Syracuse_Collatz Conjecture: Comparison of two Markov approaches towards the proof

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ABSTRACT. We analyse two probabilistic methods for the proof of the conjecture and we provide a comparison of the proofs, using the Reliability Integral Theory and the SPQR Principle.

KEYWORDS: Quality Methods, Numerical Methods, Hailstone Conjecture, SPQR.

1. Introduction

A problem posed by L. Collatz in 1937 (also called the 3x+1 mapping), states that the system of the two difference equations, involving natural numbers,

$$y_{k+1} \quad \left\{ \begin{matrix} = \frac{1}{2} y_k \ \ IF \ \ y_k \ is \ EVEN \\ = 3y_k + 1 \ \ IF \ \ y_k \ is \ ODD \end{matrix} \right\} \qquad (1)$$

given the initial condition y_0 (any integer positive number) arrives after some (n is a number not known in advance) "continued" iterations to the value $y_n=1$.

Numerical experiments confirmed the validity of the conjecture for extraordinarily large values of the starting integer y_0: it always reached 1 for all numbers up to $5.48\ 10^{18}$. (Oliveira e Silva 2008)

The system (1) can be reduced to a non-linear difference equation, as the following one

$$4y_{k+1} - 7y_k - 2 + (-1)^{y_k}\,[5y_k + 2] = 0 \qquad (2)$$

The numbers y_{k+1} of the sequence provided by the previous (Collatz) equations are sometimes named *hailstone numbers*.

It is considered a very difficult problem to be solved, in spite of its very simple definition; they say that Erdős commented that "*mathematics is not yet ready for such problems*".

In this document we compare two approaches, the 1st based on the paper *"Quantifying the degree of average contraction of Collatz orbits."* [1], and the 2nd based on the document *"Proof of the Syracuse_Collatz Conjecture."* [2]; both of them use a Markov approach to make the proof "plausible".

We will use excerpts of them to make clear the difference.

Before that we provide a little of matrix algebra.

Let u(k) an infinite dimensional row vector, with all entries $u_i(k)=0$, but one entry $u_y(k)=1$: it is a unit vector of vector space. The vector u(k) refers to the k-th iteration of a mapping T: the result of the mapping T to the vector u(k) is denoted u(k+1)=u(k)T. The vector u(k+1) is unit vector with all entries $u_j(k+1)=0$, but one entry $u_{y^*}(k+1)=1$, where we have the subindexes $y^* \neq y$. The subindexes are according to (1): if $u_y(k)=1$, then $y=y_k$ and the index y* of entry $u_{y^*}(k+1)=1$ of the vector u(k+1) has index $y^*=y_k/2$ IF y_k is *even*, and $y^*=3y_k+1$ IF y_k is *odd*.

The mapping T is provided by an infinite-dimensional matrix **P**=[a_{ij}], named **transition matrix** (with infinite rows and columns); rows and columns are indexed by the natural numbers 1, 2, 3, 4, ..., n, n+1, ...; every a_{ij} entry is 0, except

$$a_{y_k, y_{k+1}} = 1 \quad \left\{ \begin{matrix} IF \ \ y_k \ is \ EVEN \\ IF \ \ y_k \ is \ ODD \end{matrix} \right\} \quad \ldots\ldots..(3)$$

where the indexes i and j are given by (1).

Accordingly we have

u(k+1)=u(k)P (4)

In the figure 1 we show the transition matrix; the 3 by 3 matrix with rows and columns indexed by the numbers 1, 2, 4, is highlighted due to its importance:

- when the system is in the state 1, the next transition is to state 4: **1 ⇒ 4**
- when the system is in the state 2, the next transition is to state 1: **2 ⇒ 1**
- when the system is in the state 4, the next transition is to state 2: **4 ⇒ 2**

All this means that when the system enters one of those 3 states [1, 2, 4] it never leaves out of them, the system (or the process) circulates in the set [1, 2, 4] forever. It is a "*periodic process*".

state	*1*	*2*	*4*	*3*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*	*20*	....
1	0	0	**1**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	....
2	**1**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	....
4	0	**1**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	....
3	0	0	0	0	0	0	0	0	0	**1**	0	0	0	0	0	0	0	0	0	0	....
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**1**	0	0	0	0	....
6	0	0	0	**1**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	....
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	....
8	0	0	**1**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	....
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	....
10	0	0	0	0	**1**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	....
11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	....
12	0	0	0	0	0	**1**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	....
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	....
14	0	0	0	0	0	0	**1**	0	0	0	0	0	0	0	0	0	0	0	0	0	....
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	....
16	0	0	0	0	0	0	0	**1**	0	0	0	0	0	0	0	0	0	0	0	0	....
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	....
18	0	0	0	0	0	0	0	0	**1**	0	0	0	0	0	0	0	0	0	0	0	....
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	....
20	0	0	0	0	0	0	0	0	0	**1**	0	0	0	0	0	0	0	0	0	0	....
...	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	....
...	0	0	0	0	0	0	0	0	0	0	**1**	0	0	0	0	0	0	0	0	0	....
...	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	....
...	0	0	0	0	0	0	0	0	0	0	0	**1**	0	0	0	0	0	0	0	0	....
...	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	....
...	0	0	0	0	0	0	0	0	0	0	0	0	**1**	0	0	0	0	0	0	0	....
...	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	....
...	0	0	0	0	0	0	0	0	0	0	0	0	0	**1**	0	0	0	0	0	0	....
...	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	....
...	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**1**	0	0	0	0	0	....
...	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	....
...	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**1**	0	0	0	0	....

Figure 1. The transition matrix P [only a part is shown]

The matrix P can be partitioned into 4 submatrices, written simply as

$$P = \begin{bmatrix} P_{11} & 0 \\ P_{21} & P_{22} \end{bmatrix}$$

where P11 and P22 are square matrices, given more explicitly by (notice that P11 refers to the states 1, 2, 4)

$$P = \begin{bmatrix} P_{11} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} & P_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots\cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots\cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots\cdots \end{bmatrix} \\ P_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & 1 \\ \cdots & \cdots & \cdots \end{bmatrix} & P_{22} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots\cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots\cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \end{bmatrix} \end{bmatrix}$$

It is important to notice that P^3, the 3rd power of the matrix P, is such that the submatrix

$$P_{11}{}^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (5)$$

is the identity matrix; when the system reaches the set $\{1, 2, 4\}$ of the states it remains there forever. It follows that $P_{11}{}^{-1} = P_{11}{}^2$.

2. The ideas in "Quantifying the degree of average contraction of Collatz orbits. *arXiv:1612.07820.v1*"

The paper is very interesting.
We use various excerpts from the paper. The authors say:

In this paper, we provide a novel argument to support the validity of the Collatz conjecture. To anticipate our findings, we shall demonstrate that the third iterate T°3 of the Collatz map admits three fixed points, 1, 2 and 4. These latter elements define the, supposedly unique, attracting cycle conjectured by Collatz. The third iterate map is naturally defined on the mod8 congruence classes of positive integers. We thus quantify the factor of relative compression or expansion, as follows the application of T°3, on each of the eight congruence classes obtained under the mod8 operation. In the second part of the paper, we show that orbits are on average bound to asymptotically shrink in size so heading towards the deputed equilibrium. We will further enhance the resolution of the measure by reducing to an arbitrary extent the degree of imposed coarse graining, i.e. working on the congruence classes mod8^m for any chosen m. Working in this generalised setting, we will prove that the average Collatz dynamics is contracting, for m large as sought, namely shrinking the congruence classes arbitrarily close to the singletons corresponding to each integer.
Markov processes based on congruence classes invariant under application of T have been previously considered in […] and revisited by Lagarias in his comprehensive survey on the (3x + 1) problem […]. It is however the combined usage of (i) the third iterate map T°3, (ii) the representation of numbers in mod8 , (iii) the idea of employing a Markov chain constructed from T°3 via a suitably defined measure, that allows us to draw a rigorous bound for the contraction factor, which is not just heuristically guessed.
To guide the reader through the text we shall hereafter provide a schematic outline of the main steps involved in the analysis, by making explicit reference to specific key results.

- *We will begin by defining the third iterate of the Collatz map hereby named S = T°3.*
- *We will determine the action of S on integers expressed in mod8. We will obtain class-dependent, B(i; 8) (i = 0; : : : ; 7), expansion/contraction factors that exemplify the action of S, see Eqs. (3). Working in this setting we will also show that 1; 2; 4 are the only fixed points of the deterministic map S. The subsequent analysis is targeted to showing that the trajectories of S are bound to converge on average to one of the above fixed points.*
- *To this end we first introduce a finite states Markov chain which runs on the eight congruence classes B(i; 8) (i = 0; : : : ; 7). The transition probabilities are given by equation (8) and have been obtained using the S-invariant measure μ_{inv} on the classes B(j; 8m), m ≥1 and j = 0; : : : ; 8^m -1.*
- *The measure μ_{inv} is defined by equation (12) (or equivalently equation (14)). The invariance of the measure under S is proved in Theorem 5.*
- *Since the transition probabilities are computed from the S-invariant measure, it is possible to draw conclusion on the iterates of S (namely its restriction in mod8) by iterating forward the Markov process. This observation*

follows from a straightforward application of the Chapman-Kolmogorov equation, as discussed in Proposition 9. The explicit form of the stochastic matrix Q that characterises the introduced Markov chain is given in Proposition 10. The stochastic chain does not account for the specificity of 1; 2; 4, the equilibria of S. It will hence allow us to elaborate on the out-of-equilibrium dynamics of S, prior (possible) convergence to the asymptotic Collatz equilibrium.*

- *The stationary distribution of the Markov chain is computed and given by formula (29). Recall that by iterating forward the Markov chain one can inspect the equilibrium dynamics of S, in its mod8 representation, see Proposition 9.*
- *By using the expansion/contraction factors associated to each of the classes B(i; 8) (i = 0; : : : ; 7) one can show that the deterministic trajectories are on average contracting. This is substantiated by formula (30).*
- *The analysis is generalised by working on the congruence classes mod8^m, for any m. By operating in this setting, we will proof that the average Collatz dynamics is contracting, for arbitrarily large m, i.e. shrinking the size of the congruence classes as sought. Remarkably, the estimated upper bound for the contraction factor is shown to be independent on m.*

Excerpt 1. From the paper "*Quantifying … of Collatz orbits. arXiv:1612.07820.v1*"

We give the information connecting excerpt 1 with the Introduction:
The mentioned mapping T is represented (in the Introduction) by the Matrix P.
The mentioned ″*third iterate of the Collatz map hereby named* $S = T^{\circ 3}$″ is represented (in the Introduction) by the Matrix P^3.
The infinite matrix P^3 is reduced to a matrix 8 by 8 Q*, using a congruence mod8 on the integer numbers. 8 equivalence classes are obtained B(i, 8) and indicated as B(0, 8), B(1, 8), B(2, 8), B(3, 8), B(4, 8), B(5, 8), B(6, 8), B(7, 8). The matrix Q* is stochastic as given in the excerpt 2 [rows and columns are indexed by the classes B(i, 8)]

$$Q^* = \begin{pmatrix} \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix} . \qquad (27)$$

Excerpt 2. From the paper "*Quantifying … of Collatz orbits. arXiv:1612.07820.v1*"

The authors show that there is a steady state probability vector, named P_{stat}, of being in the equivalence classes B(i, 8)

P_{stat}=(1/6; 1/12; 1/6; 1/12; 1/6; 1/12; 1/6; 1/12) (29)

Excerpt 3. From the paper "*Quantifying … of Collatz orbits. arXiv:1612.07820.v1*"

From P_{stat} it is clear that the process does arrive into the set {B(1, 8), B(2, 8), B(4, 8)} BUT it does not stay there "forever" after its arrival. The process does not end in the set {1, 2, 4}, that is, process does not end in the ″*three fixed points, 1, 2 and 4.*″ (as the authors say)! Therefore the authors are forced to prove that ″*that the deterministic trajectories are on average contracting.*″

Eventually they write in the Conclusion

In this paper we have provided an analytical argument to support the validity of the so called Collatz conjecture, a long standing problem in mathematics which dates back to 1937. The analysis builds on three main pillars. In short, we (i) introduced the (forward) third iterate of the Collatz map (so to reduce the analysis of the period 3 cycle to a search for a fixed point) and considered the equivalence classes of integer numbers modulo 8; (ii) defined a Markov chain (based on a suitable non trivial measure) which runs on a set of finite states and whose transition probabilities reflect the deterministic map; (iii) showed that orbits are on average contracting, as follows strict bound that combines the visiting frequencies, as derived in the framework of the aforementioned stochastic picture, and the *contraction/expansion factors associated to each transition among classes. Notice that the conclusion reached holds for any level of imposed coarse graining, i.e. by computing the visiting frequencies on the partition in mod8m classes, with m large as wished. Despite the measure introduced cannot be extended to weight individual singletons, we can proof that the Collatz dynamics is contracting on uniform partitions of the natural numbers in classes. These partitions can be refined to approximate singletons with suited accuracy, without eventually converging to them.*

Excerpt 4. From the paper "*Quantifying … of Collatz orbits. arXiv:1612.07820.v1*"

3. The ideas in *"Proof of the Syracuse_Collatz Conjecture. 2019. <hal-02048821> and Academia.Edu"*

Some of them have been given in the Introduction, in order to make comparable the ideas of T. Carletti, D. Fanelli [1] with the ones of Galetto, F. [2].

Please see them there: the transition stochastic matrix P of figure 1, partitioned into 4 submatrices, $P = \begin{bmatrix} P_{11} & 0 \\ P_{21} & P_{22} \end{bmatrix}$ where P11 and P22 are square matrices (noticing that P11 refers to the set {1, 2, 4} of the states), given more explicitly by the following matrix in the introduction.

The process is a "*periodic*" with period 3: when the system enters one of the 3 states [1, 2, 4] it never leaves the set {1, 2, 4}, the system (or the process) circulates in the set {1, 2, 4} forever. P^3, the 3rd power of the matrix P, is such that the submatrix ${P_{11}}^3$ [see formula (5)] is the identity matrix; when the system reaches the set {1, 2, 4} of the states it remains there forever because the rectangular submatrix P_{12} in the upper right corner has only 0 entries.

The process is bound to enter the set S_1={1, 2, 4} because the rectangular submatrix P_{21} in the lower left corner has only one 1 entry [the other entries are all 0]. The "*periodic process*" circulating in the set S_1={1, 2, 4} is ruled by the submatrix P_{11}. The infinite set S_2={3, 5, 6, 7, ….} comprises all the other states. The transitions are given in figure 2.

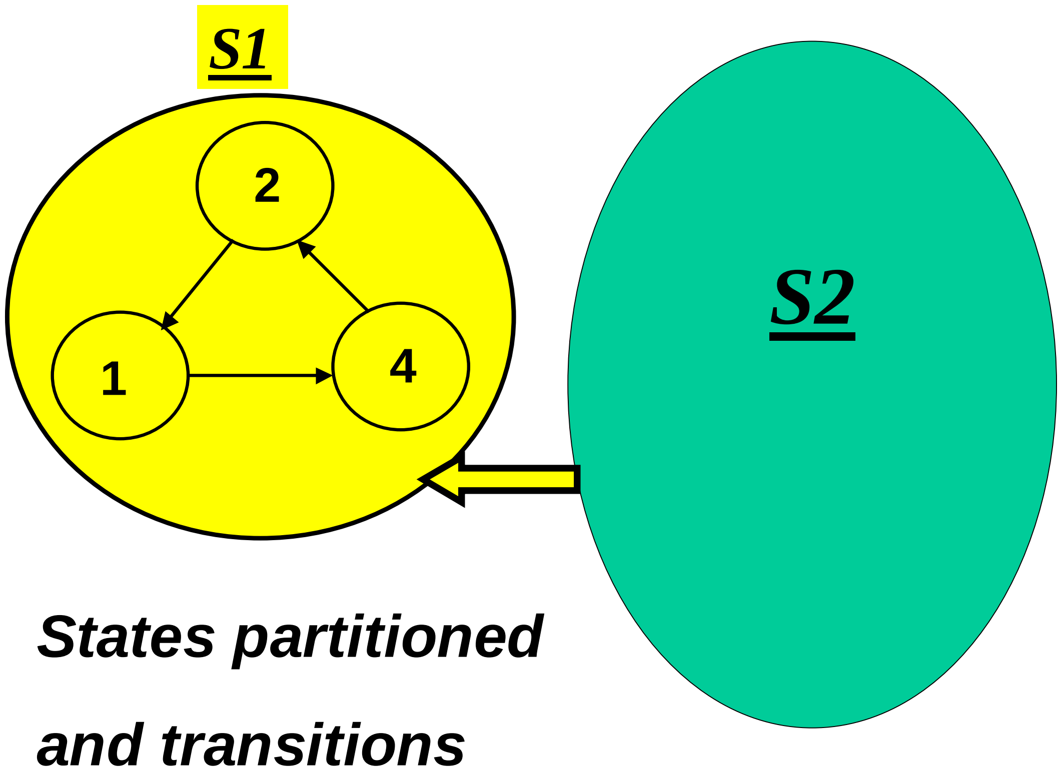

Figure 2. The two disjoint sets S_1 and S_2

By using the theory given in the books [3, 4] related to *Reliability Integral Theory [RIT]* one can find two vectors $z\alpha_1$ and $z\alpha_2$ defined, as follows,

- $z\alpha_1$ is the vector of the (steady state) probabilities of entering into one of the states {1, 2, 4} [$\in S_1$], when there is a transition S_2={3, 5, 6, ..., n, n+1, ...} $\Rightarrow$ S_1={1, 2, 4}.
- $z\alpha_2$ is the vector of the (steady state) probabilities of entering into one of the states {3, 5, 6,

..., n, n+1, ...} [$\in S_2$], when there is a transition from $S_1=\{1, 2, 4\} \Rightarrow S_2=\{3, 5, 6, ..., n, n+1, ...\}$.

$z\alpha_1$ is by definition a three-dimensional row vector [0, 0, 1] related to the set S_1; see the figure 2. The system enters into the set S_1 only through the state 4.

4. Comparison of the findings by the ideas …

We rearrange the states [8 equivalence classes are obtained B(i, 8)] of the finite process described by the previous stochastic matrix Q* (given in the excerpt 2) as follows

Matrix Q_G State->state	B(1, 8)	B(2, 8)	B(4, 8)	B(5, 8)	B(6, 8)	B(0, 8)	B(3, 8)	B(7, 8)
B(1, 8)	0.25	0.00	0.00	0.25	0.00	0.00	0.25	0.25
B(2. 8)	0.00	0.25	0.25	0.00	0.25	0.25	0.00	0.00
B(4. 8)	0.00	0.25	0.25	0.00	0.25	0.25	0.00	0.00
B(5. 8)	0.00	0.25	0.25	0.00	0.25	0.25	0.00	0.00
B(6. 8)	0.25	0.00	0.00	0.25	0.00	0.00	0.25	0.25
B(0. 8)	0.125	0.125	0.125	0.125	0.125	0.125	0.125	0.125
B(3. 8)	0.00	0.00	0.50	0.00	0.00	0.50	0.00	0.00
B(7. 8)	0.00	0.50	0.00	0.00	0.50	0.00	0.00	0.00

where the states B(1, 8), B(2, 8), B(4, 8) are in the left top corner.
The new Pstat is consequently

State	*B(1, 8)*	*B(2, 8)*	*B(4, 8)*	*B(5, 8)*	*B(6, 8)*	*B(0, 8)*	*B(3, 8)*	*B(7, 8)*
$P_{stat\_}Q_G$	*0.0833*	*0.1667*	*0.1667*	*0.0833*	*0.1667*	*0.1667*	*0.0833*	*0.0833*

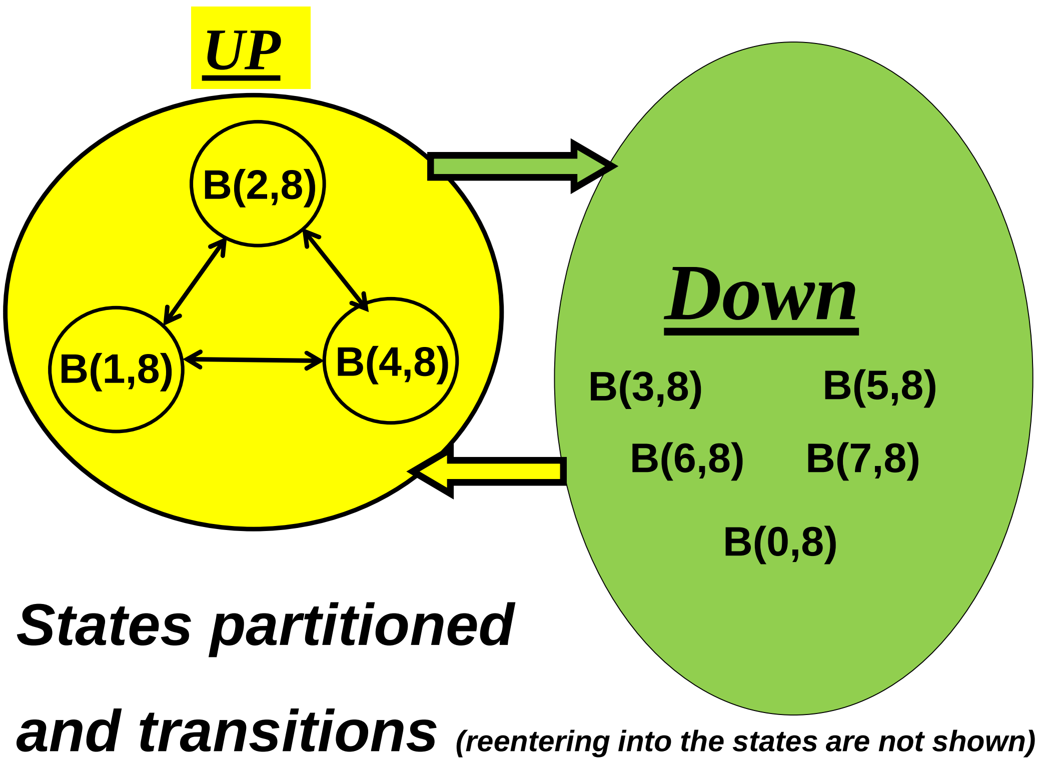

Figure 3. The two disjoint sets UP and Down (re-entering into the states are not shown)

The MTTF (Mean Time To Forward, from each UP state to the Down set) is a 3-dimensional column vector

1.333
2.000
2.000

while the MTTR (Mean Time To Return, from each Down state to the UP set) is a 5-dimensional column vector

2.316
2.737
2.526
2.263
2.368

MTTF and MTTR provide the mean number of transitions from each state of a set tot the disjoint set.

By using the theory given in the books [3, 4] related to ***Reliability Integral Theory [RIT]*** one can find two vectors $z\alpha_1$ and $z\alpha_2$ defined, as follows,

- $z\alpha_1$ is the vector of the (steady state) probabilities of entering into one of the UP, when there is a transition Down ⇒ UP.
- $z\alpha_2$ is the vector of the (steady state) probabilities of entering into one of the Down states, when there is a transition from UP ⇒ Down.

$z\alpha_1$ is a three-dimensional row vector related to the UP set (see the figure 3):

$z\alpha_1$	*0.27273*	*0.36364*	*0.36364*

so that the MUT (Mean Up Time) is MUT=1.818182

$z\alpha_2$ is a five-dimensional row vector related to the Down set (see the figure 3):

$z\alpha_2$	*0.09091*	*0.36364*	*0.36364*	*0.09091*	*0.09091*

so that the MDT (Mean Down Time) is MDT=2.545455

The MCT (Mean Cycle Time) is MCT=4.363636

The probability that the process is in the set UP={B(1, 8), B(2, 8), B(4, 8)} is A_{SS}=0.42=MUT/MCT (as it must be).

Each state has its own recurrence time [with the fraction of time spent in each state (one of the 8 equivalent classes)]:

state	*B(1, 8)*	*B(2, 8)*	*B(4, 8)*	*B(5, 8)*	*B(6, 8)*	*B(0, 8)*	*B(3, 8)*	*B(7, 8)*
recurrence time	*12*	*6*	*6*	*12*	*6*	*6*	*12*	*12*
Time spent in state	*0.0833*	*0.1667*	*0.1667*	*0.0833*	*0.1667*	*0.1667*	*0.0833*	*0.0833*

So we see that to "prove" the Collatz Conjecture one should prove that $A_{SS\{1,2,4\}}$=1 and $MUT_{\{1,2,4\}}=\infty$, from the above ideas ...

The two authors (ideas of T. Carletti, D. Fanelli [1]) write

prove the average contracting property of the deterministic map S. Roughly speaking when the map S is acted on a positive integer belonging to class $\mathcal{B}(0,8)$, it drives a contraction factor equal to $1/8$. If S is instead operated on natural numbers of the classes $\mathcal{B}(j,8)$, $j=1,2,4,5,6$, it results in a contraction of $3/4$. At variance, $S(n)$ produces an expansion with rate $9/2$, for n belonging to classes $\mathcal{B}(3,8)$ and $\mathcal{B}(7,8)$. Hence the obtained information on $\vec{P}_{stat}$ allows to estimate the degree of contraction (or expansion) f_{Q^*} that trajectories should, on average, produce:

$$f_{Q^*} \simeq \left(\frac{1}{8}\right)^{\omega_0} \left(\frac{3}{4}\right)^{\omega_1} \left(\frac{9}{2}\right)^{\omega_2} \qquad (30)$$

where $\omega_0 = 1/6$ is the probability of being in the congruence class $j=0$, $\omega_1 = 3/6 + 2/12 = 2/3$ the probability of being in the congruence classes $j = 1,2,4,5,6$ and $\omega_2 = 2/12 = 1/6$ the probability of being in the congruence classes $j=3,7$. Probabilities ω_0, ω_1, ω_2 follow equation (29), while the contraction/expansion factors $\frac{1}{8}$, $\frac{3}{4}$ and $\frac{9}{2}$ associated to each of congruence class are made explicit in Eqs. (3).

Carrying out the calculation yields $f_{Q^*} = 3/4 < 1$, thus implying in turn that the average approach to the absorbing equilibrium is contracting. Observe that this latter contracting factor is here analytically determined, at variance with previous attempts that relied on heuristic reasoning. Notice that this preliminary estimate $f_{Q^*} = 3/4$ has been obtained by just retaining the terms proportional to n in the definition of $S(n)$, see Eqs. (3), or, equivalently, working with a sufficiently large n. Accounting for the constant (n independent) contributions in (3) does not modify the conclusion that we have reached: the generic orbit is always contracting, as it is proved hereafter.

Excerpt 5. From the paper "***Quantifying … of Collatz orbits. arXiv:1612.07820.v1***"

$$\forall n \in \mathbb{N} \quad S(n) = \begin{cases} \frac{n}{8} & \text{if } n \in \mathcal{B}(0,8) \\ \frac{6n+2}{8} & \text{if } n \in \mathcal{B}(1,8) \\ \frac{6n+4}{8} & \text{if } n \in \mathcal{B}(2,8) \\ \frac{36n+20}{8} & \text{if } n \in \mathcal{B}(3,8) \\ \frac{6n+8}{8} & \text{if } n \in \mathcal{B}(4,8) \\ \frac{6n+2}{8} & \text{if } n \in \mathcal{B}(5,8) \\ \frac{6n+4}{8} & \text{if } n \in \mathcal{B}(6,8) \\ \frac{36n+20}{8} & \text{if } n \in \mathcal{B}(7,8) \ . \end{cases} \quad (3)$$

Excerpt 6. From the paper "*Quantifying … of Collatz orbits. arXiv:1612.07820.v1*"

The two authors (T. Carletti, D. Fanelli [1]) then write

> *Carrying out the calculation yields $f_{Q*}=3/4<1$, thus implying in turn that the average approach to the absorbing equilibrium is contracting. … Notice that this preliminary estimate $f_{Q*}=3/4$ has been obtained by just retaining the terms proportional to n in the definition of S(n), see Eqs. (3), or, equivalently, working with a sufficiently large n. Accounting for the constant (n independent) contributions in (3) does not modify the conclusion that we have reached: the generic orbit is always contracting, as it is proved hereafter. Consider in fact Eqs. (3) which define the map S on the classes B(i; 8). … Since {1; 2; 4} belong to the Collatz cycle, and because we are solely focusing on the dynamics that precedes the possible convergence to the Collatz cycle… Performing the calculation yields $f_{Q*}\leq 0.8926$. The dynamics of S is therefore contracting and trajectories are on average attracted towards the three fixed points as identified above, namely the entries of the Collatz cycle {1; 2; 4}.*

Excerpt 7. From the paper "*Quantifying … of Collatz orbits. arXiv:1612.07820.v1*"

The two authors (T. Carletti, D. Fanelli [1]) then consider a number 8^m of equivalent classes B(i, 8^m), generated by the congruence mod8^m. By increasing m, one generates more and more classes [process states], always with the Collatz cycle {1; 2; 4}.
The two authors prove that there is a stochastic matrix Q(m) [a 8^m by 8^m matrix] which provides a stationary row vector $P_{stat}(m)$=[a, b, a, b, …, a, b] solution of the equation $P_{stat}(m)=P_{stat}(m)Q(m)$, where a=0.1667/$8^{m-1}$ and b=0.0833/8^{m-1}. Each state has its own recurrence time [with the fraction of time spent in each state (one of the 8^m equivalent classes)]:

state	***B(1, 8^m)***	***B(2, 8^m)***	***B(4, 8^m)***	***B(5, 8^m)***	***B(6, 8^m)***
recurrence time	*12**8^{m-1}	*6**8^{m-1}	*6**8^{m-1}	*12**8^{m-1}	*6**8^{m-1}
Time spent in state	*0.0833/*8^{m-1}	*0.1667/*8^{m-1}	*0.1667/*8^{m-1}	*0.0833/*8^{m-1}	*0.1667/*8^{m-1}
state	***B(0, 8^m)***	***B(3, 8^m)***	***B(7, 8^m)***	***……***	***B(8^m-1, 8^m)***
recurrence time	*6**8^{m-1}	*12**8^{m-1}	*12**8^{m-1}	…	……
Time spent in state	*0.1667/*8^{m-1}	*0.0833/*8^{m-1}	*0.0833/*8^{m-1}	…	……

Again we see that to "prove" the Collatz Conjecture one should prove that $A_{SS\{1,2,4\}}=1$ and $MUT_{\{1,2,4\}}=\infty$, from the above ideas ...
In F. Galetto opinion, the Collatz Conjecture is not made plausible by T. Carletti and D. Fanelli [1], because as $m\rightarrow\infty$ the probability that the process is in the Collatz cycle {1; 2; 4} tends to 0 and not to 1… Perhaps F. Galetto did not understand…
Then the following *remarkable conclusion* … is in doubt …

> *The remarkable conclusion is therefore that the third iterate of the Collatz map is always contracting, when seen on the equivalence classes B(i; 8^m), for m large as sought, and that the estimated bound for the contraction factor is independent on the classes index m. In other words, we can make the number of classes as large as wished (and consequently reduce their size so to approach the singletons with arbitrary accuracy), while still detecting a contracting deterministic dynamics, with a constant (independent on m.) bound for the rate of contraction. As previously remarked when the limit for m that goes to infinity is performed, the measure of the classes, and hence the singletons, converges to zero. Despite the fact the contracting factors stays constant for any, arbitrarily large m, it seems that we cannot rule out the existence of zero measure*

orbits that violate this constraint.

Excerpt 8. From the paper "*Quantifying … of Collatz orbits. arXiv:1612.07820.v1*"

Now let's go the F. Galetto method [2], as sketched in the section 3. See the figure 2, as well.
The reader is asked to see the transition stochastic matrix P of figure 1, partitioned into 4 submatrices, $P = \begin{bmatrix} P_{11} & 0 \\ P_{21} & P_{22} \end{bmatrix}$ where P11 refers to the set $\{1, 2, 4\}$ of the states (named Collatz cycle by T. Carletti, D. Fanelli [1]). The process is a "*periodic*" with period 3: when the system enters one of the 3 states of set $\{1, 2, 4\}$ it never leaves the set, the system (or the process) circulates in the set $\{1, 2, 4\}$ forever. P^3, the 3rd power of the matrix P, is such that the submatrix ${P_{11}}^3$ [see formula (5)] is the identity matrix; when the system reaches the set $\{1, 2, 4\}$ of the states it remains there forever because the rectangular submatrix P_{12} in the upper right corner has only 0 entries. The process is bound to enter the set $S_1=\{1, 2, 4\}$ because the rectangular submatrix P_{21} in the lower left corner has only one 1 entry [the other entries are all 0]. The "*periodic process*" circulating in the set $S_1=\{1, 2, 4\}$ is ruled by the submatrix P_{11}. The infinite set $S_2=\{3, 5, 6, 7, ….\}$ comprises all the other states. The transitions are given in figure 2.
After entering S_1 the probability of being in the states $\{1, 2, 4\}$ is given by a vector π=[1/3, 1/3, 1/3] solution of the relationship $\boldsymbol{\pi=\pi P_{11}}$, which means that the mean recurrence time in each state is 3.
$z\alpha_2$ is by definition an infinite dimensional row ***null*** vector [0, 0, 0, ..., 0, ...] related to the set S_2; see the figure 2. The system never enters into the set S_2 from S_1.
This means that

- IF a person choose a number $y_0 \in S_2=\{3, 5, 6, ..., n, n+1, ...\}$
- and applies the rules $y_{k+1} \begin{cases} = \frac{1}{2} y_k \;\; IF \;\; y_k \; is \; EVEN \\ = 3y_k + 1 \;\; IF \;\; y_k \; is \; ODD \end{cases}$
- then the Markov system enters the state $y_0 \in S_2$ for the first time,
- it makes all the needed transitions within the set S_2 (figure 2)
- until it goes into the set $S_1=\{1, 2, 4\}$ (figure 2)
- *and it remains there, with probability vector π: periodic process*

5. Conclusion

Having applied the SPQR («Semper Paratus ad Qualitatem et Rationem») Principle, the author thinks that his probabilistic method is able to show the (probabilistic) proof of the Syracuse_Collatz Conjecture.
On the contrary, the Collatz Conjecture (in spite of an interesting paper) is not made plausible by T. Carletti and D. Fanelli [1], because as $m \to \infty$ the probability that the process is in the Collatz cycle $\{1; 2; 4\}$ tends to 0 and not to 1… Therefore the mean recurrence time in each state is infinite. As a matter of fact to "prove" the Collatz Conjecture one should prove that $A_{SS\{1,2,4\}}=1$ and $MUT_{\{1,2,4\}}=\infty$ ...

References

[1] T. Carletti, D. Fanelli, 2016, Quantifying the degree of average contraction of Collatz orbits. *arXiv:1612.07820.v1*

[2] Galetto, F., 2019, Proof of the Syracuse_Collatz Conjecture. 2019. *<hal-02048821>* and *Academia.Edu*

[3] Galetto, F., 2016, *Reliability and Maintenance*, *Scientific Methods, Practical Approach*, Vol-1, www.morebooks.de.

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[5] Galetto, F., 2017, The SPQR («Semper Paratus ad Qualitatem et Rationem») Principle in Action. *Engineering and Applied Sciences*.