 + C^2\Delta t^2\eta^2\alpha p^2 h^{-1}\left(2C\beta\eta p^2 h^{-1} + \frac{\omega_p}{\sqrt{\varepsilon_\infty}} + \gamma\right)^2 + C^2\Delta t\eta^2\alpha p^2 h^{-1} - 1\right),$$

$$\upsilon_4 := C^2\Delta t\eta^2\alpha p^2 h^{-1} - 1,$$

$$\upsilon_5 := 10\gamma\Delta t - 1.$$

**Proof.** Testing the first equation of (5.24) with $\boldsymbol{\vartheta}_h^n$ and the second with $2\boldsymbol{\vartheta}_h^{n+1/2}$ gives:

$$\begin{aligned}
\left(\boldsymbol{\vartheta}_h^{n+1/2}, \boldsymbol{\vartheta}_h^n\right)_{\mathcal{H}} &= (\boldsymbol{\vartheta}_h^n, \boldsymbol{\vartheta}_h^n)_{\mathcal{H}} + \Delta t c_{\alpha,h}(\boldsymbol{\vartheta}_{\boldsymbol{h}}^{\boldsymbol{n}}, \boldsymbol{\vartheta}_h^n). \\
\left(\boldsymbol{\vartheta}_h^{n+1}, 2\boldsymbol{\vartheta}_h^{n+1/2}\right)_{\mathcal{H}} &= \left(\boldsymbol{\vartheta}_h^n + \boldsymbol{\vartheta}_h^{n+1/2}, \boldsymbol{\vartheta}_h^{n+1/2}\right)_{\mathcal{H}} \\
&\quad + \Delta t c_{\alpha,h}(\boldsymbol{\vartheta}_h^{n+1/2}, \boldsymbol{\vartheta}_h^{n+1/2}).
\end{aligned}$$

Summing the two equations and using that $\left(\boldsymbol{\vartheta}_h^{n+1}, 2\boldsymbol{\vartheta}_h^{n+1/2}\right)_{\mathcal{H}} = \|\boldsymbol{\vartheta}_h^{n+1}\|_{\mathcal{H}}^2 + \|\boldsymbol{\vartheta}_h^{n+1/2}\|_{\mathcal{H}}^2 - \|\boldsymbol{\vartheta}_h^{n+1} - \boldsymbol{\vartheta}_h^{n+1/2}\|_{\mathcal{H}}^2$, we find that

$$\|\boldsymbol{\vartheta}_h^{n+1}\|_{\mathcal{H}}^2 - \|\boldsymbol{\vartheta}_h^n\|_{\mathcal{H}}^2 - \|\boldsymbol{\vartheta}_h^{n+1} - \boldsymbol{\vartheta}_h^{n+1/2}\|_{\mathcal{H}}^2 = \Delta t c_{\alpha,h}(\boldsymbol{\vartheta}_{\boldsymbol{h}}^{\boldsymbol{n}}, \boldsymbol{\vartheta}_h^n) + \Delta t c_{\alpha,h}(\boldsymbol{\vartheta}_h^{n+1/2}, \boldsymbol{\vartheta}_h^{n+1/2}) \tag{5.26}$$

Writing the variation of the energy over one time step, one has an estimate for $\|\boldsymbol{\vartheta}_h^{n+1} - \boldsymbol{\vartheta}_h^{n+1/2}\|_{\mathcal{H}}^2$.
Indeed

$$\left(\boldsymbol{\vartheta}_h^{n+1} - \boldsymbol{\vartheta}_h^{n+1/2}, \boldsymbol{\vartheta}_h'\right)_{\mathcal{H}} = \frac{1}{2}\left(\boldsymbol{\vartheta}_h^{n+1/2} - \boldsymbol{\vartheta}_h^n, \boldsymbol{\vartheta}_h'\right)_{\mathcal{H}} + \frac{\Delta t}{2} c_{\alpha,h}(\boldsymbol{\vartheta}_h^{n+1/2}, \boldsymbol{\vartheta}_h') - \Delta t c_{\alpha,h}(\boldsymbol{\vartheta}_h^n, \boldsymbol{\vartheta}_h').$$

Then using the first equation of (5.24), one finds

$$\left(\boldsymbol{\vartheta}_h^{n+1} - \boldsymbol{\vartheta}_h^{n+1/2}, \boldsymbol{\vartheta}_h'\right)_{\mathcal{H}} = \frac{\Delta t}{2} c_{\alpha,h}\left(\boldsymbol{\vartheta}_h^n, \boldsymbol{\vartheta}_h'\right) + \frac{\Delta t}{2} c_{\alpha,h}(\boldsymbol{\vartheta}_h^{n+1/2}, \boldsymbol{\vartheta}_h') - \Delta t c_{\alpha,h}(\boldsymbol{\vartheta}_h^n, \boldsymbol{\vartheta}_h').$$

Thus

$$\left(\boldsymbol{\vartheta}_h^{n+1} - \boldsymbol{\vartheta}_h^{n+1/2}, \boldsymbol{\vartheta}_h'\right)_{\mathcal{H}} = \frac{\Delta t}{2} c_{\alpha,h}\left(\boldsymbol{\vartheta}_h^{n+1/2} - \boldsymbol{\vartheta}_h^n, \boldsymbol{\vartheta}_h'\right). \tag{5.27}$$

Let us define $g_h^n := \boldsymbol{\vartheta}_h^{n+1/2} - \boldsymbol{\vartheta}_h^n \in \mathbb{W}_h^p$.
One can thus rewrite (5.27) as

$$\left(\boldsymbol{\vartheta}_h^{n+1} - \boldsymbol{\vartheta}_h^{n+1/2}, \boldsymbol{\vartheta}_h'\right)_{\mathcal{H}} = \frac{\Delta t}{2} c_{\alpha,h}\left(g_h^n, \boldsymbol{\vartheta}_h'\right). \tag{5.28}$$

One has

$$\left(g_h^n, \boldsymbol{\vartheta}_h'\right)_{\mathcal{H}} = \Delta t c_{\alpha,h}\left(\boldsymbol{\vartheta}_h^n, \boldsymbol{\vartheta}_h'\right). \tag{5.29}$$

Now we use the estimate on $c_{\alpha,h}$ given in proposition 5.6. We obtain,

$$\begin{aligned}
\|g_h^n\|_{\mathcal{H}}^2 \leq\ & \Delta t\left(2Cc\eta p^2 h^{-1} + \frac{\omega_p}{\sqrt{\varepsilon_\infty}}\right)\|\mathfrak{p}_M(\boldsymbol{\vartheta}_h)\|_{\mathcal{H}}\|\mathfrak{p}_M(g_h^n)\|_{\mathcal{H}} + \\
& \Delta t\left(2C\beta\eta p^2 h^{-1} + \frac{\omega_p}{\sqrt{\varepsilon_\infty}} + \gamma\right)\|\mathfrak{p}_H(\boldsymbol{\vartheta}_h)\|_{\mathcal{H}}\|\mathfrak{p}_H(g_h^n)\|_{\mathcal{H}} + C\Delta t\eta\alpha p h^{-\frac{1}{2}}|\boldsymbol{\vartheta}_h|_S\|g_h^n\|_{\mathcal{H}}.
\end{aligned}$$

$$\begin{aligned}\|\mathfrak{p}_M(g_h^n)\|_{\mathcal{H}}^2 &\leq \Delta t\left(2Cc\eta p^2h^{-1}+\frac{\omega_p}{\sqrt{\varepsilon_\infty}}\right)\|\mathfrak{p}_M(\boldsymbol{\vartheta}_h)\|_{\mathcal{H}}\|\mathfrak{p}_M(g_h^n)\|_{\mathcal{H}}+\\ &\quad +C\Delta t\eta\alpha ph^{-\frac{1}{2}}|\boldsymbol{\vartheta}_h|_S\|\mathfrak{p}_M(g_h^n)\|_{\mathcal{H}}.\end{aligned}$$

$$\begin{aligned}\|\mathfrak{p}_H(g_h^n)\|_{\mathcal{H}}^2 &\leq \Delta t\left(2C\beta\eta p^2h^{-1}+\frac{\omega_p}{\sqrt{\varepsilon_\infty}}\right)\|\mathfrak{p}_H(\boldsymbol{\vartheta}_h)\|_{\mathcal{H}}\|\mathfrak{p}_H(g_h^n)\|_{\mathcal{H}}\\ &\quad +C\Delta t\eta\alpha ph^{-\frac{1}{2}}|\boldsymbol{\vartheta}_h|_S\|\mathfrak{p}_H(g_h^n)\|_{\mathcal{H}}+\gamma\Delta t\|(\boldsymbol{\vartheta}_h^n)_{j\in\{7,..,9\}}\|_{\mathcal{H}}.\end{aligned}$$

This gives,

$$\begin{aligned}\|g_h^n\|_{\mathcal{H}} &\leq \Delta t\left(2Cc\eta p^2h^{-1}+\frac{\omega_p}{\sqrt{\varepsilon_\infty}}\right)\|\mathfrak{p}_M(\boldsymbol{\vartheta}_h)\|_{\mathcal{H}}+\\ &\quad \Delta t\left(2C\beta\eta p^2h^{-1}+\frac{\omega_p}{\sqrt{\varepsilon_\infty}}\right)\|\mathfrak{p}_H(\boldsymbol{\vartheta}_h)\|_{\mathcal{H}}+C\Delta t\eta\alpha ph^{-\frac{1}{2}}|\boldsymbol{\vartheta}_h|_S+\gamma\Delta t\|(\boldsymbol{\vartheta}_h^n)_{j\in\{7,..,9\}}\|_{\mathcal{H}},\end{aligned}$$

$$\begin{aligned}\|\mathfrak{p}_M(g_h^n)\|_{\mathcal{H}} \leq\ & \Delta t\left(2Cc\eta p^2h^{-1}+\tfrac{\omega_p}{\sqrt{\varepsilon_\infty}}\right)\|\mathfrak{p}_M(\boldsymbol{\vartheta}_h)\|_{\mathcal{H}}+\\ &C\Delta t\eta\alpha ph^{-\frac{1}{2}}|\boldsymbol{\vartheta}_h|_S,\end{aligned}$$

and

$$\|\mathfrak{p}_H(g_h^n)\|_{\mathcal{H}} \leq \Delta t\left(2C\beta\eta p^2h^{-1}+\tfrac{\omega_p}{\sqrt{\varepsilon_\infty}}\right)\|\mathfrak{p}_H(\boldsymbol{\vartheta}_h)\|_{\mathcal{H}}+C\Delta t\eta\alpha ph^{-\frac{1}{2}}|\boldsymbol{\vartheta}_h|_S+\gamma\Delta t\|(\boldsymbol{\vartheta}_h^n)_{j\in\{7,..,9\}}\|_{\mathcal{H}}.$$

Furthermore,

$$\left(\boldsymbol{\vartheta}_h^{n+1}-\boldsymbol{\vartheta}_h^{n+1/2},\boldsymbol{\vartheta}_h^{n+1}-\boldsymbol{\vartheta}_h^{n+1/2}\right)_{\mathcal{H}}=\frac{\Delta t}{2}c_{\alpha,h}\left(g_h^n,\boldsymbol{\vartheta}_h^{n+1}-\boldsymbol{\vartheta}_h^{n+1/2}\right). \tag{5.30}$$

We thus conclude that

$$\begin{aligned}\left(\boldsymbol{\vartheta}_h^{n+1}-\boldsymbol{\vartheta}_h^{n+1/2},\boldsymbol{\vartheta}_h^{n+1}-\boldsymbol{\vartheta}_h^{n+1/2}\right)_{\mathcal{H}} &\leq \Delta t\left(2Cc\eta p^2h^{-1}+\frac{\omega_p}{\sqrt{\varepsilon_\infty}}\right)\|\mathfrak{p}_M(g_h^n)\|_{\mathcal{H}}\|\mathfrak{p}_M(\boldsymbol{\vartheta}_h^{n+1}-\boldsymbol{\vartheta}_h^{n+1/2})\|_{\mathcal{H}}+\\ &\quad \Delta t\left(2C\beta\eta p^2h^{-1}+\frac{\omega_p}{\sqrt{\varepsilon_\infty}}\right)\|\mathfrak{p}_H(g_h^n)\|_{\mathcal{H}}\|\mathfrak{p}_H(\boldsymbol{\vartheta}_h^{n+1}-\boldsymbol{\vartheta}_h^{n+1/2})\|_{\mathcal{H}}\\ &\quad +C\Delta t\eta\alpha ph^{-\frac{1}{2}}|g_h^n|_S\|\boldsymbol{\vartheta}_h^{n+1}-\boldsymbol{\vartheta}_h^{n+1/2}\|_{\mathcal{H}}\\ &\quad +\gamma\Delta t\|(\mathfrak{p}_H(g_h^n))_{j\in\{7,..,9\}}\|_{\mathcal{H}}\|\mathfrak{p}_H(\boldsymbol{\vartheta}_h^{n+1}-\boldsymbol{\vartheta}_h^{n+1/2})\|_{\mathcal{H}}.\end{aligned}$$

This implies

$$\begin{aligned}\|\boldsymbol{\vartheta}_h^{n+1}-\boldsymbol{\vartheta}_h^{n+1/2}\|_{\mathcal{H}} &\leq (\Delta t^2\left(2Cc\eta p^2h^{-1}+\frac{\omega_p}{\sqrt{\varepsilon_\infty}}\right)^2\|\mathfrak{p}_M(\boldsymbol{\vartheta}_h)\|_{\mathcal{H}}+C\Delta t^2\eta\alpha ph^{-\frac{1}{2}}\left(2Cc\eta p^2h^{-1}+\frac{\omega_p}{\sqrt{\varepsilon_\infty}}\right)|\boldsymbol{\vartheta}_h|_S)\\ &+(\Delta t^2\left(2C\beta\eta p^2h^{-1}+\frac{\omega_p}{\sqrt{\varepsilon_\infty}}\right)^2\|\mathfrak{p}_H(\boldsymbol{\vartheta}_h)\|_{\mathcal{H}}+C\Delta t^2\eta\alpha ph^{-\frac{1}{2}}\left(2C\beta\eta p^2h^{-1}+\frac{\omega_p}{\sqrt{\varepsilon_\infty}}+\gamma\right)|\boldsymbol{\vartheta}_h|_S)\\ &\quad +C\Delta t\eta\alpha ph^{-\frac{1}{2}}|g_h^n|_S+\gamma\Delta t\|(\mathfrak{p}_H(g_h^n))_{j\in\{7,..,9\}}\|_{\mathcal{H}}.\end{aligned}$$

Finally,

$$\|\boldsymbol{\vartheta}_h^{n+1}-\boldsymbol{\vartheta}_h^{n+1/2}\|_{\mathcal{H}}^2 \leq 5(\Delta t^4\left(2Cc\eta p^2h^{-1}+\frac{\omega_p}{\sqrt{\varepsilon_\infty}}\right)^4\|\mathfrak{p}_M(\boldsymbol{\vartheta}_h^n)\|_{\mathcal{H}}^2$$
$$+5C^2\Delta t^4\eta^2\alpha^2p^2h^{-1}\left(2Cc\eta p^2h^{-1}+\frac{\omega_p}{\sqrt{\varepsilon_\infty}}\right)^2|\boldsymbol{\vartheta}_h^n|_S^2)$$
$$+5(\Delta t^4\left(2C\beta\eta p^2h^{-1}+\frac{\omega_p}{\sqrt{\varepsilon_\infty}}+\gamma\right)^4\|\mathfrak{p}_H(\boldsymbol{\vartheta}_h^n)\|_{\mathcal{H}}^2+C^2\Delta t^4\eta^2\alpha^2p^2h^{-1}\left(2C\beta\eta p^2h^{-1}+\frac{\omega_p}{\sqrt{\varepsilon_\infty}}\right)^2|\boldsymbol{\vartheta}_h^n|_S^2)$$
$$+C^2\Delta t^2\eta^2\alpha^2p^2h^{-1}|\boldsymbol{\vartheta}_h^n|_S^2+C^2\Delta t^2\eta^2\alpha^2p^2h^{-1}|\boldsymbol{\vartheta}_h^{n+1/2}|_S^2+5\gamma^2\Delta t^2\|(\mathfrak{p}_H(g_h^n))_{j\in\{7,..,9\}}\|_{\mathcal{H}}^2.$$

$$\mathcal{E}_h^{n+1}-\mathcal{E}_h^n \leq \upsilon_1\Delta t\|\mathfrak{p}_M(\boldsymbol{\vartheta}_h^n)\|_{\mathcal{H}}^2+\upsilon_2\Delta t\|\mathfrak{p}_H(\boldsymbol{\vartheta}_h^n)\|_{\mathcal{H}}^2+\upsilon_3\Delta t|\boldsymbol{\vartheta}_h^n|_S^2+\upsilon_4\Delta t|\boldsymbol{\vartheta}_h^{n+1/2}|_S^2$$
$$-\gamma\Delta t\upsilon_5\sum_{i=7}^{9}\|(\boldsymbol{\vartheta}_h^n)_j\|_{\mathcal{H}}^2-\gamma\Delta t\upsilon_5\sum_{i=7}^{9}\|(\boldsymbol{\vartheta}_h^{n+1/2})_j\|_{\mathcal{H}}^2.$$

with $\upsilon_i, i=1,\ldots,5$ defined before. Hence if $\upsilon_3<0$, $\upsilon_4<0$, $\upsilon_5<0$, $\upsilon_1<1$ and $\upsilon_2<1$, then

$$\mathcal{E}_h^{n+1}-\mathcal{E}_h^n \leq \Delta t\mathcal{E}_h^n, \tag{5.31}$$

and the conclusion follows using Grönwall's inequality. ■

**Remark 5.16** *In the previous Proposition, the dominant CFL condition is a 4/3-CFL condition (namely $\upsilon_2<1$) and is independent of the upwinding parameter $\alpha$.*

Finally, we study the fully discrete weak constraint preservation property.

**Proposition 5.17** *If for $n\in\{0,\ldots,N\}$, $\boldsymbol{\vartheta_h^n}=(E_h^n,H_h^n,J_h^n,Q_h^n)\in\mathbb{W}_h^p$ is the solution of* (5.24)-(5.25)*, and if at the initial time,*

$$-\langle\varepsilon E_h^0,\nabla p_h\rangle+\langle Q_h^0,p_h\rangle=0,\forall p_h\in\mathbb{Y}_h^p,$$

*then for all $n\in\{0,\ldots,N\}$,*

$$-\langle\varepsilon E_h^n,\nabla p_h\rangle+\langle Q_h^n,p_h\rangle=0,\forall p_h\in\mathbb{Y}_h^p,$$

*i.e. one has a weak (and discrete) preservation of the constraint* $\mathrm{div}(\varepsilon E)+Q=0$.

**Proof.** The strategy is analogous to the semi-discrete case and to the case of Leap frog scheme. Let $\mathbb{Y}_h^p\subset H_0^1(\Omega)$ being the space of piecewise continuous polynomial of degree $p$ with zero trace on the boundary. Let $p_h\in\mathbb{Y}_h^p$. Due to the continuity of $p_h$, $\nabla p_h$ has no tangential jump at the element interfaces and has zero tangential trace at the boundary of the domain. Now, we consider in the weak formulation of the RK2 scheme (5.24)-(5.25) and choose $\boldsymbol{\vartheta}_h'=(-\frac{\beta^2}{\varepsilon_0\omega_p^2}\nabla p_h,0,0,p_h)$, with $p_h\in\mathbb{Y}_h^p$. One thus has, $\forall p_h\in\mathbb{Y}_h^p$, using the tangential continuity of $\nabla p_h$ and $p_h$ at interfaces and the zero boundary condition in (5.24)

$$-\varepsilon\frac{\beta^2}{\varepsilon_0\omega_p^2}\langle\frac{E_h^{n+1/2}-E_h^n}{\Delta t},\nabla p_h\rangle \quad+\quad \frac{\beta^2}{\varepsilon_0\omega_p^2}\langle\frac{Q_h^{n+1/2}-Q_h^n}{\Delta t},p_h\rangle$$
$$=\frac{1}{2}\left(-\frac{\beta^2}{\varepsilon_0\omega_p^2}\langle J_h^n,\nabla p_h\rangle+\frac{\beta^2}{\varepsilon_0\omega_p^2}\langle J_h^n,\nabla p_h\rangle\right)=0. \tag{5.32}$$

Using (5.25), we find,

$$
\begin{aligned}
-\frac{\beta^2\varepsilon}{\varepsilon_0\omega_p^2}\langle E_h^{n+1},\nabla p_h\rangle+\frac{\beta^2}{\varepsilon_0\omega_p^2}\langle(Q_h^{n+1},p_h\rangle &= -\frac{\beta^2}{2\varepsilon_0\omega_p^2}\langle\varepsilon\left(E_h^{n+1/2}+E_h^n,\nabla p_h\rangle-\langle Q_h^{n+1/2}+Q_h^n,p_h\rangle\right)\\
&\quad+\frac{1}{2}\left(-\frac{\beta^2}{\varepsilon_0\omega_p^2}\langle J_h^{n+1/2},\nabla p_h\rangle+\frac{\beta^2}{\varepsilon_0\omega_p^2}\langle J_h^{n+1/2},\nabla p_h\rangle\right)\\
&= 0.
\end{aligned}
$$

And thus from (5.32), one deduces that

$$
-\varepsilon\langle E_h^{n+1},\nabla p_h\rangle+\langle Q_h^{n+1},p_h\rangle = -\varepsilon\langle E_h^n,\nabla p_h\rangle+\langle Q_h^n,p_h\rangle.
$$

This gives the result using the hypothesis on the initial conditions. ■

In the remainder of this paragraph, we briefly consider the case of the explicit RK4 scheme.

It writes, for all $n\in\{1,\dots,N\}$, find $\boldsymbol{\vartheta}_h^n\in\mathbb{W}_h^p$ with $(\boldsymbol{\vartheta}_h^{n+1/4},\boldsymbol{\vartheta}_h^{n+1/2},\boldsymbol{\vartheta}_h^{n+3/4})\in(\mathbb{W}_h^p)^3$ defined as follows: for all $\boldsymbol{\vartheta}_h'\in\mathbb{W}_h^p$,

$$
\left(\boldsymbol{\vartheta}_h^{n+1/4},\boldsymbol{\vartheta}_h'\right)_{\mathcal{H}} = \left(\boldsymbol{\vartheta}_h^n,\boldsymbol{\vartheta}_h'\right)_{\mathcal{H}}+\Delta t\left(c_{\alpha,h}(\boldsymbol{\vartheta}_h^n,\boldsymbol{\vartheta}_h')\right), \tag{5.33}
$$

$$
\begin{aligned}
\left(\boldsymbol{\vartheta}_h^{n+1/2},\boldsymbol{\vartheta}_h'\right)_{\mathcal{H}} &= \frac{1}{2}\left(\boldsymbol{\vartheta}_h^n+\boldsymbol{\vartheta}_h^{n+1/4},\boldsymbol{\vartheta}_h'\right)_{\mathcal{H}}\\
&\quad+\frac{\Delta t}{2}\left(c_{\alpha,h}(\boldsymbol{\vartheta}_h^{n+1/4},\boldsymbol{\vartheta}_h')\right),
\end{aligned} \tag{5.34}
$$

$$
\begin{aligned}
\left(\boldsymbol{\vartheta}_h^{n+3/4},\boldsymbol{\vartheta}_h'\right)_{\mathcal{H}} &= \frac{1}{3}\left(\boldsymbol{\vartheta}_h^n+\boldsymbol{\vartheta}_h^{n+1/4}+\boldsymbol{\vartheta}_h^{n+1/2},\boldsymbol{\vartheta}_h'\right)_{\mathcal{H}}\\
&\quad+\frac{\Delta t}{3}\left(c_{\alpha,h}(\boldsymbol{\vartheta}_h^{n+1/2},\boldsymbol{\vartheta}_h')\right),
\end{aligned} \tag{5.35}
$$

$$
\begin{aligned}
\left(\boldsymbol{\vartheta}_h^{n+1},\boldsymbol{\vartheta}_h'\right)_{\mathcal{H}} &= \frac{1}{4}\left(\boldsymbol{\vartheta}_h^n+\boldsymbol{\vartheta}_h^{n+1/4}+\boldsymbol{\vartheta}_h^{n+1/2}+\boldsymbol{\vartheta}_h^{n+3/4},\boldsymbol{\vartheta}_h'\right)_{\mathcal{H}}\\
&\quad+\frac{\Delta t}{4}\left(c_{\alpha,h}(\boldsymbol{\vartheta}_h^{n+3/4},\boldsymbol{\vartheta}_h')\right).
\end{aligned} \tag{5.36}
$$

(5.37)

We define for $n\in\{0,..,N\}$, the fully discrete energy as

$$
\mathcal{E}_h^n := \frac{1}{2}\left(\boldsymbol{\vartheta}_h^n,\boldsymbol{\vartheta}_h^n\right)_{\mathcal{H}}. \tag{5.38}
$$

Even though not presented in this paper (because the arguments are similar to a combination of extra long computations of [17] and the strategy adopted here for RK2), one could obtain with lengthy computations that under a 4/3-CFL condition, the RK4 scheme is stable. Similarly, we can prove a constraint weak preservation property, since for any $(\boldsymbol{\zeta}_h,p_h)\in\mathbb{W}_h^p\times\mathbb{Y}_h^p$,

$$
c_{\alpha,h}(\boldsymbol{\zeta}_h,\boldsymbol{\xi}_h)=0,
$$

if $\boldsymbol{\xi}_h=(-\frac{\beta^2}{\varepsilon_0\omega_p^2}\nabla p_h,0,0,p_h)$.

### 5.2.3 Some remarks on convergence estimates

Using the stability results developed in last sections and consistency estimates, one can obtain convegence results. We choose not to detail the proof here, but on shall obtain an estimate such as

$$\max_{n\in\{0,\dots,N\}} \|\boldsymbol{\vartheta}_h^n - \boldsymbol{\vartheta}(t_n)\| \leq CTh^{\min(s,p)} \tag{5.39}$$

# 6 Numerical results

Based on our analysis, we numerically investigate the stability of the given schemes. In this paper, we concentrate on giving first 2D numerical results and postpone 3D results and a more thorough analysis of the discrete stability properties of the schemes to a future work.

**Numerical setting.** We consider a 3D setting that is invariant in the $z$ direction (domain and solution) and we focus on a transverse mode i.e. $H_x = H_y = E_z = J_z = 0$. As such, the 3D problem is reduced to a 2D Maxwell Hydrodynamic problem with unknowns $(E_x, E_y, H_z, J_x, J_y, Q)$.

The convergence of the schemes presented in the last section has been previously assessed numerically, hence we do not reproduce these academic convergence tests (see e.g. [29] for these results for RK4 and LF2 in particular). Let us mention that the empirically found CFL condition for LF2 follows the theoretical predictions of the previous section. For Runge-Kutta schemes, one could numerically obtain the classical CFL condition $\Delta t \lesssim h$.

**Remark 6.1** *This discrepancy between theoretically predicted CFL and effective one is due to the energy technique proof.*

We consider the square domain $\Omega = [0,1] \times [0,1]$. The physical quantities, variables and unknowns are adimensioned using the speed of light in vacuum $c_0 = 3\text{e}8 m.s^{-1}$.

In order to test the long time behavior of the numerical solution, we choose several test cases with different initial conditions $\mathcal{U}_0 \in D(\mathcal{A})$, mesh parameters and order of approximation ($\mathbb{P}_1$ to $\mathbb{P}_4$).

**Academic constants.** We fix the adimensioned physical parameters to unitary values (with respect to the speed of light in vacuum $c_0$). In other words, $\varepsilon_L = 1$, $\varepsilon_0 = 8.85\text{e-}12\, F.m^{-1}$, $\mu = 4\pi\text{e-}7\, H.m^{-1}$ (and $c_0 = \frac{1}{\sqrt{\varepsilon_0\mu}}$), $\omega_p = c_0$, $\gamma = c_0$ and $\beta = c_0$.
First, we rely on the theory developed in section 4, especially Lemma 4.2. For $k \in \mathbb{N}^*$, we define

$$U_k = c_k \begin{pmatrix} E^k \\ -\dfrac{1}{\lambda_k \mu} \operatorname{curl} E^k \\ -\varepsilon\lambda_k E^k - \dfrac{1}{\lambda_k\mu}\lambda_{N,k}^2 E^k \\ 0 \end{pmatrix} \tag{6.1}$$

$c_k \neq 0$ is a normalization factor chosen such that

$$\|U_k\|_{\mathcal{H}} = 1,$$

| $k$ | 5 | 10 | 15 | 20 | 30 | 40 | 50 | 60 | 70 | 80 |
|---|---|---|---|---|---|---|---|---|---|---|
| $k^{-2}$ | 0.04 | 0.01 | 4.4e-3 | 2.5e-3 | 1.11e-3 | 6.25e-4 | 4e-4 | 2.77e-4 | 2.04e-4 | 1.56e-4 |
| Decay rate | 0.014 | 0.011 | 5.59e-3 | 3.25e-3 | 1.49e-3 | 8.34e-4 | 5.27e-4 | 3.58e-4 | 2.567e-4 | 2.05e-4 |
| Power decay rate | - | $-0.33$ | $-1.72$ | $-1.81$ | $-1.97$ | $-2.02$ | $-2.05$ | $-2.11$ | $-2.17$ | $-1.65$ |

Table 1: Numerical exponential rate of decay of the energy for LF2 scheme with centered fluxes and total adimensioned simulation time $T = 1000$ (physical time $3 * 10^{-5}$s).

| $k$ | 5 | 10 | 15 | 20 | 30 | 40 | 50 | 70 | 80 |
|---|---|---|---|---|---|---|---|---|---|
| $k^{-2}$ | 0.04 | 0.01 | 4.4e-3 | 2.5e-3 | 1.11e-3 | 6.25e-4 | 4e-4 | 2.04e-4 | 1.56e-4 |
| Decay rate | 0.40 | 0.58 | 0.60 | 5.5e-3 | 3.2e-3 | 0.18 | 0.65 | 2.76 | 4.86 |
| Power decay rate | - | 0.54 | 0.11 | $-16.35$ | 4.39 | 5.92 | 5.83 | 4.28 | 4.23 |

Table 2: Numerical exponential rate of decay of the energy for RK2 scheme with upwind fluxes and total adimensioned simulation time $T = 1000$ (physical time $3 * 10^{-5}$s).

where $E^k = (E_x^k, E_y^k)$.

$$\begin{aligned} E_x^k : (x,y) &\mapsto \cos(\tfrac{k\pi}{L}x)\sin(\tfrac{k\pi}{L}y), \\ E_y^k : (x,y) &\mapsto -\sin(\tfrac{k\pi}{L}x)\cos(\tfrac{k\pi}{L}y). \end{aligned}$$

The latter is an eigenvector of $A_N$ for the eigenvalue $\lambda_{N,k}^2 = 2\,(k\pi)^2$. We also denote $\lambda_k = i(\varepsilon\mu)^{-1/2}\lambda_{N,k} + i\sqrt{\frac{\mu}{\varepsilon}}\frac{\varepsilon_0\omega_p^2}{2\lambda_{N,k}} - \frac{\gamma\varepsilon_0\omega_p^2\mu}{2\lambda_{N,k}^2}$.

Doing so, we expect to observe an exponential decay rate of the energy (i.e. the energy decays as $\exp(-\nu t)$, with $\nu$ the decay rate) proportional to $k^{-2}$ for $k$ large enough . For LF2 and centered fluxes, the numerical results confirm the expected exponential decay. Furthermore, we can have a numerical estimation of the approximate energy decay rate. We observe that the rate of exponential decay decreases as $k$ increases, with an asymptotic power decay of $k^{-2}$ that corresponds to theoretical predictions (see table 1). This is in accordance with the fact that this scheme is energy preserving (in the sense that it preserves the continuous discrete energy principle at the discrete level) i.e. the scheme is non-dissipative. On the contrary, for Runge-Kutta schemes with upwind fluxes, this conclusion does not hold (see table 2). As expected, the introduction of numerical dissipation, due to upwind fluxes, changes the rate of decay. Same conclusions hold for RK4 scheme with upwind fluxes, other mesh discretization parameters and polynomial orders (we do not reproduce the detailed results here).

Then for a second type of numerical tests, we propose to use several initial conditions with various degrees of smoothness. As a simple example of initial condition we choose:

$$\begin{aligned} E_x : (x,y) &\mapsto \cos(\pi x)\sin(\pi y), \\ E_y : (x,y) &\mapsto \sin(\pi x)\cos(\pi y), \\ H_z : (x,y) &\mapsto \cos(\pi x)\sin(\pi y). \end{aligned}$$

We define $Q$ using the constraint:

$$Q = -\operatorname{div}(\varepsilon E) \tag{6.2}$$

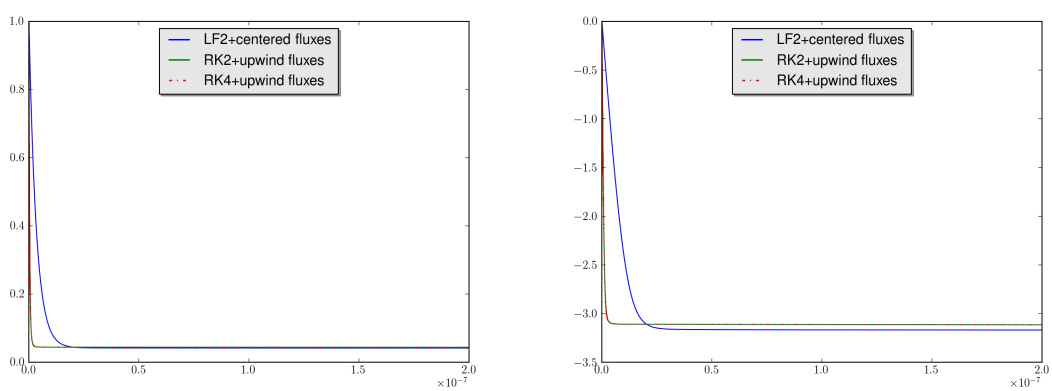
Then we consider several expressions for $(J_x, J_y)$. In particular, we investigate the case of smooth initial data to initial data that do not belong to $D(\mathcal{A})$. As smooth initial data (S), we simply choose

$$\begin{aligned} J_x^S : (x,y) &\mapsto \delta\cos(\pi x)\sin(\pi y), \\ J_y^S : (x,y) &\mapsto \delta\sin(\pi x)\cos(\pi y), \end{aligned}$$

with $\delta$ a given positive constant. Secondly, we also consider a continuous piecewise linear initial data (CPL).

$$\begin{aligned} J_x^{CPL} : (x,y) &\mapsto 1.0, \text{ if } x \leq 1/3, \\ &\quad\; 1-3(x-1/3), \text{ if } 1/3 < x < 2/3, \\ &\quad\; 0.0, \text{ if } x \geq 2/3, \\ J_y^{CPL} : (x,y) &\mapsto 1.0, \text{ if } x < 1/3, \\ &\quad\; 1-3(x-1/3), \text{ if } 1/3 \leq x < 2/3, \\ &\quad\; 0.0, \text{ if } x \geq 2/3. \end{aligned}$$

The results are summarized in figures 1 and 2. In figures 1a and 2a, we represent the evolution over time of the relative energy. In figures 1b and 2b, we represent the evolution over time $t \mapsto \log(\mathcal{E}(t)/\|\mathcal{U}_0\|^2)$. In both cases, we observe an exponential decay with saturation due to discretisation error.




(a) Discrete energy over time

(b) Representation of $\log(\frac{\mathcal{E}}{\|\mathcal{U}_0\|^2})$ over time.

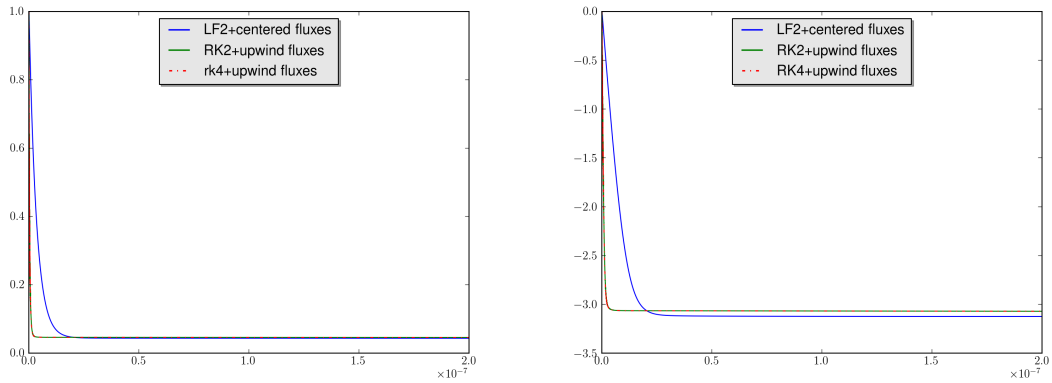
Figure 1: Energy plots for smooth initial data with $T = 2 * 10^{-7}s$, $h = 10^{-2}m$ and $\Delta t \approx 10^{-11}s$.

In order to test an initial data (NS) that does not belong to $D(\mathcal{A})$, we choose

$$\begin{aligned} J_x^{NS} : (x,y) &\mapsto \log(\sqrt{(x-\upsilon_x)^2 + (y-\upsilon_y)^2}), \\ J_y^{NS} : (x,y) &\mapsto \log(\sqrt{(x-\upsilon_x)^2 + (y-\upsilon_y)^2}), \end{aligned}$$

with a given value of $(\upsilon_x, \upsilon_y) \in ]0,1[\times]0,1[$. Here we choose $(\upsilon_x, \upsilon_y) = (\frac{1}{4}, \frac{1}{4})$.

In figures 3a, 3b and 3c, we represent the evolution over time of respectively the energy, $t \mapsto t\mathcal{E}(t)$ and $t \mapsto \log(\mathcal{E}(t))$ in this precise case. We do not observe any exponential convergence, but




(a) Discrete energy over time

(b) Representation of $\log(\frac{\mathcal{E}}{\|\mathcal{U}_0\|^2})$.

Figure 2: Energy plots for continuous and piecewise initial data over time with $T = 2 * 10^{-7}s$, $h = 10^{-2}m$ and $\Delta t \approx 10^{-11}s$.

polynomial decay. We observe that up to a given time $\hat{T} < T$ the quantity $t \mapsto t\mathcal{E}(t)$ is bounded. However, we observe a linear growth after this critical time $\hat{T}$. This behavior is due to discretisation error. Indeed, the discrete energy can be (non optimally) bounded by a sum of two contributions: $\|\mathcal{U}\|_{\mathcal{H}}$ and $\|\mathcal{U} - \mathcal{U}_h\|_{\mathcal{H}}$. The latter term can be estimated using (5.39). Therefore, for a fixed mesh size $h$ if $t$ is big enough, the (at least) linear growth will dominate over the stability decay of $t\mathcal{E}(t)$.

**Physical values of the parameters** One could also perform the same numerical experiments with physical values of the parameters. As typical values, one can use a silver medium model e.g. $\varepsilon_L = 1$, $\varepsilon_0 = 8.85\text{e-}12\, F.m^{-1}$, $\mu = 4\pi\text{e-}7\, H.m^{-1}$, $\omega_p = 1.24e16\, rad.s^{-1}$, $\gamma = 7.4e14\, Hz$ and $\beta = 1.35e06\, m.s^{-1}$. Interestingly, in this case and for all tested initial data, one numerically observes an exponential decay of $-\gamma$. As an example, we represent in Figure 4 the value of the log of the relative energy v.s. time for smooth initial data. Same plots could be obtained for other type of initial data (including data of type (6.1)). The curves show a clear exponential decay. In table 3, we computed the curves' slope for all the test cases and several discretisation parameters. The results confirm a decay rather close to $\exp(-\gamma t)$ (i.e. a decay rate close to $\gamma$). This can be understood as the physical decay since the polarization current is predominant due the respective ranges of the physical parameters. Let us point out, that, in particular, in the predicted asymptotic behavior in Lemma 4.2, the respective ranges of the physical parameters have not yet been taken into account and could impact the higher order terms.

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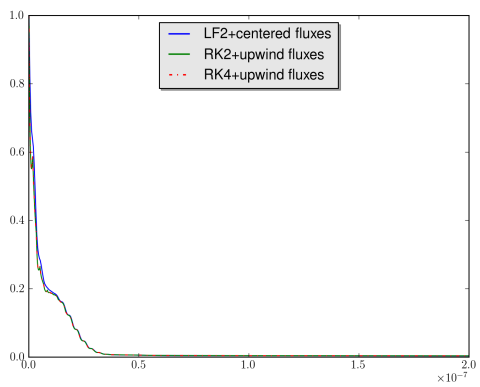
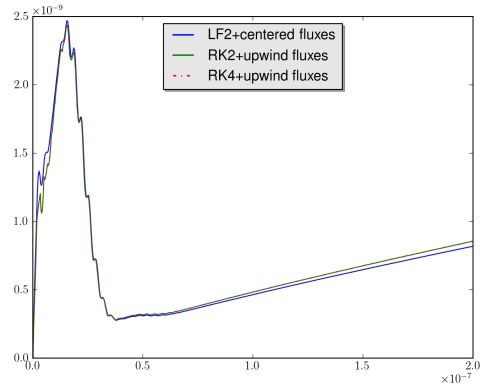
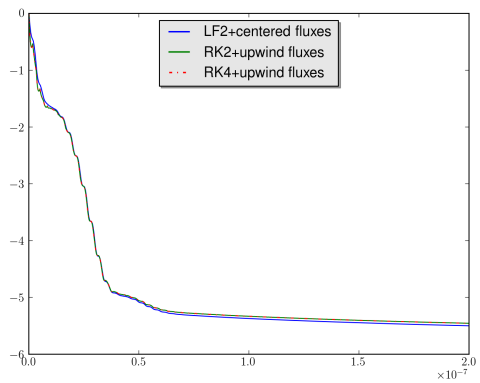



(a) Discrete energy over time




(b) Representation of $t\frac{\mathcal{E}}{\|\mathcal{U}_0\|}$ over time




(c) Representation of $\log(\frac{\mathcal{E}}{\|\mathcal{U}_0\|})$ over time

Figure 3: Energy plots for non smooth initial data with $T = 2*10^{-7}s$, $h = 10^{-2}m$ and $\Delta t \approx 10^{-11}s$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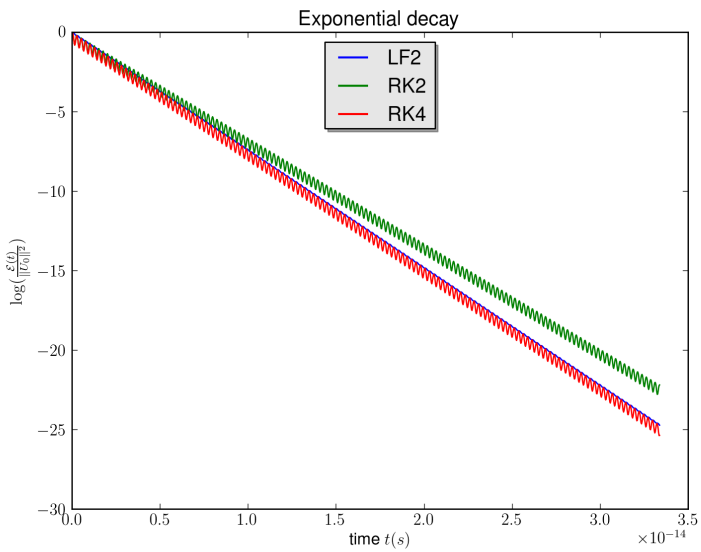


Figure 4: Exponential decay of the energy, representation of $t \mapsto \log\left(\frac{\mathcal{E}(t)}{\|\mathcal{U}_0\|^2}\right)$ over time for $LF2$ scheme with centered flux ($LF2/cent$), $RK2$ scheme with upwind flux ($RK2/up$), $RK4$ scheme with upwind flux ($RK4/up$)

| | $h$ | (6.1) ($k = 20$) | (S) | (CPL) | (NS) |
|---|---|---|---|---|---|
| $P_2$ | 2e-2 | -7.40001364e14 | -7.40001173e14 | -7.40001173e14 | -7.40001173e14 |
| $/LF2$ | 1e-2 | -7.40001305e14 | -7.40001119e14 | -7.40001119e14 | -7.40001119e14 |
| $/cent$ | 6e-3 | -7.40018936e14 | -7.40001108e14 | -7.40019042e14 | -7.40001107e14 |
| $P_2$ | 2e-2 | -6.64656315e14 | -6.64662530e14 | -6.64662531e14 | -6.64662534e14 |
| $/RK2$ | 1e-2 | -6.64656315e14 | -6.64662530e14 | -6.64662529e14 | -6.64662538e14 |
| $/up$ | 2e-3 | -6.64656313e14 | -6.64662529e14 | -6.64662527e14 | -6.64662528e14 |
| $P_2$ | 2e-2 | -7.40417381e14 | -7.40417954e14 | -7.40417954e14 | -7.40417961e14 |
| $/RK4$ | 1e-2 | -7.40417367e14 | -7.40417941e14 | -7.40417941e14 | -7.40417938e14 |
| $/up$ | 6e-3 | -7.40417353e14 | -7.40417924e14 | -7.40417924e14 | -7.40417928e14 |
| $P_4$ | 2e-2 | -7.40417360e14 | -7.40417931e14 | -7.40417930e14 | -7.40417936e14 |
| $/RK4$ | 1e-2 | -7.40417321e14 | -7.40417898e14 | -7.40417897e14 | -7.40417903e14 |
| $/up$ | 6e-3 | -7.40417288e14 | -7.40417865e14 | -7.40417865e14 | -7.40417859e14 |

Table 3: Value of the exponential decay factor.

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