Relaxation of the Cahn-Hilliard equation with singular single-well potential and degenerate mobility

Benoît Perthame, Alexandre Poulain

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Abstract

The degenerate Cahn-Hilliard equation is a standard model to describe living tissues. It takes into account cell populations undergoing short-range attraction and long-range repulsion effects. In this framework, we consider the usual Cahn-Hilliard equation with a singular single-well potential and degenerate mobility. These degeneracy and singularity induce numerous difficulties, in particular for its numerical simulation. To overcome these issues, we propose a relaxation system formed of two second order equations which can be solved with standard packages. This system is endowed with an energy and an entropy structure compatible with the limiting equation. Here, we study the theoretical properties of this system; global existence and convergence of the relaxed system to the degenerate Cahn-Hilliard equation. We also study the long-time asymptotics which interest relies on the numerous possible steady states with given mass.

1   Introduction

The Degenerate Cahn-Hilliard equation (DCH in short) is a standard model, widely used in the mechanics of living tissues, [6, 31, 13, 2, 4, 19]. It is usual to set this problem in a smooth bounded domain $\Omega \subset \mathbb{R}^d$ with the zero flux boundary condition

$$\partial_t n = \nabla \cdot \big(b(n) \nabla \left(-\gamma \Delta n + \psi'(n) \right) \big) \quad \text{in} \quad \Omega \times (0, +\infty), \tag{1}$$

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$$\frac{\partial n}{\partial \nu} = b(n)\frac{\partial\left(-\gamma\Delta n + \psi'(n)\right)}{\partial \nu} = 0 \qquad \text{on} \quad \partial\Omega \times (0, +\infty), \tag{2}$$

where ν is the outward normal vector to the boundary $\partial\Omega$ and $n = \frac{n_1}{n_1 + n_2}$ represents the relative density or volume fraction of one of the two cell types.

Degeneracy of the coefficient $b(n)$ and singularity of the potential $\psi(n)$ make this problem particularly difficult to solve numerically and in particular, to preserve the apriori bound $0 \leq n < 1$. Motivated by the use of standard software for elliptic or parabolic equations, we propose to study the following relaxed degenerate Cahn-Hilliard equation (RDHC in short)

$$\begin{cases} \partial_t n = \nabla \cdot \left(b(n)\nabla\left(\varphi + \psi'_+(n)\right)\right) & \text{in } \Omega \times (0, +\infty), \\ -\sigma\Delta\varphi + \varphi = -\gamma\Delta n + \psi'_-\left(n - \frac{\sigma}{\gamma}\varphi\right) & \text{in } \Omega \times (0, +\infty). \end{cases} \tag{3}$$

supplemented with zero-flux boundary conditions

$$\frac{\partial(\gamma n - \sigma\varphi)}{\partial \nu} = b(n)\frac{\partial\left(\varphi + \psi'_+(n)\right)}{\partial \nu} = 0 \qquad \text{on} \quad \partial\Omega \times (0, +\infty). \tag{4}$$

Our purpose is to study existence for this system, to prove that as $\sigma \to 0$, the solution of RDCH system converges to the solution of the DCH equation and study the possible long term limits to steady states.

We make the following assumptions for the different inputs of the system (3). For the mechanics of living tissues, the usual assumption is that the potential ψ is concave degenerate near $n = 0$ (short range attraction) and convex for n not too small (long range repulsion). Additionally, a singularity at $n = 1$ is desired to represent saturation by one phase [8]. For these reasons, we call the potential *single-well logarithmic* and we decompose it in a convex and a concave part $\psi_\pm$

$$\psi(n) = \psi_+(n) + \psi_-(n), \qquad \pm\psi''_\pm(n) \geq 0, \qquad 0 \leq n < 1. \tag{5}$$

The singularity is contained in the convex part of the potential and we assume that

$$\psi_+ \in C^2\big([0, 1)\big), \quad \psi'_+(1) = \infty, \tag{6}$$

and we extend the smooth concave part on $[0, 1]$ to the full line with

$$\psi_- \in C^2(\mathbb{R}) \qquad \psi_-, \ \psi'_-, \ \psi''_- \quad \text{are bounded.} \tag{7}$$

In practice, typical examples of potentials are, for some $n^* \in (0, 1)$, see [14, 11]

$$\psi(n) = -(1 - n^*)\log(1 - n) - \frac{n^3}{3} - (1 - n^*)\frac{n^2}{2} - (1 - n^*)n + k, \tag{8}$$

$$\psi(n) = \frac{1}{2}n\ln n + (1 - n)\ln(1 - n) - (n - \frac{1}{2})^2. \tag{9}$$

The potential (8) fulfills our assumptions and the convex/concave decomposition reads for $n \in [0, 1)$

$$\psi_+(n) = -(1 - n^*)\log(1 - n) - \frac{n^3}{3}, \qquad \psi_-(n) = -(1 - n^*)\frac{n^2}{2} - (1 - n^*)n + k.$$

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Potential (9) does not satisfy our assumptions because of the additional singularity at 0 (and thus is not treated here), however, it can also be decomposed as needed with

$$\psi_+(n) = \frac{1}{2}n\ln n + (1-n)\ln(1-n), \qquad \psi_-(n) = -(n-\frac{1}{2})^2.$$

The potential (8) has been used to model the interaction between cancer cells from a glioblastoma multiforme and healthy cells by Agosti *et al.* [3] and promising results have been obtained. We also use the degeneracy assumption on $b \in C^1([0,1]; \mathbb{R}^+)$,

$$b(0) = b(1) = 0, \qquad b(n) > 0 \text{ for } 0 < n < 1. \tag{10}$$

The typical expression in the applications we have in mind is $b(n) = n(1-n)^2$. Consequently, when considered as transport equations, both (1) and (3) impose formally the property that $0 \le n \le 1$. However, we need an additional technical assumption, namely that there is some cancellation at 1 and one can define

$$b(\cdot)\psi''(\cdot) \in C([0,1]; \mathbb{R}). \tag{11}$$

We implicitly assume (5)–(11) in this paper. Also, we always impose an initial condition satisfying

$$n^0 \in H^1(\Omega), \qquad 0 \le n^0 < 1 \quad \text{a.e. in } \Omega. \tag{12}$$

The assumption $n^0 \in [0,1)$ is consistent with the degeneracy of mobility at 0 which allows solutions to vanish on open sets. But the singularity of the potential at 1 and the energy bound make that $n = 1$ cannot be achieved except of a negligible set. Thanks to the boundary condition (2), the system conserves the initial mass

$$\int_\Omega n(x,t)dx = \int_\Omega n^0(x)dx =: M, \quad \forall t \ge 0.$$

We denote the flux associated with the RDCH system by

$$J_\sigma(n,\varphi) := b(n)\nabla\left(\varphi + \psi'_+(n)\right). \tag{13}$$

The system (3) comes with energy and entropy structures, namely, the energy is defined as

$$\mathcal{E}_\sigma[n_\sigma] = \int_\Omega \left[\psi_+(n_\sigma) + \frac{\gamma}{2}|\nabla(n_\sigma - \frac{\sigma}{\gamma}\varphi_\sigma)|^2 + \frac{\sigma}{2\gamma}|\varphi_\sigma|^2 + \psi_-(n_\sigma - \frac{\sigma}{\gamma}\varphi_\sigma)\right]. \tag{14}$$

The energy is bounded from below thanks to the assumptions above and satisfies

$$\frac{d}{dt}\mathcal{E}_\sigma[n_\sigma(t)] = -\int_\Omega b(n_\sigma)\big|\nabla(\varphi_\sigma + \psi'_+(n_\sigma))\big|^2 \le 0. \tag{15}$$

For the entropy, we set for $0 < n < 1$ the singular function

$$\phi''(n) = \frac{1}{b(n)}, \qquad \Phi[n] = \int_\Omega \phi\big(n(x)\big)dx. \tag{16}$$

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The entropy functional behaves as follows in the case $b(n) = n(1-n)^2$

$$\phi(n) = n\log(n), \ n \approx 0^+, \qquad \phi(n) = -\log(1-n), \ n \approx 1^-.$$

The relation holds

$$\frac{d\Phi[n_\sigma(t)]}{dt} = -\int_\Omega \gamma \left| \Delta \left(n_\sigma - \frac{\sigma}{\gamma}\varphi_\sigma \right) \right|^2 + \frac{\sigma}{\gamma}|\nabla\varphi_\sigma|^2 + \psi''_-(n_\sigma - \frac{\sigma}{\gamma}\varphi_\sigma) \left| \nabla(n_\sigma - \frac{\sigma}{\gamma}\varphi_\sigma) \right|^2 \tag{17}$$
$$+ \psi''_+(n_\sigma)|\nabla n_\sigma|^2.$$

Notice that entropy equality does not provide us with a direct a priori estimate because of the term ψ''_- can be negative. Therefore we have to combine it with the energy dissipation to write

$$\Phi[n_\sigma(T)] + \int_{\Omega_T} \left[\gamma \left| \Delta \left(n_\sigma - \frac{\sigma}{\gamma}\varphi_\sigma \right) \right|^2 + \frac{\sigma}{\gamma}|\nabla\varphi_\sigma|^2 + \psi''_+(n_\sigma)|\nabla n_\sigma|^2 \right]$$
$$\leq \Phi[n^0] + \frac{2T}{\gamma}\|\psi''_-\|_\infty \, \mathcal{E}_\sigma[n^0].$$

The first use of the Cahn-Hilliard equation is to model the spinodal decomposition occurring in binary materials during a sudden cooling [10, 9]. The bilaplacian $-\gamma\Delta^2 n$ is used to represent surface tension and the parameter γ is the square of the width of the diffuse interface between the two phases. In both equations (1) and (3), $n = n(x,t)$ is a relative quantity: for our biological application this represents a relative cell density as derived from phase field models [8] and for this reason the property $n \in [0,1)$ is relevant. The biological explanation of the fact that 1 is excluded from the interval of definition of n is due to the observation that cells tend to not form aggregates which are too dense. For instance, the two phases can be the relative density of cancer cells and the other component represents the extracellular matrix, liquid and other cells. This binary mixture tends to form aggregates in which the density of one component of the binary mixture is larger than the other component. The interest of the Cahn-Hilliard equation stems from solutions that reproduce the formation of such clusters of cells *in vivo* or on dishes. Several variants are also used. A Cahn-Hilliard-Hele-Shaw model is proposed by Lowengrub *et al* [27] to describe the avascular, vascular and metastatic stages of solid tumor growth. They proved the existence and uniqueness of a strong solution globally for $d \leq 2$ and locally for $d = 3$ as well as the long term convergence to steady state. The case with a singular potential is treated in [24]. Variants can include the coupling with fluid equations and chemotaxis, see for instance [16] and the references therein.

The analysis of the long-time behavior of the solution of the Cahn-Hilliard equation has also attracted much attention since the seminal paper [7]. A precise description of the ω-limit set has been obtained in one dimension for the case of smooth polynomial potential and constant mobility in [30]. In this work, the effect of the different parameters of the model such as the initial mass, the width of the diffuse interface are investigated. In fact, the authors show that when γ is large, the solution converges to a constant as $t \to \infty$. The same happens when the initial mass is large. However when γ is positive and small enough, the system admits nontrivial steady-states. For logarithmic potentials and constant mobility, Abels and Wilke [1] prove that solutions converge to a steady state as time goes

to infinity using the Lojasiewicz–Simon inequality. Other works have been made on the long term behavior of the solutions of some Cahn-Hilliard models including a source term [12], with dynamic boundary conditions [23], coupled with the Navier-Stokes equation [20], for non-local interactions and a reaction term [25].

Many difficulties, both analytical and numerical, arise in the context of Cahn-Hilliard equation and its variants. Because of the bilaplacian term, most of the numerical methods require to change the equation (1) into a system of two coupled equations

$$
\begin{cases}
\partial_t n = \nabla \cdot (b(n)\nabla v) \,, \\
\quad v = -\gamma \Delta n + \psi'(n).
\end{cases}
\tag{18}
$$

This system of equations has been analyzed in the case where the mobility is degenerate and the potential is a logarithmic double-well functional by Elliott and Garcke [17]. They establish the existence of weak solutions of this system. Agosti *et al* [2] establish the existence of weak solutions when ψ is a single-well logarithmic potential which is more relevant for biological applications (see [8]). They also prove that this system preserves the positivity of the cell density and the weak solutions belong to

$$
n \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap H^1(0,T;(H^1(\Omega))'), \quad J \in L^2((0,T)\times\Omega,\mathbb{R}^d) \quad \forall T > 0,
$$

The Cahn-Hilliard equation can be seen as an approximation of the famous microscopic model in [21, 22]. With our notations, it reads

$$
\partial_t n = \nabla \cdot \left[b(n)\nabla\big(K_\sigma \star n + \psi'(n)\big) \right],
$$

with a symmetric smooth kernel $K_\sigma \xrightarrow[\sigma\to0]{} \Delta\delta$. The convergence to the DCH equation has been answered recently in [15] in the case of periodic boundary conditions. Although, very similar in its form, our relaxation model undergoes different a priori estimates which allow us to study differently the limit $\sigma \to 0$ for (3).

For a full review about the mathematical analysis of the Cahn-Hilliard equation and its variants, we refer the reader to the recent book of Miranville [28].

Numerical simulations of the DCH system have been also performed in the context of double-well potentials in [18, 5]. To keep the energy inequality is a major concern in numerical methods and the survey paper by Shen *et al* [29] presents a general method applied to the present context.

Numerics is also our motivation to propose a relaxation of equation (1) in a form close to the writing (18). We recover the system (3) by introducing a new potential φ and a regularizing equation which defines v through $\nabla\varphi$. We use the decomposition (5) of the potential to keep the convex and stable part in the main equation for n, rejecting the concave and unstable part in the regularized equation. The relaxation parameter is σ and we need to verify that, in the limit $\sigma \to 0$, we recover the original DCH equation (1). This is the main purpose of the present paper.

As a first step towards the existence of solutions of (3), in section 2, we introduce a regularized problem which is not anymore degenerate and we prove the existence of weak solutions for this regularized-relaxed Cahn-Hilliard system. We show energy and entropy estimates from which we obtain a priori estimates which are used later on. In section 3, we pass to the limit in the regularization parameter ϵ

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and show the existence of weak solutions of the RDCH system. Then, in section 4, we prove the convergence as $\sigma \to 0$ to the full DCH model. Section 5 is dedicated to the study of the long term convergence of the solutions to steady-states. We end the paper with some conclusions and perspectives.

2 The regularized problem

In order to prove that the system (3), admits solutions and to precise the functional spaces, we first define a regularized problem. Then we prove existence of solutions and estimates based on energy and entropy relations.

2.1 Regularization procedure

We consider a small positive parameter $0 < \epsilon \ll 1$ and define the regularized mobility

$$
B_\epsilon(n) = \begin{cases} b(1-\epsilon) & \text{for } n \geq 1 - \epsilon, \\ b(\epsilon) & \text{for } n \leq \epsilon, \\ b(n) & \text{otherwise.} \end{cases} \tag{19}
$$

Then, there are two positive constants b_1 and B_1, such that

$$
b_1 < B_\epsilon(n) < B_1, \quad \forall n \in \mathbb{R}. \tag{20}
$$

Thus, the regularized mobility satisfies

$$
B_\epsilon \in C(\mathbb{R}, \mathbb{R}^+). \tag{21}
$$

To define a regular potential, we smooth out the singularity located at $n = 1$ which only occurs in ψ_+, see (6)–(7), and preserve the assumption (11) by setting

$$
\psi''_{+,\epsilon}(n) = \begin{cases} \psi''_+(1-\epsilon) & \text{for } n \geq 1 - \epsilon, \\ \psi''_+(\epsilon) & \text{for } n \leq \epsilon, \\ \psi''_+(n) & \text{otherwise.} \end{cases} \tag{22}
$$

It is useful to notice that, for some positive constants D_1 independent of $0 < \epsilon \leq \epsilon_0$ and D_ϵ, we have

$$
\psi_{+,\epsilon}(n) \in C^2(\mathbb{R}, \mathbb{R}) \quad \psi_{+,\epsilon}(n) \geq -D_1, \quad |\psi'_\epsilon(n)| \leq D_\epsilon(1 + |n|), \quad \forall n \in \mathbb{R}. \tag{23}
$$

See also [2] for details about the extensions needed for the potential (8).

We can now define the regularized problem

$$
\begin{cases} \partial_t n_{\sigma,\epsilon} = \nabla \cdot \left[B_\epsilon(n_{\sigma,\epsilon}) \nabla(\varphi_{\sigma,\epsilon} + \psi'_{+,\epsilon}(n_{\sigma,\epsilon})) \right], \\ -\sigma \Delta \varphi_{\sigma,\epsilon} + \varphi_{\sigma,\epsilon} = -\gamma \Delta n_{\sigma,\epsilon} + \psi'_-(n_{\sigma,\epsilon} - \dfrac{\sigma}{\gamma}\varphi_{\sigma,\epsilon}), \end{cases} \tag{24}
$$

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with zero-flux boundary conditions

$$\frac{\partial(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon})}{\partial\nu} = \frac{\partial(\varphi_{\sigma,\epsilon} + \psi'_{+,\epsilon}(n_{\sigma,\epsilon}))}{\partial\nu} = 0 \qquad \text{on } \partial\Omega \times (0, +\infty). \tag{25}$$

It is convenient to define the flux of the regularized system as

$$J_{\sigma,\epsilon} = B_\epsilon(n_{\sigma,\epsilon})\nabla\left(\varphi_{\sigma,\epsilon} + \psi'_{+,\epsilon}(n_{\sigma,\epsilon})\right).$$

2.2 Existence for the regularized problem

We can now state the existence theorem for the regularized problem (24).

Theorem 1 (Existence for $\varepsilon > 0$) *Assuming $n^0 \in H^1(\Omega)$, there exists a pair of functions $(n_{\sigma,\epsilon}, \varphi_{\sigma,\epsilon})$ such that for all $T > 0$,*

$$n_{\sigma,\epsilon} \in L^2(0, T; H^1(\Omega)), \qquad \partial_t n_{\sigma,\epsilon} \in L^2(0, T; (H^1(\Omega))'),$$
$$\varphi_{\sigma,\epsilon} \in L^2(0, T; H^1(\Omega)),$$
$$n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon} \in L^2(0, T; H^2(\Omega)), \qquad \partial_t\left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon}\right) \in L^2(0, T; (H^1(\Omega))'),$$

which satisfies the regularized-relaxed degenerate Cahn-Hilliard equation (24), (25) in the following weak sense, for all test function $\chi \in L^2(0, T; H^1(\Omega))$, it holds

$$\int_0^T <\chi, \partial_t n_{\sigma,\epsilon}> = \int_{\Omega_T} B_\epsilon(n_{\sigma,\epsilon})\nabla\left(\varphi_{\sigma,\epsilon} + \psi'_{+,\epsilon}(n_{\sigma,\epsilon})\right)\nabla\chi,$$
$$\sigma\int_{\Omega_T}\nabla\varphi_{\sigma,\epsilon}\nabla\chi + \int_{\Omega_T}\varphi_{\sigma,\epsilon}\chi = \gamma\int_{\Omega_T}\nabla n_{\sigma,\epsilon}\nabla\chi + \int_{\Omega_T}\psi'_-(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon})\chi. \tag{26}$$

Proof. We adapt the proof of the theorem 2 in [17] where the authors prove the existence of solutions of the Cahn-Hilliard system with positive mobilities. Since the regularized mobility here is positive due to (20), we can apply the same theorem. The proof of existence follows the following different stages

Step 1. Galerkin approximation. Firstly, we make an approximation of the regularized problem (24). We define the family of eigenfunctions $\{\phi_i\}_{i\in\mathbb{N}}$ of the Laplace operator subjected to zero Neumann boundary conditions.

$$-\Delta\phi_i = \lambda_i\phi_i \text{ in } \Omega \quad \text{with} \quad \nabla\phi_i \cdot \nu = 0 \text{ on } \partial\Omega.$$

The family $\{\phi_i\}_{i\in\mathbb{N}}$ form an orthogonal basis of both $H^1(\Omega)$ and $L^2(\Omega)$ and we normalize them, i.e. $(\phi_i, \phi_j)_{L^2(\Omega)} = \delta_{ij}$ to obtain an orthonormal basis. We assume that the first eigenvalue is $\lambda_1 = 0$ (which does not introduce a lack of generality).

We consider the following discretization of (24)

$$n^N(t,x) = \sum_{i=1}^{N} c_i^N(t)\phi_i(x), \quad \varphi^N(t,x) = \sum_{i=1}^{N} d_i^N(t)\phi_i(x), \tag{27}$$

$$\int_\Omega \partial_t n^N \phi_j = -\int_\Omega B_\epsilon(n^N)\nabla\left(\varphi^N + \psi'_{+,\epsilon}(n^N)\right)\nabla\phi_j, \quad \text{for } j = 1, ..., N, \tag{28}$$

$$\int_\Omega \varphi^N \phi_j = \gamma \int_\Omega \nabla\left(n^N - \frac{\sigma}{\gamma}\varphi^N\right)\nabla\phi_j + \int_\Omega \psi'_-(n^N - \frac{\sigma}{\gamma}\varphi^N)\phi_j, \quad \text{for } j = 1, ..., N, \tag{29}$$

$$n^N(0,x) = \sum_{i=1}^{N} (n_0, \phi_i)_{L^2(\Omega)}\,\phi_i. \tag{30}$$

This gives the following initial value problem for a system of ordinary differential equations, for all $j = 1, ..., N$,

$$\partial_t c_j^N = -\sum_{k=1}^{N} d_k^N \int_\Omega B_\epsilon(\sum_{i=1}^{N} c_i^N \phi_i)\nabla\phi_k\nabla\phi_j - \sum_{k=1}^{N} c_k^N \int_\Omega (B_\epsilon \psi''_{+,\epsilon})(\sum_{i=1}^{N} c_i^N \phi_i)\nabla\phi_k\nabla\phi_j, \tag{31}$$

$$d_j^N = \gamma\lambda_j c_j^N - \sigma\lambda_j d_j^N + \int_\Omega \psi'_-(\sum_{k=1}^{N}(c_k^N - \frac{\sigma}{\gamma}d_k^N)\phi_k)\phi_j, \tag{32}$$

$$c_j^N(0) = (n_0, \phi_i)_{L^2(\Omega)}. \tag{33}$$

Since the right-hand side of equation (31) depends continuously on the coefficients c_j^N, the initial value problem has a local solution.

Step 2. Energy and entropy estimates. To prove the existence of global solutions, we need to define an energy $E(t)$ and an entropy $\Phi(t)$ from the semi-discrete system (27)–(30). For the details of the derivation of the energy, we refer the reader to the section 2.3. We use here the semi-discrete version of the continuous energy dfined further in (49). Therefore, we set

$$E(t) = \int_\Omega \psi_{+,\epsilon}(n^N) + \frac{\gamma}{2}|\nabla(n^N - \frac{\sigma}{\gamma}\varphi^N)|^2 + \frac{\sigma}{2\gamma}|\varphi^N|^2 + \psi_-(n^N - \frac{\sigma}{\gamma}\varphi^N),$$

$$\frac{d}{dt}E(t) = -\int_\Omega B_\epsilon(n^N)\left|\nabla\left(\varphi^N + \psi'_{+,\epsilon}(n^N)\right)\right|^2,$$

from which we can conclude the energy inequality

$$E(T) + \int_0^T \int_\Omega B_\epsilon(n^N)\left|\nabla(\varphi^N + \psi'_{+,\epsilon}(n^N))\right|^2 \leq E(0), \tag{34}$$

where $E(0)$ is integrable due to the assumptions (23) and $E(t)$ is bounded from below thanks to the assumptions on ψ_- in (7) and the construction of $\psi_{\epsilon,+}$ in (23).

The entropy is defined differently compared to the section (49). Due to the fact that the basis functions ϕ_i are in $H^1(\Omega)$, the Laplacian does not belong to L^2. Therefore, to obtain the entropy, we go through the following steps. First define $\phi''_\epsilon(n^N) = \frac{1}{B_\epsilon(n^N)}$, and we calculate

$$\int_\Omega \partial_t \phi_\epsilon(n^N) = \int_\Omega \partial_t n^N \phi'(n^N), \quad = -\int_\Omega \nabla \varphi^N \nabla \left(n^N - \frac{\sigma}{\gamma}\varphi^N\right) + \psi''_{+,\epsilon}(n^N)|\nabla n^N|^2 + \frac{\sigma}{\gamma}|\nabla \varphi^N|^2,$$

and since

$$\nabla \varphi^N \nabla \left(n^N - \frac{\sigma}{\gamma}\varphi^N\right) = \frac{\gamma}{\sigma}\left(|\nabla n^N|^2 - |\nabla(n^N - \frac{\sigma}{\gamma}\varphi^N)|^2\right),$$

we have

$$\frac{d\Phi(n^N)}{dt} = -\int_\Omega \frac{\gamma}{\sigma}\left(|\nabla n^N|^2 - |\nabla(n^N - \frac{\sigma}{\gamma}\varphi^N)|^2\right) + \psi''_{+,\epsilon}(n^N)|\nabla n^N|^2 + \frac{\sigma}{\gamma}|\nabla \varphi^N|^2.$$

We obtain the following inequality

$$\frac{d\Phi(n^N)}{dt} + \int_\Omega \frac{\gamma}{\sigma}|\nabla n^N|^2 + \frac{\sigma}{\gamma}|\nabla \varphi^N|^2 + \psi''_{+,\epsilon}(n^N)|\nabla n^N|^2 \leq \frac{\gamma^2}{2\sigma}E(0). \tag{35}$$

Step 3. Inequalities. From (34) and (35), we deduce the following inequalities

$$\frac{\gamma}{2}\int_\Omega |\nabla(n^N - \frac{\sigma}{\gamma}\varphi^N)|^2 \leq C, \tag{36}$$

$$\frac{\sigma}{2\gamma}\int_\Omega |\varphi^N|^2 \leq C, \tag{37}$$

$$\int_0^T \int_\Omega \frac{\gamma}{\sigma}|\nabla n^N|^2 + \frac{\sigma}{\gamma}|\nabla \varphi^N|^2 \leq \Phi(0) + T\frac{\gamma^2}{2\sigma}E(0), \tag{38}$$

$$\int_0^T \int_\Omega B_\epsilon(n^N)|\nabla(\varphi^N + \psi'_{+,\epsilon}(n^N))|^2 \leq C, \tag{39}$$

which hold for small positive values of γ, σ and also for all finite time $T \geq 0$. Therefore, from these inequalities we can extract subsequences of (n^N, φ^N) such that the following convergences hold for any time $T \geq 0$ and small positive values of γ, σ.

Since, $\frac{d}{dt}\int_\Omega n^N - \frac{\sigma}{\gamma}\varphi^N = 0$, we obtain from the Poincaré-Wirtinger inequality and (36) that

$$n^N - \frac{\sigma}{\gamma}\varphi^N \rightharpoonup n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon} \text{ weakly in } L^2(0,T;H^1(\Omega)). \tag{40}$$

Using the inequalities (37)–(38), we obtain

$$\varphi^N \rightharpoonup \varphi_{\sigma,\epsilon} \text{ weakly in } L^2(0,T;H^1(\Omega)). \tag{41}$$

From the previous convergence, we conclude that $\varphi_{\sigma,\epsilon} \in L^2(0,T;H^1(\Omega))$, therefore, using elliptic regularity we know that

$$n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon} \in L^2(0,T;H^2(\Omega)) \tag{42}$$

Also from (40)–(41), we obtain

$$n^N \rightharpoonup n_{\sigma,\epsilon} \text{ weakly in } L^2(0,T;H^1(\Omega)), \tag{43}$$

This result implies that the coefficients c_j^N are bounded and a global solution to (31)–(33) exists. To be able to prove some strong convergence in $L^2(0,T;L^2(\Omega))$, we need an information about the temporal derivative $\partial_t n^N$. From the first equation of the system, we denote Π_N the projection of $L^2(\Omega)$ on the space generated by span$\{\phi_1,...,\phi_N\}$ and we have for all test functions $\phi \in L^2(0,T;H^1(\Omega))$

$$\begin{aligned}
\left|\int_{\Omega_T} \partial_t n^N \phi\right| &= \left|\int_{\Omega_T} \partial_t n^N \Pi_N \phi\right| \\
&= \left|\int_{\Omega_T} b(n^N)\nabla(\varphi^N + \psi'_{+,\epsilon}(n^N))\nabla\Pi_N\phi\right| \\
&\leq \left(B_1\int_{\Omega_T} b(n^N)\left|\nabla(\varphi^N + \psi'_{+,\epsilon}(n^N))\right|^2\right)^{\frac{1}{2}}\left(\int_{\Omega_T}|\nabla\Pi_N\phi|^2\right)^{\frac{1}{2}} \\
&\leq C||\nabla\phi||_{L^2(\Omega_T)}
\end{aligned} \tag{44}$$

which is obtained using (39). Thus we can extract a subsequence such that

$$\partial_t n^N \rightharpoonup \partial_t n_{\sigma,\epsilon} \text{ weakly in } L^2(0,T;(H^1(\Omega))'). \tag{45}$$

From (43) and (45) and using the Lions-Aubin Lemma, we obtain the strong convergence

$$n^N \to n_{\sigma,\epsilon} \text{ strongly in } L^2(0,T;L^2(\Omega)). \tag{46}$$

Next, we need to prove the strong convergence of $n^N - \frac{\sigma}{\gamma}\varphi^N$ in $L^2(0,T;H^1(\Omega))$. In order to do that we must bound the $L^2(0,T;(H^1(\Omega))')$ norm of its time derivative.

Starting from the second equation of the system (3), we can obtain an equation on the time derivative of φ (for details of this calculation we refer the reader to Proposition 4, here, in addition, we use the Galerkin discretization). Using this equation in its weak form and applying the previous Galerkin method, we obtain

$$\sigma\int_\Omega \nabla U^N \nabla\phi + \int_\Omega U^N\phi = \int_\Omega \partial_t n^N\phi - \frac{\sigma}{\gamma}\int_\Omega U^N\psi''_-(n^N - \frac{\sigma}{\gamma}\varphi^N)\phi.$$

where $U^N = \partial_t\left(n^N - \frac{\sigma}{\gamma}\varphi^N\right)$. If we replace in the previous equation the test function by U^N, we obtain

$$\sigma\int_\Omega |\nabla U^N|^2 + \int_\Omega |U^N|^2 = \int_\Omega \partial_t n^N U^N - \frac{\sigma}{\gamma}\int_\Omega |U^N|^2\psi''_-(n^N - \frac{\sigma}{\gamma}\varphi^N).$$

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From the Cauchy-Schwarz inequality, we obtain

$$0 \leq ||\nabla U^N||^2_{L^2(\Omega)} + \left(1 - \frac{\sigma}{\gamma}||\psi''_-||_\infty\right) ||U^N||^2_{L^2(\Omega)} \leq ||\partial_t n^N||_{L^2(\Omega)}||U^N||_{L^2(\Omega)}.$$

Taking into account the assumptions (7) and (44), the following inequality holds

$$||U^N||_{H^1(\Omega)} \leq C.$$

Therefore, we can extract a subsequence such that

$$\partial_t \left(n^N - \frac{\sigma}{\gamma}\varphi^N\right) \rightharpoonup \partial_t \left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon}\right) \text{ weakly in } L^2(0,T;(H^1(\Omega))'). \tag{47}$$

Using (47) and (42) and the Lions-Aubin lemma we obtain the following strong convergence

$$n^N - \frac{\sigma}{\gamma}\varphi^N \to n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon} \text{ strongly in } L^2(0,T;H^1(\Omega)). \tag{48}$$

Step 4. Limiting equation. The main difficulty to pass to the limit in the equation (29) relies mainly on the convergence of the term $\int_\Omega \psi'_-(n^N - \frac{\sigma}{\gamma}\varphi^N)\phi_j$ which is solved using the strong convergence (48) and the properties (7). Therefore, we obtain

$$\psi'_-(n^N - \frac{\sigma}{\gamma}\varphi^N) \to \psi'_-(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon}) \quad \text{a.e. in } \Omega_T.$$

We can pass to the limit in the first equation (28) by the standard manner (see [26]), using the strong convergence (46) and the properties of the mobility (21) and the potential (23). Altogether, we obtain the limiting system (26).

2.3 Energy, entropy and a priori estimates

The relaxed and regularized system (24) comes with an energy and an entropy. These provide us with estimates which are useful to prove the existence of global weak solutions of (24) and their convergence to the weak solutions of the original DHC equation or to the RDHC as ϵ and/or $\sigma \to 0$.

Being given a solution $(n_{\sigma,\epsilon}, \varphi_{\sigma,\epsilon})$ satisfying Theorem 1, we define the energy associated with the regularized potential $\psi_{+,\epsilon}$ and relaxed system as

$$\mathcal{E}_{\sigma,\epsilon}[n_{\sigma,\epsilon}] = \int_\Omega \left[\psi_{+,\epsilon}(n_{\sigma,\epsilon}) + \frac{\gamma}{2}|\nabla(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon})|^2 + \frac{\sigma}{2\gamma}|\varphi_{\sigma,\epsilon}|^2 + \psi_-(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon})\right], \tag{49}$$

where $\varphi_{\sigma,\epsilon}$ is obtained from $n_{\sigma,\epsilon}$ by solving the elliptic equation in (24). Notice that $\mathcal{E}_{\sigma,\epsilon}[n_{\sigma,\epsilon}]$ is lower bounded, uniformly in ϵ and σ, thanks to the assumptions on ψ_- in (7) and the construction of $\psi_{\epsilon,+}$ in (23).

Proposition 2 (Energy) *Consider a solution $(n_{\sigma,\epsilon}, \varphi_{\sigma,\epsilon})$ of (24)–(25) defined by Theorem 1, then, the energy of the system $\mathcal{E}_{\sigma,\epsilon}$ satisfies*

$$\frac{d}{dt}\mathcal{E}_{\sigma,\epsilon}[n_{\sigma,\epsilon}(t)] = -\int_\Omega B_\epsilon(n_{\sigma,\epsilon})|\nabla(\varphi_{\sigma,\epsilon} + \psi'_{+,\epsilon}(n_{\sigma,\epsilon}))|^2 \leq 0. \tag{50}$$

As a consequence, we obtain a first a priori estimate

$$\mathcal{E}_{\sigma,\epsilon}[n_{\sigma,\epsilon}(T)] + \int_0^T \int_\Omega B_\epsilon(n_{\sigma,\epsilon})|\nabla(\varphi_{\sigma,\epsilon} + \psi'_{+,\epsilon}(n_{\sigma,\epsilon}))|^2 = \mathcal{E}_{\sigma,\epsilon}[n^0]. \tag{51}$$

Proof. To establish the energy of the regularized system, we begin with multiplying the first equation of (24) by $\varphi_{\sigma,\epsilon} + \psi'_{+,\epsilon}(n_{\sigma,\epsilon})$. Then, we integrate on the domain Ω and use the second boundary condition (25) to obtain

$$\int_\Omega [\varphi_{\sigma,\epsilon} + \psi'_{+,\epsilon}(n_{\sigma,\epsilon})]\partial_t n_{\sigma,\epsilon} = -\int_\Omega B_\epsilon(n_{\sigma,\epsilon})|\nabla(\varphi_{\sigma,\epsilon} + \psi'_{+,\epsilon}(n_{\sigma,\epsilon}))|^2.$$

Since $\psi'_{+,\epsilon}(n_{\sigma,\epsilon})\partial_t n_{\sigma,\epsilon} = \partial_t \psi_{+,\epsilon}(n_{\sigma,\epsilon})$, to retrieve the energy equality (50) we need to focus on the calculation of $\int_\Omega \varphi_{\sigma,\epsilon}\partial_t n_{\sigma,\epsilon}$. We write

$$\int_\Omega \varphi_{\sigma,\epsilon}\partial_t n_{\sigma,\epsilon} = \int_\Omega \varphi_{\sigma,\epsilon}\partial_t[n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon}] + \frac{d}{dt}\int_\Omega \frac{\sigma}{2\gamma}|\varphi_{\sigma,\epsilon}|^2,$$

and using the second equation of (24), we rewrite the first term as

$$\int_\Omega \varphi_{\sigma,\epsilon}\partial_t[n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon}] = \int_\Omega [-\gamma\Delta(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon}) + \psi'_-(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon})]\partial_t[n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon}]$$

$$= \frac{d}{dt}\int_\Omega \frac{\gamma}{2}\nabla(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon})|^2 + \psi_-(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon}),$$

where we have used the first boundary condition (25).

Altogether, we have recovered the expression (49) and the equality (50).

We can now turn to the entropy inequality. It is classical to define the mapping $\phi_\epsilon : [0, \infty) \mapsto [0, \infty)$

$$\phi''_\epsilon(n) = \frac{1}{B_\epsilon(n)}, \qquad \phi_\epsilon(0) = \phi'_\epsilon(0) = 0, \tag{52}$$

which is well defined because $B_\epsilon \in C(\mathbb{R}, \mathbb{R}^+)$ from (20). For a nonnegative function $n(x)$, we define the entropy as

$$\Phi_\epsilon[n] = \int_\Omega \phi_\epsilon(n(x))dx.$$

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Proposition 3 (Entropy) *Consider a solution of* (24)–(25) *defined by Theorem* 1, *then the entropy of the system satisfies*

$$\frac{d\Phi_\epsilon[n_{\sigma,\epsilon}(t)]}{dt} = - \int_\Omega \gamma \left| \Delta \left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon} \right) \right|^2 + \frac{\sigma}{\gamma} |\nabla \varphi_{\sigma,\epsilon}|^2 + \psi''_-(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon}) \left| \nabla (n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon}) \right|^2 \quad (53)$$
$$+ \psi''_{+,\epsilon}(n_{\sigma,\epsilon}) |\nabla n_{\sigma,\epsilon}|^2.$$

Notice that the dissipation terms are all well defined by our definition of solution in Theorem 1. However, the equality (53) does not provide us with a direct a priori estimate because of the negative term ψ''_-, therefore we have to combine it with the energy identity to write

$$\Phi_\epsilon[n_{\sigma,\epsilon}(T)] + \int_{\Omega_T} \left[\gamma \left| \Delta \left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon} \right) \right|^2 + \frac{\sigma}{\gamma} |\nabla \varphi_{\sigma,\epsilon}|^2 + \psi''_{+,\epsilon}(n_{\sigma,\epsilon}) |\nabla n_{\sigma,\epsilon}|^2 \right]$$
$$\leq \Phi_\epsilon[n^0] + \frac{2T}{\gamma} \|\psi''_-\|_\infty \, \mathcal{E}_{\sigma,\epsilon}[n^0].$$

Proof. We compute, using the definition of ϕ''_ϵ,

$$\int_\Omega \partial_t \phi_\epsilon(n_{\sigma,\epsilon}) = \int_\Omega \partial_t n_{\sigma,\epsilon} \phi'_\epsilon(n_{\sigma,\epsilon})$$
$$= \int_\Omega \nabla \cdot \left[B_\epsilon(n_{\sigma,\epsilon}) \nabla(\varphi_{\sigma,\epsilon} + \psi'_{+,\epsilon}(n_{\sigma,\epsilon})) \right] \phi'_\epsilon(n_{\sigma,\epsilon})$$
$$= - \int_\Omega B_\epsilon(n_{\sigma,\epsilon}) \nabla(\varphi_{\sigma,\epsilon} + \psi'_{+,\epsilon}(n_{\sigma,\epsilon})) \phi''_\epsilon(n_{\sigma,\epsilon}) \nabla n_{\sigma,\epsilon} \quad (54)$$
$$= - \int_\Omega \nabla(\varphi_{\sigma,\epsilon} + \psi'_{+,\epsilon}(n_{\sigma,\epsilon})) \nabla n_{\sigma,\epsilon}$$
$$= - \int_\Omega \nabla \varphi_{\sigma,\epsilon} \nabla(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon}) + \psi''_{+,\epsilon}(n_{\sigma,\epsilon}) |\nabla n_{\sigma,\epsilon}|^2 + \frac{\sigma}{\gamma} |\nabla \varphi_{\sigma,\epsilon}|^2.$$

To rewrite the term $\int_\Omega \nabla \varphi_{\sigma,\epsilon} \nabla(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon})$, we use the second equation of the regularized system (24)

$$\varphi_{\sigma,\epsilon} = -\gamma \Delta \left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon} \right) + \psi'_-(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon}). \quad (55)$$

Using (55) and the boundary condition (25), we can rewrite the term under consideration as

$$\int_\Omega \varphi_{\sigma,\epsilon} \Delta \left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon} \right) = \int_\Omega -\gamma \left| \Delta \left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon} \right) \right|^2 + \psi'_-(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon}) \Delta \left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon} \right)$$
$$= - \int_\Omega \gamma \left| \Delta \left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon} \right) \right|^2 + \psi''_-(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon}) \left| \nabla \left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon} \right) \right|^2.$$

Injecting this equality into (54), we obtain the identity (53).

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2.4 Inequalities

From the energy and entropy properties, we can conclude the following a priori bounds, where we assume that the initial data has finite energy and entropy,

$$\frac{\sigma}{2\gamma}\int_\Omega |\varphi_{\sigma,\epsilon}(t)|^2 \leq \mathcal{E}_{\sigma,\epsilon}[n^0], \qquad \forall t \geq 0, \tag{56}$$

$$\frac{\sigma}{\gamma}\int_0^T \int_\Omega |\nabla\varphi_{\sigma,\epsilon}|^2 \leq \Phi_\epsilon[n^0] + \frac{2T}{\gamma}\|\psi''_-\|_\infty \mathcal{E}_{\sigma,\epsilon}[n^0], \qquad \forall T \geq 0, \tag{57}$$

$$\frac{\gamma}{2}\int_\Omega \left|\nabla\left(n_{\sigma,\epsilon}(t) - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon}(t)\right)\right|^2 \leq \mathcal{E}_{\sigma,\epsilon}[n^0], \qquad \forall t \geq 0, \tag{58}$$

$$\int_0^T \int_\Omega \left|\Delta\left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon}\right)\right|^2 \leq \Phi_\epsilon[n^0] + \frac{2T}{\gamma}\|\psi''_-\|_\infty \mathcal{E}_{\sigma,\epsilon}[n^0], \qquad \forall T \geq 0, \tag{59}$$

$$\int_0^T \int_\Omega B_\epsilon(n_{\sigma,\epsilon})\left|\nabla(\varphi_{\sigma,\epsilon} + \psi'_{+,\epsilon}(n_{\sigma,\epsilon}))\right|^2 \leq \mathcal{E}_{\sigma,\epsilon}[n^0], \qquad \forall T \geq 0. \tag{60}$$

Proposition 4 (Compactness of time derivatives) *Consider a solution $(n_{\sigma,\epsilon}, \varphi_{\sigma,\epsilon})$ of (24)–(25) defined by Theorem 1, then, the following inequalities hold for σ small enough*

$$\|\partial_t n_{\sigma,\epsilon}\|_{L^2(0,T;(H^1(\Omega))')} \leq C, \tag{61}$$

$$\left\|\partial_t\left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon}\right)\right\|_{L^2(0,T;(H^1(\Omega))')} \leq C. \tag{62}$$

Proof. For any test function $\chi \in L^2(0,T;H^1(\Omega))$ we obtained from (60)

$$\left|\int_{\Omega_T} \partial_t n_{\sigma,\epsilon}\chi\right| = \left|\int_{\Omega_T} B_\epsilon(n_{\sigma,\epsilon})\nabla\left(\varphi_{\sigma,\epsilon} + \psi'_{+,\epsilon}(n_{\sigma,\epsilon})\right)\nabla\chi\right|$$

$$\leq \left(\int_{\Omega_T} \left|B_\epsilon(n_{\sigma,\epsilon})\nabla\left(\varphi_{\sigma,\epsilon} + \psi'_{+,\epsilon}(n_{\sigma,\epsilon})\right)\right|^2\right)^{1/2}\|\nabla\chi\|_{L^2(\Omega_T)},$$

$$\leq C\|\nabla\chi\|_{L^2(\Omega_T)}.$$

This proves (61).

To prove (62), we start from the second equation of (3) and calculate its time derivative, we obtain

$$-\sigma\Delta U + U = \partial_t n - \frac{\sigma}{\gamma}U\psi''_-(n - \frac{\sigma}{\gamma}\varphi),$$

where $U = \partial_t\left(n - \frac{\sigma}{\gamma}\varphi\right)$. Using the previously defined regularization, we calculate the weak form of this equation from which we have

$$\sigma\int_{\Omega_T} \nabla U_{\sigma,\epsilon}\nabla\chi + \int_{\Omega_T} U_{\sigma,\epsilon}\chi = \int_{\Omega_T} \partial_t n_{\sigma,\epsilon}\chi - \frac{\sigma}{\gamma}\int_{\Omega_T} U_{\sigma,\epsilon}\psi''_-(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon})\chi,$$

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where $U_{\sigma,\epsilon} = \partial_t\left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon}\right)$. We can choose $\chi = U_{\sigma,\epsilon}$, to obtain

$$\sigma\int_{\Omega_T}|\nabla U_{\sigma,\epsilon}|^2 + \int_{\Omega_T}|U_{\sigma,\epsilon}|^2 = \int_{\Omega_T}\partial_t n_{\sigma,\epsilon}U_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\int_{\Omega_T}|U_{\sigma,\epsilon}|^2\psi_-''(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon}).$$

Using the fact that $\frac{\sigma}{\gamma}||\psi_-''||_\infty < 1$ from (7), the Cauchy-Schwarz inequality gives

$$||\nabla U_{\sigma,\epsilon}||_{L^2(\Omega_T)}^2 + \alpha||U_{\sigma,\epsilon}||_{L^2(\Omega_T)}^2 \le ||\partial_t n_{\sigma,\epsilon}||_{L^2(\Omega_T)}||U_{\sigma,\epsilon}||_{L^2(\Omega_T)},$$

where $\alpha = 1 - \frac{\sigma}{\gamma}||\psi_-''||_\infty > 0$. Altogether, we obtain the bound (62).

3 Existence: convergence as $\epsilon \to 0$

The next step is to prove the existence of global weak solutions for the RDCH system (3) by letting ε vanish. This means that for all test functions $\chi \in L^2(0,T;H^1(\Omega)) \cap L^\infty(\Omega_T)$ with $\nabla\chi \cdot \nu = 0$ on $\partial\Omega \times (0,T)$, it holds

$$\int_0^T <\chi, \partial_t n_\sigma> = \int_{\Omega_T}b(n_\sigma)\nabla\left(\varphi_\sigma + \psi_+'(n_\sigma)\right)\nabla\chi,$$

$$\sigma\int_{\Omega_T}\nabla\varphi_\sigma\nabla\chi + \int_{\Omega_T}\varphi_\sigma\chi = \gamma\int_{\Omega_T}\nabla n_\sigma\nabla\chi + \int_{\Omega_T}\psi_-'(n_\sigma - \frac{\sigma}{\gamma}\varphi_\sigma)\chi.$$

We establish the following

Theorem 5 (Existence for $\sigma > 0$, $\varepsilon = 0$) *Assume an initial condition satisfying $0 \le n^0 \le 1$, with finite energy and entropy. Then, for σ small enough, there exists a global weak solution $(n_\sigma, \varphi_\sigma)$ of the RDCH equation (3), (4) such that*

$$n_\sigma \in L^2(0,T;H^1(\Omega)), \qquad \partial_t n_\sigma \in L^2(0,T;(H^1(\Omega))'). \tag{63}$$

$$\varphi_\sigma \in L^2(0,T;H^1(\Omega)), \tag{64}$$

$$n_\sigma - \frac{\sigma}{\gamma}\varphi_\sigma \in L^2(0,T;H^2(\Omega)), \qquad \partial_t\left(n_\sigma - \frac{\sigma}{\gamma}\varphi_\sigma\right) \in L^2(0,T;(H^1(\Omega))'). \tag{65}$$

$$0 \le n_\sigma \le 1, \qquad \textit{a.e. in } \Omega_T, \tag{66}$$

and $n_\sigma < 1$ a.e. if b vanishes fast enough at 1 so that $\phi(1) = \infty$ (see (16)).

Proof. The proof relies on compactness results and the inequalities presented in section 2.4. From these inequalities, we can extract subsequences of $(n_{\sigma,\epsilon}, \varphi_{\sigma,\epsilon})$ such that the following convergences for $\epsilon \to 0$ hold for all $T > 0$.

Step 1. Weak limits. From (56) and (57), we immediately have

$$\varphi_{\sigma,\epsilon} \rightharpoonup \varphi_\sigma \text{ in } L^2((0,T);H^1(\Omega)). \tag{67}$$

Next, from (58), and the above convergence, we conclude

$$n_{\sigma,\epsilon} \rightharpoonup n_\sigma \text{ weakly in } L^2\big(0,T;H^1(\Omega)\big), \tag{68}$$

Finally from (61) and (62), we have

$$\partial_t n_{\sigma,\epsilon} \rightharpoonup \partial_t n_\sigma \text{ weakly in } L^2\big(0,T;(H^1(\Omega))'\big),$$

$$\partial_t \left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon} \right) \rightharpoonup \partial_t \left(n_\sigma - \frac{\sigma}{\gamma}\varphi_\sigma \right) \text{ weakly in } L^2\big(0,T;(H^1(\Omega))'\big).$$

Step 2. Strong convergence. Therefore, from the Lions-Aubin lemma and Proposition 4 we obtain the strong convergences

$$n_{\sigma,\epsilon} \to n_\sigma \in L^2(0,T;L^2(\Omega)). \tag{69}$$

$$n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon} \to n_\sigma - \frac{\sigma}{\gamma}\varphi_\sigma \in L^2(0,T;H^1(\Omega)). \tag{70}$$

Step 3. Bounds $0 \le n_\sigma \le 1$. To prove these bounds on n_σ, several authors have used the entropy relation. In the context of DCH equation with double-well potentials featuring singularities at $n = 1$ and $n = -1$, the solution lies a.e. in the interval $-1 < n < 1$. Elliott and Garcke [17] prove this result using the definition of the regularized entropy and by a contradiction argument. For single-well potential, Agosti *et al.* [2] used a reasoning on the measure of the set of solutions outside the set $0 \le n < 1$ and find contradictions with the boundedness of the entropy. This is the route we follow here. In the following, all functions are defined almost everywhere.

We begin by the upper bound . For $\alpha > 0$, we consider the set

$$V_\alpha^\varepsilon = \{(t,x) \in \Omega_T | n_{\sigma,\epsilon}(t,x) \ge 1 + \alpha\}.$$

Consider $A > 0$ (large) and for ε small enough, such that (because $b(1) = 0$)

$$\phi_\epsilon''(n) = \frac{1}{b(1-\epsilon)} \ge 2A \qquad \forall n \ge 1, \ \forall \epsilon > 0.$$

Thus, integrating this quantity twice, we obtain

$$\phi_\epsilon(n) \ge A(n-1)^2 \qquad \forall n \ge 1.$$

Also, from (53), we know that the entropy is uniformly bounded in ε. Therefore, we obtain

$$|V_\alpha^\varepsilon|A\alpha^2 \le \int_{\Omega_T} \phi_\epsilon(n_{\sigma,\epsilon}(t,x)) \le C(T), \qquad |V_\alpha^\varepsilon| \le \frac{C(T)}{A\alpha^2}.$$

In the limit $\varepsilon \to 0$, using Fatou's lemma and the strong convergence of $n_{\sigma,\epsilon}$, we conclude that

$$\left| \{(t,x) \in \Omega_T | n_\sigma(t,x) \ge 1 + \alpha\} \right| \le \frac{C(T)}{A\alpha^2}, \qquad \forall A > 0.$$

In other words $n_\sigma(t,x) \leq 1 + \alpha$ for all $\alpha > 0$, which means $n_\sigma(t,x) \leq 1$.

The same argument also gives $n_\sigma \geq 0$ and we do not repeat it.

The second statement, $n_\sigma < 1$ under the assumption $\phi(1) = +\infty$, is a consequence of the bound

$$\int_{\Omega_T} \phi(n_\sigma(t,x)) \leq C(T),$$

which holds true by strong convergence of $n_{\sigma,\epsilon}$ and because $\phi_\varepsilon \nearrow \phi$ as $\varepsilon \searrow 0$.

Step 4. Limiting equation. Finally, it remains to show that the limit of subsequences satisfies the RDCH equation in the weak form. Firstly, using the weak convergences (67)–(68), the strong convergence (70) and the properties of ψ'_- gathered from (7), we can pass to the limit in the standard way to obtain the second equation of the limit system.

To conclude the proof, we need to prove the following weak convergence, recalling that (60) provides a uniform L^2 bound over Ω_T, on $J_{\sigma,\epsilon}$

$$J_{\sigma,\epsilon} := B_\epsilon(n_{\sigma,\epsilon})\nabla(\varphi_{\sigma,\epsilon} + \psi'_{+,\epsilon}(n_{\sigma,\epsilon})) \rightharpoonup b(n_\sigma)\nabla(\varphi_\sigma + \psi'_+(n_\sigma)) \text{ weakly in } L^2(\Omega_T). \tag{71}$$

The convergence of $B_\epsilon(n_{\sigma,\epsilon})\nabla\varphi_{\sigma,\epsilon}$ follows from the weak convergence in $L^2(\Omega_T)$ of $\nabla\varphi_{\sigma,\epsilon}$ and the strong convergence $B_\epsilon(n_{\sigma,\epsilon}) \to b(n_\sigma)$ in all $L^p(\Omega_T)$, $1 \leq p < \infty$ which follows from (69) and the fact that $B_\epsilon(.) \to b(.)$ uniformly.

Because of the singularity $\psi'_+(1) = \infty$, we use the assumption (11) and that $B_\epsilon(\cdot)\psi''_{+,\epsilon}(\cdot) \to b(\cdot)\psi''_+(\cdot)$ uniformly and thus $B_\epsilon(n_{\sigma,\varepsilon})\psi''_{+,\epsilon}(n_{\sigma,\varepsilon}) \to b(n_{\sigma,\varepsilon})\psi''_+(n_{\sigma,\varepsilon})$ a.e. in Ω_T This achieve the proof.

It is easy to check that the energy and entropy relations (14), (17) hold, at least as inequalities. In the sequel we only use the a priori bounds coming from the limiting procedure.

4 Convergence as $\sigma \to 0$

We are now ready to study the limit of the relaxed solution n_σ towards a solution of the DCH equation, Our main result is as follows.

Theorem 6 (Limit $\sigma = 0$) *Let $(n_{\sigma,\epsilon}, \varphi_{\sigma,\epsilon})$ be a sequence of weak solutions of the RDHC system* (24) *with initial conditions n^0, $0 \leq n^0 < 1$, with finite energy and entropy. Then, as ε, $\sigma \to 0$, we can extract a subsequence of $(n_{\sigma,\epsilon}, \varphi_{\sigma,\epsilon})$ such that*

$$\varphi_{\sigma,\epsilon} \rightharpoonup -\gamma\Delta n + \psi'_-(n) \quad \text{weakly in } L^2(\Omega_T), \tag{72}$$

$$n_{\sigma,\gamma} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon} \to n \quad \text{strongly in } L^2(0,T;H^1(\Omega)), \tag{73}$$

$$n_{\sigma,\epsilon}, \nabla n_{\sigma,\epsilon} \to n, \nabla n \quad \text{strongly in } L^2(\Omega_T), \text{ and } 0 \leq n \leq 1, \tag{74}$$

and $n_\sigma < 1$ a.e. if b vanishes fast enough at 1 so that $\phi(1) = \infty$.

$$\partial_t n_{\sigma,\epsilon} \rightharpoonup \partial_t n \quad \text{weakly in } L^2(0,T;(H^1(\Omega))'). \tag{75}$$

This limit n satisfies the DCH system (1) *in the weak sense.*

We recall the definition of weak solutions; for all $\chi \in L^2(0, T; H^2(\Omega)) \cap L^\infty(\Omega_T)$ with $\nabla\chi \cdot \nu = 0$ on $\partial\Omega \times (0, T)$,

$$
\begin{cases}
\int_0^T < \chi, \partial_t n > & = \int_{\Omega_T} J \cdot \nabla\chi, \\
\int_{\Omega_T} J \cdot \nabla\chi & = \int_{\Omega_T} \gamma\Delta n \left[b'(n)\nabla n \cdot \nabla\chi + b(n)\Delta\chi \right] + (b\psi'')(n)\nabla n \cdot \nabla\chi.
\end{cases}
\tag{76}
$$

Proof. We gathered, from the energy and entropy estimates of section 2.3, the a priori bounds of the section 2.4.

Step 1. Weak limits. From the above mentioned inequalities, we can extract subsequences of $(n_{\sigma,\epsilon}, \varphi_{\sigma,\epsilon})$ such that the following convergences hold for all $T > 0$. From (56) and (57), we immediately have

$$
\sigma\varphi_{\sigma,\epsilon} \to 0 \text{ in } L^2\big((0, T); H^1(\Omega)\big).
\tag{77}
$$

Next, from (58), and the above convergence, we conclude

$$
n_{\sigma,\epsilon} \rightharpoonup n \text{ weakly in } L^2\big(0, T; H^1(\Omega)\big),
$$

and (59) gives directly

$$
\Delta(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon}) \rightharpoonup \Delta n \text{ weakly in } L^2(\Omega_T).
\tag{78}
$$

The system of equations can also be used to complement these results. We find

$$
\varphi_{\sigma,\epsilon} \rightharpoonup \varphi \text{ weakly in } L^2(\Omega_T),
$$

using the second equation of the system (24) and triangular inequality,

$$
\|\varphi_{\sigma,\epsilon}\|_{L^2(\Omega_T)} \leq \gamma\|\Delta(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon})\|_{L^2(\Omega_T)} + \|\psi'_-(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon})\|_{L^2(\Omega_T)}.
$$

Finally from (60) and the equation on $n_{\sigma,\epsilon}$ itself, we conclude (75).

Step 2. Strong convergence. We continue with proving the strong convergences in (74). From the inequality (59), we know that $\Delta\left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon}\right)$ is uniformly bounded in $L^2(\Omega_T)$. We also have the boundary conditions, $\nabla\left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi\right) \cdot \nu = 0$ and the conservation of both quantities. Therefore elliptic regularity theory gives us

$$
\|n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon}\|_{L^2(0,T;H^2(\Omega))} \leq C.
$$

Therefore strong compactness in space holds for the quantities $n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi$ and $\nabla[n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi]$. Furthermore, from the limit (77), it means that both $n_{\sigma,\epsilon}$ and $\nabla n_{\sigma,\epsilon}$ are compact in space. Compactness in time is also obtain for the quantity $n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon}$ from (62). Again from Lions-Aubin lemma, we have the strong convergence (74). The conclusion (72) follows from this results.

The bounds $0 \leq n < 1$ can be obtained as in the case $\varepsilon \to 0$, see Theorem 5 and we do not repeat the argument.

Step 3. Limiting equation. Next, we need to verify that the limit of the subsequence $n_{\sigma,\epsilon}$ satisfies the

DCH equation. The argument is different from the case $\varepsilon \to 0$ because we do not control $\nabla\varphi_{\sigma,\epsilon}$ in the case at hand. From the L^2 bound in (60), we need to identify the weak limit

$$J_{\sigma,\epsilon} := B_\epsilon(n_{\sigma,\epsilon})\nabla(\varphi_{\sigma,\epsilon} + \psi'_{+,\epsilon}(n_{\sigma,\epsilon})) \rightharpoonup b(n)\nabla(\varphi + \psi'_+(n)) \quad \text{weakly in } L^2(\Omega_T). \tag{79}$$

For a test function $\eta \in L^2(0,T; H^1(\Omega, \mathbb{R}^d)) \cap L^\infty(\Omega_T, \mathbb{R}^d)$ and $\eta \cdot \mu = 0$ on $\partial\Omega \times (0,T)$, we integrate the left-hand side to obtain

$$\int_{\Omega_T} J_{\sigma,\epsilon} \cdot \eta = \int_{\Omega_T} \gamma\Delta\left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon}\right)\nabla \cdot (B_\epsilon(n_{\sigma,\epsilon})\eta) + B_\epsilon(n_{\sigma,\epsilon})\nabla\left(\psi'_{+,\epsilon}(n_{\sigma,\epsilon}) + \psi'_-(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon})\right) \cdot \eta.$$

We have mainly two types of terms on the right-hand side $\int_{\Omega_T} \gamma\Delta\left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon}\right)\nabla \cdot (B_\epsilon(n_{\sigma,\epsilon})\eta)$ and $\int_{\Omega_T} B_\epsilon(n_{\sigma,\epsilon})\nabla\left(\psi'_{+,\epsilon}(n_{\sigma,\epsilon}) + \psi'_-(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon})\right) \cdot \eta$. Let us focus on the first term

$$\int_{\Omega_T} \gamma\Delta\left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon}\right)\nabla \cdot (B_\epsilon(n_{\sigma,\epsilon})\eta) = \int_{\Omega_T} \gamma\Delta\left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon}\right)B_\epsilon(n_{\sigma,\epsilon})\nabla \cdot \eta$$
$$+ \int_{\Omega_T} \gamma\Delta\left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon}\right)B'_\epsilon(n_{\sigma,\epsilon})\nabla n_{\sigma,\epsilon} \cdot \eta.$$

From the strong convergence (74) and the weak one (78) with the fact that $B_\epsilon(\cdot) \to b(\cdot)$ uniformly, we obtain the convergence of the first term of the right-hand side

$$\int_{\Omega_T} \gamma\Delta\left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon}\right)B_\epsilon(n_{\sigma,\epsilon})\nabla \cdot \eta \to \int_{\Omega_T} \gamma\Delta n \, b(n)\nabla \cdot \eta,$$

as $\sigma, \epsilon \to 0$ and thus we have passed to the limit in the first term of the right hand side. For the second term, we use that the derivative $B'_\epsilon(\cdot) \to b'(\cdot)$ uniformly. We also use the strong convergence of $\nabla n_{\sigma,\epsilon}$ from (74). From the results above and a generalized version of the Lebesgue dominated convergence theorem we obtain

$$\int_{\Omega_T} \gamma\Delta\left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon}\right)B'_\epsilon(n_{\sigma,\epsilon})\nabla n_{\sigma,\epsilon} \cdot \eta \to \int_{\Omega_T} \gamma\Delta n b'(n)\nabla n \cdot \eta,$$

as $\sigma, \epsilon \to 0$.

Let us now pass to the limit in $\int_{\Omega_T} B_\epsilon(n_{\sigma,\epsilon})\nabla\left(\psi'_{+,\epsilon}(n_{\sigma,\epsilon}) + \psi'_-(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon})\right) \cdot \eta$. As in the case of the convergence $\epsilon \to 0$, we have that

$$\int_{\Omega_T} B_\epsilon(n_{\sigma,\epsilon})\nabla\left(\psi'_{+,\epsilon}(n_{\sigma,\epsilon})\right) \cdot \eta,$$

using the fact that $B_\epsilon(\cdot)\psi''_{+,\epsilon}(\cdot) \to b(\cdot)\psi''_+(\cdot)$ uniformly and the strong convergence (74). Since $B_\epsilon(\cdot) \to b(\cdot)$, we have $\left(B_\epsilon\psi''_-\right)(\cdot) \to \left(b\psi''_-\right)(\cdot)$. Therefore, we pass to pass to the limit in $\int_{\Omega_T} B_\epsilon(n_{\sigma,\epsilon})\nabla\left(\psi'_-(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon})\right) \cdot \eta$ using the convergence (73). Altogether, we obtain the following convergence

$$\int_{\Omega_T} B_\epsilon(n_{\sigma,\epsilon})\nabla\left(\psi'_{+,\epsilon}(n_{\sigma,\epsilon}) + \psi'_-(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon})\right) \cdot \eta \to \int_{\Omega_T} b(n)\nabla\left(\psi'_+(n) + \psi'_-(n)\right) \cdot \eta$$

This finishes the proof of (79), i.e. that the limit solution n satisfies the weak formulation of the DCH equation (1), and also the proof of Theorem 6.

5 Long-time behavior

To complete our study of the RDCH model, we give some insights concerning the long-time behavior and convergence to steady states, $(n_\infty, \varphi_\infty)$ determined by the steady problem

$$\begin{cases} \nabla \cdot \big(b(n_\infty)\nabla \big(\varphi_\infty + \psi'_+(n_\infty)\big)\big) = 0 & \text{in } \Omega, \\ -\sigma\Delta\varphi_\infty + \varphi_\infty = -\gamma\Delta n_\infty + \psi'_-(n_\infty - \frac{\sigma}{\gamma}\varphi_\infty) \text{ in } \Omega, \\ \frac{\partial\big(n_\infty - \frac{\sigma}{\gamma}\varphi_\infty\big)}{\partial\nu} = b(n_\infty)\frac{\partial\big(\varphi_\infty + \psi'_+(n_\infty)\big)}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{80}$$

The analysis of the steady-states is not performed in this paper, however, numerical simulations can help us to have an idea of their shape for different initial situations.

The steady-states of the RDCH model present a configuration which minimizes the energy of the system. The solution obtained at the end of the simulation depends mainly on three parameters: the initial mass M, the width of the diffuse interface $\sqrt{\gamma}$ and the relaxation parameter σ.

In fact, if the initial mass is large enough, saturated aggregates are formed and we can describe two regions in the domain: the aggregates and the absence of cells. Between these two regions, the transition is smooth and the length of this interface is $\sqrt{\gamma}$. If the initial mass is small, aggregates are still formed but they are thicker and their maximum concentration does not reach 1 or the critical value $n^\star$ as in the definition of the potential (8).

The formation of aggregates happens only if γ is small enough. If γ, the initial mass M or the relaxation parameter σ is too large, the solution converges to the constant one

$$n_\infty = \frac{1}{|\Omega|}\int_\Omega n^0 dx, \quad \text{a.e. in } \Omega.$$

A surprizing fact about these observations is that the long-time behavior of the solutions of the RDCH system seems to follow the analytical description of the steady-states made by Songmu [30].

To state our convergence result of the weak solutions of the RDCH model to steady-states, we consider a global weak solution (n, φ) of the RDCH system with $\sigma > 0$, according to Theorem 5. The initial condition satisfies $0 \le n^0 < 1$ and has finite energy and entropy. so that we can use the a priori estimates from the transport structure, the energy and entropy dissipations (50) and (53) (or (14)–(17)), in particular

$$0 \le n < 1 \text{ a.e. } (0, \infty) \times \Omega. \tag{81}$$

Based on the controls provided by these relations, and using a standard method, we are going to study the large time behavior as the limit for large k of the sequence of functions

$$n_k(t, x) = n(t + k, x), \quad \text{and} \quad \varphi_k(t, x) = \varphi(t + k, x).$$

Proposition 7 (Long term convergence along subsequences) *Let (n, φ) be a weak solution of (3), (4) and initial condition n^0 with $0 \le n^0 < 1$, finite energy and entropy. Then, we can extract a*

subsequence, still denoted by index k, of (n_k, φ_k) such that

$$\lim_{k \to \infty} n_k(x,t) = n_\infty(x), \quad \lim_{k \to \infty} \varphi_k(x,t) = \varphi_\infty(x) \qquad \text{strongly in } L^2((-T,T) \times \Omega), \quad \forall T > 0, \quad (82)$$

where $(n_\infty, \varphi_\infty)$ are solutions of (80) satisfying

$$b(n_\infty) \nabla \left(\varphi_\infty + \psi'_+(n_\infty) \right) = 0. \qquad (83)$$

Proof. The proof uses the energy and entropy inequalities to obtain both uniform (in k) a priori bounds and zero entropy dissipation in the limit, which imply the result. We write these arguments in several steps.

1st step. A priori bounds from energy. Energy decay implies that $\mathcal{E}[n_k(t)]$ remains bounded in k for $t > -k$. As a consequence, the sequence (n_k, φ_k) satisfies

$$\frac{\sigma}{2\gamma} \int_\Omega |\varphi_k(t)|^2 \leq \mathcal{E}[n^0], \qquad \forall t \geq 0, \qquad (84)$$

$$\frac{\gamma}{2} \int_\Omega \left| \nabla \left(n_k(t) - \frac{\sigma}{\gamma} \varphi_k(t) \right) \right|^2 \leq \mathcal{E}[n^0], \qquad \forall t \geq 0, \qquad (85)$$

$$\int_{-T}^{T} \int_\Omega b(n_k) \left| \nabla(\varphi_k + \psi'_+(n_k)) \right|^2 := L_k(T), \qquad L_k(T) \to 0 \text{ as } k \to \infty, \qquad (86)$$

and this last line is because

$$\int_0^\infty \int_\Omega b(n) \left| \nabla(\varphi + \psi'_+(n)) \right|^2 \leq \mathcal{E}[n^0], \quad L_k(T) \leq \int_{k-T}^\infty \int_\Omega b(n) \left| \nabla(\varphi + \psi'_+(n)) \right|^2 \right] \xrightarrow[k \to \infty]{} 0.$$

2nd step. A priori bounds from entropy. Because the right hand side in the entropy balance has a positive term (since $\psi''_-(n) \leq 0, \forall n \in [0,1]$), it cannot be used as easily as the energy. However, we can integrate (17) from $k - T$ to $k + T$, and, using the control of the negative term including ψ_- as after (17), we obtain the inequality

$$\int_{-T}^{T} \left[\left| \Delta \left(n_k - \frac{\sigma}{\gamma} \varphi_k \right) \right|^2 + \frac{\sigma}{\gamma} |\nabla \varphi_k|^2 + \psi''_+(n_k) |\nabla n_k|^2 \right]$$

$$\leq \Phi[n(k-T)] - \Phi[n(k+T)] + \|\psi''_-\|_\infty \left\| \nabla \left(n_k - \frac{\sigma}{\gamma} \varphi_k \right) \right\|_{L^2((-T,T) \times \Omega)}^2$$

$$\leq \Phi[n(k-T)] - \Phi[n(k+T)] + \|\psi''_-\|_\infty \frac{4T}{\gamma} \mathcal{E}[n(k-T)]$$

$$\leq \Phi[n(k-T)] - \Phi[n(k+T)] + \|\psi''_-\|_\infty \frac{4T}{\gamma} \mathcal{E}[n^0].$$

3rd step. Extracting subsequences. From these inequalities, we can extract subsequences of (n_k, φ_k) such that for $k \to \infty$, the following convergences hold toward some functions $n_\infty(x,t)$ and $\varphi_\infty(x,t)$.

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We can conclude from inequalities (84) and the entropy control that, as $k \to \infty$,

$$\varphi_k \rightharpoonup \varphi_\infty \text{ weakly in } L^2\big(-T, T; H^1(\Omega)\big). \tag{87}$$

From the gradient bound (85), the L^2 bound in (84) and $0 \le n_k < 1$, we obtain

$$n_k - \frac{\sigma}{\gamma}\varphi_k \rightharpoonup n_\infty - \frac{\sigma}{\gamma}\varphi_\infty \text{ weakly in } L^2\big(0, T; H^1(\Omega)\big), \tag{88}$$

and thus

$$n_k \rightharpoonup n_\infty \text{ weakly in } L^2\big(0, T; H^1(\Omega)\big). \tag{89}$$

Finally, we obtain from (86) and the Cauchy-Schwarz inequality,

$$\partial_t n_k \rightharpoonup \partial_t n_\infty = 0 \text{ weakly in } L^2\big(0, T; (H^1(\Omega))'\big). \tag{90}$$

Indeed, for any test function $\phi \in C_0^\infty((-T, T) \times \Omega)$, it holds

$$\int_{-T}^T \int_\Omega \partial_t n_k \phi \, dx dt = -\int_{-T}^T \int_\Omega b(n_k) \nabla \big(\varphi_k + \psi_+'(n_k)\big) \cdot \nabla \phi,$$

$$\left| \int_{-T}^T \int_\Omega \partial_t n_k \phi \, dx dt \right|^2 \le 2T |\Omega| \|b\|_\infty \|\nabla \phi\|_\infty^2 \int_{-T}^T \int_\Omega b(n_k) \left| \nabla \big(\varphi_k + \psi_+'(n_k)\big) \right|^2 \to 0$$

as $k \to \infty$. This also shows that n_∞ only depends on x.

4th step. Strong limits. The strong compactness of n_k and φ_k follows from (85) and the entropy control. Then, time compactness of n_k, stated in (82) follows from the Lions-Aubin lemma, thanks to (90). The strong convergence of φ_k is a consequence of the elliptic equation for φ_k and of (62) which gives compactness in time of the quantity $n_k - \frac{\sigma}{\gamma}\varphi_k$. And we also have, from the strong convergence of n_k and (88), thanks to the above argument,

$$b(n_k) \nabla \big(\varphi_k + \psi_+'(n_k)\big) \to b(n_\infty) \nabla \big(\varphi_\infty + \psi_+'(n_\infty)\big) = 0, \tag{91}$$

which establishes the zero-flux equality (83).

6 Conclusion

The proposed relaxation system of the degenerate Cahn-Hilliard equation with single-well potential reduces the model to two parabolic/elliptic equations which can be solved by standard numerical solvers. The relaxation uses a regularization in space of the new unknown used to transform the original fourth-order equation into two second-order equations. This new system is a non-local relaxation of the original equation which is similar in a sense to the Cahn-Hilliard equation with a spatial interaction kernel derived in [21, 22]. We proved that in the limit of vanishing relaxation, we retrieve the original weak solutions of the DCH equation using compactness methods and estimates borrowed from energy and entropy functionals. The long-time behavior of the solutions of the RDCH system can also be

studied along the same lines. We showed that a global solution of the system converges to a steady-state as time goes to infinity, with zero flux.

The stationary states exhibit some interesting properties due to the degeneracy of the mobility. More precisely, they are split into two distinct zones: whether the mobility is zero, which is possible only in the pure phases, or the flux is null.

The RDCH system aims at the design of a numerical method to simulate the DCH equation using only second order elliptic problems. Such a numerical scheme, may depend on details of the relaxed model. For example, the solution represents a density and its numerical positivity is a desired property. Also the discrete stability is useful and a change of unknown in the RDCH system might be better adapted, using $U = \varphi - \frac{\gamma}{\sigma}n$,

$$\partial_t n = \nabla \cdot \left(b(n) \nabla \left(U + \frac{\gamma}{\sigma}n + \psi'_+(n)) \right) \right),$$
$$-\sigma \Delta U + U = -\frac{\gamma}{\sigma}n + \psi'_-(-\frac{\sigma}{\gamma}U).$$

Even though this model also consists of a parabolic transport equation coupled with an elliptic equation, the regularity is enhanced. On the one hand, in the first equation, the term $\frac{\gamma}{\sigma}n$ increases the diffusion for n. On the other hand, the second equation regularizes for the new variable U because it depends on n rather than Δn. Altogether, this model improves the regularity of the two unknowns and might be interesting for numerical purposes. In a forthcoming work, we will propose a numerical scheme based on the RDCH system, that preserves the physical properties of the solutions.

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