Exponential inequalities for the supremum of some counting processes and their square martingales

Ronan Le Guével

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R. Le Guével
Univ Rennes, CNRS, IRMAR - UMR 6625, F-35000 Rennes, France
ronan.leguevel@univ-rennes2.fr

Abstract

We establish exponential inequalities for the supremum of martingales obtained from counting processes as well as for the supremum of their square martingales. Exponential inequalities are also provided for the oscillation modulus of these martingales.

Keywords: Counting processes, concentration inequalities, exponential martingales, U-statistics.

1 Introduction

The counting processes naturally arise in a lot of applied circumstances, and the understanding of their evolution is the object of a lot of modelization problems. Exponential inequalities are of great interest in this context, particularly because they play a decisive role in the control of errors in statistics. The exponential inequalities for the distribution of random variables have been of interest for many years (see Hoeffding [1963] for one of the first result about this issue), and it is still a very active domain of research for various types of processes, like sums of i.i.d. random variables, empirical processes, U-statistics, Poisson processes, martingales and self-normalised martingales, with discrete or continuous time. For example, for discrete time processes with i.i.d. random variables, exponential inequalities have been obtained for the empirical process or for U-statistics of order two in Hanson and Wright [1971], Giné and Zinn [1992], Arcones and Giné [1993], Talagrand [1996], Ledoux [1997], Klass and Nowicki [1997], Bretagnolle [1999], Massart [2000] or Giné et al. [2000] to cite a few. We may refer also to Massart [2007] or Bercu et al. [2015] for a wide review of exponential inequalities for discrete time martingales.

In this paper we focus on counting processes and their associated square martingales in continuous time. Our aim is to provide exponential inequalities for the oscillation modulus of a counting process as well as its associated square martingale. To this end, we exhibit first local martingale properties of the exponential of some counting processes and their square martingales, we establish exponential inequalities with explicit constants for the supremum of those processes, leading to exponential bounds for the oscillation modulus.

Some results already exists for martingales in continuous time, one may refer for instance to Theorem 23.17 of Kallenberg [1997]) for semi-martingales such that $[M]_\infty \leq 1$ almost surely, or Reynaud-Bouret [2003] for the case of the Poisson process. Another framework is considered in Van De Geer [1995] or Reynaud-Bouret [2006], where exponential inequalities are obtained for more general counting processes than the Poisson process. These exponential bounds are derived from technics adapted from the empirical process, with extensions of Bernstein's exponential inequality to general martingales. As a consequence of the results of Van De Geer [1995], exponential inequalities with explicit constants have been established for the supremum of counting processes with absolutely continuous compensators in Reynaud-Bouret [2006], as well as for $\sup_{t\in[0,T]} \sup_a M_t^a$ where $(M_t^a)_{t\geq 0}$ is a countable family of martingales associated to counting processes.

The case of the square martingale is closely related to the one of U-statistics of order two which has a long history too, and exponential inequalities with explicit constants for such processes are also of main interest for statistical problems in a non-asymptotic framework. Indeed the estimator of a quadratic form may naturally be a U-statistics of order two, and the results obtained on square martingales are generally not the simple consequence of those obtained for simple martingales. In statistical problems like the estimation of a quadratic functional of a density (Laurent [2005]), or in testing problems (see Fromont and Laurent [2006] for a goodness-of-fit test in density or Fromont et al. [2011] for an adaptive test of homogeneity of a Poisson process), the keystone for controlling the statistical error is to use exponential inequalities for the right model, and many of them come from results on sequences of i.i.d. random variables. In the specific case of the Poisson process, a sharp exponential inequality with explicit constants hold for U-statistics of order two and for double integrals of Poisson processes in Houdré and Reynaud-Bouret [2003]. The Poisson process is viewed as a point process $(T_i)_{i\geq 1}$ on the real line, allowing to use the inequalities obtained for U-statistics of i.i.d. random variables like Rosenthal's inequality and Talagrand's inequality, after conditioning by the total random number of point.

In our case, we do not make any assumption about the independence of the underlying point process. We therefore shall consider more general counting processes than the Poisson process, like non-explosive Cox processes or Hawkes processes with bounded intensities for instance. Comparing to the existing literature, we use quite different proofs with stochastic calculus instead of adapting previous technics in discrete times. This leads to a more accurate tail of the distribution for large deviations, namely in $x \log x$, through inequalities with explicit constants. This also allows us to consider the supremum of double integrals of other counting processes than the Poisson process.

The remainder of this article is organized as follows: in the next section, we recall some general notations, while Section 3 is devoted to the exponential martingales of the counting processes. The exponential inequalities of our martingales and their associated square martingales are presented in Section 4. We provide some applications of the inequalities for U-statistics of order two in Section 5. We compute also the oscillation modulus of the martingales in this section. Finally, we have gathered all the proofs in Section 6.

2 Notations

Let $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ be a complete right-continous filtration, $N = (N_t)_{t\geq 0}$ be a $\mathcal{F}$-adapted counting process with a continuous compensator $\Lambda = (\Lambda_t)_{t\geq 0}$. We assume that the jumps of N are totally inaccessible and that $N - \Lambda$ is a martingale with respect to the filtration $\mathcal{F}$.

We consider also $H = (H_s)_{s\geq 0}$, a left-continuous adapted process of bounded variations, bounded by the non-random real number $\|H\|_{\infty,[c,d]}$ on the interval $[c,d]$, that is $\sup_{s\in[c,d]} |H_s| \leq \|H\|_{\infty,[c,d]}$ almost surely. If $c = 0$ and $T \geq 0$, $\|H\|_{\infty,[0,T]}$ will be written $\|H\|_{\infty,T}$ for short. The non-random real number $\|H\|_{2,[c,d]}$ will stand for a bound of the L^2 norm of H in $L^2(\Lambda([c,d]))$, that is $\int_c^d |H_u|^2 d\Lambda_u \leq \|H\|^2_{2,[c,d]} < +\infty$ almost surely.

Recall that for a stochastic process X, we define $[X]_t$ by

$$[X]_t =< X^c >_t + \sum_{s\leq t} |\Delta X_s|^2$$

where $< X^c >$ is the quadratic variation of the continuous part of X and $\Delta X_s = X_s - X_{s^-}$ is the jump of X at s. We will use the fact that if X is a martingale and H is a predictable process satisfying $\mathbb{E}[\int_0^\infty H_s^2 d[X]_s] < +\infty$, then $(\int_0^t H_s dX_s)_{t\geq 0}$ is a martingale (see [Bass, 2011, p.134]).

Recall also that for a $\mathcal{C}^2$ function f and a semi-martingale $(X_t)_{t\geq 0}$, the Itô formula ([Bass, 2011, Theorem 17.10]) entails

$$f(X_t)=f(X_0)+\int_0^t f'(X_{s^-})dX_s+\frac{1}{2}\int_0^t f''(X_{s^-})d<X^c>_s+\sum_{s\le t}[f(X_s)-f(X_{s^-})-f'(X_{s^-})\Delta X_s].$$

For $f=\exp$ and a semi-martingale X satisfying $<X^c>\equiv 0$ and $X_0=0$, this leads to

$$e^{X_t}=1+\int_0^t e^{X_{s^-}}dX_s+\sum_{s\le t}e^{X_{s^-}}[e^{\Delta X_s}-1-\Delta X_s]. \tag{1}$$

Finally we define for every $n\geq 1$

$$S_n(X)=\inf\{t>0,e^{X_{t^-}}\geq n\}$$

with the convention $\inf\varnothing=+\infty$. If $(e^{X_{t^-}})_{t\geq 0}$ is a finite left-continuous process, then $S_n(X)$ is a stopping time (see [Bass, 2010, Theorem 2.4]) satisfying $\lim_{n\to+\infty}S_n(X)=+\infty$ almost surely.

3 Martingale properties

We consider in this section the three martingales $M=(M_t)_{t\le T}$, $\tilde{M}=(\tilde{M}_t)_{t\le T}$ and $\tilde{\tilde{M}}=(\tilde{\tilde{M}}_t)_{t\le T}$ defined for $t\le T$ by

$$M_t=\int_0^t H_s d(N_s-\Lambda_s),$$

and the two double integrals with their compensator

$$\begin{aligned}\tilde{M}_t&=(\int_0^t H_s d(N_s-\Lambda_s))^2-\int_0^t H_s^2 dN_s\\&=M_t^2-\int_0^t H_s^2 dN_s\\&=\int_0^t 2M_{s^-}H_s d(N_s-\Lambda_s),\end{aligned}$$

and

$$\begin{aligned}\tilde{\tilde{M}}_t&=(\int_0^t H_s d(N_s-\Lambda_s))^2-\int_0^t H_s^2 d\Lambda_s\\&=M_t^2-\int_0^t H_s^2 d\Lambda_s\\&=\int_0^t (2M_{s^-}H_s+H_s^2)d(N_s-\Lambda_s).\end{aligned}$$

Our main goal is to establish in the next section some exponential inequalities for these three martingales. We will use Chernoff bounds in order to do that, so we are first interested by the exponential martingales associated with the three processes M, $\tilde{M}$ and $\tilde{\tilde{M}}$. We start first with the process M in the following lemma, proving that the exponential of M is a local-martingale. We follow the proof of Theorem VI.2 in Brémaud [1981] where the case of an absolutely continuous compensator Λ is treated. We may also refer to Sokol and Hansen [2012] to find in that case some conditions on the intensity and the counting process to obtain an exponential which is a martingale.

Lemma 1. *Let Z be the process defined for a fixed real number λ and all $t\le T$ by*

$$Z_t=\lambda M_t-\int_0^t(e^{\lambda H_s}-1-\lambda H_s)d\Lambda_s.$$

Then for every $n\geq 1$, the process $(\exp(Z_{t\wedge S_n(Z)}))_{t\le T}$ is a martingale.

Let us define now for $a > 0$

$$T_a = \inf_{0<t\leq T}\{|M_t| > a\}.$$

Since the jumps of N are totally inaccessible, T_a is a stopping time ([Bass, 2011, Proposition 16.3]). The next lemma sets out a stopped exponential martingale associated with the martingale $\tilde{M}$.

Lemma 2. *Let $\tilde{Z}$ be the process defined for a fixed real number λ and all $t \leq T$ by*

$$\tilde{Z}_t = \lambda\tilde{M}_t - \int_0^t (e^{2\lambda H_s M_s} - 1 - 2\lambda H_s M_s)d\Lambda_s.$$

For every positive a and every $n \geq 1$, the process $(\exp(\tilde{Z}_{t\wedge T_a \wedge Sn(\tilde{Z})})_{t\leq T}$ is a martingale.

Finally we present the analogue of Lemma 2 for the martingale $\tilde{\tilde{M}}$.

Lemma 3. *Let $\tilde{\tilde{Z}}$ be the process defined for a fixed real number λ and all $t \leq T$ by*

$$\tilde{\tilde{Z}}_t = \lambda\tilde{\tilde{M}}_t - \int_0^t (e^{\lambda H_s(H_s+2M_s)} - 1 - \lambda H_s(H_s + 2M_s))d\Lambda_s.$$

For every positive a and every $n \geq 1$, the process $(\exp(\tilde{\tilde{Z}}_{t\wedge T_a \wedge Sn(\tilde{\tilde{Z}})}))_{t\leq T}$ is a martingale.

4 Exponential inequalities

We have gathered in this section our main results, that is the exponential inequalities for the three martingales M, $\tilde{M}$ and $\tilde{\tilde{M}}$. The rates that appear in these inequalities are governed by the rate function I defined for $x \geq 0$ by

$$I(x) = (1 + x)\log(1 + x) - x.$$

We start with a technical lemma that provides a useful inequality for the proofs of the main theorems.

Lemma 4. *Let $I_t(H, \lambda)$ be defined for $t \geq 0$ by $\int_0^t (e^{\lambda H_s} - 1 - \lambda H_s)d\Lambda_s$. For $t \leq T$ and every real λ, we get the almost sure inequality*

$$|I_t(H, \lambda)| \leq \frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} g(|\lambda|\|H\|_{\infty,T}) \tag{2}$$

where $g(x) = e^x - 1 - x$. Moreover the function g satisfies for every positive A, B and x

$$\inf_{\lambda>0}(Ag(Bx) - \lambda x) = -AI(\frac{x}{AB}). \tag{3}$$

We present now in Theorem 1 an inequality for the martingale M, with its two-sided version.

Theorem 1. *For every positive x and T, we have the following inequalities:*

$$\mathbb{P}(\sup_{0\leq t\leq T} M_t \geq x) \leq \exp(-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I(\frac{\|H\|_{\infty,T}}{\|H\|_{2,T}^2} x)) \tag{4}$$

and

$$\mathbb{P}(\sup_{0\leq t\leq T} |M_t| \geq x) \leq 2\exp(-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I(\frac{\|H\|_{\infty,T}}{\|H\|_{2,T}^2} x)). \tag{5}$$

Such exponential inequalities have already been obtained for martingales with bounded jumps in Kallenberg [1997], Van De Geer [1995] or Reynaud-Bouret [2006]. In Kallenberg [1997], the bound is of the form $\exp(-\frac{Ax^2}{1+Bx})$ for some constants A and B, and is available for a semi-martingale M such that $[M]_\infty \leq 1$ almost surely, which is not our case here. In Van De Geer [1995], the bound is of the form $A\exp(-Bx)$ for some constants A and B and x large enough. Finally in Reynaud-Bouret [2006], the inequality is of the form $\mathbb{P}(\sup_{t\in[0,T]}\sup_a M_t^a \geq A\sqrt{x}+Bx) \leq \exp(-x)$ for a countable family of martingales $(M_t^a)_{t\geq 0}$. Comparing to all these results, in the case of the large deviations, that is when x tends to infinity, we get a more acccurate tail namely in $x\log x$ instead of x. When x tends to zero, these bounds are similar (up to constants), taking the form $A\exp(-Bx^2)$.

The next Theorem deals with the square martingale $\tilde{M}$. The same inequality is obtained for $-\tilde{M}$, leading to a two-sided inequality.

Theorem 2. *For every positive x and T, we have the following inequalities:*

$$\mathbb{P}(\sup_{0\leq t\leq T}\tilde{M}_t \geq x) \leq 3\exp(-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I(\frac{\|H\|_{\infty,T}}{\|H\|_{2,T}^2}\sqrt{\frac{x}{2}})) \tag{6}$$

and

$$\mathbb{P}(\sup_{0\leq t\leq T}-\tilde{M}_t \geq x) \leq 3\exp(-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I(\frac{\|H\|_{\infty,T}}{\|H\|_{2,T}^2}\sqrt{\frac{x}{2}})), \tag{7}$$

thereby we have the following two-sided exponential inequality:

$$\mathbb{P}(\sup_{0\leq t\leq T}|\tilde{M}_t| \geq x) \leq 6\exp(-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I(\frac{\|H\|_{\infty,T}}{\|H\|_{2,T}^2}\sqrt{\frac{x}{2}})). \tag{8}$$

If we compare (5) and (8), we can notice that the upper bound in (8) involves $\sqrt{x}$ instead of x in the inequality (5), leading to a sharper bound when x tends to zero, contrary to the case of the large deviations. Finally the next Theorem 3 is the analogue of Theorem 2 for the martingale $\tilde{\tilde{M}}$.

Theorem 3. *For every positive x and T, we have the following inequalities:*

$$\mathbb{P}(\sup_{0\leq t\leq T}\tilde{\tilde{M}}_t \geq x) \leq 3\exp(-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I(\frac{\|H\|_{\infty,T}^2}{\|H\|_{2,T}^2}\frac{\sqrt{1+8x/\|H\|_{\infty,T}^2}-1}{4})) \tag{9}$$

and

$$\mathbb{P}(\sup_{0\leq t\leq T}-\tilde{\tilde{M}}_t \geq x) \leq 3\exp(-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I(\frac{\|H\|_{\infty,T}^2}{\|H\|_{2,T}^2}\frac{\sqrt{1+8x/\|H\|_{\infty,T}^2}-1}{4})), \tag{10}$$

thereby we have the following two-sided exponential inequality:

$$\mathbb{P}(\sup_{0\leq t\leq T}|\tilde{\tilde{M}}_t| \geq x) \leq 6\exp(-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I(\frac{\|H\|_{\infty,T}^2}{\|H\|_{2,T}^2}\frac{\sqrt{1+8x/\|H\|_{\infty,T}^2}-1}{4})). \tag{11}$$

Comparing now (8) and (11), we observe that M and $\tilde{\tilde{M}}$ are behaving in the same way for x tending to zero, while $\tilde{M}$ appears to be much more concentrated around its expectation. When x tends to infinity, (11) provide a similar bound (up to a constant) to (8), which is quite surprising in view of the relationship $\tilde{\tilde{M}} = \tilde{M} + \int H^2 d(N-\Lambda)$. Moreover using this relationship, (5) with H^2 instead of H, (8) and $\frac{x}{2}$, lead also to an exponential inequality but less sharp than (11) because $\|H\|_{2,T}^4 \leq \|H^2\|_{2,T}^2$.

5 Examples of applications

5.1 U-statistics of order two

The main hypothesis of the previous theorems is to suppose that the counting process N has a continuous compensator Λ, which is bounded in some spaces (as well as H) through the assumption $\|H\|_{2,T} < +\infty$. If the process N admits an intensity λ, some mild assumptions on λ ensure the continuity of the compensator $\Lambda = \int \lambda(s)ds$. This allows us to consider for instance Poisson, Cox or Hawkes processes with a bounded intensity such that $N - \Lambda$ is a martingale. As an example, if the process $(\frac{1}{h}\mathbb{E}[N_{t+h} - N_t|\mathcal{F}_t])_{h,t}$ is uniformly bounded for h small enough, we know that the $\mathcal{F}$-intensity of N is bounded and $N - \Lambda$ is a martingale because the intensity is obtained by $\lambda(t) = \lim_{h\to 0^+} \frac{1}{h}\mathbb{E}[N_{t+h} - N_t|\mathcal{F}_t]$ almost surely (see formula (3.5) in Chapter 2 of Brémaud [1981]).

If N is a Poisson process, some sharp exponential inequalities have already been obtained in Houdré and Reynaud-Bouret [2003] for double stochastic integrals of the form $Z_t = \int_0^t \int_0^{y^-} h(x,y)d(N_x - \Lambda_x)d(N_y - \Lambda_y)$ where h is a bounded Borel function. The Poisson process N is viewed as a point process $(T_i)_{i\geq 1}$, so that Z_t is the U-statistic of order two for the Poisson process: $Z_t = \sum_{0\leq T_i < T_j \leq t} h(T_i, T_j)$. This allows to use the inequalities obtained for U-statistics after conditioning by the total random number of points, leading to a similar inequality to the one in Giné et al. [2000].

Such exponential inequalities for U-statistics are very useful for statistical applications. For instance the estimation of the L^2 norm $\int f^2(x)dx$ of the density of i.i.d. random variables via selection model is considered in Laurent [2005] and Fromont and Laurent [2006]. The estimator of a quadratic distance is naturally a U-statistics of order two and the exponential inequality of Houdré and Reynaud-Bouret [2003] is a main tool for the study of the property of the estimator. In the Poisson model too, as in Fromont et al. [2011] where the homogeneity of a Poisson process is tested, the method is based on an approximation of the L^2-norm of the intensity of the Poisson process seen as a point process $(T_i)_{i\geq 1}$ on the real line.

In the particular case where h is a stochastic kernel of the form $h(x,y) = H(x)H(y)$, $\tilde{M}$ may be written $\tilde{M}_t = 2Z_t$, i.e. it is a double stochastic integrals or a U-statistics of order two. Although we are not limited to the Poisson case, by the Meyer theorem (see [Protter, 2005, page 104]), the jumps of a Poisson process are totally inaccessible so that we may apply Theorem 2. Comparing to Giné et al. [2000] or Houdré and Reynaud-Bouret [2003], where the supremum of $(Z_t)_{t\geq 0}$ is not considered, the inequality (6) provide different bounds for the large deviations (with an additional $\log x$ in our inequality) as well as for the small deviations. Indeed in Giné et al. [2000] or Houdré and Reynaud-Bouret [2003], the bound is of the form $L\exp(-\frac{1}{L}\min(\frac{x^{1/2}}{A^{1/2}}, \frac{x^{2/3}}{B^{2/3}}, \frac{x}{C}, \frac{x^2}{D^2}))$ for some explicit constants A, B, C, D and L.

5.2 Oscillation modulus control

The main theorems of the previous section provide also an upper bound for the oscillation modulus of the three martingales. We consider then c, d and x three non-negative real numbers, and the counting process $N_t^c = N_{t+c} - N_c$ whose compensator is $\Lambda_{t+c} - \Lambda_c$. The followong theorem gives upper bounds for the oscillation modulus of the martingales M and $\tilde{M}$.

Theorem 4. *For every non-negative x, c and d, we have the following inequality for the oscillation modulus of M:*

$$\mathbb{P}(\sup_{(s,t)\in[c,d]^2} |M_t - M_s| \geq x) \leq 2\exp(-\frac{\|H\|_{2,[c,d]}^2}{\|H\|_{\infty,[c,d]}^2} I(\frac{\|H\|_{\infty,[c,d]}}{\|H\|_{2,[c,d]}^2}\frac{x}{2})). \tag{12}$$

For the martingale $\tilde{M}$, we get the following exponential upper bound

$$\mathbb{P}(\sup_{(s,t)\in[c,d]^2} |(\int_s^t H_u d(N_u - \Lambda_u))^2 - \int_s^t H_u^2 dN_u| \geq x) \leq 10 \exp(-\frac{\|H\|_{2,[c,d]}^2}{\|H\|_{\infty,[c,d]}^2} I(\frac{\|H\|_{\infty,[c,d]}}{\|H\|_{2,[c,d]}^2}\sqrt{\frac{x}{8}})), \quad (13)$$

leading to the exponential inequality for the oscillation modulus of $\tilde{M}$:

$$\begin{aligned}\mathbb{P}(\sup_{(s,t)\in[c,d]^2} |\tilde{M}_t - \tilde{M}_s| \geq x) &\leq 10 \exp(-\frac{\|H\|_{2,[c,d]}^2}{\|H\|_{\infty,[c,d]}^2} I(\frac{\|H\|_{\infty,[c,d]}}{\|H\|_{2,[c,d]}^2}\sqrt{\frac{x}{16}}))\\ &+ 2\exp(-\frac{\|H\|_{2,d}^2}{\|H\|_{\infty,d}^2} I(\frac{\sqrt{\|H\|_{\infty,[c,d]}\|H\|_{\infty,d}}}{\|H\|_{2,d}\|H\|_{2,[c,d]}}\sqrt{\frac{x}{8}}))\\ &+ 2\exp(-\frac{\|H\|_{2,[c,d]}^2}{\|H\|_{\infty,[c,d]}^2} I(\frac{\sqrt{\|H\|_{\infty,[c,d]}\|H\|_{\infty,d}}}{\|H\|_{2,d}\|H\|_{2,[c,d]}}\sqrt{\frac{x}{8}})). \quad (14)\end{aligned}$$

In view of Theorems 1 and 2, the previous inequalities show that considering the oscillation modulus instead of the martingales M and $\tilde{M}$ themselves does not affect the exponential bounds, except for the constants. We obtain in Theorem 4 explicit constants with respect to the integrand H as well as the interval $[c,d]$, which may be useful for the applications.

6 Proofs

Proof of Lemma 1 The process Z is defined as $\lambda M_t - \int_0^t (e^{\lambda H_s} - 1 - \lambda H_s) d\Lambda_s$ where λ is a fixed real number. Z is of bounded variations because H and M are of bounded variations, and the continuity of Λ entails the equality $\Delta Z_s = \lambda H_s \Delta N_s$. We get then from (1) that

$$\begin{aligned} e^{Z_t} &= 1 + \int_0^t e^{Z_{s^-}} dZ_s + \sum_{s\leq t} e^{Z_{s^-}} [e^{\lambda H_s \Delta N_s} - 1 - \lambda H_s \Delta N_s] \\ &= 1 + \int_0^t e^{Z_{s^-}} [\lambda dM_s - (e^{\lambda H_s} - 1 - \lambda H_s) d\Lambda_s] + \int_0^t e^{Z_{s^-}} (e^{\lambda H_s} - 1 - \lambda H_s) dN_s \\ &= 1 + \int_0^t e^{Z_{s^-}} (e^{\lambda H_s} - 1) d(N_s - \Lambda_s). \end{aligned}$$

For $n \geq 1$, the stopping time $S_n(Z)$ is defined by

$$S_n(Z) = \inf\{t > 0, e^{Z_{t^-}} \geq n\}.$$

Since $e^{Z_{t^-}}$ is a finite left-continuous process, $\lim_{n\to+\infty} S_n(Z) = +\infty$ almost surely. Moreover, for every $t \leq T$,

$$e^{Z_{t\wedge S_n(Z)}} = 1 + \int_0^t e^{Z_{s^-}} (e^{\lambda H_s} - 1) \mathbf{1}_{s\leq S_n(Z)\wedge T} d(N_s - \Lambda_s).$$

To conclude, the result follows from the inequality

$$\mathbb{E}[\int_0^\infty e^{2Z_{s^-}} (e^{\lambda H_s} - 1)^2 \mathbf{1}_{s\leq S_n(Z)\wedge T} dN_s] \leq n^2 (e^{\lambda \|H\|_{\infty,T}} + 1)^2 \mathbb{E}[N_T] < +\infty \quad \blacksquare$$

Proof of Lemma 2 We proceed as in the proof of Lemma 1. The process $\tilde{Z}$ is defined as $\lambda\tilde{M}_t - \int_0^t (e^{2\lambda H_s M_s} - 1 - 2\lambda H_s M_s) d\Lambda_s$ for a fixed real λ. $\tilde{Z}$ is of bounded variations because H and M are of bounded variations too, and since Λ is continuous, we may compute $\Delta\tilde{Z}_s = 2\lambda H_s M_{s^-}\Delta N_s$. We get from (1) that

$$\begin{aligned}
e^{\tilde{Z}_t} &= 1 + \int_0^t e^{\tilde{Z}_{s^-}} d\tilde{Z}_s + \sum_{s\le t} e^{\tilde{Z}_{s^-}}[e^{2\lambda H_s M_{s^-}\Delta N_s} - 1 - 2\lambda H_s M_{s^-}\Delta N_s] \\
&= 1 + \lambda\int_0^t e^{\tilde{Z}_{s^-}} d\tilde{M}_s + \int_0^t e^{\tilde{Z}_{s^-}}(e^{2\lambda H_s M_{s^-}} - 1 - 2\lambda H_s M_{s^-}) d(N_s - \Lambda_s) \\
&= 1 + \int_0^t e^{\tilde{Z}_{s^-}}(e^{2\lambda H_s M_{s^-}} - 1) d(N_s - \Lambda_s)
\end{aligned}$$

and

$$e^{\tilde{Z}_{t\wedge T_a\wedge S_n(\tilde{Z})}} = 1 + \int_0^t e^{\tilde{Z}_{s^-}}(e^{2\lambda H_s M_{s^-}} - 1)\mathbf{1}_{s\le T_a\wedge S_n(\tilde{Z})} d(N_s - \Lambda_s).$$

It remains to show that $\mathbb{E}[\int_0^{+\infty} e^{2\tilde{Z}_{s^-}}(e^{2\lambda H_s M_{s^-}} - 1)^2 \mathbf{1}_{s\le T\wedge T_a\wedge S_n(\tilde{Z})} dN_s] < +\infty$. For all $s \le T\wedge T_a\wedge S_n(\tilde{Z})$,

$$|2\lambda H_s M_{s^-}| \le 2|\lambda|\|H\|_{\infty,T}|M_{s^-}| \le 2|\lambda|(a + \|H\|_{\infty,T})\|H\|_{\infty,T},$$

$$(e^{2\lambda H_s M_{s^-}} - 1)^2 \le (e^{2|\lambda|(a+\|H\|_{\infty,T})\|H\|_{\infty,T}} + 1)^2.$$

As a consequence, we obtain with the fact that $e^{2\tilde{Z}_{s^-}} \le n^2$ for $s \le S_n(\tilde{Z})$

$$\mathbb{E}[\int_0^{+\infty} e^{2Z_{s^-}}(e^{2\lambda H_s M_{s^-}} - 1)^2 \mathbf{1}_{s\le T\wedge T_a\wedge S_n(\tilde{Z})} dN_s] \le n^2(e^{2|\lambda|(a+\|H\|_{\infty,T})\|H\|_{\infty,T}} + 1)^2\mathbb{E}[N_T] < +\infty \quad \blacksquare$$

Proof of Lemma 3 We follow the steps of the proof of Lemma 2, adapting the computations to this case. The process $\tilde{\tilde{Z}}$ is defined as $\lambda\tilde{\tilde{M}}_t - \int_0^t (e^{\lambda H_s(H_s+2M_s)} - 1 - \lambda H_s(H_s + 2M_s)) d\Lambda_s$ for a fixed real λ. The process $\tilde{\tilde{Z}}$ is again of bounded variations because H and M are of bounded variations, and the continuity of Λ entails the equality $\Delta\tilde{\tilde{Z}}_s = \lambda H_s(H_s + 2M_{s^-})\Delta N_s$. Then (1) yields

$$\begin{aligned}
e^{\tilde{\tilde{Z}}_t} &= 1 + \int_0^t e^{\tilde{\tilde{Z}}_{s^-}} d\tilde{\tilde{Z}}_s + \sum_{s\le t} e^{\tilde{\tilde{Z}}_{s^-}}[e^{\lambda H_s(H_s+2M_{s^-})\Delta N_s} - 1 - \lambda H_s(H_s + 2M_{s^-})\Delta N_s] \\
&= 1 + \lambda\int_0^t e^{\tilde{\tilde{Z}}_{s^-}} d\tilde{\tilde{M}}_s + \int_0^t e^{\tilde{\tilde{Z}}_{s^-}}(e^{\lambda H_s(H_s+2M_{s^-})} - 1 - \lambda H_s(H_s + 2M_{s^-})) d(N_s - \Lambda_s) \\
&= 1 + \int_0^t e^{\tilde{\tilde{Z}}_{s^-}}(e^{\lambda H_s(H_s+2M_{s^-})} - 1) d(N_s - \Lambda_s)
\end{aligned}$$

and

$$\exp(\tilde{\tilde{Z}}_{t\wedge T_a\wedge S_n(\tilde{\tilde{Z}})}) = 1 + \int_0^t e^{\tilde{\tilde{Z}}_{s^-}}(e^{\lambda H_s(H_s+2M_{s^-})} - 1)\mathbf{1}_{s\le T_a\wedge S_n(\tilde{\tilde{Z}})} d(N_s - \Lambda_s).$$

The proof is complete showing that $\mathbb{E}[\int_0^{+\infty} e^{2\tilde{\tilde{Z}}_{s^-}}(e^{\lambda H_s(H_s+2M_{s^-})} - 1)^2 \mathbf{1}_{s\le T\wedge T_a\wedge S_n(\tilde{\tilde{Z}})} dN_s] < +\infty$. For all $s \le T\wedge T_a\wedge S_n(\tilde{\tilde{Z}})$,

$$|H_s(H_s + 2M_{s^-})| \le \|H\|_{\infty,T}^2 + 2\|H\|_{\infty,T}|M_{s^-}| \le \|H\|_{\infty,T}^2 + 2(a + \|H\|_{\infty,T})\|H\|_{\infty,T},$$

$$(e^{\lambda H_s(H_s+2M_{s-})}-1)^2 \leq (e^{|\lambda|(\|H\|^2_{\infty,T}+2(a+\|H\|_{\infty,T})\|H\|_{\infty,T})}+1)^2.$$

Combining with the inequality $e^{2\widetilde{\widetilde{Z}}_{s-}} \leq n^2$, this entails

$$\mathbb{E}[\int_0^{+\infty} e^{2\widetilde{\widetilde{Z}}_{s-}}(e^{\lambda H_s(H_s+2M_{s-})}-1)^2 \mathbf{1}_{s\leq T\wedge T_a\wedge S_n(\widetilde{\widetilde{Z}})} dN_s] \leq n^2(e^{|\lambda|(\|H\|^2_{\infty,T}+2(a+\|H\|_{\infty,T})\|H\|_{\infty,T})}+1)^2\mathbb{E}[N_T] < +\infty \blacksquare$$

Proof of Lemma 4 Let $s \leq t \leq T$ and $\lambda \in \mathbb{R}$. We use the following inequality:

$$\begin{aligned}
\left|e^{\lambda H_s}-1-\lambda H_s\right| &= \left|\sum_{j\geq 2}\frac{(\lambda H_s)^j}{j!}\right| \\
&= \left|\frac{(\lambda H_s)^2}{2!}+H_s^2\sum_{j\geq 3}\frac{\lambda^j H_s^{j-2}}{j!}\right| \\
&\leq \frac{(|\lambda|H_s)^2}{2!}+H_s^2\sum_{j\geq 3}\frac{|\lambda|^j\|H\|^{j-2}_{\infty,T}}{j!} \\
&= H_s^2(\frac{\lambda^2}{2}+\frac{1}{\|H\|^2_{\infty,T}}\sum_{j\geq 3}\frac{|\lambda|^j\|H\|^j_{\infty,T}}{j!}),
\end{aligned}$$

that is

$$\left|e^{\lambda H_s}-1-\lambda H_s\right| \leq \frac{H_s^2}{\|H\|^2_{\infty,T}}\sum_{j\geq 2}\frac{|\lambda|^j\|H\|^j_{\infty,T}}{j!}. \tag{15}$$

Integrating with respect to $d\Lambda_s$ we obtain

$$|I_t(H,\lambda)| \leq \frac{\|H\|^2_{2,T}}{\|H\|^2_{\infty,T}}g(|\lambda|\|H\|_{\infty,T})$$

where $g(x) = e^x - 1 - x$. For the proof of (3), consider the function h defined for $\lambda > 0$ by $h(\lambda) = Ag(B\lambda) - \lambda x$. Since $h'(\lambda) = AB(e^{B\lambda}-1) - x$, we get that the minimum of h is reached for $\lambda = \frac{1}{B}\log(1+\frac{x}{AB}) =: \lambda_0$ and $h(\lambda_0) = -AI(\frac{x}{AB})$ $\blacksquare$

Proof of Theorem 1 Recall that $I_t(H,\lambda)$ is defined by $\int_0^t(e^{\lambda H_s}-1-\lambda H_s)d\Lambda_s$. We define the process Z as in Lemma 1 by $Z_t = \lambda M_t - I_t(H,\lambda)$ and for $n \geq 1$, the stopping time $S_n(Z)$ is defined by $S_n(Z) = \inf\{t > 0, e^{Z_{t-}} \geq n\}$. Since $(S_n(Z))_{n\geq 1}$ is a non-decreasing sequence of stopping times with $\lim_{n\to+\infty} S_n(Z) = +\infty$ almost surely, we get by monotony

$$\mathbb{P}(\sup_{0\leq t\leq T} M_t \geq x) = \lim_{n\to+\infty}\mathbb{P}(\sup_{0\leq t\leq T\wedge S_n(Z)} M_t \geq x) = \sup_{n\geq 1}\mathbb{P}(\sup_{0\leq t\leq T} M_{t\wedge S_n(Z)} \geq x).$$

Using Lemma 4 (2), we obtain for all $\lambda > 0$, $x > 0$ and $n \geq 1$,

$$\begin{aligned}
\mathbb{P}(\sup_{0\leq t\leq T} M_{t\wedge S_n(Z)} \geq x) &= \mathbb{P}(\sup_{0\leq t\leq T} e^{\lambda M_{t\wedge S_n(Z)} - I_{t\wedge S_n(Z)}(H,\lambda) + I_{t\wedge S_n(Z)}(H,\lambda)} \geq e^{\lambda x}) \\
&\leq \mathbb{P}(e^{\frac{\|H\|^2_{2,T}}{\|H\|^2_{\infty,T}}g(\lambda\|H\|_{\infty,T})}\sup_{0\leq t\leq T} e^{Z_{t\wedge S_n(Z)}} \geq e^{\lambda x}).
\end{aligned}$$

Doob's maximal inequality and Lemma 1 then lead to

$$\mathbb{P}(\sup_{0\le t\le T} M_{t\wedge S_n(Z)} \ge x) \le \exp(\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} g(\lambda\|H\|_{\infty,T}) - \lambda x)$$

for every $\lambda > 0$ with $g(x) = e^x - 1 - x$, so taking the limit in n and the infimum in λ, we get by (3)

$$\begin{aligned}\mathbb{P}(\sup_{0\le t\le T} M_t \ge x) &\le \inf_{\lambda>0} \exp(\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} g(\lambda\|H\|_{\infty,T}) - \lambda x)\\ &= \exp(-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I(\frac{\|H\|_{\infty,T}}{\|H\|_{2,T}^2} x))\end{aligned}$$

that is (4). Applying this inequality with $-H$ instead of H, we obtain also

$$\mathbb{P}(\sup_{0\le t\le T} -M_t \ge x) \le \exp(-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I(\frac{\|H\|_{\infty,T}}{\|H\|_{2,T}^2} x)).$$

Then (5) follows from the inequality

$$\mathbb{P}(\sup_{0\le t\le T} |M_t| \ge x) \le \mathbb{P}(\sup_{0\le t\le T} M_t \ge x) + \mathbb{P}(\sup_{0\le t\le T} -M_t \ge x) \quad \blacksquare$$

Proof of Theorem 2 Let us begin with the proof of (6). We define $\tilde{Z}$ as in Lemma 2 by $\tilde{Z}_t = \lambda\tilde{M}_t - I_t(2HM,\lambda)$, thereby $(S_n(\tilde{Z}))_{n\ge 1}$ is a sequence of non-decreasing stopping times such that $\lim_{n\to+\infty} S_n(\tilde{Z}) = +\infty$ almost surely. We proceed then as in the proof of Theorem 1. For all positive λ, a and x

$$\begin{aligned}\mathbb{P}(\sup_{0\le t\le T} \tilde{M}_t \ge x) &= \sup_{n\ge 1}\mathbb{P}(\sup_{0\le t\le T} \tilde{M}_{t\wedge S_n(\tilde{Z})} \ge x) &(16)\\ &\le \mathbb{P}(T_a < T) + \sup_{n\ge 1}\mathbb{P}(\sup_{0\le t\le T} \tilde{M}_{t\wedge T_a\wedge S_n(\tilde{Z})} \ge x \cap T_a \ge T)\\ &\le \mathbb{P}(\sup_{0\le t\le T} |M_t| \ge a) + \sup_{n\ge 1}\mathbb{P}(\sup_{0\le t\le T} e^{\lambda\tilde{M}_{t\wedge T_a\wedge S_n(\tilde{Z})}} \ge e^{\lambda x}). &(17)\end{aligned}$$

Using the inequality (15), we get for $t \le T$ and $\lambda > 0$

$$I_{t\wedge T_a\wedge S_n(\tilde{Z})}(2HM,\lambda) = \int_0^{t\wedge T_a\wedge S_n(\tilde{Z})} (e^{2\lambda H_s M_s} - 1 - 2\lambda H_s M_s) d\Lambda_s \le \frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} g(2\lambda a\|H\|_{\infty,T}).$$

Since T_a is a bounded stopping time, Lemma 2 and Doob's maximal inequality yield for every $\lambda > 0$ and $n \ge 1$

$$\begin{aligned}\mathbb{P}(\sup_{0\le t\le T} e^{\lambda\tilde{M}_{t\wedge T_a\wedge S_n(\tilde{Z})}} \ge e^{\lambda x}) &\le \mathbb{P}(\sup_{0\le t\le T} e^{\tilde{Z}_{t\wedge T_a\wedge S_n(\tilde{Z})}} \ge e^{\lambda x - \frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} g(2\lambda a\|H\|_{\infty,T})})\\ &\le \exp(\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} g(2\lambda a\|H\|_{\infty,T}) - \lambda x)\end{aligned}$$

whereby

$$\sup_{n\geq 1}\mathbb{P}(\sup_{0\leq t\leq T} e^{\lambda\tilde{M}_{t\wedge T_a\wedge S_n(\tilde{Z})}}\geq e^{\lambda x})\leq \inf_{\lambda>0}\exp(\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2}g(2\lambda a\|H\|_{\infty,T})-\lambda x)$$
$$=\exp(-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2}I(\frac{\|H\|_{\infty,T}}{2a\|H\|_{2,T}^2}x))$$

thanks to (3). Coming back to the inequality (17), Theorem 1 then entail for every $a>0$,

$$\mathbb{P}(\sup_{0\leq t\leq T}\tilde{M}_t\geq x)\leq 2e^{-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2}I(\frac{\|H\|_{\infty,T}}{\|H\|_{2,T}^2}a)}+e^{-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2}I(\frac{\|H\|_{\infty,T}}{2a\|H\|_{2,T}^2}x)}.$$

We choose $a=\sqrt{\frac{x}{2}}$ in order to obtain (6). For the proof of (7), we consider $\tilde{Z}_t=-\lambda\tilde{M}_t-I_t(2HM,-\lambda)$ for $\lambda>0$. We get similarly, thanks to Lemma 4 and Lemma 2

$$\mathbb{P}(\sup_{0\leq t\leq T} e^{-\lambda\tilde{M}_{t\wedge T_a\wedge S_n(\tilde{Z})}}\geq e^{\lambda x})\leq \mathbb{P}(\sup_{0\leq t\leq T} e^{\tilde{Z}_{t\wedge T_a\wedge S_n(\tilde{Z})}}\geq e^{\lambda x-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2}g(2\lambda a\|H\|_{\infty})})\leq e^{\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2}g(2\lambda a\|H\|_{\infty,T})-\lambda x}$$

and the end of the proof is similar to the one of (6). To conclude, (8) follows from the inequality

$$\mathbb{P}(\sup_{0\leq t\leq T}|\tilde{M}_t|\geq x)\leq \mathbb{P}(\sup_{0\leq t\leq T}\tilde{M}_t\geq x)+\mathbb{P}(\sup_{0\leq t\leq T}-\tilde{M}_t\geq x)\quad\blacksquare$$

Proof of Theorem 3 This proof is similar to the one of Theorem 2. Let us begin showing the inequality (9). We introduce $\tilde{\tilde{Z}}$ as in Lemma 3 with $\tilde{\tilde{Z}}_t=\lambda\tilde{\tilde{M}}_t-I_t(H(H+2M),\lambda)$ and its associated sequence of stopping times $S_n(\tilde{\tilde{Z}})$ to obtain for all positive a, λ and x

$$\mathbb{P}(\sup_{0\leq t\leq T}\tilde{\tilde{M}}_t\geq x)\leq \mathbb{P}(\sup_{0\leq t\leq T}|M_t|\geq a)+\sup_{n\geq 1}\mathbb{P}(\sup_{0\leq t\leq T}\exp(\lambda\tilde{\tilde{M}}_{t\wedge T_a\wedge S_n(\tilde{\tilde{Z}})})\geq e^{\lambda x}).\tag{18}$$

Using the inequality (15), we get for $t\leq T$ and $\lambda>0$

$$\int_0^{t\wedge T_a\wedge S_n(\tilde{\tilde{Z}})}(e^{\lambda H_s(H_s+2M_s)}-1-\lambda H_s(H_s+2M_s))d\Lambda_s\leq \frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2}g(\lambda\|H\|_{\infty,T}(\|H\|_{\infty,T}+2a)).$$

Then Lemma 3 and Doob's maximal inequality yield for every $\lambda>0$ and $n\geq 1$

$$\mathbb{P}(\sup_{0\leq t\leq T}\exp(\lambda\tilde{\tilde{M}}_{t\wedge T_a\wedge S_n(\tilde{\tilde{Z}})})\geq e^{\lambda x})\leq \mathbb{P}(\sup_{0\leq t\leq T}\exp(\tilde{\tilde{Z}}_{t\wedge T_a\wedge S_n(\tilde{\tilde{Z}})})\geq e^{\lambda x-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2}g(\lambda\|H\|_{\infty,T}(\|H\|_{\infty,T}+2a))})$$
$$\leq \exp(\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2}g(\lambda\|H\|_{\infty,T}(\|H\|_{\infty,T}+2a))-\lambda x).$$

As a consequence

$$\sup_{n\geq 1}\mathbb{P}(\sup_{0\leq t\leq T}\exp(\tilde{\tilde{M}}_{t\wedge T_a\wedge S_n(\tilde{\tilde{Z}})})\geq e^{\lambda x})\leq \inf_{\lambda>0}\exp(\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2}g(\lambda\|H\|_{\infty,T}(\|H\|_{\infty,T}+2a))-\lambda x)$$
$$=\exp(-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2}I(\frac{\|H\|_{\infty,T}}{\|H\|_{2,T}^2(2a+\|H\|_{\infty,T})}x)).$$

thanks to (3). The inequality (18) and Theorem 1 then entail for every $a > 0$,

$$\mathbb{P}(\sup_{0 \le t \le T} \tilde{\tilde{M}}_t \ge x) \le 2e^{-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I(\frac{\|H\|_{\infty,T}}{\|H\|_2^2} a)} + e^{-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I(\frac{\|H\|_{\infty,T}}{\|H\|_{2,T}^2 (2a+\|H\|_{\infty,T})} x)}.$$

We choose $a = \frac{x}{2a+\|H\|_{\infty,T}}$ i.e. $a = \frac{-\|H\|_{\infty,T}+\sqrt{\|H\|_{\infty,T}^2+8x}}{4}$ in order to get (9). For the proof of (10), let $\tilde{\tilde{Z}}$ be defined by $\tilde{\tilde{Z}}_t = -\lambda \tilde{\tilde{M}}_t - I_t(H(H+2M), -\lambda)$ for $\lambda > 0$. We obtain similarly with Lemma 4 and Lemma 3

$$\begin{aligned}
\mathbb{P}(\sup_{0 \le t \le T} \exp(-\lambda \tilde{\tilde{M}}_{t \wedge T_a \wedge S_n(\tilde{\tilde{Z}})}) \ge e^{\lambda x}) &\le \mathbb{P}(\sup_{0 \le t \le T} \exp(\tilde{\tilde{Z}}_{t \wedge T_a \wedge S_n(\tilde{\tilde{Z}})}) \ge e^{\lambda x - \frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} g(\lambda \|H\|_{\infty,T}(\|H\|_{\infty,T}+2a))}) \\
&\le \exp(\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} g(\lambda \|H\|_{\infty,T}(\|H\|_{\infty,T}+2a)) - \lambda x)
\end{aligned}$$

and the end of the proof is similar to the one of (9). To conclude, (11) also comes from the inequality

$$\mathbb{P}(\sup_{0 \le t \le T} |\tilde{\tilde{M}}_t| \ge x) \le \mathbb{P}(\sup_{0 \le t \le T} \tilde{\tilde{M}}_t \ge x) + \mathbb{P}(\sup_{0 \le t \le T} -\tilde{\tilde{M}}_t \ge x) \quad \blacksquare$$

Proof of Theorem 4 Let us prove (12) first. We use the relationship $M_t - M_s = \int_c^t H_u(dN_u - \Lambda_u) - \int_c^s H_u(dN_u - \Lambda_u)$ to get

$$\begin{aligned}
\sup_{(s,t) \in [c,d]} |M_t - M_s| &\le 2 \sup_{t \in [c,d]} |\int_c^t H_u(dN_u - \Lambda_u)| \\
&= 2 \sup_{t \in [0,d-c]} \left| \int_0^t H_{u+c}(dN_u^c - d\Lambda_u^c) \right|.
\end{aligned}$$

Since N^c satisfies the same assumptions than N, we may apply (5) with N^c, Λ^c and the process $u \mapsto H_{u+c}$ in order to obtain

$$\mathbb{P}(\sup_{(s,t) \in [c,d]^2} |M_t - M_s| \ge x) \le 2 \exp(-\frac{\|H\|_{2,[c,d]}^2}{\|H\|_{\infty,[c,d]}^2} I(\frac{\|H\|_{\infty,[c,d]}}{\|H\|_{2,[c,d]}^2} \frac{x}{2})),$$

that is (12). Let us prove (13) now. We shall consider the following relationship

$$(\int_s^t H_u d(N_u - \Lambda_u))^2 - \int_s^t H_u^2 dN_u = \tilde{M}_t^c - \tilde{M}_s^c - 2(M_t - M_s) \int_c^s H_u(N_u - \Lambda_u)$$

where $\tilde{M}_t^c = (\int_c^t H_u d(N_u - \Lambda_u))^2 - \int_c^t H_u^2 d\Lambda_u = (\int_0^{t-c} H_{u+c} d(N_u^c - \Lambda_u^c))^2 - \int_0^{t-c} H_{u+c}^2 d\Lambda_u^c$. This yields for $a > 0$

$$\begin{aligned}
&\mathbb{P}(\sup_{(s,t) \in [c,d]^2} |(\int_s^t H_u d(N_u - \Lambda_u))^2 - \int_s^t H_u^2 dN_u| \ge x) \\
&\le \mathbb{P}(2 \sup_{t \in [c,d]} |\tilde{M}_t^c| \ge \frac{x}{2}) + \mathbb{P}(\sup_{s \in [c,d]} |\int_c^s H_u(N_u - \Lambda_u)| \ge a) + \mathbb{P}(\sup_{(s,t) \in [c,d]^2} |M_t - M_s| \ge \frac{x}{4a}).
\end{aligned}$$

We get then from (8), (5) and (12)

$$\mathbb{P}(2\sup_{t\in[c,d]}|\tilde{M}_t^c|\geq\frac{x}{2})\leq 6\exp(-\frac{\|H\|_{2,[c,d]}^2}{\|H\|_{\infty,[c,d]}^2}I(\frac{\|H\|_{\infty,[c,d]}}{\|H\|_{2,[c,d]}^2}\sqrt{\frac{x}{8}})),$$

$$\mathbb{P}(\sup_{s\in[c,d]}|\int_c^s H_u(N_u-\Lambda_u)|\geq a)\leq 2\exp(-\frac{\|H\|_{2,[c,d]}^2}{\|H\|_{\infty,[c,d]}^2}I(\frac{\|H\|_{\infty,[c,d]}}{\|H\|_{2,[c,d]}^2}a))$$

and

$$\mathbb{P}(\sup_{(s,t)\in[c,d]^2}|M_t-M_s|\geq\frac{x}{4a})\leq 2\exp(-\frac{\|H\|_{2,[c,d]}^2}{\|H\|_{\infty,[c,d]}^2}I(\frac{\|H\|_{\infty,[c,d]}}{\|H\|_{2,[c,d]}^2}\frac{x}{8a})).$$

If we choose $a=\sqrt{\frac{x}{8}}$, we obtain

$$\mathbb{P}(\sup_{(s,t)\in[c,d]^2}|(\int_s^t H_u d(N_u-\Lambda_u))^2-\int_s^t H_u^2 dN_u|\geq x)\leq 10\exp(-\frac{\|H\|_{2,[c,d]}^2}{\|H\|_{\infty,[c,d]}^2}I(\frac{\|H\|_{\infty,[c,d]}}{\|H\|_{2,[c,d]}^2}\sqrt{\frac{x}{8}})), \quad (19)$$

that is (13). To conclude with the oscillation modulus of $\tilde{M}$, we may use similarly

$$\tilde{M}_t-\tilde{M}_s=(\int_s^t H_u d(N_u-\Lambda_u))^2-\int_s^t H_u^2 dN_u+2(M_t-M_s)M_s$$

and

$$\begin{aligned}\mathbb{P}(\sup_{(s,t)\in[c,d]}|\tilde{M}_t-\tilde{M}_s|\geq x)\leq{}&\mathbb{P}(\sup_{(s,t)\in[c,d]^2}|(\int_s^t H_u d(N_u-\Lambda_u))^2-\int_s^t H_u^2 dN_u|\geq\frac{x}{2})\\&+\mathbb{P}(\sup_{s\in[0,d]}|M_s|\geq a)+\mathbb{P}(\sup_{(s,t)\in[c,d]^2}|M_t-M_s|\geq\frac{x}{4a}).\end{aligned}$$

Using (13)), (5), (12) and choosing $a=\sqrt{\frac{x}{8}\frac{\|H\|_{\infty,[c,d]}}{\|H\|_{\infty,d}}\frac{\|H\|_{2,d}}{\|H\|_{2,[c,d]}}}$ we get

$$\begin{aligned}\mathbb{P}(\sup_{(s,t)\in[c,d]}|\tilde{M}_t-\tilde{M}_s|\geq x)\leq{}&10\exp(-\frac{\|H\|_{2,[c,d]}^2}{\|H\|_{\infty,[c,d]}^2}I(\frac{\|H\|_{\infty,[c,d]}}{\|H\|_{2,[c,d]}^2}\sqrt{\frac{x}{16}}))\\&+2\exp(-\frac{\|H\|_{2,d}^2}{\|H\|_{\infty,d}^2}I(\frac{\sqrt{\|H\|_{\infty,[c,d]}\|H\|_{\infty,d}}}{\|H\|_{2,d}\|H\|_{2,[c,d]}}\sqrt{\frac{x}{8}}))\\&+2\exp(-\frac{\|H\|_{2,[c,d]}^2}{\|H\|_{\infty,[c,d]}^2}I(\frac{\sqrt{\|H\|_{\infty,[c,d]}\|H\|_{\infty,d}}}{\|H\|_{2,d}\|H\|_{2,[c,d]}}\sqrt{\frac{x}{8}}))\quad\blacksquare\end{aligned}$$

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