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Abstract

In the context of bivariate random variables $(Y^{(1)}, Y^{(2)})$, the marginal expected shortfall, defined as $\mathbb{E}(Y^{(1)}|Y^{(2)} \geqslant Q_2(1-p))$ for p small, where Q_2 denotes the quantile function of $Y^{(2)}$, is an important risk measure, which finds applications in areas like, e.g., finance and environmental science. We consider estimation of the marginal expected shortfall when the random variables of main interest $(Y^{(1)}, Y^{(2)})$ are observed together with a random covariate X, leading to the concept of the conditional marginal expected shortfall. The asymptotic behavior of an estimator for this conditional marginal expected shortfall is studied for a wide class of conditional bivariate distributions, with heavy-tailed marginal conditional distributions, and where p tends to zero at an intermediate rate. The finite sample performance is evaluated on a small simulation experiment. The practical applicability of the proposed estimator is illustrated on flood claim data.

1 Introduction

In the past years, many risk measures have been introduced in the literature, and these have been used to determine the amount of an asset to be kept in reserve in the financial framework. The most famous of these are the Value-At-Risk (VaR) defined for a random variable X as the $p-$quantile

$$Q(p) := \inf\{x \geqslant 0 : \mathbb{P}(X \leqslant x) \geqslant p\}, \text{ for } p \in (0,1),$$

and the Conditional Tail Expectation (CTE) defined as

$$CTE_p[X] = \mathbb{E}(X|X > Q(p)), \text{ for } p \in (0,1).$$

The latter risk measure is more conservative than the VaR for a same level of degree of confidence (see Landsman and Valdez, 2003) and it also satisfies the desirable property of being a coherent risk measure as defined by Artzner et al. (1999). For all these reasons, the CTE has been extensively studied and also it has been generalized to the multivariate framework, see, e.g., Cai and Li (2005), Cai et al. (2015), and Di Bernardino and Prieur (2018). More precisely, if $(Y^{(1)}, Y^{(2)})$ denotes a pair of risk factors, the CTE can be extended into $\mathbb{E}(Y^{(1)}|Y^{(2)} > Q_2(p))$, where $Q_2(p)$ is the $p-$quantile of the risk $Y^{(2)}$. In such a multivariate context, this risk measure is well-known as the Marginal Expected Shortfall (MES). It was introduced by Acharya et al. (2010), and used to measure the contribution of a financial institution to an overall systemic risk. The ongoing global credit crisis and other former financial crises have demonstrated the vital aspect of adequate risk measurement. For a financial firm, the MES is defined as its short-run expected equity loss conditional on the market taking a loss greater than its VaR. The MES is very simple to compute and therefore easy for regulators to consider. When estimating this risk measure, one often has the availability of additional information given by covariates, and these are important to take into account in order to obtain more precise estimates. This leads to the concept of conditional marginal expected shortfall.

In this paper, we will consider the estimation of the conditional marginal expected shortfall when the random variables of main interest $(Y^{(1)}, Y^{(2)})$ are recorded together with a random covariate $X \in \mathbb{R}^d$. We will denote by $F_j(.|x)$ the continuous conditional distribution function of $Y^{(j)}$, $j = 1, 2$, given $X = x$, and use the notation $\overline{F}_j(.|x)$ for the conditional survival function and $U_j(.|x)$ for the associated tail quantile function defined as $U_j(.|x) = \inf\{y : F_j(y|x) \geqslant 1 - 1/.\}$. Also, we will denote by f_X the density function of the covariate X and by x_0 a reference position such that $x_0 \in \mathrm{Int}(S_X)$, the interior of the support $S_X \subset \mathbb{R}^d$ of f_X, which is assumed to be non-empty. Our aim will be to estimate the conditional marginal expected shortfall, given $X = x_0$, and defined as

$$\theta_p = \mathbb{E}\left[Y^{(1)} \middle| Y^{(2)} \geqslant U_2\left(\frac{1}{p}\middle| x_0\right); x_0 \right],$$

where p is small.

The remainder of the paper is organized as follows. In Section 2, we introduce our estimator for the conditional marginal expected shortfall and we establish its main asymptotic properties. Simulations are provided in Section 3 to illustrate the efficiency of our estimator, while in Section 4 the method is applied to a dataset of flood insurance claims. All the proofs of the results are postponed to Section 5.

2 Estimator and asymptotic properties

We assume that $Y^{(1)}$ and $Y^{(2)}$ follow a conditional Pareto-type model.

Assumption $(\mathcal{D})$ *For all $x \in S_X$, the conditional survival functions of $Y^{(j)}$, $j = 1, 2$, satisfy*

$$\overline{F}_j(y|x) \quad = \quad A_j(x) y^{-1/\gamma_j(x)} \left(1 + \frac{1}{\gamma_j(x)} \delta_j(y|x)\right),$$

where $A_j(x) > 0$, $\gamma_j(x) > 0$, and $|\delta_j(.|x)|$ is normalised regularly varying with index $-\beta_j(x)$, $\beta_j(x) > 0$, i.e.,

$$\delta_j(y|x) \quad = \quad B_j(x) \exp\left(\int_1^y \frac{\varepsilon_j(u|x)}{u} du\right),$$

with $B_j(x) \in \mathbb{R}$ and $\varepsilon_j(y|x) \to -\beta_j(x)$ as $y \to \infty$. Moreover, we assume $y \to \varepsilon_j(y|x)$ to be a continuous function.

Under Assumption $(\mathcal{D})$, $F_1(.|x)$ and $F_2(.|x)$ have density functions. Indeed, straightforward differentiation gives

$$f_j(y|x) = \frac{A_j(x)}{\gamma_j(x)} y^{-1/\gamma_j(x)-1} \left[1 + \left(\frac{1}{\gamma_j(x)} - \varepsilon_j(y|x)\right) \delta_j(y|x)\right], j = 1, 2. \tag{2.1}$$

Now, let $(Y_i^{(1)}, Y_i^{(2)}, X_i)$, $i = 1, \ldots, n$, be independent copies of $(Y^{(1)}, Y^{(2)}, X)$. We consider estimating the conditional marginal expected shortfall when $p \to 0$ at an intermediate rate, i.e., $p = k/n$, where $k, n \to \infty$ such that $k/n \to 0$. A natural idea is then to study

$$\widehat{\theta}_n := \frac{1}{k} \sum_{i=1}^{n} K_{h_n}(x_0 - X_i) Y_i^{(1)} \mathbb{1}_{\{Y_i^{(2)} \geqslant \widehat{U}_2(n/k|x_0)\}},$$

where $\widehat{U}_2(.|x_0)$ is an estimator for $U_2(.|x_0)$, to be introduced later, and $K_{h_n}(.) := K(./h_n)/h_n^d$, with K a joint density function on $\mathbb{R}^d$, h_n a positive non-random sequence of bandwidths with $h_n \to 0$ if $n \to \infty$, and $\mathbb{1}_A$ the indicator function on the event A.

To simplify the situation, let us assume for the moment that $U_2(.|x_0)$ is known and consider

$$\widetilde{\theta}_n := \frac{1}{k} \sum_{i=1}^{n} K_{h_n}(x_0 - X_i) Y_i^{(1)} \mathbb{1}_{\{Y_i^{(2)} \geqslant U_2(n/k|x_0)\}}.$$

Clearly, assuming $F_1(y|x_0)$ is strictly increasing in y, we have

$$\begin{aligned}
\widetilde{\theta}_n &= \int_0^\infty \frac{1}{k} \sum_{i=1}^{n} K_{h_n}(x_0 - X_i) \mathbb{1}_{\{Y_i^{(1)} \geqslant s\}} \mathbb{1}_{\{Y_i^{(2)} \geqslant U_2(n/k|x_0)\}} ds \\
&= \int_0^\infty \frac{1}{k} \sum_{i=1}^{n} K_{h_n}(x_0 - X_i) \mathbb{1}_{\{Y_i^{(1)} \geqslant s,\, \overline{F}_2(Y_i^{(2)}|x_0) \leqslant k/n\}} ds \\
&= \int_0^\infty \frac{1}{k} \sum_{i=1}^{n} K_{h_n}(x_0 - X_i) \mathbb{1}_{\{\overline{F}_1(Y_i^{(1)}|x_0) \leqslant (k/n)[(n/k)\overline{F}_1(s|x_0)],\, \overline{F}_2(Y_i^{(2)}|x_0) \leqslant k/n\}} ds \\
&= -U_1(n/k|x_0) \int_0^\infty \frac{1}{k} \sum_{i=1}^{n} K_{h_n}(x_0 - X_i) \mathbb{1}_{\{\overline{F}_1(Y_i^{(1)}|x_0) \leqslant (k/n) s_n(u),\, \overline{F}_2(Y_i^{(2)}|x_0) \leqslant k/n\}} du^{-\gamma_1(x_0)},
\end{aligned}$$

where $s_n(u) := \frac{n}{k}\, \overline{F}_1(u^{-\gamma_1(x_0)} U_1(n/k|x_0)|x_0)$. Note that under $(\mathcal{D})$, we have $s_n(u) \to u$ as $n \to \infty$.

The key statistic to consider is thus, for $x_0 \in \mathrm{Int}(S_X)$,

$$T_n(y_1, y_2|x_0) := \frac{1}{k} \sum_{i=1}^{n} K_{h_n}(x_0 - X_i) \mathbb{1}_{\{\overline{F}_1(Y_i^{(1)}|x_0) \leqslant k/n\, y_1,\, \overline{F}_2(Y_i^{(2)}|x_0) \leqslant k/n\, y_2\}},$$

where $y_1, y_2 > 0$.

As a first main result we study the weak convergence of the process

$$\left\{ \sqrt{k h_n^d}\, \frac{T_n(y_1, y_2|x_0) - \mathbb{E}(T_n(y_1, y_2|x_0))}{y_1^\eta},\ \ y_1, y_2 \in (0, T] \right\}, \tag{2.2}$$

for any $T > 0$, finite, and $0 \leqslant \eta < 1/2 - \chi$, where $\chi > 0$ small. This will require some further assumptions.

In order to deal with the regression context, f_X and the functions appearing in $F_1(y|x)$ and $F_2(y|x)$ are assumed to satisfy the following Hölder conditions. Let $\|.\|$ denote some norm on

$\mathbb{R}^d$.

Assumption $(\mathcal{H})$ *There exist positive constants* M_{f_X}, M_{A_j}, M_{γ_j}, M_{B_j}, M_{ε_j}, η_{f_X}, η_{A_j}, η_{γ_j}, η_{B_j} *and* η_{ε_j}, *where* $j = 1, 2$, *such that for all* $x, z \in S_X$*:*

$$\begin{aligned} |f_X(x) - f_X(z)| &\leqslant M_{f_X} \|x - z\|^{\eta_{f_X}}, \\ |A_j(x) - A_j(z)| &\leqslant M_{A_j} \|x - z\|^{\eta_{A_j}}, \\ |\gamma_j(x) - \gamma_j(z)| &\leqslant M_{\gamma_j} \|x - z\|^{\eta_{\gamma_j}}, \\ |B_j(x) - B_j(z)| &\leqslant M_{B_j} \|x - z\|^{\eta_{B_j}}, \\ \sup_{y \geqslant 1} |\varepsilon_j(y|x) - \varepsilon_j(y|z)| &\leqslant M_{\varepsilon_j} \|x - z\|^{\eta_{\varepsilon_j}}. \end{aligned}$$

We also impose a condition on the kernel function K, which is a standard condition in local estimation.

Assumption $(\mathcal{K})$ *K is a bounded density function on $\mathbb{R}^d$, with support S_K included in the unit ball in $\mathbb{R}^d$.*

Next, a uniform convergence result is needed for the joint conditional distribution of $(Y^{(1)}, Y^{(2)})$. Let $R_t(y_1, y_2|x) := t\mathbb{P}(\overline{F}_1(Y^{(1)}|x) \leqslant y_1/t,\ \overline{F}_2(Y^{(2)}|x) \leqslant y_2/t | X = x)$.

Assumption $(\mathcal{R})$ *For all $x \in S_X$ we have*

$$\lim_{t \to \infty} R_t(y_1, y_2|x) = R(y_1, y_2|x),$$

uniformly in $y_1, y_2 \in (0, T]$, for any $T > 0$, and $x \in B(x_0, h_n)$.

The weak convergence of (2.2) is then established in the following theorem. Throughout the paper weak convergence is denoted by '$\rightsquigarrow$'.

Theorem 2.1. *Assume $(\mathcal{D})$, $(\mathcal{H})$, $(\mathcal{K})$, $(\mathcal{R})$ with $x \to R(y_1, y_2|x)$ being a continuous function, $x_0 \in Int(S_X)$ with $f_X(x_0) > 0$, and $y \to F_j(y|x_0)$, $j = 1, 2$, are strictly increasing. Consider sequences $k \to \infty$ and $h_n \to 0$ as $n \to \infty$, in such a way that $k/n \to 0$, $kh_n^d \to \infty$ and $h_n^{\eta_{\gamma_1} \wedge \eta_{\gamma_2} \wedge \eta_{\varepsilon_1} \wedge \eta_{\varepsilon_2}} \ln n/k \to 0$. Then for $\eta \in [0, 1/2 - \chi)$, where $\chi > 0$, small, we have,*

$$\sqrt{kh_n^d}\, \frac{T_n(y_1, y_2|x_0) - \mathbb{E}(T_n(y_1, y_2|x_0))}{y_1^\eta} \rightsquigarrow \frac{W(y_1, y_2)}{y_1^\eta}, \tag{2.3}$$

in $D((0, T]^2)$, for any $T > 0$, where $W(y_1, y_2)$ is a zero centered Gaussian process with covariance function

$$\mathbb{E}(W(y_1, y_2) W(\bar{y}_1, \bar{y}_2)) = \|K\|_2^2 f_X(x_0) R(y_1 \wedge \bar{y}_1, y_2 \wedge \bar{y}_2 | x_0).$$

We also introduce the following weak convergence result for a related process. This process will be useful in establishing the asymptotic properties of the quantile estimator $\widehat{U}_2(n/k|x_0)$. Let $\widehat{f}_n(x_0) := 1/n \sum_{i=1}^n K_{h_n}(x_0 - X_i)$ be a classical kernel density estimator.

Theorem 2.2. *Assume* $(\mathcal{D})$, $(\mathcal{H})$, $(\mathcal{K})$, *and* $x_0 \in Int(S_X)$ *with* $f_X(x_0) > 0$. *Consider sequences* $k \to \infty$ *and* $h_n \to 0$ *as* $n \to \infty$, *in such a way that* $k/n \to 0$, $kh_n^d \to \infty$, $h_n^{\eta_{\varepsilon_2}} \ln n/k \to 0$, $\sqrt{kh_n^d}\, h_n^{\eta_{f_X} \wedge \eta_{A_2}} \to 0$, $\sqrt{kh_n^d}\, h_n^{\eta_{\gamma_2}} \ln n/k \to 0$, $\sqrt{kh_n^d}|\delta_2(U_2(n/k|x_0)|x_0)|h_n^{\eta_{B_2}} \to 0$, *and* $\sqrt{kh_n^d}|\delta_2(U_2(n/k|x_0)|x_0)|h_n^{\eta_{\varepsilon_2}} \ln n/k \to 0$. *Then, we have*

$$\sqrt{kh_n^d}\left(\frac{T_n(\infty, y_2|x_0)}{\widehat{f}_n(x_0)} - y_2\right) \rightsquigarrow \frac{W(\infty, y_2)}{f_X(x_0)}$$

in $D((0,T])$, *for any* $T > 0$, *where* $W(\infty, y_2)$ *is a zero centered Gaussian process with covariance function*

$$\mathbb{E}(W(\infty, y_2)W(\infty, \bar{y}_2)) = \|K\|_2^2 f_X(x_0)(y_2 \wedge \bar{y}_2).$$

The joint weak convergence of the above two processes can be established by showing the joint finite dimensional weak convergence of them, combined with joint tightness. The joint finite dimensional convergence can be established by using the Cramér-Wold device (van der Vaart, 1998, p. 16). This is a standard, but tedious, calculation which is for brevity omitted from the paper. Note that the joint tightness follows from the individual tightness (similarly to Lemma 1 in Bai and Taqqu, 2013).

Now, generalize $\widetilde{\theta}_n$ to $\widetilde{\theta}_n(y_2)$, defined as

$$\widetilde{\theta}_n(y_2) = \frac{1}{k}\sum_{i=1}^n K_{h_n}(x_0 - X_i)Y_i^{(1)} 1\!\!1_{\{Y_i^{(2)} \geqslant U_2(n/(ky_2)|x_0)\}}.$$

Assuming $F_1(y|x_0)$ strictly increasing in y, we have

$$\widetilde{\theta}_n(y_2) = -U_1\left(\frac{n}{k}\middle|x_0\right)\int_0^\infty T_n(s_n(u), y_2|x_0)du^{-\gamma_1(x_0)}.$$

Proposition 2.1. *Assume* $(\mathcal{D})$, $(\mathcal{H})$, $(\mathcal{K})$, $(\mathcal{R})$ *with* $x \to R(y_1, y_2|x)$ *being a continuous function,* $x_0 \in Int(S_X)$ *with* $f_X(x_0) > 0$, *and* $y \to F_j(y|x_0)$, $j = 1, 2$, *are strictly increasing. Consider sequences* $k \to \infty$ *and* $h_n \to 0$ *as* $n \to \infty$, *in such a way that* $k/n \to 0$, $kh_n^d \to \infty$ *and* $h_n^{\eta_{\gamma_1} \wedge \eta_{\gamma_2} \wedge \eta_{\varepsilon_1} \wedge \eta_{\varepsilon_2}} \ln n/k \to 0$. *Then, for* $\gamma_1(x_0) < 1/2 - \kappa$, *with* $\kappa > 0$ *small, we have*

$$\sup_{\frac{1}{2} \leqslant y_2 \leqslant 2}\left|\frac{\sqrt{kh_n^d}}{U_1(n/k|x_0)}\left[\widetilde{\theta}_n(y_2) - \mathbb{E}(\widetilde{\theta}_n(y_2))\right] + \int_0^\infty W(u, y_2)du^{-\gamma_1(x_0)}\right| \xrightarrow{\mathbb{P}} 0.$$

The main result of this paper is the asymptotic normality of $\overline{\theta}_n := \widehat{\theta}_n/\widehat{f}_n(x_0)$, which is an estimator for the conditional marginal expected shortfall $\theta_{k/n}$. Note that $\widehat{\theta}_n = \widetilde{\theta}_n(\widehat{e}_n)$, where $\widehat{e}_n := \frac{n}{k}\overline{F}_2(\widehat{u}_n U_2(\frac{n}{k}|x_0)|x_0)$ with $\widehat{u}_n := \widehat{U}_2(\frac{n}{k}|x_0)/U_2(\frac{n}{k}|x_0)$. To estimate $U_2(.|x_0)$ we use $\widehat{U}_2(.|x_0) := \inf\{y : \widehat{F}_{n,2}(y|x_0) \geqslant 1 - 1/.\}$ with

$$\widehat{F}_{n,2}(y|x_0) := \frac{\sum_{i=1}^n K_{h_n}(x_0 - X_i) 1\!\!1_{\{Y_i^{(2)} \leqslant y\}}}{\sum_{i=1}^n K_{h_n}(x_0 - X_i)},$$

the empirical kernel estimator of the unknown conditional distribution function of $Y^{(2)}$ given $X = x_0$. See for instance Daouia et al. (2011).

In order to obtain the weak convergence of $\overline{\theta}_n$, we need to introduce the following second order condition.

Assumption $(\mathcal{S})$. *There exist $\beta > \gamma_1(x_0)$ and $\tau < 0$ such that, as $t \to \infty$*

$$\sup_{x \in B(x_0,h_n)} \sup_{0<y_1<\infty, \frac{1}{2} \leqslant y_2 \leqslant 2} \frac{|R_t(y_1,y_2|x) - R(y_1,y_2|x_0)|}{y_1^{\beta} \wedge 1} = O(t^{\tau}).$$

Theorem 2.3. *Assume $(\mathcal{D})$, $(\mathcal{H})$, $(\mathcal{K})$, $(\mathcal{S})$ with $x \to R(y_1,y_2|x)$ being a continuous function, and $y \to F_j(y|x_0)$, $j = 1, 2$, are strictly increasing. Let $x_0 \in Int(S_X)$ such that $f_X(x_0) > 0$. Consider sequences $k = \lfloor n^{\alpha}\ell_1(n) \rfloor$ and $h_n = n^{-\Delta}\ell_2(n)$, where ℓ_1 and ℓ_2 are slowly varying functions at infinity, with $\alpha \in (0,1)$ and*

$$\max\left(\frac{\alpha}{d + 2\gamma_1(x_0)(\eta_{A_1} \wedge \eta_{\gamma_1})}, \frac{\alpha}{d + 2(1-\gamma_1(x_0))(\eta_{A_2} \wedge \eta_{\gamma_2} \wedge \eta_{B_2} \wedge \eta_{\varepsilon_2} \wedge \eta_{f_X})},\right.$$
$$\left.\frac{\alpha}{d} - \frac{2(1-\alpha)\gamma_1^2(x_0)\beta_1(x_0)}{d + d(\beta_1(x_0)+\varepsilon)\gamma_1(x_0)}, \frac{\alpha - 2(1-\alpha)(\gamma_1(x_0) \wedge (\beta_2(x_0)\gamma_2(x_0)) \wedge (-\tau))}{d}\right) < \Delta < \frac{\alpha}{d}.$$

Then, for $\gamma_1(x_0) < 1/2 - \kappa$, with $\kappa > 0$ small, we have

$$\sqrt{kh_n^d}\left(\frac{\overline{\theta}_n}{\theta_{k/n}} - 1\right) \rightsquigarrow -(1-\gamma_1(x_0))\frac{W(\infty,1)}{f_X(x_0)} + \frac{1}{f_X(x_0)} \frac{\int_0^{\infty} W(s,1) ds^{-\gamma_1(x_0)}}{\int_0^{\infty} R(s,1|x_0) ds^{-\gamma_1(x_0)}}.$$

The conditions on k and h_n in Theorem 2.3 are due to the method of proof of the auxiliary result given in Lemma 5.4. Also in Cai et al. (2015) one needed a condition on the growth of k, but in the context without covariates. Note that due to the conditions $k, n \to \infty$ with $k/n \to 0$, the $Y^{(2)}$ quantile is intermediate, and the estimator $\bar{\theta}_n$ cannot be used for extrapolation outside the $Y^{(2)}$ data range. The situation where $p < 1/n$ will be investigated in future work.

3 Simulation experiment

In this section we evaluate the finite sample behavior of the proposed estimator with a simulation experiment. We simulate from the following models:

Model 1. We consider the logistic copula model

$$C(u_1,u_2|x) = e^{-[(-\ln u_1)^x + (-\ln u_2)^x]^{1/x}}, \quad u_1,u_2 \in [0,1], x \geqslant 2. \tag{3.1}$$

We take $X \sim U[2,10]$, and combine this copula model with Fréchet distributions for $Y^{(1)}$ and $Y^{(2)}$:

$$F_j(y) = e^{-y^{-1/\gamma_j}}, \quad y > 0,$$

$j = 1, 2$. We set $\gamma_1 = 0.25$ and $\gamma_2 = 0.5$. This model satisfies $(\mathcal{S})$ with $R(y_1,y_2|x) = y_1 + y_2 - (y_1^x + y_2^x)^{1/x}$, $\tau = -1$ and $\beta = 1 - \varepsilon$ for some small $\varepsilon > 0$.

Model 2. The conditional distribution of $(Y^{(1)}, Y^{(2)})$ given $X = x$ is that of

$$(|Z_1|^{\gamma_1(x)}, |Z_2|^{\gamma_2(x)}),$$

where (Z_1, Z_2) follow a bivariate standard Cauchy distribution with density function

$$f(z_1, z_2) = \frac{1}{2\pi}(1 + z_1^2 + z_2^2)^{-3/2}, \quad (z_1, z_2) \in \mathbb{R}^2.$$

We take $X \sim U[0,1]$, and set

$$\begin{aligned} \gamma_1(x) &= 0.4\,[0.1 + \sin(\pi x)]\left[1.1 - 0.5e^{-64(x-0.5)^2}\right], \\ \gamma_2(x) &= 0.1 + 0.1x. \end{aligned}$$

This model satisfies $(\mathcal{S})$ with $R(y_1, y_2|x) = y_1 + y_2 - \sqrt{y_1^2 + y_2^2}$, $\tau = -1$ and $\beta = 2$ (see, e.g., Cai et al., 2015, in the context without covariates).

Model 3. We consider the logistic copula model from (3.1), with $X \sim U[2,10]$, combined with conditional Burr distributions for $Y^{(1)}$ and $Y^{(2)}$:

$$F_j(y|x) = 1 - \left(\frac{\beta_j}{\beta_j + y^{\tau_j(x)}}\right)^{\lambda_j}, \quad y > 0; \beta_j, \lambda_j, \tau_j(x) > 0,$$

$j = 1, 2$. We set $\beta_1 = \beta_2 = 1$, $\lambda_1 = 1$, $\lambda_2 = 0.5$, and

$$\tau_1(x) = 2e^{0.2x}, \qquad \tau_2(x) = 8/\sin(0.3x).$$

Similarly to Model 1, this model satisfies $(\mathcal{S})$.

The marginal conditional distributions in the above models are standard heavy-tailed distributions that satisfy $(\mathcal{D})$, see, e.g., Beirlant, Joossens and Segers (2009), Table 1.

Concerning the kernel function K, we take the bi-quadratic function

$$K(x) = \frac{15}{16}(1 - x^2)^2 \mathbb{1}_{\{x \in [-1,1]\}}.$$

To compute our estimator $\overline{\theta}_n$, the bandwidth h_n need to be chosen. To this aim, we use the cross validation criterion introduced by Yao (1999), and used in an extreme value context by Daouia et al. (2011), Daouia, Gardes and Girard (2013) and Escobar-Bach, Goegebeur and Guillou (2018):

$$h_{cv} := \underset{h_n \in \mathcal{H}}{\operatorname{argmin}} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\mathbb{1}_{\left\{Y_i^{(2)} \leqslant Y_j^{(2)}\right\}} - \widehat{F}_{n,2,-i}\left(Y_j^{(2)} \middle| X_i\right)\right)^2,$$

where $\mathcal{H}$ is the grid of values defined as $R_X \times \{0.05, 0.10, \ldots, 0.30\}$, with R_X the range of the covariate X, and

$$\widehat{F}_{n,2,-i}(y|x) := \frac{\sum_{k=1,k\neq i}^{n} K_{h_n}(x - X_k)\,\mathbb{1}_{\left\{Y_k^{(2)} \leqslant y\right\}}}{\sum_{k=1,k\neq i}^{n} K_{h_n}(x - X_k)}.$$

The boxplots of the ratios between the estimates $\overline{\theta}_n$ and the true values $\theta_{k/n}$ based on 500 replications are given in Figure 1, for three different values of the covariate, $x_0 = 3$ (first row), $x_0 = 5$ (second row), $x_0 = 7$ (third row), two different sample sizes, $n = 500$ (left) and $n = 1000$ (right), and some specific values of k, in case of Model 1. Figures 2 and 3 are constructed similarly but for Model 2 and Model 3, respectively. The values of $\theta_{k/n}$ are computed with numerical integration.

From these figures we can draw the following conclusions:

- Overall the estimator $\overline{\theta}_n$ performs quite well, but obviously the performance depends on the model and also on the position x_0. The best results are obtained for Model 1, where the dependence structure depends on x but the marginal distributions are covariate independent. Model 2 has covariate dependent marginal distributions, but $R(y_1, y_2|x)$ does not depend on x, and for Model 3 both the margins and $R(y_1, y_2|x)$ depend on x. These models are more challenging than Model 1, but the estimator continues to perform well.

- The estimator behaves as expected in k and n, namely, for a fixed n the variance decreases with k and for a fixed sample fraction (as percentage of n) the variance decreases with n.

- The estimator seems to be not too much sensitive on the value of the covariate x_0 in case of Model 1. On the contrary, for Model 2, it depends a lot on the value of the covariate, the estimation being the best for $x_0 = 0.8$, with almost no bias. Model 3 is in between, with some improvement in the variability of the estimates when the covariate increases, which may be explained by the fact that $\gamma_1(x)$ decreases in x.

Next, in Figure 4 we provide some normal quantile plots of $\sqrt{kh_n} \ln \overline{\theta}_n/\theta_{k/n}$, with k taken as 5% of n and h_n obtained from the above mentioned cross-validation criterion. The rows of Figure 4 correspond with Models 1-3, respectively, while the columns represent the sample sizes, $n = 500$ and $n = 1000$, respectively. For all models and sample sizes, the normal quantile plots show a quite linear pattern, confirming the validity of the normal approximation. Moreover, with increasing n the normal approximation improves slightly.

4 Application to flood insurance claim data

In this section we illustrate the practical applicability of the method on a dataset of flood insurance claims. Recently, the Federal Emergency Management Agency (FEMA) has released millions of records from the National Flood Insurance Program (NFIP). In particular, this database contains approximately 2.4 million damage claims dating back to 1978, where for each claim one has information on the date of the flood, location of the property (latitude and longitude), claim amount, and on insurance policy and building characteristics. As such, it provides important information for policymakers, researchers, insurers and prospective homebuyers. The dataset is publicly available on `https://www.fema.gov/media-library/assets/documents/180374`. For our purposes we consider the data from the year 2000 on, and define Y_1 as the sum of the amount paid on the building claim, the content claim and the increased cost of compliance claim, Y_2 is taken as the sum of the insured amount for the building and content, while the covariate X consists of X_1 : latitude, X_2 : longitude and X_3 : date of loss. Interest is in estimating the expected claim amount conditional on an insured capital that exceeds a high

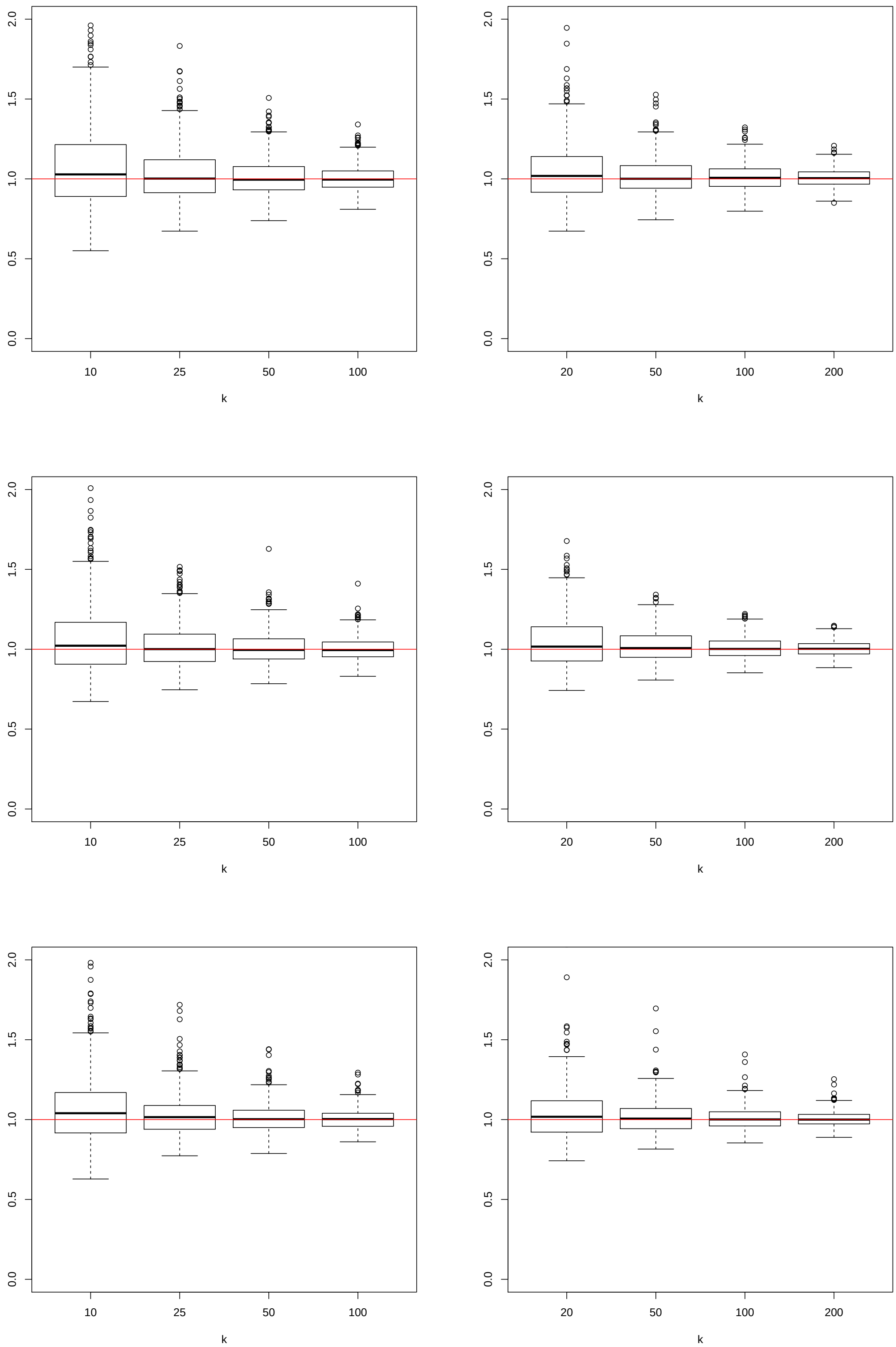

Figure 1: Model 1, boxplots of $\overline{\theta}_n/\theta_{k/n}$ for 500 simulations of size $n = 500$ (left) and $n = 1000$ (right), at $x_0 = 3$ (first row), $x_0 = 5$ (second row) and $x_0 = 7$ (third row). The values of k are taken as 2%, 5%, 10% and 20% of n.

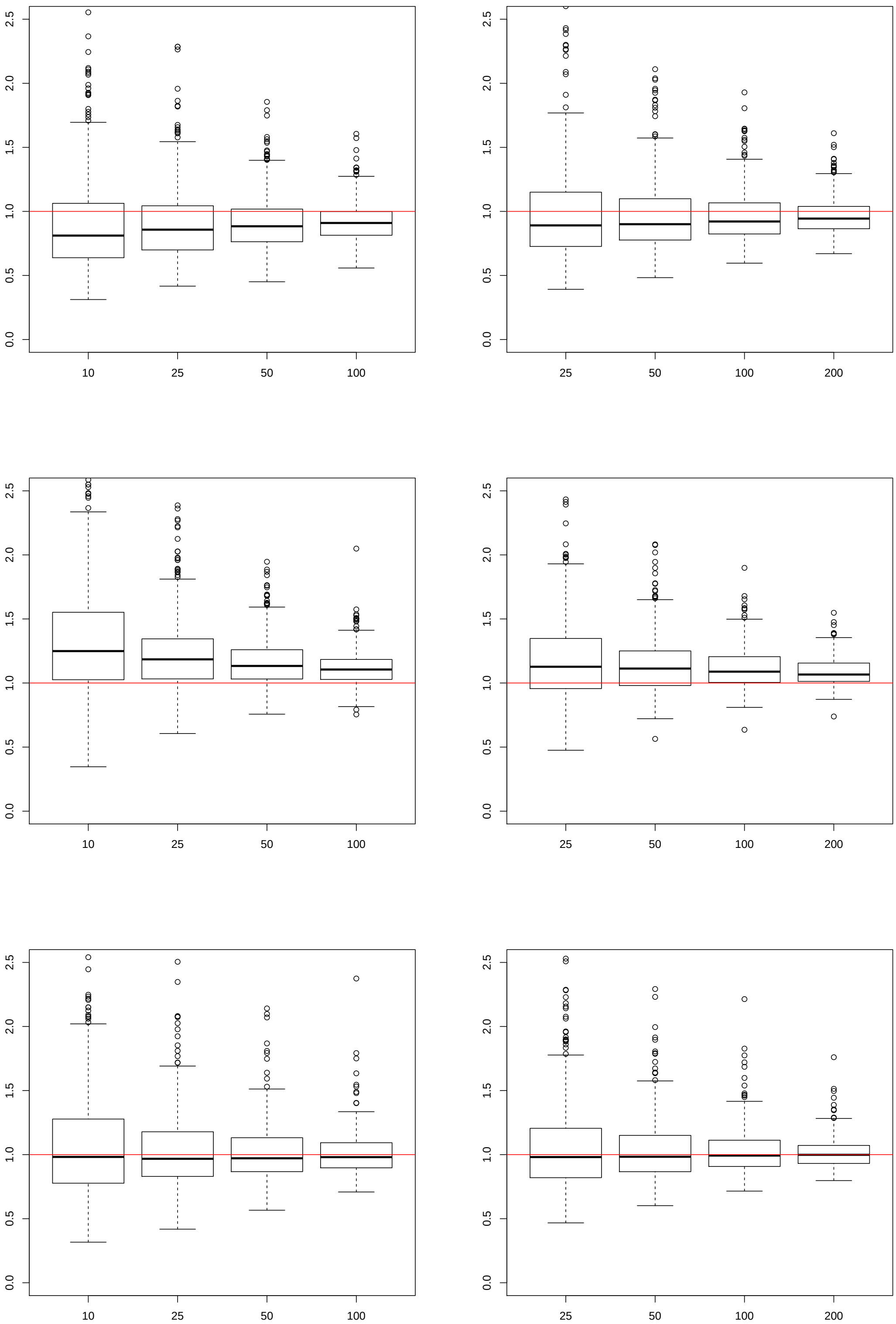

Figure 2: Model 2, boxplots of $\overline{\theta}_n/\theta_{k/n}$ for 500 simulations of size $n = 500$ (left) and $n = 1000$ (right), at $x_0 = 0.3$ (first row), $x_0 = 0.5$ (second row) and $x_0 = 0.8$ (third row). The values of k are taken as 2%, 5%, 10% and 20% of n.

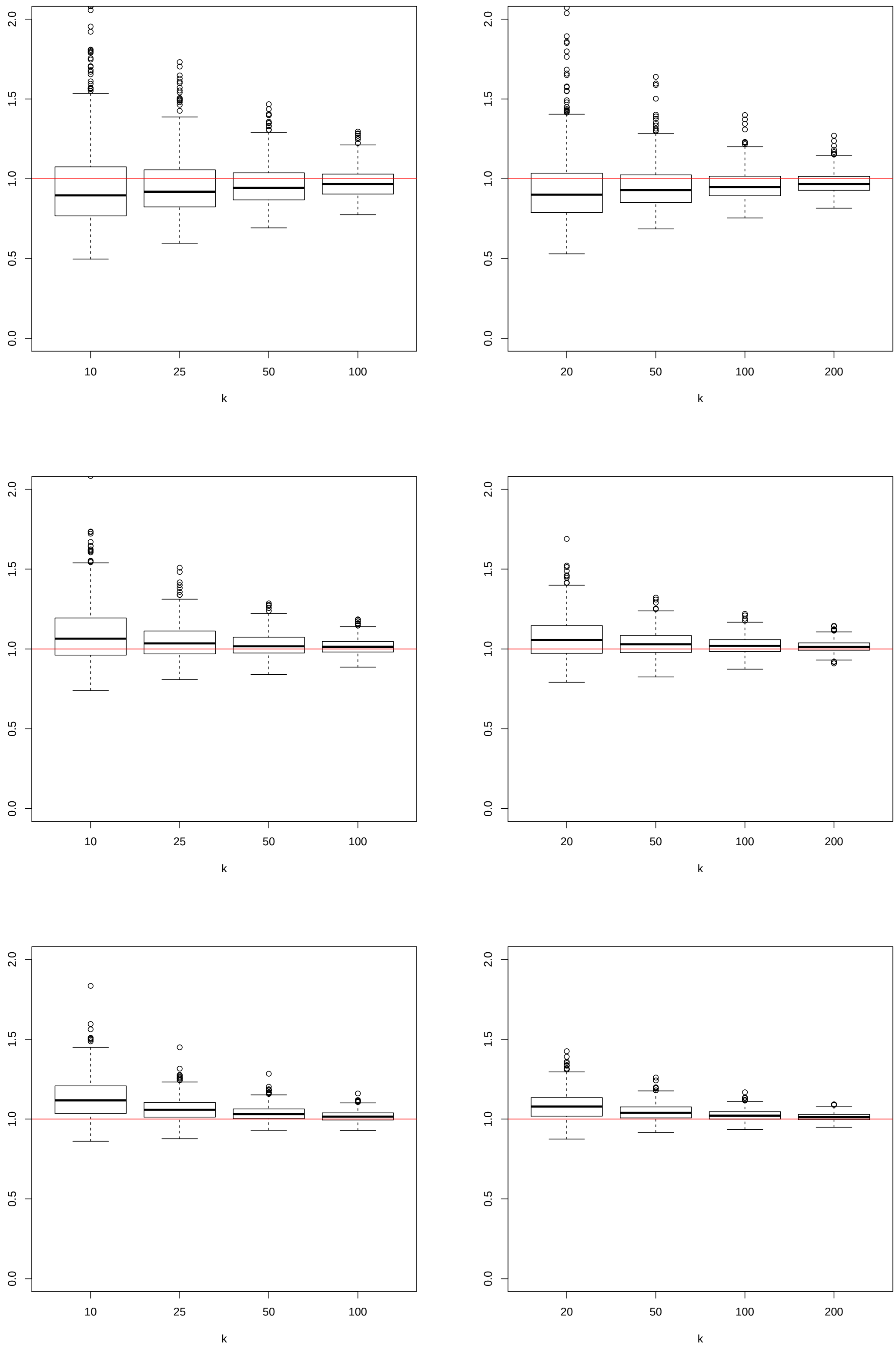

Figure 3: Model 3, boxplots of $\overline{\theta}_n/\theta_{k/n}$ for 500 simulations of size $n = 500$ (left) and $n = 1000$ (right), at $x_0 = 3$ (first row), $x_0 = 5$ (second row) and $x_0 = 7$ (third row). The values of k are taken as 2%, 5%, 10% and 20% of n.

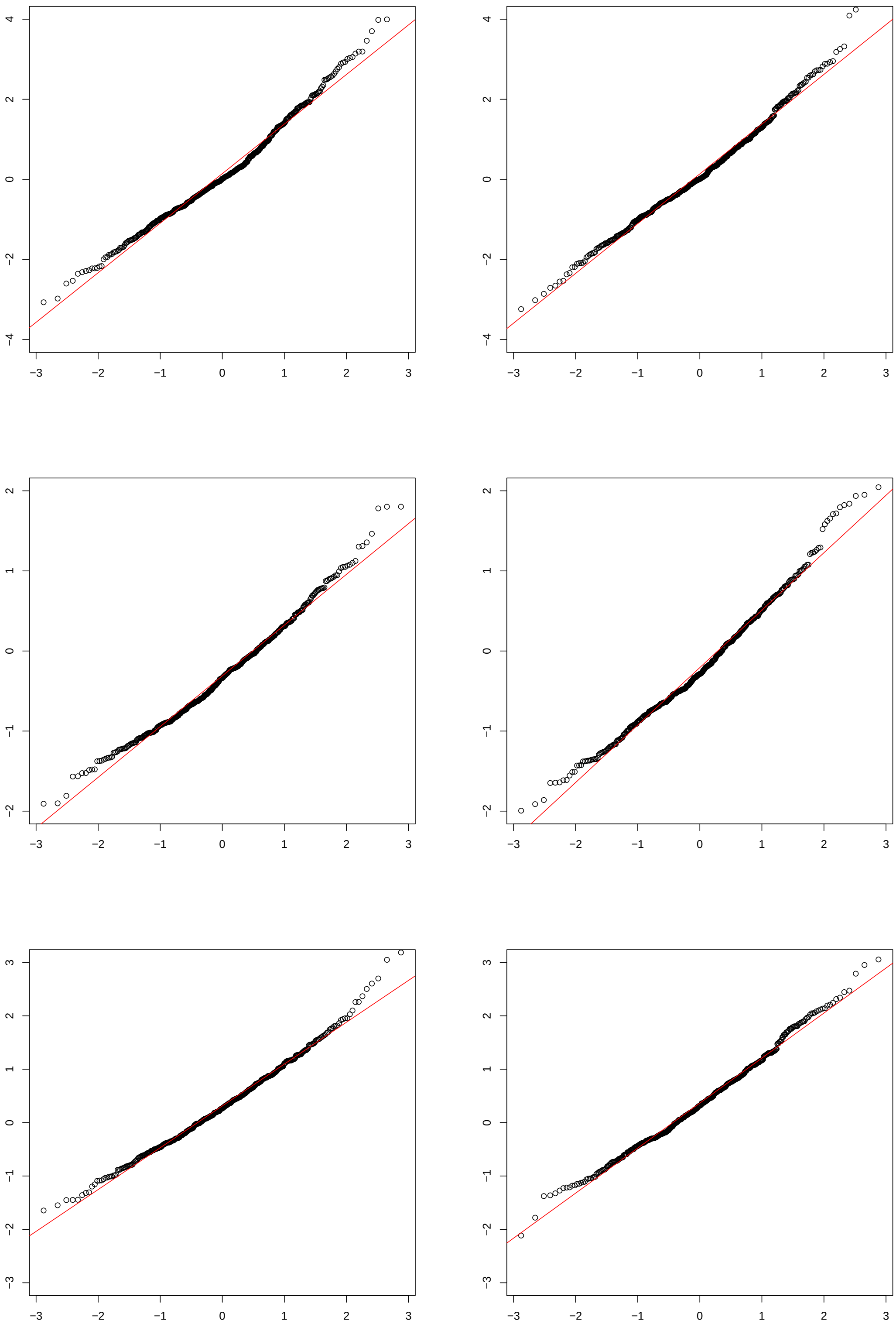

Figure 4: Normal quantile plots of $\sqrt{kh_n}\ln\overline{\theta}_n/\theta_{k/n}$. Top: Model 1, $x_0 = 3$, middle: Model 2, $x_0 = 0.3$, and bottom: Model 3, $x_0 = 5$. The quantile plots are constructed with k taken as 5% of n, with $n = 500$ (left) and $n = 1000$ (right).

quantile, and for a given location and time. The estimation method was implemented with the same cross-validation criterion as in the simulation section, including the same choice for $\mathcal{H}$, after standardizing the covariates to the interval $[0,1]$. As for the kernel function, we used the bi-quadratic kernel, generalized to the case $d=3$, as follows

$$K_{h_n}(x) = \frac{1}{h_n^3} K\left(\frac{\|x\|}{h_n}\right),$$

where $x \in \mathbb{R}^3$, and $\|.\|$ denotes the Euclidean norm. In order to verify the Pareto-type behavior of $Y^{(1)}$ and $Y^{(2)}$, we constructed the local Hill plots of the $Y^{(1)}$ and $Y^{(2)}$ data, respectively, for which the X coordinate is in a neighborhood of (latitude, longitude)=(33.84,-84.45), and date of loss equal to 2018, July, see Figure 5. The location under consideration is in the city of Atlanta. In these plots we show the local Hill estimates $H_k^{(j)}(x_0) := \frac{1}{k}\sum_{i=1}^k \ln \widetilde{Y}^{(j)}_{n_{x_0}-i+1,n_{x_0}} - \ln \widetilde{Y}^{(j)}_{n_{x_0}-k,n_{x_0}}$ as a function of k, where $\widetilde{Y}^{(j)}_{i,n_{x_0}}$, $i=1,\ldots,n_{x_0}$, are the order statistics of the $Y^{(j)}$ data for which the X coordinate belongs to $B(x_0,h_n)$, and n_{x_0} is the number of observations in $B(x_0,h_n)$. For both $Y^{(1)}$ and $Y^{(2)}$ the Hill estimate is clearly positive for the smaller k values supporting the assumption of underlying conditional Pareto-type distributions. For $Y^{(1)}$, total claim amount, the Hill plot shows a stable estimate for $\gamma_1(x_0)$ of about 0.3 when k is in the range 50-200. This satisfies the theoretical requirement that $\gamma_1(x_0) < 0.5-\kappa$, with $\kappa>0$, small. For $Y^{(2)}$, capital insured, the Hill plot shows some systematic pattern beyond $k=150$, which is due to the occurrence of repeated values for this variable. Despite this, the local Hill plot also suggests an underlying conditional Pareto-type distribution. Similar local Hill plots were obtained at other locations and for other time points. Next we illustrate the estimation of the conditional marginal expected shortfall at the above mentioned location, for the period 2008 till present, and using $k=1\%$ (solid line) and $k=10\%$ (dashed-dotted line) of n, respectively, see Figure 6. As expected, the conditional marginal expected shortfall at quantile level $k/n=0.01$ shows more variability than the one at level $k/n=0.10$, due to the smaller amount of data available to estimate the former, but otherwise they show the same pattern. The plot shows clearly the catastrophic Atlanta flood in 2009, September, resulting from multiple days of prolonged rainfall. The height of the event was on September 20-21 where 10 to 20 inches of rain occurred in less than 24 hours, which led to flash flooding, with flooded river basins remaining swollen for weeks. For this period, the difference between the two levels of the conditional marginal expected shortfall is larger than at other time points included in the analysis, which can be probably explained by the increased frequency of very large damage claims.

5 Appendix

Lemma 5.1. *Assume $(\mathcal{D})$ and $(\mathcal{H})$ and $x_0 \in Int(S_X)$. Let $(t_n)_{n\geqslant 1}$ and $(h_n)_{n\geqslant 1}$ be arbitrary sequences satisfying $t_n \to \infty$ and $h_n \to 0$ such that $h_n^{\eta_{\gamma_j}\wedge\eta_{\varepsilon_j}} \ln t_n \to 0$, as $n\to\infty$, and $0 \leqslant \eta < 1$. Then*

$$\left|\frac{t_n \overline{F}_j(U_j(t_n/y|x_0)|x)}{y^\eta} - y^{1-\eta}\right| \to 0, \ \text{as } n\to\infty,$$

uniformly in $y \in (0,T]$ and $x \in B(x_0,h_n)$.

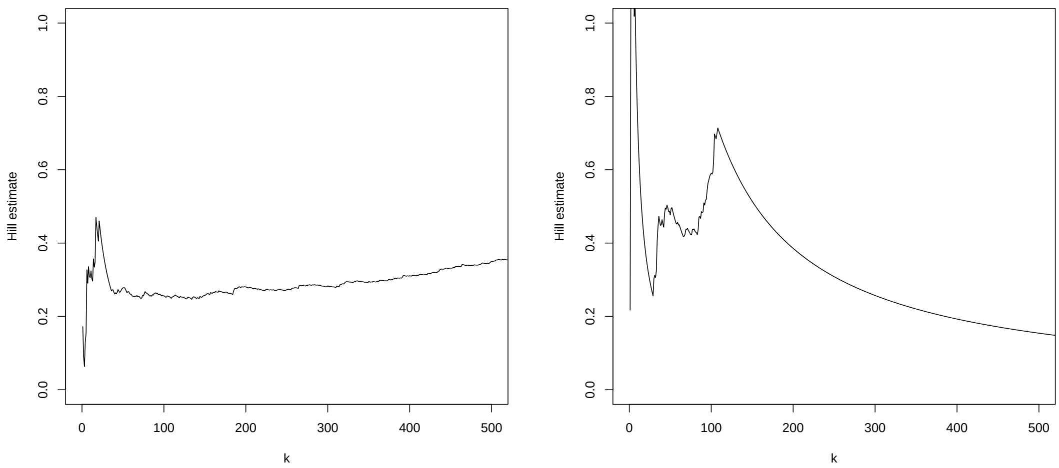

Figure 5: FEMA claim data: local Hill plots for $Y^{(1)}$, total claim amount (left), and $Y^{(2)}$, capital insured (right).

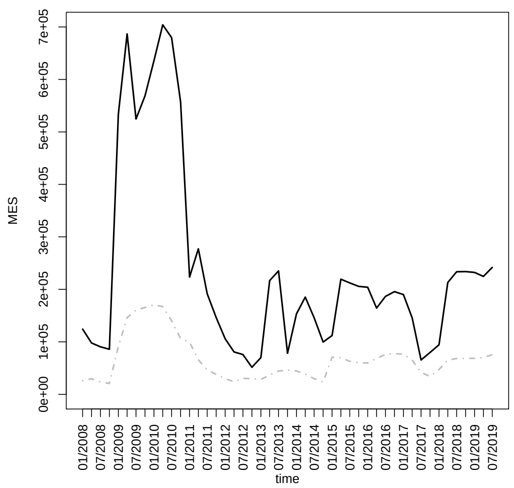

Figure 6: FEMA claim data: $\overline{\theta}_n$ with $k = 1\%$ (solid line) and $k = 10\%$ (dashed-dotted line) of n, as a function of time, for location (latitude, longitude)=(33.84,-84.45).

Proof

First note that, by continuity of $y \to F_j(y|x)$,

$$t_n \overline{F}_j(U_j(t_n/y|x_0)|x) = y\ \frac{\overline{F}_j(U_j(t_n/y|x_0)|x)}{\overline{F}_j(U_j(t_n/y|x_0)|x_0)}.$$

Then, from condition $(\mathcal{D})$, and a straightforward decomposition,

$$\begin{aligned}
&\left|\frac{t_n \overline{F}_j(U_j(t_n/y|x_0)|x)}{y^\eta} - y^{1-\eta}\right| \\
&\leqslant\ y^{1-\eta}\left\{\left|\frac{A_j(x)}{A_j(x_0)} - 1\right| (U_j(t_n/y|x_0))^{1/\gamma_j(x_0)-1/\gamma_j(x)} \frac{1+\frac{1}{\gamma_j(x)}\delta_j(U_j(t_n/y|x_0)|x)}{1+\frac{1}{\gamma_j(x_0)}\delta_j(U_j(t_n/y|x_0)|x_0)}\right. \\
&\quad + \left|(U_j(t_n/y|x_0))^{1/\gamma_j(x_0)-1/\gamma_j(x)} - 1\right| \frac{1+\frac{1}{\gamma_j(x)}\delta_j(U_j(t_n/y|x_0)|x)}{1+\frac{1}{\gamma_j(x_0)}\delta_j(U_j(t_n/y|x_0)|x_0)} \\
&\quad \left. + \left|\frac{1+\frac{1}{\gamma_j(x)}\delta_j(U_j(t_n/y|x_0)|x)}{1+\frac{1}{\gamma_j(x_0)}\delta_j(U_j(t_n/y|x_0)|x_0)} - 1\right|\right\}.
\end{aligned}$$

Each of the absolute differences in the right-hand side of the above display can be handled by condition $(\mathcal{H})$. Obviously, for some constant C,

$$\left|\frac{A_j(x)}{A_j(x_0)} - 1\right| \leqslant C h_n^{\eta_{A_j}}, \quad \text{for} \ \ x \in B(x_0, h_n).$$

Next, using the inequality $|e^z - 1| \leqslant e^{|z|}|z|$, we have, for some constant C (not necessarily equal to the one introduced above), and $x \in B(x_0, h_n)$,

$$\left|(U_j(t_n/y|x_0))^{1/\gamma_j(x_0)-1/\gamma_j(x)} - 1\right| \leqslant e^{C h_n^{\eta_{\gamma_j}} \ln U_j(t_n/y|x_0)} C h_n^{\eta_{\gamma_j}} \ln U_j(t_n/y|x_0).$$

For distributions satisfying $(\mathcal{D})$, one easily verifies that

$$U_j(t_n|x_0) = (A_j(x_0))^{\gamma_j(x_0)} t_n^{\gamma_j(x_0)} (1 + a_j(t_n|x_0)) \tag{5.1}$$

where $|a_j(.|x_0)|$ is regularly varying with index equal to $-\gamma_j(x_0)\beta_j(x_0)$. Hence, for some constants C_1 and C_2, not depending on y, one gets for $x \in B(x_0, h_n)$ and n large,

$$\left|(U_j(t_n/y|x_0))^{1/\gamma_j(x_0)-1/\gamma_j(x)} - 1\right| \leqslant C_1 t_n^{C_2 h_n^{\eta_{\gamma_j}}} y^{-C_2 h_n^{\eta_{\gamma_j}}} \left(h_n^{\eta_{\gamma_j}} \ln t_n - h_n^{\eta_{\gamma_j}} \ln y\right).$$

Finally, for n large,

$$\begin{aligned}
&\left|\frac{1+\frac{1}{\gamma_j(x)}\delta_j(U_j(t_n/y|x_0)|x)}{1+\frac{1}{\gamma_j(x_0)}\delta_j(U_j(t_n/y|x_0)|x_0)} - 1\right| \\
&\leqslant\ C|\delta_j(U_j(t_n/y|x_0)|x_0)| \left\{\left|\frac{\delta_j(U_j(t_n/y|x_0)|x)}{\delta_j(U_j(t_n/y|x_0)|x_0)} - 1\right| + \left|\frac{1}{\gamma_j(x)} - \frac{1}{\gamma_j(x_0)}\right|\right\}.
\end{aligned}$$

By the assumptions on δ_j we obtain

$$\left|\frac{\delta_j(U_j(t_n/y|x_0)|x)}{\delta_j(U_j(t_n/y|x_0)|x_0)} - 1\right| \leqslant \left|\frac{B_j(x)}{B_j(x_0)} - 1\right| e^{\int_1^{U_j(t_n/y|x_0)} \frac{\varepsilon_j(u|x)-\varepsilon_j(u|x_0)}{u}du} + \left|e^{\int_1^{U_j(t_n/y|x_0)} \frac{\varepsilon_j(u|x)-\varepsilon_j(u|x_0)}{u}du} - 1\right|,$$

and, hence, using $(\mathcal{H})$, for $x \in B(x_0, h_n)$ and n large,

$$\left|\frac{1 + \frac{1}{\gamma_j(x)}\delta_j(U_j(t_n/y|x_0)|x)}{1 + \frac{1}{\gamma_j(x_0)}\delta_j(U_j(t_n/y|x_0)|x_0)} - 1\right| \leqslant C_1 \left[h_n^{\eta_{\gamma_j} \wedge \eta_{B_j}} + t_n^{C_2 h_n^{\eta_{\varepsilon_j}}} y^{-C_2 h_n^{\eta_{\varepsilon_j}}} \left(h_n^{\eta_{\varepsilon_j}} \ln t_n - h_n^{\eta_{\varepsilon_j}} \ln y\right)\right].$$

Combining the above results establishes the lemma. ■

Lemma 5.2. *Assume $(\mathcal{D})$, $(\mathcal{H})$, $(\mathcal{K})$ and $(\mathcal{R})$ with $x \to R(y_1, y_2|x)$ being a continuous function, and $x_0 \in Int(S_X)$ such that $f_X(x_0) > 0$. Consider sequences $k \to \infty$ and $h_n \to 0$ as $n \to \infty$ in such a way that $k/n \to 0$ and $h_n^{\eta_{\gamma_1} \wedge \eta_{\gamma_2} \wedge \eta_{\varepsilon_1} \wedge \eta_{\varepsilon_2}} \ln n/k \to 0$. Then, as $n \to \infty$*

$$\begin{aligned}
\mathbb{E}(T_n(y_1, y_2|x_0)) &\to f_X(x_0)R(y_1, y_2|x_0), \\
kh_n^d \mathbb{V}ar(T_n(y_1, y_2|x_0)) &\to \|K\|_2^2 f_X(x_0)R(y_1, y_2|x_0).
\end{aligned}$$

Proof

We have

$$\begin{aligned}
\mathbb{E}(T_n(y_1, y_2|x_0)) &= \frac{n}{k}\mathbb{E}\left[K_{h_n}(x_0 - X)1\!\!1_{\{\overline{F}_1(Y^{(1)}|x_0) \leqslant k/n\, y_1,\, \overline{F}_2(Y^{(2)}|x_0) \leqslant k/n\, y_2\}}\right] \\
&= \frac{n}{k}\int_{S_K} K(v)\mathbb{P}(\overline{F}_1(Y^{(1)}|x_0) \leqslant k/n\, y_1,\, \overline{F}_2(Y^{(2)}|x_0) \leqslant k/n\, y_2|X = x_0 - h_n v) \\
&\qquad \times f_X(x_0 - h_n v)dv \\
&= \int_{S_K} K(v)R(y_1, y_2|x_0 - h_n v)f_X(x_0 - h_n v)dv \\
&\quad + \int_{S_K} K(v)\left[\frac{n}{k}\mathbb{P}(\overline{F}_1(Y^{(1)}|x_0) \leqslant k/n\, y_1,\, \overline{F}_2(Y^{(2)}|x_0) \leqslant k/n\, y_2|X = x_0 - h_n v)\right. \\
&\qquad \left. - R(y_1, y_2|x_0 - h_n v)\right] f_X(x_0 - h_n v)dv \\
&=: T_{1,n} + T_{2,n}.
\end{aligned}$$

Concerning $T_{1,n}$, by the continuity of $f_X(x)$ and $R(y_1, y_2|x)$ at x_0, we have that f_X and R are bounded in a neighborhood of x_0, and hence, by Lebesgue's dominated convergence theorem

$$T_{1,n} \to f_X(x_0)R(y_1, y_2|x_0), \text{ as } n \to \infty.$$

As for $T_{2,n}$,

$$\begin{aligned}
|T_{2,n}| &\leqslant \sup_{v \in S_K}\left|\frac{n}{k}\mathbb{P}(\overline{F}_1(Y^{(1)}|x_0) \leqslant k/n\, y_1,\, \overline{F}_2(Y^{(2)}|x_0) \leqslant k/n\, y_2|X = x_0 - h_n v)\right. \\
&\qquad \left. - R(y_1, y_2|x_0 - h_n v)\right| \int_{S_K} K(v)f_X(x_0 - h_n v)dv,
\end{aligned}$$

and note that

$$
\begin{aligned}
&\mathbb{P}(\overline{F}_1(Y^{(1)}|x_0) \leqslant k/n\; y_1,\, \overline{F}_2(Y^{(2)}|x_0) \leqslant k/n\; y_2|X = x_0 - h_n v) \\
&= \mathbb{P}\Bigg(\overline{F}_1(Y^{(1)}|x_0 - h_n v) \leqslant \frac{k}{n}\frac{n}{k}\overline{F}_1(U_1(n/(ky_1)|x_0)|x_0 - h_n v), \\
&\qquad \overline{F}_2(Y^{(2)}|x_0 - h_n v) \leqslant \frac{k}{n}\frac{n}{k}\overline{F}_2(U_2(n/(ky_2)|x_0)|x_0 - h_n v)|X = x_0 - h_n v\Bigg).
\end{aligned}
$$

Then, by the result of Lemma 5.1 and the uniformity of the convergence in Assumption $(\mathcal{R})$, we have that $T_{2,n} \to 0$ as $n \to \infty$.

Now, consider the variance. We have

$$
\begin{aligned}
kh_n^d \mathbb{V}ar(T_n(y_1, y_2|x_0)) &= \frac{nh_n^d \mathbb{V}ar(K_{h_n}(x_0 - X)\mathbb{1}_{\{\overline{F}_1(Y^{(1)}|x_0)\leqslant k/n\; y_1,\, \overline{F}_2(Y^{(2)}|x_0)\leqslant k/n\; y_2\}})}{k} \\
&= \|K\|_2^2 \frac{n}{k}\mathbb{E}\left[\frac{1}{h_n^d\|K\|_2^2}K^2\left(\frac{x_0 - X}{h_n}\right)\mathbb{1}_{\{\overline{F}_1(Y^{(1)}|x_0)\leqslant k/n\; y_1,\, \overline{F}_2(Y^{(2)}|x_0)\leqslant k/n\; y_2\}}\right] \\
&\quad -\frac{kh_n^d}{n}\left\{\frac{n}{k}\mathbb{E}\left[K_{h_n}(x_0 - X)\mathbb{1}_{\{\overline{F}_1(Y^{(1)}|x_0)\leqslant k/n\; y_1,\, \overline{F}_2(Y^{(2)}|x_0)\leqslant k/n\; y_2\}}\right]\right\}^2,
\end{aligned}
$$

from which the result follows. ∎

5.1 Proof of Theorem 2.1

To prove the result we will make use of empirical process theory with changing function classes, see for instance van der Vaart and Wellner (1996). To this aim we start by introducing some notation. Let P be the distribution measure of $(Y^{(1)}, Y^{(2)}, X)$, and denote the expected value under P, the empirical version and empirical process as follows

$$
Pf := \int f dP, \quad \mathbb{P}_n f := \frac{1}{n}\sum_{i=1}^{n} f\left(Y_i^{(1)}, Y_i^{(2)}, X_i\right), \quad \mathbb{G}_n f := \sqrt{n}(\mathbb{P}_n - P)f,
$$

for any real-valued measurable function $f : \mathbb{R}^2 \times \mathbb{R}^d \to \mathbb{R}$. For a function class $\mathcal{F}$, let $N_{[]}(\varepsilon, \mathcal{F}, L_2(P))$, denote the minimal number of $\varepsilon-$brackets needed to cover $\mathcal{F}$. The bracketing integral is then defined as

$$
J_{[]}(\delta, \mathcal{F}, L_2(P)) = \int_0^\delta \sqrt{\ln N_{[]}(\varepsilon, \mathcal{F}, L_2(P))}d\varepsilon.
$$

We introduce our sequence of classes $\mathcal{F}_n$ on $\mathbb{R}^2 \times \mathbb{R}^d$ as

$$
\mathcal{F}_n := \left\{(u, z) \to f_{n,y}(u, z),\;\; y \in (0, T]^2\right\}
$$

where

$$
f_{n,y}(u, z) := \sqrt{\frac{nh_n^d}{k}}K_{h_n}(x_0 - z)\frac{\mathbb{1}_{\{\overline{F}_1(u_1|x_0)\leqslant k/n\, y_1,\, \overline{F}_2(u_2|x_0)\leqslant k/n\, y_2\}}}{y_1^\eta}.
$$

Denote also by F_n an envelope function of the class $\mathcal{F}_n$. Now, according to Theorem 19.28 in van der Vaart (1998), the weak convergence of the stochastic process (2.3) follows from the following four conditions. Let ρ_{x_0} be a semimetric, possibly depending on x_0, making $(0,T]^2$ totally bounded. We have to prove that

$$\begin{aligned}\sup_{\rho_{x_0}(y,\bar{y})\leqslant\delta_n} P(f_{n,y}-f_{n,\bar{y}})^2 &\longrightarrow 0 \text{ for every } \delta_n \searrow 0, &(5.2)\\ PF_n^2 &= O(1), &(5.3)\\ PF_n^2 1\!\!1_{\{F_n>\varepsilon\sqrt{n}\}} &\longrightarrow 0 \text{ for every } \varepsilon>0, &(5.4)\\ J_{[]}(\delta_n,\mathcal{F}_n,L_2(P)) &\longrightarrow 0 \text{ for every } \delta_n\searrow 0, &(5.5)\end{aligned}$$

along with the pointwise convergence of the covariance function.

We start with verifying condition (5.2), with $\rho_{x_0}(y,\bar{y}) := |y_1-\bar{y}_1|+|y_2-\bar{y}_2|$. Denote $A_{n,y} := \{\overline{F}_1(Y^{(1)}|x_0)\leqslant k/n\, y_1, \overline{F}_2(Y^{(2)}|x_0)\leqslant k/n\, y_2\}$. We have then

$$\begin{aligned}P(f_{n,y}-f_{n,\bar{y}})^2 &= \frac{nh_n^d}{k}\,\mathbb{E}\left[K_{h_n}^2(x_0-X)\left(\frac{1\!\!1_{A_{n,y}}}{y_1^\eta}-\frac{1\!\!1_{A_{n,\bar{y}}}}{\bar{y}_1^\eta}\right)^2\right]\\ &= \frac{nh_n^d}{k}\,\mathbb{E}\left[K_{h_n}^2(x_0-X)\mathbb{E}\left[\left(\frac{1\!\!1_{A_{n,y}}}{y_1^\eta}-\frac{1\!\!1_{A_{n,\bar{y}}}}{\bar{y}_1^\eta}\right)^2\middle| X\right]\right]. \qquad (5.6)\end{aligned}$$

We consider now three cases.

Case 1: $y_1\wedge\bar{y}_1\leqslant\delta_n$. Assume without loss of generality that $y_1\leqslant\bar{y}_1$. By expanding the square in the above conditional expectation and using the fact that, e.g., $A_{n,y}\subset\{\overline{F}_1(Y^{(1)}|x_0)\leqslant k/n\, y_1\}$, we obtain the following inequality

$$\begin{aligned}\mathbb{E}\left[\left(\frac{1\!\!1_{A_{n,y}}}{y_1^\eta}-\frac{1\!\!1_{A_{n,\bar{y}}}}{\bar{y}_1^\eta}\right)^2\middle| X=x\right] &\leqslant \frac{3P(\overline{F}_1(Y^{(1)}|x_0)\leqslant k/n\, y_1|X=x)}{y_1^{2\eta}}\\ &\quad+\frac{P(\overline{F}_1(Y^{(1)}|x_0)\leqslant k/n\, \bar{y}_1|X=x)}{\bar{y}_1^{2\eta}},\end{aligned}$$

which, after substituting in (5.6) leads to

$$\begin{aligned}P(f_{n,y}-f_{n,\bar{y}})^2 &\leqslant 3\frac{n}{k}\int_{S_K}K^2(v)\frac{P(\overline{F}_1(Y^{(1)}|x_0)\leqslant k/n\, y_1|X=x_0-h_nv)}{y_1^{2\eta}}f_X(x_0-h_nv)dv\\ &\quad+\frac{n}{k}\int_{S_K}K^2(v)\frac{P(\overline{F}_1(Y^{(1)}|x_0)\leqslant k/n\, \bar{y}_1|X=x_0-h_nv)}{\bar{y}_1^{2\eta}}f_X(x_0-h_nv)dv.\end{aligned}$$

Now note that

$$P(\overline{F}_1(Y^{(1)}|x_0)\leqslant k/n\, y_1|X=x_0-h_nv)=\overline{F}_1\left(U_1(n/(ky_1)|x_0)|x_0-h_nv\right),$$

which, together with the result of Lemma 5.1, motivates the following decomposition

$$\begin{aligned}
P(f_{n,y}-f_{n,\bar{y}})^2 \leqslant\ & 3y_1^{1-2\eta}\int_{S_K} K^2(v)f_X(x_0-h_nv)dv \\
&+3\int_{S_K} K^2(v)\left[\frac{1}{y_1^{2\eta}}\frac{n}{k}\overline{F}_1\left(U_1(n/(ky_1)|x_0)|x_0-h_nv\right)-y_1^{1-2\eta}\right]f_X(x_0-h_nv)dv \\
&+\bar{y}_1^{1-2\eta}\int_{S_K} K^2(v)f_X(x_0-h_nv)dv \\
&+\int_{S_K} K^2(v)\left[\frac{1}{\bar{y}_1^{2\eta}}\frac{n}{k}\overline{F}_1\left(U_1(n/(k\bar{y}_1)|x_0)|x_0-h_nv\right)-\bar{y}_1^{1-2\eta}\right]f_X(x_0-h_nv)dv.
\end{aligned}$$

Using Lemma 5.1 and the fact that $\rho_{x_0}(y,\bar{y})\leqslant\delta_n$ which implies $\bar{y}_1\leqslant 2\delta_n$, we get

$$P(f_{n,y}-f_{n,\bar{y}})^2 \leqslant 5\delta_n^{1-2\eta}\int_{S_K} K^2(v)f_X(x_0-h_nv)dv+o(1),$$

where the $o(1)$ term does not depend on y_1 and $\bar{y}_1$.

Case 2: $y_1\wedge\bar{y}_1>\delta_n$ and $y_2\wedge\bar{y}_2\leqslant\delta_n$. Assume without loss of generality that $y_2\leqslant\bar{y}_2$. Similarly to the approach followed in Case 1, we obtain

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{1\!\!1_{A_{n,y}}}{y_1^{\eta}}-\frac{1\!\!1_{A_{n,\bar{y}}}}{\bar{y}_1^{\eta}}\right)^2\middle| X=x\right] \leqslant\ & \frac{3P(\overline{F}_2(Y^{(2)}|x_0)\leqslant k/n\ y_2|X=x)}{(y_1\wedge\bar{y}_1)^{2\eta}} \\
&+\frac{P(\overline{F}_2(Y^{(2)}|x_0)\leqslant k/n\ \bar{y}_2|X=x)}{(y_1\wedge\bar{y}_1)^{2\eta}},
\end{aligned}$$

and thus

$$\begin{aligned}
P(f_{n,y}-f_{n,\bar{y}})^2 \leqslant\ & \frac{3y_2}{(y_1\wedge\bar{y}_1)^{2\eta}}\int_{S_K} K^2(v)f_X(x_0-h_nv)dv \\
&+\frac{3y_2^{2\eta}}{(y_1\wedge\bar{y}_1)^{2\eta}}\int_{S_K} K^2(v)\left[\frac{1}{y_2^{2\eta}}\frac{n}{k}\overline{F}_2\left(U_2(n/(ky_2)|x_0)|x_0-h_nv\right)-y_2^{1-2\eta}\right]f_X(x_0-h_nv)dv \\
&+\frac{\bar{y}_2}{(y_1\wedge\bar{y}_1)^{2\eta}}\int_{S_K} K^2(v)f_X(x_0-h_nv)dv \\
&+\frac{\bar{y}_2^{2\eta}}{(y_1\wedge\bar{y}_1)^{2\eta}}\int_{S_K} K^2(v)\left[\frac{1}{\bar{y}_2^{2\eta}}\frac{n}{k}\overline{F}_2\left(U_2(n/(k\bar{y}_2)|x_0)|x_0-h_nv\right)-\bar{y}_2^{1-2\eta}\right]f_X(x_0-h_nv)dv.
\end{aligned}$$

Again by Lemma 5.1 and using that $\bar{y}_2\leqslant 2\delta_n$ we have that

$$P(f_{n,y}-f_{n,\bar{y}})^2 \leqslant 5\delta_n^{1-2\eta}\int_{S_K} K^2(v)f_X(x_0-h_nv)dv+o(1),$$

where the $o(1)$ term does not depend on y_2 and $\bar{y}_2$.

Case 3: $y_1 \wedge \bar{y}_1 > \delta_n$ *and* $y_2 \wedge \bar{y}_2 > \delta_n$. Let $y \vee \bar{y}$ denote the vector with the component-wise maxima of y and $\bar{y}$, and similarly $y \wedge \bar{y}$ is the vector with the component-wise mimima of y and $\bar{y}$. Then

$$P(f_{n,y} - f_{n,\bar{y}})^2 \leqslant \frac{nh_n^d}{k} \mathbb{E}\left[K_{h_n}^2(x_0 - X)\mathbb{E}\left[\left(\frac{\mathbb{1}_{A_{n,y\vee\bar{y}}}}{(y_1 \wedge \bar{y}_1)^\eta} - \frac{\mathbb{1}_{A_{n,y\wedge\bar{y}}}}{(y_1 \vee \bar{y}_1)^\eta}\right)^2 \middle| X\right]\right].$$

Note that

$$\left(\frac{\mathbb{1}_{A_{n,y\vee\bar{y}}}}{(y_1 \wedge \bar{y}_1)^\eta} - \frac{\mathbb{1}_{A_{n,y\wedge\bar{y}}}}{(y_1 \vee \bar{y}_1)^\eta}\right)^2 = \left(\frac{1}{y_1^\eta} - \frac{1}{\bar{y}_1^\eta}\right)^2 \mathbb{1}_{A_{n,y\wedge\bar{y}}} + \frac{1}{(y_1 \wedge \bar{y}_1)^{2\eta}}(\mathbb{1}_{A_{n,y\vee\bar{y}}} - \mathbb{1}_{A_{n,y\wedge\bar{y}}}), \tag{5.7}$$

which leads to

$$\begin{aligned}
&P(f_{n,y} - f_{n,\bar{y}})^2 \\
&\leqslant \frac{(y_1^\eta - \bar{y}_1^\eta)^2}{(y_1\bar{y}_1)^{2\eta}} \frac{n}{k} \int_{S_K} K^2(v) P\left(\overline{F}_1(Y^{(1)}|x_0) \leqslant k/n\, y_1 \wedge \bar{y}_1, \overline{F}_2(Y^{(2)}|x_0) \leqslant k/n\, y_2 \wedge \bar{y}_2 \middle| X = x_0 - h_n v\right) \\
&\qquad \times f_X(x_0 - h_n v) dv \\
&\quad + \frac{1}{(y_1 \wedge \bar{y}_1)^{2\eta}} \frac{n}{k} \int_{S_K} K^2(v) \left[P\left(\overline{F}_1(Y^{(1)}|x_0) \leqslant k/n\, y_1 \vee \bar{y}_1, \overline{F}_2(Y^{(2)}|x_0) \leqslant k/n\, y_2 \vee \bar{y}_2 \middle| X = x_0 - h_n v\right)\right. \\
&\quad \left. - P\left(\overline{F}_1(Y^{(1)}|x_0) \leqslant k/n\, y_1 \wedge \bar{y}_1, \overline{F}_2(Y^{(2)}|x_0) \leqslant k/n\, y_2 \wedge \bar{y}_2 \middle| X = x_0 - h_n v\right)\right] f_X(x_0 - h_n v) dv \\
&=: Q_{1,n} + Q_{2,n}.
\end{aligned}$$

As for $Q_{1,n}$, we easily obtain

$$Q_{1,n} \leqslant \frac{(y_1^\eta - \bar{y}_1^\eta)^2}{(y_1\bar{y}_1)^{2\eta}} \int_{S_K} K^2(v) \frac{n}{k} \overline{F}_1\left(U_1(n/(k\, y_1 \wedge \bar{y}_1)|x_0)|x_0 - h_n v\right) f_X(x_0 - h_n v) dv.$$

Now, by the mean value theorem, applied to $(y_1^\eta - \bar{y}_1^\eta)^2$, and a decomposition motivated by Lemma 5.1,

$$\begin{aligned}
&Q_{1,n} \\
&\leqslant (y_1 \wedge \bar{y}_1)^{-1-2\eta}(y_1 - \bar{y}_1)^2 \int_{S_K} K^2(v) f_X(x_0 - h_n v) dv \\
&\quad + (y_1 \wedge \bar{y}_1)^{-2}(y_1 - \bar{y}_1)^2 \int_{S_K} K^2(v) \left[\frac{1}{(y_1 \wedge \bar{y}_1)^{2\eta}} \frac{n}{k} \overline{F}_1\left(U_1(n/(k\, y_1 \wedge \bar{y}_1)|x_0) \middle| x_0 - h_n v\right) - (y_1 \wedge \bar{y}_1)^{1-2\eta}\right] \\
&\qquad \times f_X(x_0 - h_n v) dv.
\end{aligned}$$

This then gives

$$Q_{1,n} \leqslant \delta_n^{1-2\eta} \int_{S_K} K^2(v) f_X(x_0 - h_n v) dv + o(1),$$

where the $o(1)$ term does not depend on y_1 and $\bar{y}_1$.

Concerning $Q_{2,n}$, we have the following inequality

$$
\begin{aligned}
& Q_{2,n} \\
\leqslant & \frac{1}{(y_1 \wedge \bar{y}_1)^{2\eta}} \frac{n}{k} \int_{S_K} K^2(v) P\left(k/n\, y_1 \wedge \bar{y}_1 \leqslant \overline{F}_1(Y^{(1)}|x_0) \leqslant k/n\, y_1 \vee \bar{y}_1 \Big| X = x_0 - h_n v\right) f_X(x_0 - h_n v) dv \\
& + \frac{1}{(y_1 \wedge \bar{y}_1)^{2\eta}} \frac{n}{k} \int_{S_K} K^2(v) P\left(k/n\, y_2 \wedge \bar{y}_2 \leqslant \overline{F}_2(Y^{(2)}|x_0) \leqslant k/n\, y_2 \vee \bar{y}_2 \Big| X = x_0 - h_n v\right) f_X(x_0 - h_n v) dv. \\
=: & \; Q_{2,1,n} + Q_{2,2,n}.
\end{aligned}
$$

We only give details about $Q_{2,1,n}$, the term $Q_{2,2,n}$ can be handled analogously. Direct computations give

$$
Q_{2,1,n} = \frac{1}{(y_1 \wedge \bar{y}_1)^{2\eta}} \frac{n}{k} \int_{S_K} K^2(v) \int_{U_1(n/(k(y_1 \vee \bar{y}_1))|x_0)}^{U_1(n/(k(y_1 \wedge \bar{y}_1))|x_0)} f_1(y|x_0 - h_n v) dy f_X(x_0 - h_n v) dv,
$$

and, after substituting $u = (n/k)\overline{F}_1(y|x_0)$, we have

$$
Q_{2,1,n} = \frac{1}{(y_1 \wedge \bar{y}_1)^{2\eta}} \int_{S_K} K^2(v) \int_{y_1 \wedge \bar{y}_1}^{y_1 \vee \bar{y}_1} \frac{f_1(U_1(n/(ku)|x_0)|x_0 - h_n v)}{f_1(U_1(n/(ku)|x_0)|x_0)} du f_X(x_0 - h_n v) dv.
$$

Using (2.1) and arguments similar to those used in the proof of Lemma 5.1 one obtains for n large and some small $\kappa > 0$,

$$
\frac{f_1(U_1(n/(ku)|x_0)|x_0 - h_n v)}{f_1(U_1(n/(ku)|x_0)|x_0)} \leqslant C u^{-\kappa},
$$

where C does not depend on u. Then, for n large enough,

$$
\begin{aligned}
Q_{2,1,n} & \leqslant \frac{C}{(y_1 \wedge \bar{y}_1)^{2\eta}} \int_{y_1 \wedge \bar{y}_1}^{y_1 \vee \bar{y}_1} u^{-\kappa} du \int_{S_K} K^2(v) f_X(x_0 - h_n v) dv \\
& \leqslant \frac{C}{(y_1 \wedge \bar{y}_1)^{2\eta}} (y_1 \wedge \bar{y}_1)^{-\kappa} (y_1 \vee \bar{y}_1 - y_1 \wedge \bar{y}_1) \\
& \leqslant C \delta_n^{1-2\eta-\kappa}.
\end{aligned}
$$

Combining all the above we have verified (5.2).

Now, we move to the proof of (5.3). A natural envelope function of the class $\mathcal{F}_n$ is

$$
F_n(u, z) := \sqrt{\frac{n h_n^d}{k}} K_{h_n}(x_0 - z) \frac{\mathbb{1}_{\{\overline{F}_1(u_1|x_0) \leqslant kT/n\}}}{[(n/k)\, \overline{F}_1(u_1|x_0)]^{\eta}}.
$$

This yields

$$
\begin{aligned}
PF_n^2 &= \left(\frac{n}{k}\right)^{1-2\eta} h_n^d\, \mathbb{E}\left(K_{h_n}^2(x_0-X)\mathbb{E}\left[\frac{1\!\!1_{\{\overline{F}_1(Y^{(1)}|x_0)\leqslant kT/n\}}}{(\overline{F}_1(Y^{(1)}|x_0))^{2\eta}} \middle| X \right]\right)\\
&= \left(\frac{n}{k}\right)^{1-2\eta} \int_{S_K} K^2(v)\mathbb{E}\left[\frac{1\!\!1_{\{\overline{F}_1(Y^{(1)}|x_0)\leqslant kT/n\}}}{(\overline{F}_1(Y^{(1)}|x_0))^{2\eta}} \middle| X=x_0-h_nv \right] f_X(x_0-h_nv)dv\\
&= \left(\frac{n}{k}\right)^{1-2\eta} \int_{S_K} K^2(v) \int_{U_1(n/(kT)|x_0)}^{\infty} \frac{1}{(\overline{F}_1(y|x_0-h_nv))^{2\eta}} dF_1(y|x_0-h_nv) f_X(x_0-h_nv)dv\\
&\quad + \left(\frac{n}{k}\right)^{1-2\eta} \int_{S_K} K^2(v) \int_{U_1(n/(kT)|x_0)}^{\infty} \frac{1}{(\overline{F}_1(y|x_0-h_nv))^{2\eta}}\\
&\qquad\qquad \times \left\{ \left(\frac{\overline{F}_1(y|x_0-h_nv)}{\overline{F}_1(y|x_0)} \right)^{2\eta} - 1 \right\} dF_1(y|x_0-h_nv) f_X(x_0-h_nv)dv\\
&=: Q_{3,n}(T) + Q_{4,n}(T).
\end{aligned}
$$

Concerning $Q_{3,n}(T)$ we obtain by direct integration and a slight adjustment of Lemma 5.1, for large n

$$
\begin{aligned}
Q_{3,n}(T) &= \frac{1}{1-2\eta}\left(\frac{n}{k}\right)^{1-2\eta} \int_{S_K} K^2(v)[\overline{F}_1(U_1(n/(kT)|x_0)|x_0-h_nv)]^{1-2\eta} f_X(x_0-h_nv)dv\\
&= \frac{T^{1-2\eta}}{1-2\eta}\int_{S_K} K^2(v) f_X(x_0-h_nv)dv\\
&\quad + \frac{1}{1-2\eta}\int_{S_K} K^2(v)\left[\left(\frac{n}{k}\overline{F}_1(U_1(n/(kT)|x_0)|x_0-h_nv) \right)^{1-2\eta} - T^{1-2\eta} \right] f_X(x_0-h_nv)dv\\
&\leqslant CT^{1-2\eta-\kappa}, \qquad (5.8)
\end{aligned}
$$

for $\kappa < 1-2\eta$.

Concerning $Q_{4,n}(T)$, combining $(\mathcal{D})$ with $(\mathcal{H})$ gives the following bound, for n large and $y \geqslant U_1(n/(kT)|x_0)$,

$$
\begin{aligned}
\left| \left(\frac{\overline{F}_1(y|x_0-h_nv)}{\overline{F}_1(y|x_0)} \right)^{2\eta} - 1 \right| &\leqslant C_1 \left(h_n^{\eta_{A_1}} + y^{C_2 h_n^{\eta_{\gamma_1}}} h_n^{\eta_{\gamma_1}} \ln y + |\delta_1(y|x_0)| h_n^{\eta_{B_1}} \right.\\
&\qquad \left. + |\delta_1(y|x_0)| y^{C_3 h_n^{\eta_{\varepsilon_1}}} h_n^{\eta_{\varepsilon_1}} \ln y \right). \qquad (5.9)
\end{aligned}
$$

Each of the terms in the right-hand side of the above inequality needs now be used in $Q_{4,n}(T)$, leading to the terms $Q_{4,j,n}(T)$, $j=1,\ldots,4$, studied below. First

$$
Q_{4,1,n}(T) := h_n^{\eta_{A_1}} \left(\frac{n}{k}\right)^{1-2\eta} \int_{S_K} K^2(v) \int_{U_1(n/(kT)|x_0)}^{\infty} \frac{1}{(\overline{F}_1(y|x_0-h_nv))^{2\eta}} dF_1(y|x_0-h_nv) f_X(x_0-h_nv)dv.
$$

This term is clearly of smaller order than $Q_{3,n}(T)$ studied above and hence $Q_{4,1,n}(T) = O(1)$. For the second term in the right-hand side of (5.9) we need to study

$$
Q_{4,2,n}(T) := h_n^{\eta_{\gamma_1}} \left(\frac{n}{k}\right)^{1-2\eta} \int_{S_K} K^2(v) \int_{t_n(T)}^{\infty} y^{\xi_{1,n}} \ln y \frac{1}{(\overline{F}_1(y|x_0-h_nv))^{2\eta}} dF_1(y|x_0-h_nv) f_X(x_0-h_nv)dv
$$

where $t_n(T) := U_1(n/(kT)|x_0)$ and $\xi_{1,n} := C_2 h_n^{\eta\gamma_1}$. Let $p_n(y) := \xi_{1,n} y^{\xi_{1,n}-1}\ln y + y^{\xi_{1,n}-1}$. Applying integration by parts on the inner integral gives, for n large enough,

$$
\begin{aligned}
Q_{4,2,n}(T) &= \left(\frac{n}{k}\right)^{1-2\eta} \frac{h_n^{\eta\gamma_1}\ln(t_n(T))[t_n(T)]^{\xi_{1,n}}}{1-2\eta}\int_{S_K} K^2(v)\left[\overline{F}_1(t_n(T)|x_0-h_nv)\right]^{1-2\eta} f_X(x_0-h_nv)dv \\
&\quad + \left(\frac{n}{k}\right)^{1-2\eta}\frac{h_n^{\eta\gamma_1}}{1-2\eta}\int_{S_K} K^2(v)\int_{t_n(T)}^{\infty} p_n(y)\left[\overline{F}_1(y|x_0-h_nv)\right]^{1-2\eta} dy f_X(x_0-h_nv)dv \\
&=: Q_{4,2,1,n}(T) + Q_{4,2,2,n}(T).
\end{aligned}
$$

We obtain, for n large enough

$$
\begin{aligned}
Q_{4,2,1,n}(T) &\leqslant C h_n^{\eta\gamma_1}\ln(t_n(T))[t_n(T)]^{\xi_{1,n}} T^{1-2\eta-\kappa} \\
&= O(1),
\end{aligned}
$$

by (5.1) and the fact that $h_n^{\eta\gamma_1}\ln(n/k)\to 0$ as $n\to\infty$.
Now consider $Q_{4,2,2,n}(T)$. We have

$$
\begin{aligned}
Q_{4,2,2,n}(T) &= \frac{h_n^{\eta\gamma_1}T^{1-2\eta}}{1-2\eta}\int_{S_K} K^2(v)\int_{t_n(T)}^{\infty} p_n(y)\left(\frac{\overline{F}_1(y|x_0-h_nv)}{\overline{F}_1(y|x_0)}\right)^{1-2\eta}\left(\frac{\overline{F}_1(y|x_0)}{\overline{F}_1(t_n(T)|x_0)}\right)^{1-2\eta} dy \\
&\qquad \times f_X(x_0-h_nv)dv.
\end{aligned}
$$

For n large and $y \geqslant t_n(T)$, with $\xi_{2,n} = C h_n^{\eta\varepsilon_1}$,

$$
\left(\frac{\overline{F}_1(y|x_0-h_nv)}{\overline{F}_1(y|x_0)}\right)^{1-2\eta} \leqslant C y^{\xi_{1,n}}\left(1 + y^{\xi_{2,n}} h_n^{\eta\varepsilon_1}\ln y\right).
$$

Substituting $u = y/t_n(T)$ we get

$$
\begin{aligned}
Q_{4,2,2,n}(T) &\leqslant C h_n^{\eta\gamma_1}T^{1-2\eta}[t_n(T)]^{1+\xi_{1,n}}\int_{S_K} K^2(v)\int_1^{\infty} p_n(t_n(T)u)u^{\xi_{1,n}}\left(1 + (t_n(T)u)^{\xi_{2,n}} h_n^{\eta\varepsilon_1}\ln(t_n(T)u)\right) \\
&\qquad \times \left(\frac{\overline{F}_1(t_n(T)u|x_0)}{\overline{F}_1(t_n(T)|x_0)}\right)^{1-2\eta} du f_X(x_0-h_nv)dv.
\end{aligned}
$$

Since $\overline{F}_1(.|x_0)$ is regularly varying, we can apply the Potter bound (see, e.g., de Haan and Ferreira, 2006, Proposition B.1.9), and obtain, for n large enough

$$
\begin{aligned}
Q_{4,2,2,n}(T) &\leqslant C h_n^{\eta\gamma_1}T^{1-2\eta}[t_n(T)]^{2\xi_{1,n}}\int_{S_K} K^2(v) f_X(x_0-h_nv)dv \\
&\quad \times \int_1^{\infty}\left(\xi_{1,n}u^{\xi_{1,n}-1}\ln(t_n(T)) + \xi_{1,n}u^{\xi_{1,n}-1}\ln u + u^{\xi_{1,n}-1}\right) u^{\xi_{1,n}-(1/\gamma_1(x_0)-\delta)(1-2\eta)} \\
&\qquad \times \left(1 + (t_n(T)u)^{\xi_{2,n}} h_n^{\eta\varepsilon_1}\ln(t_n(T)u)\right) du,
\end{aligned}
$$

where $0 < \delta < 1/\gamma_1(x_0)$. After tedious computations one gets

$$
\begin{aligned}
Q_{4,2,2,n}(T) &\leqslant C T^{1-2\eta} h_n^{\eta\gamma_1}[t_n(T)]^{2\xi_{1,n}}\left\{1 + h_n^{\eta\gamma_1}\ln(t_n(T)) + [t_n(T)]^{\xi_{2,n}} h_n^{\eta\varepsilon_1}\ln(t_n(T))\right\} \\
&= O(1),
\end{aligned}
$$

by (5.1) and the fact that $h_n^{\eta_{\gamma_1}\wedge\eta_{\varepsilon_1}} \ln(n/k) \to 0$ as $n \to \infty$. Hence, $Q_{4,2,n}(T) = O(1)$.

Finally, the two last terms $Q_{4,3,n}(T)$ and $Q_{4,4,n}(T)$ can be dealt with similarly as the two previous ones since

$$\begin{aligned} Q_{4,3,n}(T) &:= h_n^{\eta_{B_1}} \left(\frac{n}{k}\right)^{1-2\eta} \int_{S_K} K^2(v) \int_{t_n(T)}^{\infty} \frac{|\delta_1(y|x_0)|}{(\overline{F}_1(y|x_0-h_nv))^{2\eta}} dF_1(y|x_0-h_nv) f_X(x_0-h_nv)dv \\ &\leqslant \left(\sup_{y\geqslant t_n(T)} |\delta_1(y|x_0)|\right) h_n^{\eta_{B_1}} \left(\frac{n}{k}\right)^{1-2\eta} \\ &\quad \times \int_{S_K} K^2(v) \int_{t_n(T)}^{\infty} \frac{1}{(\overline{F}_1(y|x_0-h_nv))^{2\eta}} dF_1(y|x_0-h_nv) f_X(x_0-h_nv)dv \qquad (5.10) \end{aligned}$$

and

$$\begin{aligned} Q_{4,4,n}(T) &:= h_n^{\eta_{\varepsilon_1}} \left(\frac{n}{k}\right)^{1-2\eta} \int_{S_K} K^2(v) \int_{t_n(T)}^{\infty} \frac{|\delta_1(y|x_0)| y^{\xi_{2,n}} \ln y}{(\overline{F}_1(y|x_0-h_nv))^{2\eta}} dF_1(y|x_0-h_nv) f_X(x_0-h_nv)dv \\ &\leqslant \left(\sup_{y\geqslant t_n(T)} |\delta_1(y|x_0)|\right) h_n^{\eta_{\varepsilon_1}} \left(\frac{n}{k}\right)^{1-2\eta} \\ &\quad \times \int_{S_K} K^2(v) \int_{t_n(T)}^{\infty} \frac{y^{\xi_{2,n}} \ln y}{(\overline{F}_1(y|x_0-h_nv))^{2\eta}} dF_1(y|x_0-h_nv) f_X(x_0-h_nv)dv. \qquad (5.11) \end{aligned}$$

This yields $Q_{4,3,n}(T) = O(1)$ and $Q_{4,4,n}(T) = O(1)$. Combining all these results, we get (5.3).

Now, we want to show (5.4). To this aim, for any $\alpha \in (0, 1/\eta - 2)$, we have

$$\begin{aligned} PF_n^2 \mathbb{1}_{\{F_n > \varepsilon\sqrt{n}\}} &\leqslant \frac{1}{\varepsilon^\alpha n^{\alpha/2}} PF_n^{2+\alpha} \\ &= \frac{1}{\varepsilon^\alpha n^{\alpha/2}} \left(\frac{nh_n^d}{k}\right)^{1+\frac{\alpha}{2}} \mathbb{E}\left(K_{h_n}^{2+\alpha}(x_0-X) \frac{\mathbb{1}_{\{\overline{F}_1(Y^{(1)}|x_0)\leqslant kT/n\}}}{[(n/k)\,\overline{F}_1(Y^{(1)}|x_0)]^{\eta(2+\alpha)}}\right) \\ &= \frac{1}{\varepsilon^\alpha} \frac{1}{(kh_n^d)^{\alpha/2}} \left(\frac{n}{k}\right)^{1-\eta(2+\alpha)} \int_{S_K} K^{2+\alpha}(v) \mathbb{E}\left(\frac{\mathbb{1}_{\{\overline{F}_1(Y^{(1)}|x_0)\leqslant kT/n\}}}{[\overline{F}_1(Y^{(1)}|x_0)]^{\eta(2+\alpha)}} \middle| X = x_0 - h_nv\right) \\ &\qquad \times f_X(x_0-h_nv)dv \\ &= \frac{1}{\varepsilon^\alpha} \frac{1}{(kh_n^d)^{\alpha/2}} \left(\frac{n}{k}\right)^{1-\eta(2+\alpha)} \\ &\quad \times \left\{ \int_{S_K} K^{2+\alpha}(v) \int_{t_n(T)}^{\infty} \frac{1}{(\overline{F}_1(y|x_0-h_nv))^{\eta(2+\alpha)}} dF_1(y|x_0-h_nv) f_X(x_0-h_nv)dv \right. \\ &\quad + \int_{S_K} K^{2+\alpha}(v) \int_{t_n(T)}^{\infty} \frac{1}{(\overline{F}_1(y|x_0-h_nv))^{\eta(2+\alpha)}} \\ &\qquad \left. \times \left[\left(\frac{\overline{F}_1(y|x_0-h_nv)}{\overline{F}_1(y|x_0)}\right)^{\eta(2+\alpha)} - 1\right] dF_1(y|x_0-h_nv) f_X(x_0-h_nv)dv \right\}. \end{aligned}$$

The terms into brackets can be studied similarly as $Q_{j,n}(T), j = 3, 4$, and thus (5.4) is established since $kh_n^d \to \infty$.

Finally we verify (5.5). Without loss of generality assume $T = 1$ and consider, for $a, \theta, \tilde{\theta} < 1$, the classes

$$\begin{aligned}
\mathcal{F}_n^{(1)}(a) &:= \{f_{n,y} \in \mathcal{F}_n : y_1 \leqslant a\}, \\
\mathcal{F}_n^{(2)}(a) &:= \{f_{n,y} \in \mathcal{F}_n : y_1 > a, y_2 \leqslant a\}, \\
\mathcal{F}_n(\ell, m) &:= \{f_{n,y} \in \mathcal{F}_n : \theta^{\ell+1} \leqslant y_1 \leqslant \theta^\ell, \tilde{\theta}^{m+1} \leqslant y_2 \leqslant \tilde{\theta}^m\},
\end{aligned}$$

where $\ell = 0, \ldots, \lfloor \ln a / \ln \theta \rfloor$ and $m = 0, \ldots, \left\lfloor \ln a / \ln \tilde{\theta} \right\rfloor$. We start by showing that $\mathcal{F}_n^{(1)}(a)$ is an $\varepsilon-$bracket, for n sufficiently large. Clearly

$$\begin{aligned}
0 \leqslant f_{n,y}(u, z) &\leqslant \sqrt{\frac{n h_n^d}{k}} K_{h_n}(x_0 - z) \frac{1\!\!1_{\{\overline{F}_1(u_1|x_0) \leqslant k/n\, y_1\}}}{[(n/k)\, \overline{F}_1(u_1|x_0)]^\eta} \\
&\leqslant h_n^{d/2} (n/k)^{1/2-\eta} K_{h_n}(x_0 - z) \frac{1\!\!1_{\{\overline{F}_1(u_1|x_0) \leqslant k/n\, a\}}}{(\overline{F}_1(u_1|x_0))^\eta} := u_{1,n}(u, z).
\end{aligned}$$

Then

$$\begin{aligned}
P u_{1,n}^2 &= \left(\frac{n}{k}\right)^{1-2\eta} \int_{S_K} K^2(v) \int_{t_n(a)}^\infty \frac{1}{(\overline{F}_1(y|x_0))^{2\eta}} dF_1(y|x_0 - h_n v) f_X(x_0 - h_n v) dv \\
&= Q_{3,n}(a) + Q_{4,n}(a),
\end{aligned}$$

using the same decomposition as for PF_n^2. Thus, one can obtain the result from the above analysis of $Q_{3,n}(T)$ and $Q_{4,n}(T)$, taking into account that the various constants involved in these will not depend on a.

Concerning $Q_{3,n}(a)$, according to (5.8), for n large

$$Q_{3,n}(a) \leqslant C a^{1-2\eta-\kappa},$$

where C does not depend on a. Now, taking $a = \varepsilon^{3/(1-2\eta)}$, for n large enough and ε small we have $|Q_{3,n}(a)| \leqslant \varepsilon^2$.

Concerning $Q_{4,n}(a)$, we use the same decomposition as for $Q_{4,n}(T)$ based on (5.9), which entails that, for n large enough, ε small and some small $\zeta > 0$

$$\begin{aligned}
Q_{4,1,n}(a) &\leqslant \varepsilon^2, \\
Q_{4,2,1,n}(a) &\leqslant C h_n^{\eta\gamma_1} \ln(t_n(a)) [t_n(a)]^{\xi_{1,n}} a^{1-2\eta-\kappa} \\
&\leqslant C(1 + |\ln a|) a^{-\zeta} a^{1-2\eta-\kappa} \\
&\leqslant C a^{1-2\eta-2\kappa},
\end{aligned}$$

with C a constant not depending on a, since from (5.1) and for n large,

$$h_n^{\eta\gamma_1} \ln t_n(a) \leqslant C(1 + |\ln a|).$$

Also, for n large, and some small $\zeta > 0$

$$\begin{aligned}
Q_{4,2,2,n}(a) &\leqslant C a^{1-2\eta} h_n^{\eta\gamma_1} [t_n(a)]^{2\xi_{1,n}} \left\{1 + h_n^{\eta\gamma_1} \ln(t_n(a)) + [t_n(a)]^{\xi_{2,n}} h_n^{\eta\varepsilon_1} \ln(t_n(a))\right\} \\
&\leqslant C a^{1-2\eta} h_n^{\eta\gamma_1} a^{-\zeta} (1 + |\ln a| + a^{-\zeta}(1 + |\ln a|)) \\
&\leqslant C a^{1-2\eta-\kappa},
\end{aligned}$$

where C does not depend on a. Hence, for n large and ε small we obtain $Q_{4,2,2,n}(a) \leqslant \varepsilon^2$. Using (5.10) and (5.11), we have also $Q_{4,3,n}(a) \leqslant \varepsilon^2$ and $Q_{4,4,n}(a) \leqslant \varepsilon^2$. Combining all the terms we get $Pu_{1,n}^2 \leqslant \varepsilon^2$ for n large.

Next consider $\mathcal{F}_n^{(2)}(a)$. Then

$$0 \leqslant f_{n,y}(u,z) \leqslant \sqrt{\frac{nh_n^d}{k}} K_{h_n}(x_0 - z) \frac{1\!\!1_{\{\overline{F}_2(u_2|x_0) \leqslant k/n\ a\}}}{a^\eta} =: u_{2,n}(u,z),$$

and

$$\begin{aligned} Pu_{2,n}^2 &= \frac{1}{a^{2\eta}} \frac{n}{k} \int_{S_K} K^2(v) \overline{F}_2 \left(U_2 \left(\frac{n}{ka} \Big| x_0 \right) \Big| x_0 - h_n v \right) f_X(x_0 - h_n v) dv \\ &\leqslant \varepsilon^2, \end{aligned}$$

when n is large enough and for ε small.

Finally, we consider $\mathcal{F}_n(\ell, m)$. We obtain the following bounds

$$\begin{aligned} \underline{u}_n(u,z) := \sqrt{\frac{nh_n^d}{k}} K_{h_n}(x_0 - z) \frac{1\!\!1_{\{\overline{F}_1(u_1|x_0) \leqslant k/n\ \theta^{\ell+1},\, \overline{F}_2(u_2|x_0) \leqslant k/n\ \tilde{\theta}^{m+1}\}}}{\theta^{\ell\eta}} \quad &\leqslant \quad f_{n,y}(u,z) \leqslant \\ \sqrt{\frac{nh_n^d}{k}} K_{h_n}(x_0 - z) \frac{1\!\!1_{\{\overline{F}_1(u_1|x_0) \leqslant k/n\ \theta^{\ell},\, \overline{F}_2(u_2|x_0) \leqslant k/n\ \tilde{\theta}^{m}\}}}{\theta^{(\ell+1)\eta}} \quad &=: \overline{u}_n(u,z). \end{aligned}$$

Then

$$\begin{aligned} P(\overline{u}_n - \underline{u}_n)^2 &= \frac{nh_n^d}{k} \mathbb{E} \left[K_{h_n}^2(x_0 - X) \left(\frac{1\!\!1_{\{\overline{F}_1(Y^{(1)}|x_0) \leqslant k/n\ \theta^{\ell},\, \overline{F}_2(Y^{(2)}|x_0) \leqslant k/n\ \tilde{\theta}^{m}\}}}{\theta^{(\ell+1)\eta}} \right.\right. \\ &\qquad \left.\left. - \frac{1\!\!1_{\{\overline{F}_1(Y^{(1)}|x_0) \leqslant k/n\ \theta^{\ell+1},\, \overline{F}_2(Y^{(2)}|x_0) \leqslant k/n\ \tilde{\theta}^{m+1}\}}}{\theta^{\ell\eta}} \right)^2 \right]. \end{aligned}$$

The difference of the indicator functions can be decomposed as in (5.7), and subsequent calculations follow arguments similar to those used in the verification of (5.2), Case 3. Taking $\theta = 1 - \varepsilon^3$ and $\tilde{\theta} = 1 - a$, gives for n large enough and ε small that $P(\overline{u}_n - \underline{u}_n)^2 \leqslant \varepsilon^2$.

Combining the above, for n large and ε small one obtains that the cover number by bracketing is of the order $\varepsilon^{-4-3/(1-2\eta)}$, and hence (5.5) is satisfied.

To conclude the proof, we comment on the pointwise convergence of the covariance function, which is given by $Pf_{n,y}f_{n,\bar{y}} - Pf_{n,y}Pf_{n,\bar{y}}$. We have

$$\begin{aligned} Pf_{n,y}f_{n,\bar{y}} &= \frac{\|K\|_2^2}{(y_1\bar{y}_1)^\eta} \frac{n}{k} \mathbb{E} \left[\frac{1}{\|K\|_2^2 h_n^d} K^2 \left(\frac{x_0 - X}{h_n} \right) 1\!\!1_{A_{n,y \wedge \bar{y}}} \right] \\ &\to \|K\|_2^2 f_X(x_0) \frac{R(y_1 \wedge \bar{y}_1, y_2 \wedge \bar{y}_2 | x_0)}{(y_1\bar{y}_1)^\eta}, \end{aligned}$$

as $n \to \infty$, by the arguments used in the proof of Lemma 5.2. Also

$$\begin{aligned} P f_{n,y} &= \sqrt{\frac{k h_n^d}{n}} \frac{1}{y_1^\eta} \frac{n}{k} \mathbb{E}\left[K_{h_n}(x_0 - X) 1\!\!1_{A_{n,y}}\right] \\ &\to 0, \end{aligned}$$

as $n \to \infty$. ■

5.2 Proof of Theorem 2.2

Recall that

$$T_n(\infty, y_2|x_0) = \frac{1}{k}\sum_{i=1}^{n} K_{h_n}(x_0 - X_i) 1\!\!1_{\{\overline{F}_2(Y_i^{(2)}|x_0) \leqslant \frac{k}{n} y_2\}}.$$

We follow the lines of proof of Theorem 2.1. We introduce the sequence of classes $\widetilde{\mathcal{F}}_n$ on $\mathbb{R} \times \mathbb{R}^d$ as

$$\widetilde{\mathcal{F}}_n := \{(u, z) \to \widetilde{f}_{n,y}(u, z), y \in (0, T]\}$$

where

$$\widetilde{f}_{n,y}(u, z) := \sqrt{\frac{n h_n^d}{k}} K_{h_n}(x_0 - z) 1\!\!1_{\{\overline{F}_2(u|x_0) \leqslant \frac{k}{n} y\}}.$$

We have to verify the conditions (5.2)-(5.5) in the proof of Theorem 2.1 for the new functions $\widetilde{f}_{n,y}$, and with $\rho_{x_0}(y, \overline{y}) := |y - \overline{y}|$. Without loss of generality, we may assume that $y > \overline{y}$. Thus, we have

$$\begin{aligned} P\left(\widetilde{f}_{n,y} - \widetilde{f}_{n,\overline{y}}\right)^2 &= \frac{n h_n^d}{k} \mathbb{E}\left[K_{h_n}^2(x_0 - X)\left(1\!\!1_{\{\overline{F}_2(Y^{(2)}|x_0) \leqslant \frac{k}{n} y\}} - 1\!\!1_{\{\overline{F}_2(Y^{(2)}|x_0) \leqslant \frac{k}{n} \overline{y}\}}\right)\right] \\ &= \frac{n}{k}\int_{S_K} K^2(v)\left[\overline{F}_2\left(U_2\left(\frac{n}{ky}\middle|x_0\right)\middle|x_0 - h_n v\right) - \overline{F}_2\left(U_2\left(\frac{n}{k\overline{y}}\middle|x_0\right)\middle|x_0 - h_n v\right)\right] \\ &\qquad \times f_X(x_0 - h_n v) dv \\ &= (y - \overline{y})\int_{S_K} K^2(v) f_X(x_0 - h_n v) dv \\ &\quad + \int_{S_K} K^2(v)\left[\frac{n}{k}\overline{F}_2\left(U_2\left(\frac{n}{ky}\middle|x_0\right)\middle|x_0 - h_n v\right) - y\right] f_X(x_0 - h_n v) dv \\ &\quad - \int_{S_K} K^2(v)\left[\frac{n}{k}\overline{F}_2\left(U_2\left(\frac{n}{k\overline{y}}\middle|x_0\right)\middle|x_0 - h_n v\right) - \overline{y}\right] f_X(x_0 - h_n v) dv \\ &\leqslant \delta_n \int_{S_K} K^2(v) f_X(x_0 - h_n v) dv + o(1), \end{aligned}$$

with a $o(1)-$term which is uniform in y and $\overline{y}$ by Lemma 5.1. This yields (5.2).

Now, concerning (5.3) we can use the following envelope function of the class $\widetilde{\mathcal{F}}_n$

$$\widetilde{F}_n(u, z) := \sqrt{\frac{n h_n^d}{k}} K_{h_n}(x_0 - z) 1\!\!1_{\{\overline{F}_2(u|x_0) \leqslant \frac{k}{n} T\}}$$

from which we deduce that

$$P\widetilde{F}_n^2 = \frac{n}{k}\int_{S_K} K^2(v)\overline{F}_2\left(U_2\left(\frac{n}{kT}\Big|x_0\right)\Big|x_0-h_nv\right)f_X(x_0-h_nv)dv = O(1).$$

Next condition (5.4) is also a direct consequence of the definition of the envelope since

$$\begin{aligned} P\widetilde{F}_n^2 1\!\!1_{\{\widetilde{F}_n>\varepsilon\sqrt{n}\}} &\leqslant \frac{1}{\varepsilon^\alpha n^{\alpha/2}}P\widetilde{F}_n^{2+\alpha} \\ &\leqslant \frac{1}{\varepsilon^\alpha (kh_n^d)^{\alpha/2}}\frac{n}{k}\int_{S_K} K^{2+\alpha}(v)\overline{F}_2\left(U_2\left(\frac{n}{kT}\Big|x_0\right)\Big|x_0-h_nv\right)f_X(x_0-h_nv)dv = o(1) \end{aligned}$$

as soon as $kh_n^d \to \infty$.

Finally, concerning (5.5), again without loss of generality we assume $T=1$ and divide $[0,1]$ into m intervals of length $1/m$. Then, for $y \in [(i-1)/m, i/m]$ we have the bounds

$$\underline{u}_n(u,z) := \sqrt{\frac{nh_n^d}{k}}K_{h_n}(x_0-z)1\!\!1_{\{\overline{F}_2(u|x_0)\leqslant \frac{k}{n}\frac{i-1}{m}\}} \leqslant \widetilde{f}_{n,y}(u,z) \leqslant \sqrt{\frac{nh_n^d}{k}}K_{h_n}(x_0-z)1\!\!1_{\{\overline{F}_2(u|x_0)\leqslant \frac{k}{n}\frac{i}{m}\}} =: \overline{u}_n(u,z)$$

from which we deduce that

$$\begin{aligned} P(\underline{u}_n-\overline{u}_n)^2 &= \frac{1}{m}\int_{S_K} K^2(v)f_X(x_0-h_nv)dv \\ &\quad + \int_{S_K} K^2(v)\left[\frac{n}{k}\overline{F}_2\left(U_2\left(\frac{n}{k}\frac{m}{i}\Big|x_0\right)\Big|x_0-h_nv\right)-\frac{i}{m}\right]f_X(x_0-h_nv)dv \\ &\quad - \int_{S_K} K^2(v)\left[\frac{n}{k}\overline{F}_2\left(U_2\left(\frac{n}{k}\frac{m}{i-1}\Big|x_0\right)\Big|x_0-h_nv\right)-\frac{i-1}{m}\right]f_X(x_0-h_nv)dv \\ &\leqslant \varepsilon^3\int_{S_K} K^2(v)f_X(x_0-h_nv)dv + 2\varepsilon^3 \end{aligned}$$

when $m = \lceil \frac{1}{\varepsilon^3}\rceil$. If ε is small and n large, then $P(\underline{u}_n-\overline{u}_n)^2 \leqslant \varepsilon^2$.

The pointwise convergence of the covariance function can be verified with arguments similar to those used in the proof of Theorem 2.1.

Consequently

$$\sqrt{kh_n^d}\,[T_n(\infty,y_2|x_0)-\mathbb{E}(T_n(\infty,y_2|x_0))] \rightsquigarrow W(\infty,y_2),$$

in $D((0,T])$.

Now, remark that

$$\begin{aligned} \mathbb{E}(T_n(\infty,y_2|x_0)) &= y_2 f_X(x_0) + O\left(h_n^{\eta_{f_X}}\right) \\ &\quad + f_X(x_0)\int_{S_K} K(v)\left[\frac{n}{k}\overline{F}_2\left(U_2\left(\frac{n}{ky_2}\Big|x_0\right)\Big|x_0-h_nv\right)-y_2\right]dv \\ &\quad + \int_{S_K} K(v)\left[\frac{n}{k}\overline{F}_2\left(U_2\left(\frac{n}{ky_2}\Big|x_0\right)\Big|x_0-h_nv\right)-y_2\right][f_X(x_0-h_nv)-f_X(x_0)]\,dv. \end{aligned}$$

Following the lines of proof of Lemma 5.1, we deduce that

$$\left|\frac{n}{k}\overline{F}_2\left(U_2\left(\frac{n}{ky_2}\Big|x_0\right)\Big|x_0-h_n v\right)-y_2\right| \leqslant C\left\{h_n^{\eta_{A_2}}+h_n^{\eta_{\gamma_2}}\ln\frac{n}{k}+|\delta_2(U_2(n/k|x_0)|x_0)|\left(h_n^{\eta_{B_2}}+h_n^{\eta_{\varepsilon_2}}\ln\frac{n}{k}\right)\right\}$$

from which we obtain

$$\begin{aligned}\mathbb{E}(T_n(\infty,y_2|x_0)) &= y_2 f_X(x_0)+O\left(h_n^{\eta_{f_X}\wedge\eta_{A_2}}\right)+O\left(h_n^{\eta_{\gamma_2}}\ln\frac{n}{k}\right)+O\left(|\delta_2(U_2(n/k|x_0)|x_0)|h_n^{\eta_{B_2}}\right)\\ &\quad +O\left(|\delta_2(U_2(n/k|x_0)|x_0)|h_n^{\eta_{\varepsilon_2}}\ln\frac{n}{k}\right)\end{aligned}$$

with $O-$terms which are uniform in $y_2\in(0,T]$. This implies that, under the assumptions of Theorem 2.2, we have

$$\sqrt{kh_n^d}\left[T_n(\infty,y_2|x_0)-y_2 f_X(x_0)\right]\rightsquigarrow W(\infty,y_2), \tag{5.12}$$

in $D((0,T])$.

Finally,

$$\sqrt{kh_n^d}\left(\frac{T_n(\infty,y_2|x_0)}{\widehat{f}_n(x_0)}-y_2\right)=\sqrt{kh_n^d}\left(\frac{T_n(\infty,y_2|x_0)}{f_X(x_0)}-y_2\right)-\frac{T_n(\infty,y_2|x_0)}{\widehat{f}_n(x_0)f_X(x_0)}\sqrt{\frac{k}{n}}\sqrt{nh_n^d}\left(\widehat{f}_n(x_0)-f_X(x_0)\right),$$

from which Theorem 2.2 follows. ■

5.3 Proof of Proposition 2.1

We use the decomposition

$$\sup_{\frac12\leqslant y_2\leqslant 2}\left|\frac{\sqrt{kh_n^d}}{U_1(n/k|x_0)}\left[\widetilde{\theta}_n(y_2)-\mathbb{E}(\widetilde{\theta}_n(y_2))\right]+\int_0^\infty W(u,y_2)du^{-\gamma_1(x_0)}\right|\leqslant I_1(T)+\sum_{i=2}^4 I_{i,n}(T),$$

where

$$\begin{aligned}I_1(T) &:= \sup_{\frac12\leqslant y_2\leqslant 2}\left|\int_T^\infty W(u,y_2)du^{-\gamma_1(x_0)}\right|,\\ I_{2,n}(T) &:= \sup_{\frac12\leqslant y_2\leqslant 2}\left|\int_T^\infty \sqrt{kh_n^d}\left[T_n(s_n(u),y_2|x_0)-\mathbb{E}\left(T_n(s_n(u),y_2|x_0)\right)\right]du^{-\gamma_1(x_0)}\right|,\\ I_{3,n}(T) &:= \sup_{\frac12\leqslant y_2\leqslant 2}\left|\int_0^T\left\{\sqrt{kh_n^d}\left[T_n(s_n(u),y_2|x_0)-\mathbb{E}\left(T_n(s_n(u),y_2|x_0)\right)\right]-W(s_n(u),y_2)\right\}du^{-\gamma_1(x_0)}\right|,\\ I_{4,n}(T) &:= \sup_{\frac12\leqslant y_2\leqslant 2}\left|\int_0^T\left[W(s_n(u),y_2)-W(u,y_2)\right]du^{-\gamma_1(x_0)}\right|.\end{aligned}$$

Similarly to the proof of Proposition 2 in Cai et al. (2015), it is sufficient to show that for any $\varepsilon>0$, there exists $T_0=T_0(\varepsilon)$ such that

$$\mathbb{P}(I_1(T_0)>\varepsilon)<\varepsilon, \tag{5.13}$$

and $n_0 = n_0(T_0)$ such that, for any $n > n_0$

$$\mathbb{P}(I_{j,n}(T_0) > \varepsilon) < \varepsilon, \text{ for } j = 2, 3 \text{ and } 4.$$

Clearly

$$I_1(T) \leqslant \sup_{u \geqslant T, \frac{1}{2} \leqslant y_2 \leqslant 2} |W(u, y_2)| T^{-\gamma_1(x_0)}.$$

Since a rescaled version of our Gaussian process $W(.,.)$ gives the one in Cai et al. (2015), according to their Lemma 2, we have $\sup_{0<u<\infty, \frac{1}{2} \leqslant y_2 \leqslant 2} |W(u, y_2)| < \infty$ with probability one. This implies that there exists $T_1 = T_1(\varepsilon)$ such that

$$\mathbb{P}\left(\sup_{0<u<\infty, \frac{1}{2} \leqslant y_2 \leqslant 2} |W(u, y_2)| > T_1^{\gamma_1(x_0)} \varepsilon \right) < \varepsilon,$$

from which we deduce that, for any $T > T_1$

$$\mathbb{P}\left(I_1(T) > \varepsilon\right) \leqslant \mathbb{P}\left(\sup_{0<u<\infty, \frac{1}{2} \leqslant y_2 \leqslant 2} |W(u, y_2)| > T_1^{\gamma_1(x_0)} \varepsilon \right) < \varepsilon.$$

Consequently (5.13) holds for $T_0 > T_1$.

We continue with the term $I_{2,n}(T)$. We have

$$\begin{aligned}
&\mathbb{P}\left(I_{2,n}(T) > \varepsilon\right) \\
&\leqslant \mathbb{P}\left(\sup_{y_1 \geqslant T, \frac{1}{2} \leqslant y_2 \leqslant 2} \left| \sqrt{k h_n^d} \left[T_n(s_n(y_1), y_2 | x_0) - \mathbb{E}(T_n(s_n(y_1), y_2 | x_0)) \right] \right| > \varepsilon T^{\gamma_1(x_0)} \right) \\
&= \mathbb{P}\left(\sup_{y_1 \geqslant T, \frac{1}{2} \leqslant y_2 \leqslant 2} \left| \sum_{i=1}^n \left[\frac{K}{\|K\|_\infty} \left(\frac{x_0 - X_i}{h_n} \right) \mathbb{1}_{\{\overline{F}_1(Y_i^{(1)} | x_0) \leqslant \frac{k}{n} s_n(y_1), \overline{F}_2(Y_i^{(2)} | x_0) \leqslant \frac{k}{n} y_2\}} \right. \right. \right. \\
&\qquad \left. \left. \left. - \mathbb{E}\left(\frac{K}{\|K\|_\infty} \left(\frac{x_0 - X}{h_n} \right) \mathbb{1}_{\{\overline{F}_1(Y^{(1)} | x_0) \leqslant \frac{k}{n} s_n(y_1), \overline{F}_2(Y^{(2)} | x_0) \leqslant \frac{k}{n} y_2\}} \right) \right] \right| > \frac{\varepsilon T^{\gamma_1(x_0)}}{\|K\|_\infty} \sqrt{k h_n^d} \right) \\
&\leqslant \frac{\|K\|_\infty}{\varepsilon T^{\gamma_1(x_0)} \sqrt{k h_n^d}} \mathbb{E}\left\{ \sup_{y_1 \geqslant T, \frac{1}{2} \leqslant y_2 \leqslant 2} \left| \sum_{i=1}^n \left[\frac{K}{\|K\|_\infty} \left(\frac{x_0 - X_i}{h_n} \right) \mathbb{1}_{\{\overline{F}_1(Y_i^{(1)} | x_0) \leqslant \frac{k}{n} s_n(y_1), \overline{F}_2(Y_i^{(2)} | x_0) \leqslant \frac{k}{n} y_2\}} \right. \right. \right. \\
&\qquad \left. \left. \left. - \mathbb{E}\left(\frac{K}{\|K\|_\infty} \left(\frac{x_0 - X}{h_n} \right) \mathbb{1}_{\{\overline{F}_1(Y^{(1)} | x_0) \leqslant \frac{k}{n} s_n(y_1), \overline{F}_2(Y^{(2)} | x_0) \leqslant \frac{k}{n} y_2\}} \right) \right] \right| \right\}.
\end{aligned}$$

Consider the class of functions

$$\begin{aligned}
g_{n,y}(u, z) \quad := \quad & \frac{K}{\|K\|_\infty} \left(\frac{x_0 - z}{h_n} \right) \mathbb{1}_{\{\overline{F}_1(u_1 | x_0) \leqslant \frac{k}{n} s_n(y_1), \overline{F}_2(u_2 | x_0) \leqslant \frac{k}{n} y_2\}} \\
& - \mathbb{E}\left(\frac{K}{\|K\|_\infty} \left(\frac{x_0 - X}{h_n} \right) \mathbb{1}_{\{\overline{F}_1(Y^{(1)} | x_0) \leqslant \frac{k}{n} s_n(y_1), \overline{F}_2(Y^{(2)} | x_0) \leqslant \frac{k}{n} y_2\}} \right),
\end{aligned}$$

with $y_1 \geqslant T$ and $1/2 \leqslant y_2 \leqslant 2$, and with envelope function

$$G_n(u, z) := \frac{K}{\|K\|_\infty} \left(\frac{x_0 - z}{h_n} \right) \mathbb{1}_{\{\overline{F}_2(u_2 | x_0) \leqslant \frac{2k}{n}\}} + \mathbb{E}\left(\frac{K}{\|K\|_\infty} \left(\frac{x_0 - X}{h_n} \right) \mathbb{1}_{\{\overline{F}_2(Y^{(2)} | x_0) \leqslant \frac{2k}{n}\}} \right).$$

This class of functions satisfies the conditions of Theorem 7.3 in Wellner (2005) with $\sigma^2 = O(kh_n^d/n)$ and $PG_n^2 = O(kh_n^d/n)$ for n large, and thus, for some constant C,

$$\mathbb{P}\left(I_{2,n}(T) > \varepsilon\right) \leqslant \frac{C}{\varepsilon T^{\gamma_1(x_0)}}$$

for n large enough. We have then that for every ε there is a $T = T(\varepsilon)$ such that for n large enough

$$\mathbb{P}\left(I_{2,n}(T) > \varepsilon\right) \leqslant \varepsilon.$$

Now, to study $I_{3,n}(T)$, remark that for any $T > 0, \exists\, n_1 = n_1(T) : \forall n > n_1 : s_n(T) < T + 1$. Hence for $n > n_1$ and any $\eta_0 \in (\gamma_1(x_0), 1/2 - \kappa)$:

$$\begin{aligned}\mathbb{P}\left(I_{3,n}(T) > \varepsilon\right) \leqslant \mathbb{P}\Bigg(& \sup_{0<y_1\leqslant T+1, \frac{1}{2}\leqslant y_2\leqslant 2}\left|\frac{\sqrt{kh_n^d}[T_n(y_1,y_2|x_0) - \mathbb{E}(T_n(y_1,y_2|x_0))] - W(y_1,y_2)}{y_1^{\eta_0}}\right| \\ & \times \left|\int_0^T [s_n(u)]^{\eta_0} du^{-\gamma_1(x_0)}\right| > \varepsilon\Bigg).\end{aligned}$$

According to Lemma 3 in Cai et al. (2015)

$$\left|\int_0^T [s_n(u)]^{\eta_0} du^{-\gamma_1(x_0)}\right| \longrightarrow \frac{\gamma_1(x_0)}{\eta_0 - \gamma_1(x_0)} T^{\eta_0 - \gamma_1(x_0)},$$

which, combining with our Theorem 2.1, entails that there exists $n_2(T) > n_1(T)$ such that $\forall n > n_2(T)$, $\mathbb{P}(I_{3,n}(T) > \varepsilon) < \varepsilon$.

Finally, concerning $I_{4,n}(T)$, we first remark that according to Lemma 2 in Cai et al. (2015), we have for $\eta_0 \in (\gamma_1(x_0), 1/2)$ and any $T > 0$, with probability one,

$$\sup_{0<y_1\leqslant T, \frac{1}{2}\leqslant y_2\leqslant 2} \frac{|W(y_1,y_2)|}{y_1^{\eta_0}} < \infty.$$

Then, applying Lemma 3 in Cai et al. (2015) with $S = T, S_0 = T + 1$ and $g = W$, we deduce that there exists $n_3(T)$ such that for $n > n_3(T)$ we have $\mathbb{P}(I_{4,n}(T) > \varepsilon) < \varepsilon$.
This achieves the proof of Proposition 2.1. ■

In order to prove Theorem 2.3 we need some auxiliary results. Define for $u > 0$ and $v \in S_K$

$$\begin{aligned}\tilde{s}_n(u) &:= \frac{n}{k}\,\overline{F}_1\left(u^{-\gamma_1(x_0)} U_1\left(\frac{n}{k}\Big|x_0\right)\Big|x_0 - h_n v\right),\\ t_n(y_2) &:= \frac{n}{k}\,\overline{F}_2\left(U_2\left(\frac{n}{ky_2}\Big|x_0\right)\Big|x_0 - h_n v\right).\end{aligned}$$

Lemma 5.3. *Assume $(\mathcal{D})$ and $(\mathcal{H})$ and $x_0 \in Int(S_X)$. Consider sequences $k \to \infty$ and $h_n \to 0$ as $n \to \infty$, in such a way that $k/n \to 0$ and $h_n^{\eta_{\varepsilon_1} \wedge \eta_{\gamma_1}} \ln \frac{n}{k} \to 0$. Then, we have, for any $u \leqslant T_n \to \infty$*

such that $kT_n/n \to 0$ and $0<\varepsilon<\beta_1(x_0)$, that

$$\begin{aligned}\left|\widetilde{s}_n(u)-u\right| &\leqslant Cu\left\{h_n^{\eta_{A_1}}+h_n^{\eta_{\gamma_1}}\ln\frac{n}{k}+h_n^{\eta_{\gamma_1}}|\ln u|u^{\pm C h_n^{\eta_{\gamma_1}}}\right.\\ &+\left|\delta_1\left(U_1\left(\frac{n}{k}\Big|x_0\right)\Big|x_0\right)\right|\left[1+u^{\pm C h_n^{\eta_{\gamma_1}}}h_n^{\eta_{\gamma_1}}|\ln u|\right]\\ &\times\left[u^{\gamma_1(x_0)\beta_1(x_0)}\left(1+u^{\pm\gamma_1(x_0)\varepsilon}\right)\left(h_n^{\eta_{B_1}}+u^{-C h_n^{\eta_{\varepsilon_1}}}h_n^{\eta_{\varepsilon_1}}\left(|\ln u|+\ln\frac{n}{k}\right)\right)\right.\\ &\left.\left.+\,u^{\gamma_1(x_0)(\beta_1(x_0)\pm\varepsilon)}+\left|u^{\gamma_1(x_0)\beta_1(x_0)}-1\right|\right]\right\},\end{aligned}$$

where $u^{\pm\bullet}$ means $u^{\bullet}$ if u is greater than 1, and $u^{-\bullet}$ if u is smaller than 1.

Proof

Using Assumption $(\mathcal{D})$, we have

$$\begin{aligned}\widetilde{s}_n(u) &= \frac{\overline{F}_1\left(u^{-\gamma_1(x_0)}U_1\left(\frac{n}{k}\Big|x_0\right)\Big|x_0-h_nv\right)}{\overline{F}_1\left(U_1\left(\frac{n}{k}\Big|x_0\right)\Big|x_0\right)}\\ &= \frac{A_1(x_0-h_nv)}{A_1(x_0)}\left(U_1\left(\frac{n}{k}\Big|x_0\right)\right)^{\frac{1}{\gamma_1(x_0)}-\frac{1}{\gamma_1(x_0-h_nv)}}u^{\frac{\gamma_1(x_0)}{\gamma_1(x_0-h_nv)}}\\ &\quad\times\frac{1+\frac{1}{\gamma_1(x_0-h_nv)}\delta_1\left(u^{-\gamma_1(x_0)}U_1\left(\frac{n}{k}\Big|x_0\right)\Big|x_0-h_nv\right)}{1+\frac{1}{\gamma_1(x_0)}\delta_1\left(U_1\left(\frac{n}{k}\Big|x_0\right)\Big|x_0\right)}.\end{aligned}$$

This implies that

$$\begin{aligned}\left|\widetilde{s}_n(u)-u^{\frac{\gamma_1(x_0)}{\gamma_1(x_0-h_nv)}}\right| &\leqslant u^{\frac{\gamma_1(x_0)}{\gamma_1(x_0-h_nv)}}\left\{\left|\frac{A_1(x_0-h_nv)}{A_1(x_0)}-1\right|\left(U_1\left(\frac{n}{k}\Big|x_0\right)\right)^{\frac{1}{\gamma_1(x_0)}-\frac{1}{\gamma_1(x_0-h_nv)}}\right.\\ &\quad\times\left|\frac{1+\frac{1}{\gamma_1(x_0-h_nv)}\delta_1\left(u^{-\gamma_1(x_0)}U_1\left(\frac{n}{k}\Big|x_0\right)\Big|x_0-h_nv\right)}{1+\frac{1}{\gamma_1(x_0)}\delta_1\left(U_1\left(\frac{n}{k}\Big|x_0\right)\Big|x_0\right)}\right|\\ &\quad+\left|\left(U_1\left(\frac{n}{k}\Big|x_0\right)\right)^{\frac{1}{\gamma_1(x_0)}-\frac{1}{\gamma_1(x_0-h_nv)}}-1\right|\\ &\quad\times\left|\frac{1+\frac{1}{\gamma_1(x_0-h_nv)}\delta_1\left(u^{-\gamma_1(x_0)}U_1\left(\frac{n}{k}\Big|x_0\right)\Big|x_0-h_nv\right)}{1+\frac{1}{\gamma_1(x_0)}\delta_1\left(U_1\left(\frac{n}{k}\Big|x_0\right)\Big|x_0\right)}\right|\\ &\quad\left.+\left|\frac{1+\frac{1}{\gamma_1(x_0-h_nv)}\delta_1\left(u^{-\gamma_1(x_0)}U_1\left(\frac{n}{k}\Big|x_0\right)\Big|x_0-h_nv\right)}{1+\frac{1}{\gamma_1(x_0)}\delta_1\left(U_1\left(\frac{n}{k}\Big|x_0\right)\Big|x_0\right)}-1\right|\right\}\\ &=: u^{\frac{\gamma_1(x_0)}{\gamma_1(x_0-h_nv)}}\{T_1+T_2+T_3\}.\end{aligned}$$

Using Assumption $(\mathcal{H})$ and the inequality $|e^x-1|\leqslant|x|\,e^{|x|}$, we deduce that, for n large,

$$\left|\frac{A_1(x_0-h_nv)}{A_1(x_0)}-1\right| \leqslant C\,h_n^{\eta_{A_1}} \tag{5.14}$$

$$\left|\left(U_1\left(\frac{n}{k}\Big|x_0\right)\right)^{\frac{1}{\gamma_1(x_0)}-\frac{1}{\gamma_1(x_0-h_nv)}}-1\right| \leqslant C\,h_n^{\eta_{\gamma_1}}\ln\frac{n}{k}. \tag{5.15}$$

Now, direct computations yield, for n large,

$$T_3 \leqslant C\left|\delta_1\left(U_1\left(\frac{n}{k}\Big|x_0\right)\Big|x_0\right)\right|\left\{\left|\frac{\gamma_1(x_0)}{\gamma_1(x_0-h_nv)}-1\right|\left|\frac{\delta_1\left(u^{-\gamma_1(x_0)}U_1\left(\frac{n}{k}\big|x_0\right)\big|x_0-h_nv\right)}{\delta_1\left(U_1\left(\frac{n}{k}\big|x_0\right)\big|x_0\right)}\right|\right.$$
$$\left.+\left|\frac{\delta_1\left(u^{-\gamma_1(x_0)}U_1\left(\frac{n}{k}\big|x_0\right)\big|x_0-h_nv\right)}{\delta_1\left(u^{-\gamma_1(x_0)}U_1\left(\frac{n}{k}\big|x_0\right)\big|x_0\right)}\frac{\delta_1\left(u^{-\gamma_1(x_0)}U_1\left(\frac{n}{k}\big|x_0\right)\big|x_0\right)}{\delta_1\left(U_1\left(\frac{n}{k}\big|x_0\right)\big|x_0\right)}-1\right|\right\}.$$

Using the assumed form for $\delta_1(y|x)$, $(\mathcal{H})$, and the uniform bound from Proposition B.1.10 in de Haan and Ferreira (2006) with $0<\varepsilon<\beta_1(x_0)$, we obtain, for n large, that

$$T_3 \leqslant C\left|\delta_1\left(U_1\left(\frac{n}{k}\Big|x_0\right)\Big|x_0\right)\right|\left\{h_n^{\eta_{\gamma_1}}+u^{\gamma_1(x_0)\beta_1(x_0)}\left(1+u^{\pm\gamma_1(x_0)\varepsilon}\right)\right.$$
$$\left.\times\left[h_n^{\eta_{B_1}}+u^{-Ch_n^{\eta_{\varepsilon_1}}}h_n^{\eta_{\varepsilon_1}}\left(|\ln u|+\ln\frac{n}{k}\right)\right]+u^{\gamma_1(x_0)(\beta_1(x_0)\pm\varepsilon)}+\left|u^{\gamma_1(x_0)\beta_1(x_0)}-1\right|\right\} \quad (5.16)$$

Since

$$\left|\widetilde{s}_n(u)-u\right| \leqslant \left|\widetilde{s}_n(u)-u^{\frac{\gamma_1(x_0)}{\gamma_1(x_0-h_nv)}}\right|+u\left|u^{\frac{\gamma_1(x_0)-\gamma_1(x_0-h_nv)}{\gamma_1(x_0-h_nv)}}-1\right|$$
$$\leqslant \left|\widetilde{s}_n(u)-u^{\frac{\gamma_1(x_0)}{\gamma_1(x_0-h_nv)}}\right|+Cu^{1\pm Ch_n^{\eta_{\gamma_1}}}h_n^{\eta_{\gamma_1}}|\ln u|, \quad (5.17)$$

combining (5.14), (5.15), (5.16) with (5.17), Lemma 5.3 is established. ■

Lemma 5.4. *Assume $(\mathcal{D})$, $(\mathcal{H})$, $\gamma_1(x_0)<1$ and $x_0\in Int(S_X)$. For sequences $k=\lfloor n^\alpha\ell_1(n)\rfloor$ and $h_n=n^{-\Delta}\ell_2(n)$, where ℓ_1 and ℓ_2 are slowly varying functions at infinity, with $\alpha\in(0,1)$ and*

$$\max\left(\frac{\alpha}{d+2\gamma_1(x_0)(\eta_{A_1}\wedge\eta_{\gamma_1})},\frac{\alpha}{d+2(1-\gamma_1(x_0))(\eta_{A_2}\wedge\eta_{\gamma_2}\wedge\eta_{B_2}\wedge\eta_{\varepsilon_2})},\frac{\alpha}{d}-\frac{2(1-\alpha)\gamma_1^2(x_0)\beta_1(x_0)}{d+d(\beta_1(x_0)+\varepsilon)\gamma_1(x_0)},\right.$$
$$\left.\frac{\alpha-2(1-\alpha)\gamma_1(x_0)}{d}\right)<\Delta<\frac{\alpha}{d},$$

one has that

$$\sup_{v\in S_K}\sup_{\frac{1}{2}\leqslant y_2\leqslant 2}\sqrt{kh_n^d}\left|\int_0^\infty\left[R\left(\widetilde{s}_n(u),t_n(y_2)|x_0\right)-R(u,y_2|x_0)\right]du^{-\gamma_1(x_0)}\right|\longrightarrow 0$$

and

$$\sup_{\frac{1}{2}\leqslant y_2\leqslant 2}\sqrt{kh_n^d}\left|\int_0^\infty\left[R\left(s_n(u),y_2|x_0\right)-R(u,y_2|x_0)\right]du^{-\gamma_1(x_0)}\right|\longrightarrow 0.$$

Proof

We use the following decomposition along with the Lipschitz property of the function R:

$$
\begin{aligned}
&\sqrt{kh_n^d}\left|\int_0^\infty \left[R\left(\tilde{s}_n(u), t_n(y_2)|x_0\right) - R(u, y_2|x_0)\right] du^{-\gamma_1(x_0)}\right| \\
&\qquad \leqslant \sqrt{kh_n^d}\left|\int_0^{\delta_n} \left[R\left(\tilde{s}_n(u), t_n(y_2)|x_0\right) - R(u, y_2|x_0)\right] du^{-\gamma_1(x_0)}\right| \\
&\qquad\quad +\sqrt{kh_n^d}\left|\int_{\delta_n}^{T_n} \left[R\left(\tilde{s}_n(u), t_n(y_2)|x_0\right) - R(u, y_2|x_0)\right] du^{-\gamma_1(x_0)}\right| \\
&\qquad\quad +\sqrt{kh_n^d}\left|\int_{T_n}^{\infty} \left[R\left(\tilde{s}_n(u), t_n(y_2)|x_0\right) - R(u, y_2|x_0)\right] du^{-\gamma_1(x_0)}\right| \\
&\qquad \leqslant \sqrt{kh_n^d}\left|\int_0^{\delta_n} R\left(\tilde{s}_n(u), t_n(y_2)|x_0\right) du^{-\gamma_1(x_0)}\right| \\
&\qquad\quad +\sqrt{kh_n^d}\left|\int_0^{\delta_n} R(u, y_2|x_0) du^{-\gamma_1(x_0)}\right| \\
&\qquad\quad -\sqrt{kh_n^d}\int_{\delta_n}^{T_n} \left[|\tilde{s}_n(u) - u| + |t_n(y_2) - y_2|\right] du^{-\gamma_1(x_0)} \\
&\qquad\quad +2 \sup_{u \geqslant 0, \frac{1}{2}-\zeta \leqslant y_2 \leqslant 2+\zeta} R(u, y_2|x_0)\sqrt{kh_n^d}\, T_n^{-\gamma_1(x_0)} \\
&\qquad =: T_1 + T_2 + T_3 + T_4,
\end{aligned}
$$

for $\zeta > 0$ small and where $\delta_n \to 0$ and $T_n \to \infty$, as $n \to \infty$.

Now, since $R(y_1, y_2|x_0) \leqslant y_1 \wedge y_2$, using Lemma 5.3, and assuming $h_n^{\eta_{\varepsilon_1} \wedge \eta_{\gamma_1}} |\ln \delta_n| \to 0$, we obtain after tedious calculations, for n large,

$$
\begin{aligned}
T_1 + T_2 &\leqslant -2\sqrt{kh_n^d}\int_0^{\delta_n} u\, du^{-\gamma_1(x_0)} - \sqrt{kh_n^d}\int_0^{\delta_n} |\tilde{s}_n(u) - u|\, du^{-\gamma_1(x_0)} \\
&\leqslant C\sqrt{kh_n^d \delta_n^{1-\gamma_1(x_0)}}. \qquad (5.18)
\end{aligned}
$$

As for T_3, using again Lemma 5.3 and following the lines of proof of Lemma 5.1, we have, for n large,

$$
\begin{aligned}
T_3 &\leqslant -\sqrt{kh_n^d}\int_0^{T_n} |\tilde{s}_n(u) - u|\, du^{-\gamma_1(x_0)} - \sqrt{kh_n^d}\int_{\delta_n}^{T_n} |t_n(y_2) - y_2|\, du^{-\gamma_1(x_0)} \\
&\leqslant C\sqrt{kh_n^d}\, T_n^{1-\gamma_1(x_0)} \left\{h_n^{\eta_{A_1}} + h_n^{\eta_{\gamma_1}} \ln\frac{n}{k} + h_n^{\eta_{\gamma_1}} \ln T_n + \left|\delta_1\left(U_1\left(\frac{n}{k}\middle|x_0\right)\middle|x_0\right)\right| T_n^{(\beta_1(x_0)+\varepsilon)\gamma_1(x_0)}\right\} \\
&\quad +C\sqrt{kh_n^d}\, \delta_n^{-\gamma_1(x_0)} \left\{h_n^{\eta_{A_2}} + h_n^{\eta_{\gamma_2}} \ln\frac{n}{k} + \left|\delta_2\left(U_2\left(\frac{n}{k}\middle|x_0\right)\middle|x_0\right)\right| \left[h_n^{\eta_{B_2}} + h_n^{\eta_{\varepsilon_2}} \ln\frac{n}{k}\right]\right\} \qquad (5.19)
\end{aligned}
$$

assuming $h_n^{\eta_{\varepsilon_1} \wedge \eta_{\gamma_1}} \ln T_n \to 0$.

Finally

$$
T_4 \leqslant C\sqrt{kh_n^d}\, T_n^{-\gamma_1(x_0)}. \qquad (5.20)
$$

Take $\delta_n = h_n^{\xi}$ and $T_n = n^{\kappa}$, with ξ and κ positive numbers, and $0 < \varepsilon < \beta_1(x_0)$. Combining (5.18), (5.19) and (5.20), the first part of Lemma 5.4 follows if the sequences δ_n and T_n are chosen such that

$$\begin{aligned}
\alpha-\Delta\left[d-2\xi\gamma_1(x_0)+2(\xi\wedge\eta_{A_2}\wedge\eta_{B_2}\wedge\eta_{\gamma_2}\wedge\eta_{\varepsilon_2})\right] &< 0,\\
\alpha-\Delta d-2\kappa\gamma_1(x_0) &< 0,\\
\alpha-\Delta d+2\kappa(1-\gamma_1(x_0))-2\Delta(\eta_{A_1}\wedge\eta_{\gamma_1}) &< 0,\\
\alpha-\Delta d-2(1-\alpha)\gamma_1(x_0)\beta_1(x_0)+2\kappa[1+(\beta_1(x_0)+\varepsilon)\gamma_1(x_0)-\gamma_1(x_0)] &< 0.
\end{aligned}$$

Note that this is possible if we proceed as follows:
- α and Δ are chosen as stated in Lemma 5.4;
- κ is chosen such that

$$\frac{\alpha-\Delta d}{2\gamma_1(x_0)}<\kappa<\min\left(1-\alpha,\frac{2\Delta(\eta_{A_1}\wedge\eta_{\gamma_1})-(\alpha-\Delta d)}{2(1-\gamma_1(x_0))},\frac{2(1-\alpha)\gamma_1(x_0)\beta_1(x_0)-(\alpha-\Delta d)}{2[1-\gamma_1(x_0)+(\beta_1(x_0)+\varepsilon)\gamma_1(x_0)]}\right);$$

- ξ is chosen such that

$$\frac{\alpha-\Delta d}{2\Delta(1-\gamma_1(x_0))}<\xi<\eta_{A_2}\wedge\eta_{\gamma_2}\wedge\eta_{B_2}\wedge\eta_{\varepsilon_2}.$$

Note that the choices of κ and ξ only depend on those of α and Δ.

The second part of Lemma 5.4 is similar, although simpler. Indeed, a decomposition of the quantity of interest this time into two parts yields

$$\begin{aligned}
&\sqrt{kh_n^d}\left|\int_0^\infty\left[R\left(s_n(u),y_2|x_0\right)-R(u,y_2|x_0)\right]du^{-\gamma_1(x_0)}\right|\\
\leqslant\;& \sqrt{kh_n^d}\left|\int_0^{T_n}\left[R\left(s_n(u),y_2|x_0\right)-R(u,y_2|x_0)\right]du^{-\gamma_1(x_0)}\right|\\
&+\sqrt{kh_n^d}\left|\int_{T_n}^\infty\left[R\left(s_n(u),y_2|x_0\right)-R(u,y_2|x_0)\right]du^{-\gamma_1(x_0)}\right|\\
\leqslant\;& -\sqrt{kh_n^d}\int_0^{T_n}|s_n(u)-u|\,du^{-\gamma_1(x_0)}+2\sup_{u\geqslant 0,\frac12\leqslant y_2\leqslant 2}R(u,y_2|x_0)\sqrt{kh_n^d}\,T_n^{-\gamma_1(x_0)}\\
\leqslant\;& -\sqrt{kh_n^d}\frac{|\delta_1(U_1(\frac{n}{k}|x_0)|x_0)|}{|\gamma_1(x_0)+\delta_1(U_1(\frac{n}{k}|x_0)|x_0)|}\int_0^{T_n}u\left|\frac{\delta_1(u^{-\gamma_1(x_0)}U_1(\frac{n}{k}|x_0)|x_0)}{\delta_1(U_1(\frac{n}{k}|x_0)|x_0)}-1\right|du^{-\gamma_1(x_0)}+C\sqrt{kh_n^d}\,T_n^{-\gamma_1(x_0)}\\
\leqslant\;& C\sqrt{kh_n^d}\left|\delta_1\left(U_1\left(\frac{n}{k}\Big|x_0\right)\Big|x_0\right)\right|T_n^{1-\gamma_1(x_0)+(\beta_1(x_0)+\varepsilon)\gamma_1(x_0)}+C\sqrt{kh_n^d}\,T_n^{-\gamma_1(x_0)}.
\end{aligned}$$

This achieves the proof of Lemma 5.4. ■

Lemma 5.5. *Assume $(\mathcal{D})$, $(\mathcal{H})$, $(\mathcal{K})$, $x_0\in Int(S_X)$ with $f_X(x_0)>0$ and $y\to F_2(y|x_0)$ is strictly increasing. Consider sequences $k\to\infty$ and $h_n\to 0$ as $n\to\infty$, in such a way that $k/n\to 0$, $kh_n^d\to\infty$, $h_n^{\eta_{\varepsilon_2}}\ln n/k\to 0$, $\sqrt{kh_n^d}\,h_n^{\eta_{f_X}\wedge\eta_{A_2}}\to 0$, $\sqrt{kh_n^d}\,h_n^{\eta_{\gamma_2}}\ln n/k\to 0$, $\sqrt{kh_n^d}|\delta_2(U_2(n/k|x_0)|x_0)|h_n^{\eta_{B_2}}\to 0$, and $\sqrt{kh_n^d}|\delta_2(U_2(n/k|x_0)|x_0)|h_n^{\eta_{\varepsilon_2}}\ln n/k\to 0$. Then, for any sequence u_n satisfying*

$$\sqrt{kh_n^d}\left(\frac{\overline{F}_2(U_2(n/k|x_0)|x_0)}{\overline{F}_2(u_n|x_0)}-1\right)\to c\in\mathbb{R},$$

as $n \to \infty$, we have

$$\sqrt{nh_n^d \overline{F}_2(u_n|x_0)} \left(\frac{\widehat{\overline{F}}_{n,2}(u_n|x_0)}{\overline{F}_2(u_n|x_0)} - 1 \right) \rightsquigarrow \frac{W(\infty, 1)}{f_X(x_0)}.$$

Proof

First remark that

$$\frac{\frac{1}{n}\sum_{i=1}^n K_{h_n}(x_0 - X_i) 1\!\!1_{\{Y_i^{(2)} > u_n\}}}{\overline{F}_2(u_n|x_0)} = \frac{\overline{F}_2(U_2(n/k|x_0)|x_0)}{\overline{F}_2(u_n|x_0)} T_n\left(\infty, \frac{n}{k}\overline{F}_2(u_n|x_0) \Big| x_0\right), \quad a.s. \ .$$

Since, with $r_n := \sqrt{nh_n^d \overline{F}_2(u_n|x_0)}$,

$$\begin{aligned}
&\left| r_n \left[\frac{\overline{F}_2(U_2(n/k|x_0)|x_0)}{\overline{F}_2(u_n|x_0)} T_n\left(\infty, \frac{n}{k}\overline{F}_2(u_n|x_0) \Big| x_0\right) - f_X(x_0) \right] - W(\infty, 1) \right| \\
&\quad \leqslant \left| \sqrt{kh_n^d} \left[T_n\left(\infty, \frac{n}{k}\overline{F}_2(u_n|x_0) \Big| x_0\right) - \frac{n}{k}\overline{F}_2(u_n|x_0) f_X(x_0) \right] - W\left(\infty, \frac{n}{k}\overline{F}_2(u_n|x_0)\right) \right| \\
&\qquad + \sqrt{kh_n^d} \left| \sqrt{\frac{\overline{F}_2(u_n|x_0)}{\overline{F}_2(U_2(n/k|x_0)|x_0)}} - 1 \right| \left| T_n\left(\infty, \frac{n}{k}\overline{F}_2(u_n|x_0) \Big| x_0\right) - \frac{n}{k}\overline{F}_2(u_n|x_0) f_X(x_0) \right| \\
&\qquad + \left| W\left(\infty, \frac{n}{k}\overline{F}_2(u_n|x_0)\right) - W(\infty, 1) \right| \\
&\qquad + r_n \left| \frac{\overline{F}_2(U_2(n/k|x_0)|x_0)}{\overline{F}_2(u_n|x_0)} - 1 \right| \left| T_n\left(\infty, \frac{n}{k}\overline{F}_2(u_n|x_0) \Big| x_0\right) - \frac{n}{k}\overline{F}_2(u_n|x_0) f_X(x_0) \right|,
\end{aligned}$$

we have by (5.12) combined with the Skorohod construction that

$$r_n \left(\frac{\frac{1}{n}\sum_{i=1}^n K_{h_n}(x_0 - X_i) 1\!\!1_{\{Y_i^{(2)} > u_n\}}}{\overline{F}_2(u_n|x_0)} - f_X(x_0) \right) \rightsquigarrow W(\infty, 1).$$

Then

$$\begin{aligned}
r_n \left(\frac{\widehat{\overline{F}}_{n,2}(u_n|x_0)}{\overline{F}_2(u_n|x_0)} - 1 \right) &= r_n \left(\frac{\frac{1}{n}\sum_{i=1}^n K_{h_n}(x_0 - X_i) 1\!\!1_{\{Y_i^{(2)} > u_n\}}}{\overline{F}_2(u_n|x_0) f_X(x_0)} - 1 \right) \\
&\quad + r_n \frac{f_X(x_0) - \hat{f}_n(x_0)}{f_X(x_0)\hat{f}_n(x_0)} \frac{\frac{1}{n}\sum_{i=1}^n K_{h_n}(x_0 - X_i) 1\!\!1_{\{Y_i^{(2)} > u_n\}}}{\overline{F}_2(u_n|x_0)} \\
&\rightsquigarrow \frac{W(\infty, 1)}{f_X(x_0)}.
\end{aligned}$$

■

Lemma 5.6. *Assume $(\mathcal{D})$, $(\mathcal{H})$, $(\mathcal{K})$, $x_0 \in Int(S_X)$ with $f_X(x_0) > 0$ and $y \to F_2(y|x_0)$ is strictly increasing. Consider sequences $k \to \infty$ and $h_n \to 0$ as $n \to \infty$, in such a way that $k/n \to 0$, $kh_n^d \to \infty$, $h_n^{\eta_{\varepsilon_2}} \ln n/k \to 0$, $\sqrt{kh_n^d}\, h_n^{\eta_{f_X} \wedge \eta_{A_2}} \to 0$, $\sqrt{kh_n^d}\, h_n^{\eta_{\gamma_2}} \ln n/k \to 0$, $\sqrt{kh_n^d} |\delta_2(U_2(n/k|x_0)|x_0)| \to 0$. Then, as $n \to \infty$, we have*

$$\sqrt{kh_n^d}\, (\hat{u}_n - 1) \rightsquigarrow \frac{\gamma_2(x_0) W(\infty, 1)}{f_X(x_0)}.$$

Proof

To prove the lemma we will use the idea of Wretman (1978), applied to our situation. We have, for $z \in \mathbb{R}$, and u_n from Lemma 5.5 taken as $U_2(n/k|x_0)(1 + z/\sqrt{kh_n^d})$, that

$$\begin{aligned}
&\mathbb{P}\left(\sqrt{kh_n^d}\,(\widehat{u}_n - 1) \leqslant z\right) \\
&\quad = \mathbb{P}\left(\sqrt{nh_n^d \overline{F}_2(u_n|x_0)}\left(\frac{\widehat{\overline{F}}_{n,2}(u_n|x_0)}{\overline{F}_2(u_n|x_0)} - 1\right) \leqslant \sqrt{nh_n^d \overline{F}_2(u_n|x_0)}\left(\frac{\overline{F}_2(U_2(n/k|x_0)|x_0)}{\overline{F}_2(u_n|x_0)} - 1\right)\right).
\end{aligned}$$

We have that in the present context

$$a_n := \sqrt{nh_n^d \overline{F}_2(u_n|x_0)}\left(\frac{\overline{F}_2(U_2(n/k|x_0)|x_0)}{\overline{F}_2(u_n|x_0)} - 1\right) \to \frac{z}{\gamma_2(x_0)}.$$

Let H_n denote the distribution function of $\sqrt{nh_n^d \overline{F}_2(u_n|x_0)}(\widehat{\overline{F}}_{n,2}(u_n|x_0)/\overline{F}_2(u_n|x_0) - 1)$, and H is the distribution function of $W(\infty, 1)/f_X(x_0)$. Then by Lemma 5.5 and by continuity of H one has that $H_n(a_n) \to H(z/\gamma_2(x_0))$, as $n \to \infty$, hence the result of the lemma. ■

5.4 Proof of Theorem 2.3

Let $\mathbb{E}_n(y) := \mathbb{E}(\widetilde{\theta}_n(y)/U_1(n/k|x_0))$. We have the following decomposition:

$$\begin{aligned}
\sqrt{kh_n^d}\left(\frac{\widehat{\theta}_n}{f_X(x_0)\theta_{k/n}} - 1\right) &= \frac{U_1(n/k|x_0)}{\theta_{k/n}} \frac{\sqrt{kh_n^d}}{f_X(x_0)}\left(\frac{\widehat{\theta}_n}{U_1(n/k|x_0)} - \mathbb{E}_n(1)\right) \\
&\quad + \frac{U_1(n/k|x_0)}{\theta_{k/n}} \frac{\sqrt{kh_n^d}}{f_X(x_0)}\left(\mathbb{E}_n(1) - \frac{f_X(x_0)\theta_{k/n}}{U_1(n/k|x_0)}\right) \\
&= \frac{U_1(n/k|x_0)}{\theta_{k/n}} \frac{\sqrt{kh_n^d}}{f_X(x_0)}\left(\frac{\widetilde{\theta}_n(\widehat{e}_n)}{U_1(n/k|x_0)} - \mathbb{E}_n(\widehat{e}_n)\right) \\
&\quad + \frac{U_1(n/k|x_0)}{\theta_{k/n}} \frac{\sqrt{kh_n^d}}{f_X(x_0)}\left(\mathbb{E}_n(\widehat{e}_n) - \mathbb{E}_n(1)\right) \\
&\quad + \frac{U_1(n/k|x_0)}{\theta_{k/n}} \frac{\sqrt{kh_n^d}}{f_X(x_0)}\left(\mathbb{E}_n(1) - \frac{f_X(x_0)\theta_{k/n}}{U_1(n/k|x_0)}\right) \\
&=: T_1 + T_2 + T_3.
\end{aligned}$$

First, remark that the common factor of the three terms, $U_1(n/k|x_0)/\theta_{k/n}$ can be handled in a similar way as in Proposition 1 in Cai et al. (2015), i.e., as $n \to \infty$

$$\frac{U_1(n/k|x_0)}{\theta_{k/n}} \longrightarrow \frac{-1}{\int_0^\infty R(s, 1|x_0)ds^{-\gamma_1(x_0)}}.$$

Thus the three terms without this factor need to be studied.

We start with T_1. Note that

$$\sqrt{kh_n^d}(\widehat{e}_n - 1) = -\frac{f_2(\tilde{u}_n U_2(n/k|x_0)|x_0)U_2(n/k|x_0)}{\overline{F}_2(U_2(n/k|x_0)|x_0)}\sqrt{kh_n^d}(\widehat{u}_n - 1),$$

where $\tilde{u}_n$ is a random value between $\widehat{u}_n$ and 1. By the continuous mapping theorem we have then

$$\frac{f_2(\tilde{u}_n U_2(n/k|x_0)|x_0)U_2(n/k|x_0)}{\overline{F}_2(U_2(n/k|x_0)|x_0)} \xrightarrow{\mathbb{P}} \frac{1}{\gamma_2(x_0)},$$

and hence by Lemma 5.6

$$\sqrt{kh_n^d}(\widehat{e}_n - 1) \rightsquigarrow -W(\infty, 1)/f_X(x_0). \tag{5.21}$$

This implies that

$$\mathbb{P}\left(|\widehat{e}_n - 1| > (kh_n^d)^{-1/4}\right) \to 0.$$

Hence, with probability tending to one,

$$\begin{aligned}
&\left|\frac{\sqrt{kh_n^d}}{f_X(x_0)}\left(\frac{\widetilde{\theta}_n(\widehat{e}_n)}{U_1(n/k|x_0)} - \mathbb{E}_n(\widehat{e}_n)\right) + \frac{1}{f_X(x_0)}\int_0^\infty W(s,1)ds^{-\gamma_1(x_0)}\right| \\
\leqslant \; & \sup_{|y-1|\leqslant (kh_n^d)^{-1/4}}\left|\frac{\sqrt{kh_n^d}}{f_X(x_0)}\left(\frac{\widetilde{\theta}_n(y)}{U_1(n/k|x_0)} - \mathbb{E}_n(y)\right) + \frac{1}{f_X(x_0)}\int_0^\infty W(s,y)ds^{-\gamma_1(x_0)}\right| \\
&+ \frac{1}{f_X(x_0)}\sup_{|y-1|\leqslant (kh_n^d)^{-1/4}}\left|\int_0^\infty [W(s,y) - W(s,1)]ds^{-\gamma_1(x_0)}\right|.
\end{aligned}$$

The first term of the right-hand side tends to 0 in probability by our Proposition 2.1, whereas the second term can be handled similarly as in the proof of Proposition 3 in Cai et al. (2015). Consequently

$$T_1 \rightsquigarrow \frac{1}{\int_0^\infty R(s,1|x_0)ds^{-\gamma_1(x_0)}}\frac{1}{f_X(x_0)}\int_0^\infty W(s,1)ds^{-\gamma_1(x_0)}. \tag{5.22}$$

Next step consists to look at T_2. To this aim, remark that for y equal either to 1 or $\widehat{e}_n$, we have

$$
\begin{aligned}
&\int_0^\infty \mathbb{E}\left(T_n(s_n(u), y|x_0)\right) du^{-\gamma_1(x_0)} \\
= \quad & \int_0^\infty \int_{S_K} K(v) R_{\frac{n}{k}}\left(\widetilde{s}_n(u), t_n(y)|x_0 - h_n v\right) f_X(x_0 - h_n v) dv du^{-\gamma_1(x_0)} \\
= \quad & \int_0^\infty \int_{S_K} K(v) R\left(\widetilde{s}_n(u), t_n(y)|x_0\right) f_X(x_0 - h_n v) dv du^{-\gamma_1(x_0)} \\
& + \int_0^\infty \int_{S_K} K(v) \left[R_{\frac{n}{k}}\left(\widetilde{s}_n(u), t_n(y)|x_0 - h_n v\right) - R\left(\widetilde{s}_n(u), t_n(y)|x_0\right)\right] f_X(x_0 - h_n v) dv du^{-\gamma_1(x_0)} \\
= \quad & \int_0^\infty R\left(u, y|x_0\right) du^{-\gamma_1(x_0)} \int_{S_K} K(v) f_X(x_0 - h_n v) dv \\
& + \int_{S_K} K(v) \int_0^\infty \left[R\left(\widetilde{s}_n(u), t_n(y)|x_0\right) - R(u, y|x_0)\right] du^{-\gamma_1(x_0)} f_X(x_0 - h_n v) dv \\
& + \int_0^\infty \int_{S_K} K(v) \left[R_{\frac{n}{k}}\left(\widetilde{s}_n(u), t_n(y)|x_0 - h_n v\right) - R\left(\widetilde{s}_n(u), t_n(y)|x_0\right)\right] f_X(x_0 - h_n v) dv du^{-\gamma_1(x_0)} \\
=: \quad & \widetilde{T}_{2,1} + \widetilde{T}_{2,2} + \widetilde{T}_{2,3}.
\end{aligned}
$$

By Lemma 5.4, Assumptions $(\mathcal{S})$ and $(\mathcal{H})$ we obtain

$$
\begin{aligned}
\widetilde{T}_{2,1} &= f_X(x_0) \int_0^\infty R\left(u, y|x_0\right) du^{-\gamma_1(x_0)} + O_{\mathbb{P}}\left(h_n^{\eta_{f_X}}\right), \\
\widetilde{T}_{2,2} &= o_{\mathbb{P}}\left(\frac{1}{\sqrt{k h_n^d}}\right), \\
|\widetilde{T}_{2,3}| &\leqslant - \sup_{x \in B(x_0, h_n)} \sup_{0 < y_1 < \infty, \frac{1}{2} \leqslant y_2 \leqslant 2} \frac{|R_{n/k}(y_1, y_2|x) - R(y_1, y_2|x_0)|}{y_1^\beta \wedge 1} \\
& \quad \times \int_{S_K} K(v) \int_0^\infty \left([\widetilde{s}_n(u)]^\beta \wedge 1\right) du^{-\gamma_1(x_0)} f_X(x_0 - h_n v) dv \\
&= O_{\mathbb{P}}\left(\left(\frac{n}{k}\right)^\tau\right).
\end{aligned}
$$

Note that the integral appearing in the bound for $|\widetilde{T}_{2,3}|$ is finite for n large, as $\widetilde{s}_n(u) \leqslant C u^{1-\xi}$ for $u \in (0, 1/2], \xi \in (0, (\beta - \gamma_1(x_0))/\beta)$ and n large. Consequently, under our assumptions and using the homogeneity of the $R-$function and the mean value theorem combining with (5.21), we have

$$
\begin{aligned}
& \frac{\sqrt{k h_n^d}}{f_X(x_0)} \left(\mathbb{E}_n(\widehat{e}_n) - \mathbb{E}_n(1)\right) \\
= \quad & \frac{\sqrt{k h_n^d}}{f_X(x_0)} \left(\int_0^\infty \mathbb{E}\left(T_n(s_n(u), 1|x_0)\right) du^{-\gamma_1(x_0)} - \int_0^\infty \mathbb{E}\left(T_n(s_n(u), \widehat{e}_n|x_0)\right) du^{-\gamma_1(x_0)}\right) \\
= \quad & \sqrt{k h_n^d} \left(\int_0^\infty R\left(u, 1|x_0\right) du^{-\gamma_1(x_0)} - \int_0^\infty R\left(u, \widehat{e}_n|x_0\right) du^{-\gamma_1(x_0)}\right) + o_{\mathbb{P}}(1) \\
= \quad & \sqrt{k h_n^d} \left(1 - \widehat{e}_n^{1-\gamma_1(x_0)}\right) \int_0^\infty R\left(u, 1|x_0\right) du^{-\gamma_1(x_0)} + o_{\mathbb{P}}(1) \\
\rightsquigarrow \quad & (1 - \gamma_1(x_0)) \frac{W(\infty, 1)}{f_X(x_0)} \int_0^\infty R\left(u, 1|x_0\right) du^{-\gamma_1(x_0)}.
\end{aligned}
$$

This implies that

$$T_2 \rightsquigarrow -(1-\gamma_1(x_0))\frac{W(\infty,1)}{f_X(x_0)}. \tag{5.23}$$

Finally, for T_3 we have,

$$\begin{aligned}
&\frac{\sqrt{kh_n^d}}{f_X(x_0)}\left(\mathbb{E}_n(1)-\frac{f_X(x_0)\theta_{k/n}}{U_1(n/k|x_0)}\right)\\
=\;&\frac{\sqrt{kh_n^d}}{f_X(x_0)}\left(-\int_0^\infty \mathbb{E}(T_n(s_n(u),1|x_0))du^{-\gamma_1(x_0)}-\frac{f_X(x_0)\theta_{k/n}}{U_1(n/k|x_0)}\right)\\
=\;&\sqrt{kh_n^d}\int_0^\infty \left[R_{n/k}(s_n(u),1|x_0)-R(u,1|x_0)\right]du^{-\gamma_1(x_0)}+o(1)\\
=\;&\sqrt{kh_n^d}\int_0^\infty \left[R_{n/k}(s_n(u),1|x_0)-R(s_n(u),1|x_0)\right]du^{-\gamma_1(x_0)}\\
&+\sqrt{kh_n^d}\int_0^\infty \left[R(s_n(u),1|x_0)-R(u,1|x_0)\right]du^{-\gamma_1(x_0)}+o(1)\\
=:\;&\widetilde{T}_{3,1}+\widetilde{T}_{3,2}+o(1),
\end{aligned}$$

where

$$\begin{aligned}
|\widetilde{T}_{3,1}| &\leqslant \sqrt{kh_n^d}\sup_{x\in B(x_0,h_n)}\sup_{0<y_1<\infty,\frac{1}{2}\leqslant y_2\leqslant 2}\frac{|R_{n/k}(y_1,y_2|x)-R(y_1,y_2|x_0)|}{y_1^\beta\wedge 1}\left|\int_0^\infty\left([s_n(u)]^\beta\wedge 1\right)du^{-\gamma_1(x_0)}\right|\\
&= O\left(\sqrt{kh_n^d}\left(\frac{n}{k}\right)^\tau\right),\\
\widetilde{T}_{3,2} &= o(1).
\end{aligned}$$

Overall, we have then

$$T_3 = o(1). \tag{5.24}$$

Combining (5.22), (5.23) and (5.24), and following the argument as at the end of the proof of Theorem 2.2, we can establish the result of Theorem 2.3. ■

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