Ample completions of oriented matroids and complexes of uniform oriented matroids

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Distance and routing labeling schemes for cube-free median graphs

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Abstract. Distance labeling schemes are schemes that label the vertices of a graph with short labels in such a way that the distance between any two vertices u and v can be determined efficiently by merely inspecting the labels of u and v , without using any other information. Similarly, routing labeling schemes label the vertices of a graph in a such a way that given the labels of a source node and a destination node, it is possible to compute efficiently the port number of the edge from the source that heads in the direction of the destination. One of important problems is finding natural classes of graphs admitting distance and/or routing labeling schemes with labels of polylogarithmic size. In this paper, we show that the class of cube-free median graphs on n nodes enjoys distance and routing labeling schemes with labels of $O(\log^3 n)$ bits.

1. INTRODUCTION

Classical network representations are usually global in nature. In order to derive a useful piece of information, one must access to a global data structure representing the entire network even if the needed information only concerns few nodes. Nowadays, with networks getting bigger and bigger, the need for locality is more important than ever. Indeed, in several cases, global representations are impractical and network representation must be distributed. The notion of (distributed) labeling scheme has been introduced [18, 46, 55, 56, 40] in order to meet this need. A *(distributed) labeling scheme* is a scheme maintaining global information on a network using local data structures (or labels) assigned to nodes of the network. Their goal is to locally store some useful information about the network in order to answer specific query concerning a pair of nodes by only inspecting the labels of the two nodes. Motivation of such localized data structure in distributed computing is surveyed and widely discussed in [55]. The predefined queries can be of various types such as distance, adjacency, or routing. The quality of a labeling scheme is measured by the size of the labels of nodes and the time required to answer queries. Trees with n vertices admit adjacency and routing labeling schemes with size of labels and query time $O(\log n)$ and distance labeling schemes with size of labels and query time $O(\log^2 n)$, and this is asymptotically optimal. Finding natural classes of graphs admitting distance and/or routing labeling schemes with labels of polylogarithmic size is an important and challenging problem.

In this paper, we design distance and routing schemes for the subclass of median graphs containing no cubes (hypercube graphs of dimension three). In our schemes, the labels have $O(\log^3 n)$ bits¹ and the time complexity of the queries is in $O(\log^2 n)$. Median graphs constitutes the most important class in metric graph theory [9]. This importance is explained by the bijections between median graphs and discrete structures arising and playing important roles in completely different areas of research in mathematics and theoretical computer science: in fact, median graphs, 1-skeletons of CAT(0) cube complexes from geometric group theory [44, 58], domains of event structures from concurrency [64], median algebras from universal algebra [11], and solution sets of 2-SAT formulae from complexity theory [51, 59] are all the same.

The remaining part of this note is organized in the following way. In the next Section 2 we introduce the most important and general notions used in this paper. In Section 3 we review the main results on distance and routing labeling schemes and the main results on median graphs

¹All logarithms in this paper are in base 2

related to the paper. In Section 4 we recall or establish some properties of general median graphs used in our labeling schemes. In Section 5 we present the most important geometric and structural properties of cube-free median graphs, which are the essence of our distance and routing schemes and which do not hold for general median graphs. Sections 6 and 7 describe our distance and routing labeling schemes for cube-free median graphs and analyse their size, time complexity of queries, and the complexity of their construction.

2. PRELIMINARIES

2.1. Basic notions. In this subsection, we recall some basic notions from graph theory. All graphs $G = (V, E)$ occurring in this note are undirected, simple, and connected. In our algorithmic results we will also suppose that they are finite. The *closed neighborhood* of a vertex v is denoted by $N[v]$ and consists of v and the vertices adjacent to v . The *(open) neighborhood* $N(v)$ of v is the set $N[v] \setminus \{v\}$. The *degree* $\deg(v)$ of v is the number of vertices in its open neighborhood. We will write $u \sim v$ if two vertices u and v are adjacent and $u \not\sim v$ if u and v are not adjacent. We will denote by $G[S]$ the subgraph of G induced by a subset of vertices S of V . If it is clear from the context, we will use the same notation S for the set S and the subgraph $G[S]$ induced by S .

The *distance* $d_G(u, v)$ between two vertices u and v is the length of a shortest (u, v) -path, and the *interval* $I(u, v)$ between u and v consists of all the vertices on shortest (u, v) -paths, that is, of all vertices (metrically) *between* u and v :

$$I(u, v) := \{x \in V : d_G(u, x) + d_G(x, v) = d_G(u, v)\}.$$

A subgraph H of G (or the corresponding vertex set) is called *convex* if it includes the interval of G between any pair of its vertices. A subgraph H of G is said to be *gated* if for every vertex $v \notin V(H)$, there exists a vertex $v' \in V(H)$ such that for all $u \in V(H)$, $d_G(v, u) = d_G(v, v') + d_G(v', u)$ (v' is called the *gate* of v in H). For a vertex x of a gated subgraph H of G , the set (or the subgraph induced by this set) $F(x) = \{v \in V : x \text{ is the gate of } v \text{ in } H\}$ is called the *fiber* of x with respect to H . From the definition it follows that the fibers $\{F(x) : x \in H\}$ define a partition of the vertex set of G . Notice also that gated sets of a graph enjoy the finite *Helly property*, that is, every finite family of gated sets that pairwise intersect has a nonempty intersection.

A graph $G = (V, E)$ is *isometrically embeddable* into a graph $H = (W, F)$ if there exists a mapping $\varphi : V \rightarrow W$ such that $d_H(\varphi(u), \varphi(v)) = d_G(u, v)$ for all vertices $u, v \in V$.

The *m -dimensional hypercube* Q_m is the graph whose vertex-set consists of all subsets of an m -set $X := \{1, \dots, m\}$ and in which two vertices A and B are linked by an edge if and only if $|A \Delta B| = 1$.

For a vertex $u \in V$ of a graph $G = (V, E)$, let $M(u) := \sum_{v \in V} d_G(u, v)$. A vertex $m \in V$ minimizing the function M is called a *median vertex* of G . It is well known that any tree T has either a single median vertex or two adjacent median vertices. Moreover, a vertex v is a median vertex of T if and only if any subtree of $T \setminus \{v\}$ contains at most a half of vertices of T . For this reason, a median vertex of a tree is often called a *centroid*.

A graph G is called *median* if the intersection $I(x, y) \cap I(y, z) \cap I(z, x)$ is a singleton for each triplet x, y, z of vertices. The unique vertex $m(x, y, z) \in I(x, y) \cap I(y, z) \cap I(z, x)$ is called the *median* of x, y, z . Median graphs are bipartite. Basic examples of median graphs are trees, hypercubes, rectangular grids, and Hasse diagrams of distributive lattices and of median semilattices [9]. The *star* $\text{St}(z)$ of a vertex z of a median graph G is the union of all hypercubes of G containing z . If G is a tree and z has degree r , then $\text{St}(z)$ is the closed neighborhood of z and is isomorphic to $K_{1,r}$. The *dimension* $\dim(G)$ of a median graph G is the largest dimension of a hypercube of G .

A *cube-free median graph* is a median graph G of dimension 2, i.e., a median graph not containing 3-cubes as isometric subgraphs. Two illustrations of cube-free median graphs are given in Figure 1.

The left figure will be used as a running example to illustrate the main definitions. Even if cube-free median graphs are the skeletons of 2-dimensional CAT(0) cube complexes, their combinatorial structure is rather intricate. For example, cube-free median graphs are not necessarily planar: for this, take the Cartesian product $K_{1,n} \times K_{1,m}$ of the stars $K_{1,n}$ and $K_{1,m}$ for $n, m \geq 5$.

FIGURE 1. Two cube-free median graphs. The left graph will be used as a running example.

2.2. Distance and routing labeling schemes. Let $G = (V, E)$ be a finite graph. The *ports* of a vertex $u \in V$ are the unique (with respect to u) numbers given to the oriented edges around u , i.e., the edges $\vec{uv}$ with $v \in N(u)$. If $uv \in E$, then the *port from u to v* , denoted $\text{port}(u, v)$, is the number given to $\vec{uv}$. More generally, for arbitrary vertices u, v of G , $\text{port}(u, v)$ denote any value $\text{port}(u, v')$ such that $uv' \in E$ and $v' \in I(u, v)$. A *graph with ports* is a graph to which vertices and edges are given ports. All the graphs in this paper are supposed to be graphs with ports.

A *labeling scheme* on a graph family $\mathcal{G}$ consists of an encoding function and a decoding function. The encoding function is given a total knowledge of a graph $G \in \mathcal{G}$ and gives labels to its vertices in order to allow the decoding function to answer a predefined question (query) with knowledge of a restricted number of labels only. The encoding and decoding functions highly depend on the family $\mathcal{G}$ and on the type of queries: adjacency, distance, or routing queries.

More formally, a *distance labeling scheme* on a graph family $\mathcal{G}$ consists of an *encoding function* $C_G : V(G) \rightarrow \{0, 1\}^*$ that gives to every vertex of a graph G of $\mathcal{G}$ a label, and of a *decoding function* $D_G : \{0, 1\}^* \times \{0, 1\}^* \rightarrow \mathbb{N}$ that, given the labels of two vertices u and v of G , can compute efficiently the distance $d_G(u, v)$ between them. In a *routing labeling scheme*, the encoding function $C'_G : V(G) \rightarrow \{0, 1\}^*$ gives labels such that the decoding function $D'_G : \{0, 1\}^* \times \{0, 1\}^* \rightarrow \mathbb{N}$ is able, given the labels of a *source* u and a *target* v , to decide which port of u to take to get closer to v .

We continue by recalling the distance labeling scheme for trees proposed by Peleg in [55]. First, as we noticed above, if T is a tree with n vertices and m is a median vertex of T , then the removal of m splits T in subtrees with at most $\frac{n}{2}$ vertices each. The distance between any two vertices u and v from different subtrees of $T \setminus \{m\}$ is $d_T(u, m) + d_T(m, v)$. Therefore, each vertex of T can keep in its label the distance to m . Hence, it remains to recover the information necessary to compute the distance between two vertices in the same subtree of $T \setminus \{m\}$. This can be done by recursively applying to each subtree T' of $T \setminus \{m\}$ the same procedure as for T . Consequently, the label of each vertex v of T consists of the distances from v to the roots of all subtrees occurring in the recursive calls and containing v . Since from step to step the size of such subtrees is divided by at least 2, v belongs to $\log n$ subtrees, thus the label of each vertex v of T has size $\log^2 n$.

3. RELATED WORK

In this section we review some known results on distance and routing labeling schemes and on median graphs.

3.1. Distance and routing labeling schemes.

3.1.1. Distance labeling schemes. The notion of Distance Labeling Schemes (DLS) was first introduced in a series of papers by Peleg et al. [55, 56, 40]. Before these works, some closely related notions already existed such as embeddings in a squashed cube [62] (equivalent to distance labeling schemes with labels of size $\log n$ times the dimension of the cube) or implicit representation of graphs [46] (labeling schemes for adjacency requests).

One of the main results for DLS is that general graphs support distance labeling schemes with labels of size $O(n)$ bits [62, 40, 5]. This scheme is asymptotically optimal since it is easy to show that $\Omega(n)$ bits labels are needed for general graphs. Another important result is that there exists a distance labeling scheme for the class of trees with $O(\log^2 n)$ bits labels [55, 6]. Several classes of graphs containing trees also enjoy a distance labeling scheme with $O(\log^2 n)$ bit labels such as bounded tree-width graphs [40], distance-hereditary graphs [38], bounded clique-width graphs [30], and non-positively curved plane graphs [26]. A lower bound of $\Omega(\log^2 n)$ bits on the label length is known for trees [40, 6], implying that all the results mentioned above are optimal as well. Other families of graphs have been considered such as interval graphs, permutation graphs, and their generalizations [14, 39] for which an optimal bound of $\Theta(\log n)$ bits was given, and planar graphs for which there is a lower bound of $\Omega(n^{\frac{1}{3}})$ bits [40] and an upper bound of $O(\sqrt{n})$ bits [42].

Other results concern approximate distance labeling schemes, i.e., schemes that gives an approximation of the distance up to an additive factor and/or a multiplicative factor (often called stretch). For arbitrary graphs, the most impactful result is due to Thorup and Zwick [61]. They proposed a $(2k - 1)$ -multiplicative distance labeling scheme, for each integer $k \geq 1$, with labels of $O(n^{1/k} \log^2 n)$ bits. In [37], it is proved that trees (and bounded tree-width graphs as well) admit a $(1 + 1/\log n)$ -multiplicative DLS with labels of $O(\log n \log \log n)$ bits, and this is tight in terms of label length and approximation. They also design some $O(1)$ -additive DLS with $O(\log^2 n)$ bit labels for several families of graphs, including the graphs with bounded longest induced cycle, and, more generally, the graphs of bounded tree-length. For δ -hyperbolic graph, there is a $O(\delta \log n)$ -additive scheme with $O(\log^2 n)$ bit labels [25]. Finally, some works deal with affine approximation that combines a multiplicative factor and an additive factor [1]. Notice that graphs of bounded tree-length have bounded hyperbolicity and, more importantly, they can be embedded into trees with bounded distortion, depending of the tree-length. This provides an alternative view on the last result of [37]. Interestingly, one can easily show that every exact DLS for all those families of graphs needs labels of $\Omega(n)$ bits in the worst-case [37]. This can be explained by the fact that such properties as hyperbolicity, tree-length, and quasi-isometricity to a tree are global (coarse) geometric properties, thus allowing an arbitrary local behavior, and therefore, arbitrary errors for reporting small distances.

An alternative to approximate all distances is to report exact distance only for some subsets of all pairs of nodes. This work was mainly concentrated on reporting all large distances (i.e., distances larger than D) or of all small distances (distances smaller than D) for a threshold D . For example, Bollobàs et al. [17] introduced the notion of D -preserving DLS, which is a DLS that reports exact distances only for pairs of nodes at distance at least D (notice also that the existing distance labeling schemes for δ -hyperbolic graphs report large distances with a much better accuracy than small distances for a threshold value related to δ and $\log n$). They presented such labeling schemes with labels of size $O(\frac{n}{D} \log^2 n)$. This was later improved to $O(\frac{n}{D} \log^2 D)$ and a lower bound of $\Omega(\frac{n}{D})$ was also provided [4]. For rooted trees, [47] introduced the notion of DLS for short distances, i.e.,

that reports the distance to the common ancestor, and so the distance, for nodes at distance at most D . The best known upper bound for the size of labels of such scheme is $\log n + O(D \log(D \log(n/D)))$ [36] and for $D \geq 2$ there is a lower bound of $\log n + \Omega(\log \log n)$ [3].

3.1.2. Routing labeling scheme. Routing is one of the basic tasks that a distributed network must be able to perform. The design of efficient Routing Labeling Scheme (RLS) is a well studied subject. For a general overview of this area, we refer the reader to the book [54]. One trivial way to produce an exact RLS, i.e., a routing via shortest path, is to store a complete routing table at each node of the network. This table specifies, for any destination, the port leading to a shortest path to that destination. This gives an exact RLS with labels of size $O(n \log d)$ bits for graphs of maximum degree d that is optimal for general graphs [41]. For trees, there exists exact RLS with labels of size $(1 + o(1)) \log n$ [35, 60]. Exact RLS with labels of polylogarithmic size also exist for graphs of bounded tree-width, clique-width or chordality [33] and for non-positively curved plane graphs [26]. For the families of graph excluding a fixed minor (including planar and bounded genus graphs), there is an exact RLS with labels of size $O(\sqrt{n} \log^2 n / \log \log n)$ [33].

To obtain RLS for general graphs with $o(n)$ bits label, one has to abandon the requirement that packets are always routed via shortest paths, and settle instead for the requirement that packets are routed on paths which are close to optimal like the results for DLS [31, 34, 60]. A 3-multiplicative RLS that uses labels of size $\tilde{O}(n^{2/3})$ was obtained in [31], and a 5-multiplicative RLS with labels of size $\tilde{O}(n^{1/2})$ was obtained in [34]. The authors of [60] later improved these results by giving a $(4k - 5)$ -multiplicative RLS with only $\tilde{O}(kn^{1/k})$ bit labels, for every $k \geq 2$. There are also some results on affine stretch RLS [1].

3.2. Median graphs. Median graphs and related median structures (median algebras and median complexes) have an extensive literature. The term of median graphs was introduced by [52] while the concept have arisen before in works on distributed lattices [16, 7]. These structures have been investigated in several contexts by quite a number of authors for more than half a century. Median structures are still being rediscovered in various disguises and several surveys exist listing their numerous characterizations and their properties [9, 48, 49]. In this subsection we briefly review some characterizations of median graphs and the bijection between median graphs and CAT(0) cube complexes. We also recall some results, related to the subject of this paper, about the distance and shortest path problems in median graphs and CAT(0) cube complexes. For a survey of results on median graphs and their bijections with median algebras, median semilattices, and solution spaces of 2-SAT formulae, see [9, 49]. For a comprehensive presentation of median graphs and CAT(0) cube complexes as domains of event structures, see the long version of [20].

3.2.1. Characterizations and properties of median graphs. A median graph is a graph in which every triplet of vertices has a unique median, i.e., a vertex simultaneously lying on shortest paths between any pair of the triplet. It is not immediately clear from the definition, but median graphs are intimately related to hypercubes: median graphs can be obtained from hypercubes by amalgams and median graphs are themselves isometric subgraphs of hypercubes [12, 50]. Even more, by a nice result of Bandelt [8], *median graphs are exactly the retracts of hypercubes*.

The canonical isometric embedding of a median graph G into a (smallest) hypercube can be determined by the so called *Djoković-Winkler* (“parallelism”) relation Θ on the edges of G [32, 63]. For median graphs, the equivalence relation Θ can be defined as follows. First say that two edges uv and xy are in relation Θ' if they are opposite edges of a 4-cycle $uvxy$ in G . Then let Θ be the reflexive and transitive closure of Θ' . Any equivalence class of Θ constitutes a cutset of the median graph G , which determines one factor of the canonical hypercube [50]. The cutset (equivalence class) $\Theta(xy)$ containing an edge xy defines a convex split $\{W(x, y), W(y, x)\}$ of G [50], where $W(x, y) = \{z \in V : d_G(z, x) < d_G(z, y)\}$ and $W(y, x) = V \setminus W(x, y)$ (we call the complementary

convex sets $W(x, y)$ and $W(y, x)$ *halfspaces*). Conversely, for every convex split of a median graph G there exists at least one edge xy such that $\{W(x, y), W(y, x)\}$ is the given split. We denote by $\{\Theta_i : i \in I\}$ the equivalence classes of the relation Θ (in [13], they were called parallelism classes). For an equivalence class $\Theta_i, i \in I$, we denote by $\{A_i, B_i\}$ the associated convex split. We say that Θ_i *separates* the vertices x and y if $x \in A_i, y \in B_i$ or $x \in B_i, y \in A_i$. Then the isometric embedding φ of G into a hypercube is obtained by taking a basepoint v , setting $\varphi(v) = \emptyset$ and for any other vertex u , letting $\varphi(u)$ be all parallelism classes of Θ which separate u from v .

Notice that this embeddings into a hypercube can be performed for all bipartite graphs for which for every edge xy , the sets $W(x, y)$ and $W(y, x)$ are convex. In fact, this completely characterizes the graphs isometrically embeddable into hypercubes due to a result of Djoković [32]. The difference between median graphs and general isometric subgraphs of hypercubes is that in median graphs the convex sets are gated (we will provide the simple proof of this folklore result below). Therefore, all halfspaces $\{W(x, y), W(y, x)\}$ of a median graph G are gated and therefore satisfy the Helly property. In fact, this *Helly property of halfspaces characterizes median graphs* [51].

In median graphs not only halfspaces are convex (and gated) but also their boundaries $\{\partial W(x, y), \partial W(y, x)\}$ are convex [50], where $\partial W(x, y)$ consists of all vertices $x' \in W(x, y)$ having a neighbor y' in $W(y, x)$. Then clearly $y' \in \partial W(y, x)$ and such neighbor y' of x' is unique.

3.2.2. Median graphs and $CAT(0)$ cube complexes. Due to the abundance of hypercubes, to each median graph G one can associate a cube complex $X(G)$ and expect that $X(G)$ has strong structural properties. $X(G)$ is obtained by replacing every subgraph of G which is a hypercube by a solid unit cube of the same dimension. Then G can be recovered as the 1-skeleton $X^{(1)}(G)$ of $X(G)$, i.e., as the graph having the 0-cubes of $X(G)$ as vertices and the 1-cubes of $X(G)$ as edges. The cube complex $X(G)$ can be endowed with several intrinsic metrics. The intrinsic ℓ_1 -metric of $X(G)$ extends the standard graph metric d_G of G . Another important metric on a cube complex X is the intrinsic ℓ_2 -metric defined by letting the distance between two points $x, y \in X$ be equal to the greatest lower bound on the ℓ_2 -length of the paths joining them. Here a path in X from x to y is a sequence $x = x_0, x_1, \dots, x_{k-1}, x_k = y$ such that any two consecutive points x_i, x_{i+1} belong to a common cube of X . Then X endowed with the ℓ_2 -metric is a geodesic metric space.

An important class of cube complexes studied in geometric group theory and combinatorics is the class of $CAT(0)$ cube complexes. $CAT(0)$ geodesic metric spaces are usually defined via the nonpositive curvature comparison axiom of Cartan–Alexandrov–Toponogov [19]. However for cube complexes (and more generally for cell complexes) the $CAT(0)$ property can be defined in a very simple and intuitive way by the property that *ℓ_2 -geodesics between any two points are unique*. $CAT(0)$ spaces can be characterized in several different natural ways and have many strong geometric and topological properties, see for example [19]. Gromov [44] gave a beautiful combinatorial characterization of $CAT(0)$ cube complexes, which can be also taken as their definition: *A cube complex X endowed with the ℓ_2 -metric is $CAT(0)$ if and only if X is simply connected and whenever three $(k+2)$ -cubes of X share a common k -cube containing x and pairwise share common $(k+1)$ -cubes, then they are contained in a $(k+3)$ -cube of X .*

We continue with the promised bijection between $CAT(0)$ cube complexes and median graphs established in [24, 57]: *Median graphs are exactly the 1-skeletons of $CAT(0)$ cube complexes*. The proof of this result presented in [24] is based on the following local-to-global characterization of median graphs: *A graph G is a median graph if and only if its cube complex is simply connected and G satisfies the 3-cube condition: if three squares of G pairwise intersect in an edge and all three intersect in a vertex, then they belong to a 3-cube.*

3.2.3. Distance problems in median graphs and $CAT(0)$ cube complexes. Gromov’s characterization was used to show that several cube complexes arising in applications are $CAT(0)$. Billera, Holmes, and Vogtmann [15] proved that the space of trees (encoding all tree topologies with a given set of

leaves) is a CAT(0) cube complex. The spaces of trees are particular bouquets (stars) of cubes. Abrams, Ghrist and Peterson [2, 43] considered the continuous space of all possible positions of a reconfigurable system, called a state complex, and showed that in many cases this state complex is CAT(0). Billera et al. [15] formulated the problem of computing the geodesic (the unique shortest path) between two points in the space of trees. In the robotics literature, geodesics in the CAT(0) state complex correspond to the motion planning to get the robot from one position to another one with minimal power consumption. A polynomial-time algorithm for geodesic problem in the space of trees was provided in [53]. A linear-time algorithm for computing distances in CAT(0) square complexes (2-dimensional cube complexes) was proposed in [29]. Finally, very recently Hayashi [45] designed the first polynomial-time algorithm for geodesic problem in all CAT(0) cube complexes.

Returning to median graphs, computing the distance or a shortest path between two vertices constitute more tractable problems and, to our knowledge, no special algorithms were designed. If we come to labeling schemes for median graphs, the following is known. First, any median graph G on n vertices has at most $n \log n$ edges, thus its arboricity is at most $\log n$. As a consequence, median graphs admit adjacency schemes of size $O(\log^2 n)$ per vertex. As we noticed in [28], one $\log n$ factor can be replaced by the dimension d of the largest cube of G . Compact distance and routing labeling schemes can be obtained for some subclasses of cube-free median graphs. One particular class is that of squaregraphs: these are plane graphs in which all inner vertices have degree ≥ 4 . For squaregraphs, distance and routing labeling schemes with labels of size $O(\log^2 n)$ follow from a more general result of [26] for plane graphs of nonpositive curvature. Another such class of graphs is that of partial double trees [10]. Those are exactly the median graphs which can be isometrically embedded into a Cartesian product of two trees and can be characterized as the cube-free median graphs in which all links are bipartite graphs. The isometric embedding of partial double trees into a product of two trees immediately leads to distance labeling schemes with $O(\log^2 n)$ labels. Finally, with a technically involved proof, it was shown in [27] that there exists a constant M such that any cube-free median graph G with maximum degree Δ can be isometrically embedded into a Cartesian product of at most $\epsilon(\Delta) := M\Delta^{26}$ trees. This immediately shows that cube-free median graph admit distance labeling schemes with labels of length $O(\epsilon(\Delta) \log^2 n)$. Compared with the $O(\log^3 n)$ -labeling scheme obtained in the current paper, the disadvantage of the $O(\epsilon(\Delta) \log^2 n)$ -labeling scheme is the dependence from the maximum degree Δ of G .

However, the situation is even worse for high dimensional median graphs: the paper [27] presents an example of a 5-dimensional median graph/CAT(0) cube complex with uniformly bounded degrees which cannot be embedded into a Cartesian product of a finite number of trees. Therefore, for general finite median graphs the function $\epsilon(\Delta)$ does not exist. This in some sense explains the difficulty of designing polylogarithmic distance labeling schemes for general median graphs. Nevertheless, we do not have any indication to believe that such schemes do not exist.

4. FIBERS IN MEDIAN GRAPHS

In this section, we recall the properties of median graphs and of the fibers of their gated subgraphs. They will be used in our labeling schemes and some of them could be potentially useful for designing distance labeling schemes for general median graphs. Since all those results are dispersed in the literature and time, we present them with (usually, short and unified) proofs. To motivate the investigation of fibers, in the next subsection we present two approaches for designing distance schemes in median graphs.

4.1. Two ideas of distance schemes for median graphs. Let $G = (V, E)$ be a median graph with n vertices. Similarly to trees, one can first envisage the following recursive approach. Let m be a median vertex of G and for each vertex v of G let's keep in its label the distance $d_G(v, m)$ to m . For any neighbor x of m , the halfspace $W(x, m) = \{v \in V : d_G(x, v) < d_G(x, m)\}$ (which we can call

a *halfspace at m*) contains at most $n/2$ vertices and induces a gated (and thus median) subgraph of G . (For a tree T and a neighbor x of m , $W(x, m)$ is the subtree of $T \setminus \{m\}$ containing x .) Thus, we can recursively call the algorithm to the subgraph induced by each halfspaces at m . There are $\log n$ levels of recursion calls, however the size of labels of vertices is no longer polylogarithmic and, even worse, the resulting labels do not provide a distance labeling scheme for G . This is due to the fact that, differently from the subtrees of $T \setminus \{m\}$, the halfspaces at m are not pairwise disjoint. Therefore, the separation of vertices for which $d_G(u, v)$ can be computed as $d_G(u, m) + d_G(m, v)$ or via a recursive call is not longer done via a membership test to different halfspaces.

To circumvent this difficulty, instead of considering the halfspaces at m , we can consider the fibers of the star $\text{St}(m)$ of the median vertex m . One can show that $\text{St}(m)$ is gated, moreover, all fibers $F(x), x \in \text{St}(m)$ are also gated. As a result, the fibers $\mathcal{F}_m := \{F(x) : x \in \text{St}(m)\}$ of $\text{St}(m)$ partition the vertex-set of G into gated (and thus median) subgraphs of G . In case of trees T , this is exactly the partition into subtrees of $T \setminus \{m\}$ plus the vertex m . Since m is a median vertex of G , each fiber has at most $n/2$ vertices. Consequently, for each vertex v of G one can keep in its label the distance $d_G(v, m)$ and make a recursive call to the (gated and thus median) subgraphs induced by the fibers $F(x)$ of $\text{St}(m)$. This way, each vertex belongs to at most $\log n$ subgraphs occurring in recursive calls, thus the labels of vertices have size $\log^2 n$. However, this is not yet a distance labeling scheme because the distance between two vertices u and v belonging to distinct fibers $F(x)$ and $F(y)$ of $\text{St}(m)$ is not always $d_G(u, m) + d_G(m, v)$. One can show that $d_G(u, v) = d_G(u, m) + d_G(m, v)$ if the cubes Q_x and Q_y in $\text{St}(m)$ spanned by the pairs $\{x, m\}$ and $\{y, m\}$ intersect only in the vertex m , however $d_G(u, v)$ can be arbitrarily smaller than $d_G(u, m) + d_G(m, v)$ if Q_x and Q_y intersect in a cube of dimension ≥ 1 . It is not clear how to manage this problem for general median graphs, however the additional properties of fibers of cube-free median graphs established in the next Section 5 allow us to complete this labeling scheme to a distance labeling scheme of size $O(\log^3 n)$.

4.2. Properties of median graphs. In this subsection we recall some well-known properties of median graphs.

Lemma 1. *Any median graph $G = (V, E)$ satisfies the following quadrangle condition:*

For any vertices $u, v, w, z \in V$ such that $d_G(u, z) = k + 1$, $v, w \sim z$, and $d_G(u, v) = d_G(u, w) = k$, there is a unique vertex $x \sim v, w$ such that $d_G(u, x) = k - 1$.

Proof. Let x be the median of the triplet u, v, w . Then x must be adjacent to v and w . Since $x \in I(u, v) \cap I(u, w)$, necessarily $d_G(u, x) = k - 1$. Since any vertex x' adjacent to v, w and having distance $k - 1$ to u is a median of u, v, w , we conclude that $x' = x$, concluding the proof. $\square$

The following result is a particular case of the local-to-global characterization of convexity and gatedness in weakly modular graphs established in [23]:

Lemma 2. *For a median graph $G = (V, E)$ and a subset of vertices A of G , the following properties are equivalent:*

- (i) $H := G[A]$ is connected and A is locally convex, i.e., if $x, y \in A$ and $d_G(x, y) = 2$, then $I(x, y) \subseteq A$;
- (ii) A is convex;
- (iii) A is gated.

Proof. (i) $\Rightarrow$ (ii): Let u and v be any two vertices of A . We show that $I(u, v) \subseteq A$ by induction on the distance $d_H(u, v)$ between u and v in H . If $d_H(u, v) = 2$, then the property holds by local convexity of A . Let $d_H(u, v) = k \geq 3$ and suppose that $I(u', v') \subseteq A$ for any two vertices $u', v' \in A$ such that $d_H(u', v') \leq k - 1$. Pick any vertex $x \in I(u, v)$. Let u' be the neighbor of u on a shortest (u, v) -path of G passing via x . Let also u'' be the neighbor of u on a shortest (u, v) -path of H . Since $d_H(u'', v) = k - 1$, by induction hypothesis, $I(u'', v) \subseteq A$. Since G is bipartite and $u \sim u''$,

$|d_G(u, v) - d_G(u'', v)| = 1$. If $d_G(u'', v) > d_G(u, v)$, then $x \in I(u, v) \subset I(u'', v) \subset A$ and we are done. Now, let $d_G(u'', v) = d_G(u, v) - 1 = d_G(u', v)$ and $d_H(u, v) = d_G(u, v)$. By quadrangle condition there exists a vertex $z \sim u', u''$ at distance $k - 2$ from v . Since $z \in I(u'', v) \subset A$ and $u' \sim u, z$, by local convexity of A we deduce that u' belongs to A . Since $d_H(u', v) = d_G(u'', v) = k - 1$, by induction hypothesis, $I(u', v) \subset A$. Since $x \in I(u', v)$, x belongs to A and we are done.

(ii) $\Rightarrow$ (iii): Assume by way of contradiction that A is convex but not gated. Then there exists a vertex $u \in V \setminus A$ which does not have a gate in A . Let x be a closest to u vertex of A . Since x is not the gate of u , there exists a vertex $y \in A$ such that $x \notin I(u, y)$. Let m be the median of the triplet u, x, y . Since $x \notin I(u, y)$, $m \neq x$. Since $m \in I(x, y)$ and H is convex, m belongs to A . Since $m \in I(x, u)$ and $m \neq x$, $d_G(u, m) < d_G(u, x)$, contrary to the choice of x .

(iii) $\Rightarrow$ (i): Any gated set A induces a connected subgraph. To prove that a gated set A is locally convex, pick $x, y \in A$ with $d_G(x, y) = 2$ and a common neighbor u of x, y . If $u \notin A$, then obviously u does not have a gate in A because $I(u, x) \cap A = \{x\}$ and $I(u, y) \cap A = \{y\}$. $\square$

4.3. Properties of fibers in median graphs. We continue with properties of stars and fibers of stars of median graphs.

Combinatorially, the stars of median graphs may have quite an arbitrary structure: by a result of [12], there is a bijection between the stars of median graphs and arbitrary graphs. Namely, given an arbitrary graph H , the *simplex graph* $\sigma(H)$ of H has a vertex v_σ for each clique of G (i.e., empty set, vertices, edges, triangles, etc.) and two vertices v_σ and $v_{\sigma'}$ are adjacent in $\sigma(H)$ if and only if the cliques σ and σ' differ only in a vertex. It was shown in [12] that the simplex graph $\sigma(H)$ of any graph H is a median graph. Moreover, one can easily show that the star in $\sigma(H)$ of the vertex $v_\emptyset$ coincides with the whole graph $\sigma(H)$. Vice-versa, any star $\text{St}(z)$ of a median graph can be realized as the simplex graph $\sigma(H)$ of the graph H having the neighbors of z as the set of vertices and two such neighbors u', u'' of z are adjacent in H if and only if z, u', u'' belong to a common square of G .

Next, we consider stars $\text{St}(z)$ of median graphs from the metric point of view.

Lemma 3. *For any vertex z of a median graph G , the star $\text{St}(z)$ is a gated subgraph of G .*

Proof. We will only sketch the proof (for a complete proof, see Theorem 6.17 of [21] and its proof for a more general class of graphs). By Lemma 2 it suffices to show that $\text{St}(z)$ is locally convex. Let $x, y \in \text{St}(z)$ be two vertices at distance two and let $v \sim x, y$. Then $Q_x = I(x, z)$ and $Q_y = I(y, z)$ are two cubes of $\text{St}(z)$. We can suppose without loss of generality that $v \notin Q_x \cup Q_y$. This implies that $x, y \in I(z, v)$, i.e., we can suppose that $d_G(x, z) = d_G(y, z) = k$ and $d_G(z, v) = k + 1$. By quadrangle condition, there exists u such that $d_G(u, z) = k - 1$ and $u \sim x, y$. Necessarily $I(u, z)$ is a $(k - 1)$ -cube Q_u included in the k -cubes Q_x and Q_y . Therefore z has a neighbor x' such that $I(x, x')$ is a $(k - 1)$ -cube disjoint from Q_u and which together with Q_u gives Q_x . Analogously, z has a neighbor y' such that $I(y, y')$ is a $(k - 1)$ -cube disjoint from Q_u and which together with Q_u gives Q_y . By quadrangle condition there exists $v' \sim x', y'$ at distance $k - 1$ to v . Then one can show that $I(v, v')$ induces a $(k - 1)$ -cube, which together with the k -cubes Q_x and Q_y define the $(k + 1)$ -cube $Q_v = I(v, z)$. This establishes that v belongs to $\text{St}(z)$. $\square$

The following property of median graphs is also well-known in more general contexts. The graphs satisfying this property are called *fiber-complemented* [22].

Lemma 4. *For any gated subgraph H of a median graph G , the fibers $F(x), x \in V(H)$, are gated.*

Proof. Each fiber $F(x)$ induces a connected subgraph of G , thus it suffices to show that $F(x)$ is locally convex. Pick $u, v \in F(x)$ with $d_G(u, v) = 2$ and let z be any common neighbor of u and v . Suppose by way of contradiction that $z \in F(y)$ for $y \in V(H), y \neq x$. Then $x \in I(u, y) \cap I(v, y)$ and $y \in I(z, x)$. This implies in particular that $x \sim y$, $u, v \in I(z, x)$, and $z \in I(u, y) \cap I(v, y)$. By quadrangle condition, there exists $x' \sim u, v$, one step closer to x than u and v . Then $z, x' \in I(u, y)$

and by quadrangle condition there exists a vertex $y' \sim x', z$ one step closer to y than x' and z . But then the vertices u, v, z, x', y' induce a $K_{2,3}$, which is a forbidden subgraph of median graphs. $\square$

Lemma 4 has two corollaries. First, from this lemma and Lemma 3 we obtain:

Corollary 1. *For any vertex z of a median graph G , the fibers of the star $\text{St}(z)$ are gated.*

Since edges of a median graph G are gated, applying Lemma 4 for edges of G , we obtain:

Corollary 2. *For any edge uv of a median graph G , the halfspaces $W(u, v)$ and $W(v, u)$ are gated.*

That the halfspaces of a median graph are convex was established first by Mulder [50]. He also proved that the boundaries of halfspaces are convex (the boundary of the halfspace $W(u, v)$ is the set $\partial W(u, v) = \{z' \in W(u, v) : \exists z'' \in W(v, u), z'' \sim z'\}$). We will prove this property for boundaries of fibers of arbitrary gated subgraphs of a median graph.

Let H be a gated subgraph of a median graph $G = (V, E)$ and let $\mathcal{F}(H) = \{F(x) : x \in V(H)\}$ be the partition of V into the fibers of H . We will call two fibers $F(x)$ and $F(y)$ *neighboring* (notation $F(x) \sim F(y)$) if there exists an edge $x'y'$ of G with one end x' in $F(x)$ and another end y' in $F(y)$. If $F(x)$ and $F(y)$ are neighboring fibers of H , then denote by $\partial_y F(x)$ the set of all vertices $x' \in F(x)$ having a neighbor y' in $F(y)$ and call $\partial_y F(x)$ the *boundary of $F(x)$ relative to $F(y)$* .

Lemma 5. *Let H be a gated subgraph of a median graph $G = (V, E)$. Two fibers $F(x)$ and $F(y)$ of H are neighboring if and only if $x \sim y$. If $F(x) \sim F(y)$, then the boundary $\partial_y F(x)$ of $F(x)$ relative to $F(y)$ induces a gated subgraph of G of dimension $\leq \dim(G) - 1$.*

Proof. If $x \sim y$, then clearly $F(x) \sim F(y)$. Conversely, suppose that $F(x) \sim F(y)$, i.e., there exists an edge $x'y'$ of G such that $x' \in F(x)$ and $y' \in F(y)$. Since $F(x)$ and $F(y)$ are convex and G is bipartite, necessarily $x' \in I(y', x)$ and $y' \in I(x', y)$. Since $x' \in F(x), y' \in F(y)$ and H is gated, we deduce that $x \in I(x', y)$ and $y \in I(y', x)$. From all this we conclude that $d_G(x', x) = d_G(y', y)$ and that $d_G(x, y) = d_G(x', y') = 1$. This establishes the first assertion.

To prove the second assertion, let $F(x) \sim F(y)$ and we have to prove that $\partial_y F(x)$ is gated. First by induction on $k = d_G(x, x')$, we can show that $I(x', x) \subseteq \partial_y F(x)$ for any vertex x' of $\partial_y F(x)$. For this it suffices to show that any neighbor x'' of x' in $I(x', x)$ belongs to $\partial_y F(x)$. Let y' be the neighbor of x' in $\partial_x F(y)$. Then $x'', y' \in I(x', y)$, $x'', y' \sim x'$, and $d_G(x', y) = k + 1$, thus by quadrangle condition there exists a vertex $y'' \sim y', x''$ at distance $k - 1$ from y . Since $y'' \in I(y', y) \subset F(y)$, we conclude that $x'' \in \partial_y F(x)$. Thus $I(x', x) \subseteq \partial_y F(x)$, yielding that the subgraph induced by $\partial_y F(x)$ is connected.

By Lemma 2 it remains to show that $\partial_y F(x)$ is locally convex. Pick $x', x'' \in \partial_y F(x)$ at distance two and let $u \sim x', x''$. Since $F(x)$ is convex, $u \in F(x)$. Let y' and y'' be the neighbors of x' and x'' , respectively, in $F(y)$. Let v be the gate of u in $F(y)$ (by Lemma 4, $F(y)$ is gated). Since $d_G(u, y') = d_G(u, y'') = 2$ (because G is bipartite) and $v \in I(u, y') \cap I(u, y'')$, we conclude that v is adjacent to u, y' , and y'' . Hence $v \in I(y', y'') \subset F(y)$, yielding $u \in \partial_y F(x)$. This finishes the proof that $\partial_y F(x)$ is gated. If $\dim(\partial_y F(x)) = \dim(G) = d$, then $\partial_y F(x)$ contains a d -dimensional cube Q' . Then the neighbors in $\partial_x F(y)$ of Q' induce a d -cube Q'' . But then Q' together with Q'' induce a $(d + 1)$ -cube of G , a contradiction. Thus $\dim(\partial_y F(x)) \leq d - 1$. $\square$

For a vertex x of a gated subgraph H of G and its fiber $F(x)$, the union of all boundaries $\partial_y F(x)$ over all $F(y) \sim F(x), y \in V(H)$, is called the *total boundary* of the fiber $F(x)$ and is denoted by $\partial^* F(x)$. The boundaries $\partial_y F(x)$ constituting $\partial^* F(x)$ are called *branches* of $\partial^* F(x)$.

Lemma 6. *Let H be a gated subgraph of a median graph G of dimension d . Then the total boundary $\partial^* F(x)$ of any fiber $F(x)$ of H does not contain d -dimensional cubes.*

Proof. Suppose by way of contradiction that $\partial^* F(x)$ contains a d -dimensional cube Q . Since Q is a gated subgraph of G , we can consider the gate x' of x in Q and the furthest from x vertex of Q

(this is the vertex of Q opposite to x'). Denote this furthest from x vertex of Q by x'' . Suppose that $x'' \in \partial_y F(x) \subset \partial^* F(x)$. Since $\partial_y F(x)$ is gated (Lemma 5) and $Q \subset I(x'', x)$, Q is included in the boundary $\partial_y F(x)$. This contradicts Lemma 5 that $\partial_y F(x)$ has dimension $\leq d - 1$. $\square$

Lemma 7. *Let H be a gated subgraph of a median graph G . Then the total boundary $\partial^* F(x)$ of any fiber $F(x)$ of H is an isometric subgraph of G .*

Proof. Pick $u, v \in \partial^* F(x)$, say $u \in \partial_y F(x)$ and $v \in \partial_z F(x)$. Let w be the median of the triplet x, u, v . Since $w \in I(u, x) \subseteq \partial_y F(x) \subset \partial^* F(x)$ we deduce that $I(v, w) \subset \partial^* F(x)$. Analogously, we can show that $I(v, w) \subset \partial^* F(x)$. Since $w \in I(u, v)$ and $I(u, w) \cup I(w, v) \subset \partial^* F(x)$, the vertices u and v can be connected in $\partial^* F(x)$ by a shortest path passing via w . $\square$

We conclude this section with an additional property of fibers of stars of median vertices of G . Recall, that m is a median vertex of G if m minimizes the function $M(x) = \sum_{v \in V} d_G(x, v)$.

Lemma 8. *Let m be a median vertex of a median graph G with n vertices. Then any fiber $F(x)$ of the star $\text{St}(m)$ of m has at most $n/2$ vertices.*

Proof. Suppose by way of contradiction that $|F(x)| > n/2$ for some vertex $x \in \text{St}(m)$. Let u be a neighbor of m in $I(x, m)$. If $v \in F(x)$, then $x \in I(v, m)$ and $u \in I(x, m)$, and we conclude that $u \in I(v, m)$. Consequently, $F(x) \subseteq W(u, m)$, whence $|W(u, m)| > n/2$. Therefore $|W(m, u)| = n - |W(u, m)| < n/2$. But this contradicts the fact that m is a median of G . Indeed, since $u \sim m$, one can easily show that $M(u) - M(m) = |W(m, u)| - |W(u, m)| < 0$. $\square$

Unfortunately, the total boundary $\partial^* F(x)$ of a fiber does not always induce a median subgraph. Therefore, even if $\partial^* F(x)$ is an isometric subgraph of G of dimension $\leq \dim(G) - 1$, one cannot recursively apply the algorithm to the subgraphs induced by the total boundaries $\partial^* F(x)$. However, if G is 2-dimensional (i.e., G is cube-free), then the total boundaries of fibers are isometric subtrees of G and one can use for them distance and routing schemes for trees. Even in this case, we still need an additional property of total boundaries, which we will establish in the next section.

5. FIBERS IN CUBE-FREE MEDIAN GRAPHS

In this section, we establish additional properties of fibers of stars and of their total boundaries in cube-free median graphs $G = (V, E)$. Using them we can show that for any pair u, v of vertices of G , the following trichotomy holds: the distance $d_G(u, v)$ either can be computed as $d_G(u, m) + d_G(m, v)$, or as the sum of distances from u, v to appropriate vertices u', v' of $\partial^* F(x)$ plus the distance between u', v' in $\partial^* F(x)$, or via a recursive call to the fiber containing u and v .

5.1. Classification of fibers. From now on, let $G = (V, E)$ be a cube-free median graph. Then the star $\text{St}(z)$ of any vertex z of G is the union of all squares and edges containing z . Specifying the bijection between stars of median graphs and simplex graphs of arbitrary graphs mentioned above, the stars of cube-free median graphs correspond to simplex graphs of triangle-free graphs.

Let z be an arbitrary vertex of G and let $\mathcal{F}_z = \{F(x) : x \in \text{St}(z)\}$ denote the partition of V into the fibers of $\text{St}(z)$. We distinguish two types of fibers: the fiber $F(x)$ is called a *panel* if x is adjacent to z and $F(x)$ is called a *cone* if x has distance two to z . The interval $I(x, z)$ is the edge xz if $F(x)$ is a panel and is a square $Q_x := (x, y', z, y'')$ if $F(x)$ is a cone. In the second case, since y' and y'' are the only neighbors of x in $\text{St}(z)$, by Lemma 5 we deduce that the cone $F(x)$ is adjacent to the panels $F(y')$ and $F(y'')$ and that $F(x)$ is not adjacent to any other panel or cone. By the same lemma, any panel $F(y)$ is not adjacent to any other panel, but $F(y)$ is adjacent to all cones $F(x)$ such that the square Q_x contains the edge yz . For an illustration, see Figure 4.

FIGURE 2. A star $\text{St}(z)$ (in gray) and its fibers (cones in blue, and panels in red).

5.2. Total boundaries of fibers are quasigated. For a set A , an *imprint* of a vertex $u \notin A$ on A is a vertex $a \in A$ such that $I(u, a) \cap A = \{a\}$. Denote by $\Upsilon(u, A)$ the set of all imprints of u on A . The most important property of imprints is that for any vertex $z \in A$, there exists a shortest (u, z) -path passing via an imprint, i.e., that $I(u, z) \cap \Upsilon(u, A) \neq \emptyset$. Therefore, if the set $\Upsilon(u, A)$ has constant size, one can store in the label of u the distances to the vertices of $\Upsilon(u, A)$. Using this, for any $z \in A$, one can compute $d_G(u, z)$ as $\min\{d_G(u, a) + d_G(a, z) : a \in \Upsilon(u, A)\}$. Note that a set A is gated if and only if any vertex $u \notin A$ has a unique imprint on A . Following this, we will say that a set A is *k-gated* if for any vertex $u \notin A$, $|\Upsilon(u, A)| \leq k$. In particular, we will say that a set A is *quasigated* if $|\Upsilon(u, A)| \leq 2$ for any vertex $u \notin A$. The main goal of this subsection is to show that the total boundaries of fibers are quasigated.

Let T be a tree with a distinguished vertex r in G . The vertex r is called the *root* of T and T is called a *rooted tree*. We will say that a rooted tree T has *gated branches* if for any vertex x of T the unique path $P(x, r)$ of T connecting x to the root r is a gated subgraph of G .

Lemma 9. *For every fiber $F(x)$ of a star $\text{St}(z)$ of a cube-free median graph G , the total boundary $\partial^*F(x)$ is an isometric tree with gated branches.*

Proof. From Lemmas 6 and 7 it follows that $\partial^*F(x)$ is an isometric tree rooted at the vertex x . For any vertex $v \in \partial^*F(x)$ there exists a fiber $F(y) \sim F(x)$ of $\text{St}(z)$ such that v belongs to the boundary $\partial_y F(x)$ of $F(x)$ relative to $F(y)$. Since, by Lemma 5, $\partial_y F(x)$ is a gated subtree of G and $v, x \in \partial_y F(x)$, the unique path $P(v, x)$ connecting v and x in $\partial_y F(x)$ is a convex subpath of $\partial_y F(x)$, and therefore a convex subpath of the whole graph G . Since convex subgraphs are gated, $P(x, v)$ is a gated path of G belonging to $\partial^*F(x)$. $\square$

By Lemma 9, $\partial^*F(x)$ has gated branches, however $\partial^*F(x)$ is not necessarily gated itself. Since a panel $F(x)$ may be adjacent to an arbitrary number of cones, one can think that the imprint-set $\Upsilon(u, \partial^*F(x))$ of a vertex u of $F(x)$ may have an arbitrarily large size. The following lemma shows that this is not the case, namely that $|\Upsilon(u, \partial^*F(x))| \leq 2$. This is one of the key ingredients in the design of the distance and routing labeling schemes presented in Sections 6 and 7. Unfortunately, this property is no longer true for median graphs of dimension greater than 2.

Lemma 10. *Let T be a rooted tree with gated branches of a cube-free median graph $G = (V, E)$. Then T is quasigated.*

Proof. Let r be the root of T . Pick any $u \in V \setminus V(T)$ and suppose by way of contradiction that $\Upsilon(u, T)$ contains three distinct imprints x_1, x_2 , and x_3 . Since T has gated branches, neither of the vertices x_1, x_2, x_3 belong to the path of T between the root r and another vertex from this triplet.

In particular, r is different from x_1, x_2, x_3 . Suppose additionally that among all rooted trees T' with gated branches of G and such that $|\Upsilon(u, T')| \geq 3$, the tree T has the minimal number of vertices. This minimality choice (and the fact that any subtree of T containing r is also a rooted tree with gated branches) implies that T is exactly the union of the three gated paths $P(r, x_1), P(r, x_2)$, and $P(r, x_3)$. Therefore, x_1, x_2 , and x_3 are the leaves of T .

Let y_i be the neighbor of x_i in the path $P(r, x_i)$, $i = 1, 2, 3$. Since G is bipartite, either $x_i \in I(y_i, u)$ or $y_i \in I(x_i, u)$. Since $x_i \in \Upsilon(u, T)$, necessarily $x_i \in I(y_i, u)$. Let T'_i be the subtree of T obtained by removing the leaf x_i . From the minimality choice of T , we cannot replace T by the subtree T'_i . This means that $|\Upsilon(u, T'_i)| \leq 2$. Since $x_j, x_k \in \Upsilon(u, T'_i)$ for $\{i, j, k\} = \{1, 2, 3\}$, necessarily $I(y_i, u) \cap \{x_j, x_k\} \neq \emptyset$ holds.

First, notice that $x_1, x_2, x_3 \in I(u, r)$. Indeed, let z_i denote the median of the triplet x_i, u, r . If $z_i \neq x_i$, since $z_i \in I(x_i, r) = P(x_i, r) \subset T$ and $z_i \in I(u, x_i)$, we obtain a contradiction with the inclusion of x_i in $\Upsilon(u, T)$. Thus $z_i = x_i$, yielding $x_i \in I(u, z_i)$.

Now, suppose without loss of generality that $d_G(r, x_3) = \max\{d_G(r, x_i) : i = 1, 2, 3\} := k$. Since $I(y_3, u) \cap \{x_1, x_2\} \neq \emptyset$ by what has been shown above, we can suppose without loss of generality that $x_2 \in I(y_3, u)$. Since $x_3 \in I(y_3, u)$, from these two inclusions we obtain that $d_G(x_3, u) + 1 = d_G(y_3, x_2) + d_G(x_2, u)$. Then $d_G(x_3, u) \geq d_G(x_2, u)$, and we conclude that $d_G(x_3, u) = d_G(x_2, u)$ and $d_G(y_3, x_2) = 1$ (i.e., $y_2 = y_3 =: y$). Since $x_2, x_3 \in I(r, u)$, we also have $d_G(x_3, r) = d_G(x_2, r)$. We distinguish two cases:

Case 1. $d_G(x_1, r) = k$.

Since all three vertices x_1, x_2, x_3 have the same distance k to r , we can apply to x_1 the same analysis as to x_3 and deduce that the neighbor y_1 of x_1 in T coincides with one of the vertices y_2, y_3 . Since $y_2 = y_3 = y$, we conclude that the vertices x_1, x_2, x_3 have the same neighbor y in T . Since y is closer to r than each of the vertices x_1, x_2, x_3 and since $x_1, x_2, x_3 \in I(r, u)$, we conclude that $x_1, x_2, x_3 \in I(y, u)$. Applying the quadrangle condition three times, we can find three vertices $x_{i,j}, i, j \in \{1, 2, 3\}, i \neq j$, such that $x_{i,j} \sim x_i, x_j$ and $d_G(x_{i,j}, u) = k - 1$ (see Figure 3, left). If two of the vertices $x_{1,2}, x_{2,3}$, and $x_{3,1}$ coincide, then we will get a forbidden $K_{2,3}$. Thus $x_{1,2}, x_{2,3}$, and $x_{3,1}$ are pairwise distinct. Since G is bipartite, this implies that $d_G(x_i, x_{j,k}) = 3$ for $\{i, j, k\} = \{1, 2, 3\}$. Since $x_{1,2}, x_{2,3} \in I(x_2, u)$, by quadrangle condition there exists a vertex w such that $w \sim x_{1,2}, x_{2,3}$ and $d_G(w, u) = k - 2$. Since G is bipartite, $d_G(w, x_{3,1})$ equals to 3 or to 1. If $d_G(w, x_{3,1}) = 3 = d(y, w)$, then the triplet $y, w, x_{3,1}$ has two medians x_1 and x_3 , which is impossible, because G is median. Thus $d_G(w, x_{3,1}) = 1$, i.e., $w \sim x_{3,1}$. Then one can easily see that the vertices $y, x_1, x_2, x_3, x_{1,2}, x_{2,3}, x_{3,1}, w$ define an isometric 3-cube of G , contrary to the assumption that G is cube-free. This finishes the analysis of Case 1.

Case 2. $d_G(x_1, r) < k$.

This implies that $d_G(r, x_1) \leq k - 1 = d_G(r, y)$. Let r' be the neighbor of r in the (r, y) -path of T . Notice that $r' \notin I(r, x_1) = P(r, x_1)$. Indeed, otherwise, $r' \in P(r, x_1) \cap P(r, x_2) \cap P(r, x_3)$ and we can replace the tree T by the subtree T' rooted at r' and consisting of the subpaths of $P(r, x_i)$ comprised between r' and x_i , $i = 1, 2, 3$. Clearly T' is a rooted tree with gated branches and $x_1, x_2, x_3 \in \Upsilon(u, T')$, contrary to the minimality choice of the counterexample T . Thus $r' \notin P(r, x_1)$.

Let also $P(r, x_1) = (r, v_1, \dots, v_{m-1}, v_m =: x_1)$. Notice that r may coincide with y_1 and x_1 may coincide with v_1 . Since $v_1, r' \in I(r, u)$, applying the quadrangle condition we will find a vertex $v'_2 \sim v_1, r'$ at distance $d_G(r, u) - 2$ from u . Since $r' \notin I(r, x_1)$, $v'_2 \neq v_2$. Since $v_2, v'_2 \in I(v_1, u)$, by quadrangle condition we will find $v'_3 \sim v_2, v'_2$ at distance $d_G(r, u) - 3$ from u . Again, since $r' \notin I(r, x_1)$, $v'_3 \neq v_3$. Continuing this way, we will find the vertices $v'_2, v'_3, \dots, v'_m, v'_{m+1} =: x'_1$ forming an (r', x'_1) -path $P(r', x'_1)$ and such that $v'_{i+1} \sim v_i, v'_i$, $v'_{i+1} \neq v_{i+1}$, and v'_{i+1} is one step closer to u than v_i and v'_i (see Figure 3, right). From its construction, the path $P(r', x'_1)$ is a shortest path. We assert that $P(r', x'_1)$ is gated. If this is not the case, by Lemma 2 and since

FIGURE 4. Illustration of the classification of pairs of vertices: the pairs of empty circle points are roommates, of disk points are 2-neighboring, of empty square points are 1-neighboring, and of full square and diamond points are separated.

We continue with distance formulae for separated, 2-neighboring, and 1-neighboring vertices. The illustration of each of the formulae is provided in Figure 5.

Lemma 11. *For vertices u and v belonging to the fibers $F(x)$ and $F(y)$ of $\text{St}(z)$, respectively, the following conditions are equivalent:*

- (i) u and v are separated;
- (ii) $I(x, z) \cap I(y, z) = \{z\}$;
- (iii) $d_G(u, v) = d_G(u, z) + d_G(z, v)$, i.e., $z \in I(u, v)$.

Proof. (i) $\iff$ (ii): Notice that u and v are separated if and only if $x \neq y$ and either $F(x)$ and $F(y)$ both are panels, or both are cones not having a neighboring panel, or one is a cone and another is a panel and the cone and the panel are not neighboring. If $F(x)$ and $F(y)$ are panels, then $I(x, z) = \{x, z\}$ and $I(y, z) = \{y, z\}$, thus $I(x, z) \cap I(y, z) = \{z\}$. If $F(x)$ and $F(y)$ are cones, then $I(x, z)$ and $I(y, z)$ are two squares Q_x and Q_y . By Lemma 5, Q_x and Q_y intersect in an edge wz if and only if $F(w)$ is a panel neighboring $F(x)$ and $F(y)$, i.e., if and only if u and v are not separated. Finally, if $F(x)$ is a cone and $F(y)$ is a panel, then $I(x, z)$ is the square Q_x and $I(y, z)$ is the edge yz . Then $F(x)$ and $F(y)$ are not neighboring if and only if the edge yz is not an edge of the square Q_x , i.e., if and only if $I(x, z) \cap I(y, z) = \{z\}$.

(ii) $\iff$ (iii): First, suppose that $I(x, z) \cap I(y, z) = \{z\}$. To show that $z \in I(u, v)$ it suffices to prove that z is the median of the triplet u, v, z . Suppose by way of contradiction that the median of u, v, z is the vertex w different from z . Let s be a neighbor of z in $I(z, w)$. Then obviously $s \in \text{St}(z)$. Since $I(x, z) \cap I(y, z) = \{z\}$, s does not belong to at least one of the intervals $I(x, z)$ and $I(y, z)$, say $s \notin I(x, z)$. This implies that $d_G(s, x) = d_G(z, x) + 1$. Since x is the gate of u in $\text{St}(z)$ and $s \in \text{St}(z)$, necessarily $x \in I(u, s)$. This implies that there is a shortest (s, u) -path passing via z and x , i.e., $d_G(s, u) = 1 + d_G(z, u)$. On the other hand, since $s \in I(z, w) \subset I(z, u)$, we conclude that $d_G(z, u) = 1 + d_G(s, u)$. Comparing the two equalities, we obtain a contradiction. This establishes that (ii) $\implies$ (iii).

Conversely, suppose that $z \in I(u, v)$. This implies that z is the median of the triplet u, v, z and that $I(u, z) \cap I(v, z) = \{z\}$. Since x is the gate of u and y is the gate of v in $\text{St}(z)$, we conclude that $x \in I(u, z)$ and $y \in I(v, z)$. Consequently, $I(x, z) \subseteq I(u, z)$ and $I(y, z) \subseteq I(v, z)$, proving that $I(x, z) \cap I(y, z) = \{z\}$. This establishes (iii) $\implies$ (ii). $\square$

Remark 5.1. *The equivalence (ii) $\iff$ (iii) of Lemma 11 holds in the setting of general median graphs.*

Lemma 12. *Let u and v be two 1-neighboring vertices such that u belongs to the panel $F(x)$ and v belongs to the cone $F(y)$. Let u_1 and u_2 be the two imprints of u on the total boundary $\partial^* F(x)$ (it may happen that $u_1 = u_2$) and let v^+ be the gate of v in $F(x)$. Then,*

$$d_G(u, v) = \min\{d_G(u, u_1) + d_{\partial^* F(x)}(u_1, v^+), d_G(u, u_2) + d_{\partial^* F(x)}(u_2, v^+)\} + d_G(v^+, v).$$

Proof. By Lemma 4 $F(x)$ is gated. Hence there must exist a shortest (u, v) -path passing via v^+ . The vertices u_1, u_2 , and v^+ belong to the total boundary $\partial^* F(x)$ of $F(x)$. Since, by Lemma 7, $\partial^* F(x)$ is an isometric tree and since, by Lemma 10, u has at most two imprints u_1 and u_2 in $\partial^* F(x)$, we conclude that $d_G(u, v^+) = \min\{d_G(u, u_1) + d_{\partial^* F(x)}(u_1, v^+), d_G(u, u_2) + d_{\partial^* F(x)}(u_2, v^+)\}$. Consequently, there is a shortest (u, v) -path passing first via one of the vertices u_1, u_2 and then via v^+ , establishing the asserted property. $\square$

FIGURE 5. An illustration of Lemmas 11, 12 and 13: examples of shortest paths (in red) between separated, 1-neighboring, and 2-neighboring vertices u and v . The total boundaries of the panels appear in blue.

Lemma 13. *Let u and v be two 2-neighboring vertices belonging to the cones $F(x)$ and $F(y)$, respectively, and let $F(w)$ be the panel neighboring $F(x)$ and $F(y)$. Let u^+ and v^+ be the gates of u and v in $F(w)$. Then*

$$d_G(u, v) = d_G(u, u^+) + d_{\partial^* F(w)}(u^+, v^+) + d_G(v^+, v).$$

Proof. Since the halfspace $W(w, z)$ is convex and $u, v \in F(x) \cup F(w) \cup F(y) \subset W(w, z)$, any shortest (u, v) -path $P(u, v)$ is contained in $W(w, z)$. We assert that $P(u, v) \subset F(x) \cup F(w) \cup F(y)$. Indeed, since $u \in F(x)$, $v \in F(y)$ and the fibers $F(x), F(y)$ are not neighboring, while moving from u to v along $P(u, v)$, we have to leave $F(x)$ and enter a panel neighboring $F(x)$. But the

cone $F(x)$ has only two neighboring panels: $F(w)$ and eventually a panel $F(w') \subset W(z, w)$. Since $P(u, v) \subset W(w, z)$, necessarily $P(u, v)$ must enter $F(w)$ (and not $F(w')$). Analogously, one can show that while moving from v to u along $P(u, v)$ when we leave $F(y)$ we must enter the same panel $F(w)$. Consequently, since the fibers $F(x)$, $F(w)$, and $F(y)$ are gated, the path $P(u, v)$ must be included in their union.

Next we show that u^+ and v^+ belong to a common shortest (u, v) -path. Indeed, by what has been shown above, any shortest (u, v) -path intersects $F(w)$, in particular, there exists a vertex $s \in I(u, v) \cap F(w)$. Since u^+ is the gate of u in $F(w)$ and v^+ is the gate of v in $F(w)$, we deduce that $u^+ \in I(u, s)$ and $v^+ \in I(v, s)$. Since $s \in I(u, v)$, there exists a shortest path from u to v passing via u^+ , s , and v^+ . This shows that $d_G(u, v) = d_G(u, u^+) + d_G(u^+, v^+) + d_G(v^+, v)$. Since $\partial^* F(w)$ is an isometric tree, $d_G(u^+, v^+) = d_{\partial^* F(w)}(u^+, v^+)$, establishing the required equality $d_G(u, v) = d_G(u, u^+) + d_{\partial^* F(w)}(u^+, v^+) + d_G(v^+, v)$. $\square$

6. DISTANCE LABELING SCHEME FOR CUBE-FREE MEDIAN GRAPHS

Let $G = (V, E)$ be a cube-free median graph with n vertices and let m be a median vertex of G . Let u, v be any pair of vertices of G for which we have to compute the distance $d_G(u, v)$. Applying Lemmas 11, 12, and 13 of previous section with m instead of z , the distance $d_G(u, v)$ can be computed once u and v are separated, 1-neighboring, or 2-neighboring and once u and v keep in their labels the distances to m , to the respective gates u^+ and v^+ , and to the imprints u_1 and u_2 if u belongs to a panel. It also requires keeping in the labels of u and v the information necessary to compute each of the distances $d_{\partial^* F(x)}(u_1, v^+)$, $d_{\partial^* F(x)}(u_2, v^+)$, $d_{\partial^* F(w)}(u^+, v^+)$. Since the total boundaries are isometric trees, this can be done by keeping in the label of u the labels of u_1, u_2 , and u^+ in a distance labeling scheme for a tree (as well as keeping in the label of v such a label of v^+). This shows that $d_G(u, v)$ can be computed in all cases except when u and v are roommates, i.e., they belong to a common fiber $F(x)$ of $\text{St}(m)$. Since $F(x)$ is gated and thus median, we can apply the same recursive procedure to each fiber $F(x)$ instead of G . Therefore, $d_G(u, v)$ is computed in the first recursive call when u and v will no longer belong to the same fiber of the current median vertex. Since at each step the division into fibers is performed with respect to a median, $|F(x)| \leq n/2$ by Lemma 8, thus the tree of recursive calls has logarithmic depth.

In this section, we present the formal description of the distance labeling scheme. The encoding scheme is described by the algorithm `DISTANCE_ENCODING` presented in Subsection 6.2. Subsection 6.3 presents the algorithm `DISTANCE` used for answering distance queries. In Subsection 6.1 we formally present the distance labeling schemes for trees and stars.

6.1. Distance and routing labelings for trees and stars.

6.1.1. Trees. We present the distance labeling scheme (`Dist_Enc_Tree`, `Dist_Tree`) for trees, which we briefly described it in Subsection 2.2. The procedure `Dist_Enc_Tree` that gives a label $\text{LD}_T(v)$ to every vertex v of a tree T works as follows:

- (1) Give to every vertex v a unique identifier $\text{id}(v)$;
- (2) Find a median vertex m of T ;
- (3) For every vertex v of T , concatenate $(\text{id}(m), d_T(v, m))$ to the current label of v ;
- (4) Repeat Step 2 for each subtree with at least two vertices, created by the removal of m .

Given two labels $\text{LD}_T(u)$ and $\text{LD}_T(v)$, the procedure `Dist_Tree` can find the last common separator m of u and v (i.e., the common median vertex stored latest in their labels) and return $d_T(u, m) + d_T(v, m)$ as the distance $d_T(u, v)$. The encoding `Rout_Enc_Tree` for routing in trees is similar, just replace $(\text{id}(m), d_T(u, m))$ at step (3) by $(\text{id}(m), \text{port}(u, m), \text{port}(m, u))$. Then, the decoding function `Rout_Tree`(u, v), returns $\text{port}(u, m)$ (stored in the label of u) if $m \neq u$, or $\text{port}(m, v)$ (stored in the label of v) otherwise (where m is again the last common separator of u and v).

6.1.2. *Stars.* We present the distance labeling scheme for stars $\text{St}(z)$ of any median graph G . It is based on the fact that median graphs are isometrically embeddable into hypercubes and that $\text{St}(z)$ is gated, and thus is an isometric median subgraph of G . So, we can suppose that $\text{St}(z)$ is isometrically embedded into a hypercube. Let $\varphi : \text{St}(z) \rightarrow Q_d$ be such an isometric embedding so that $\varphi(z) = \emptyset$. Consequently, for each vertex x of $\text{St}(z)$, $\varphi(x)$ is a set of cardinality equal to the dimension of the cube $I(x, z)$, thus $\varphi(x)$ has size at most $\log n$, where $n = |\text{St}(z)|$. For any two vertices, x and y of $\text{St}(z)$, $d_{\text{St}(z)}(x, y) = |\varphi(x) \Delta \varphi(y)| := |(\varphi(x) \cup \varphi(y)) \setminus (\varphi(x) \cap \varphi(y))|$.

Using the isometric embedding φ , we can describe a simple encoding $\text{Enc_Star}(\text{St}(z))$ of the vertices of $\text{St}(z)$ which can be used to answer distance and routing queries. For a vertex $x \in \text{St}(z)$, let $L_{\text{St}(z)}(x) = \varphi(x)$. Then $\text{Enc_Star}(\text{St}(z))$ gives to z the label $\emptyset$ and to every neighbor of z a unique label in $\{1, \dots, \deg(z)\}$. For any vertex x at distance k from z , $I(x, z)$ contains exactly k neighbors of z and the labels of these neighbors completely define $\varphi(x)$ and $L_{\text{St}(z)}(x)$.

Giving unique labels to the neighbors of z require $\lceil \log(\deg(z)) \rceil$ bits and thus, in the worst case, $\text{Enc_Star}(\text{St}(z))$ gives labels of length $O(\deg(x) \log(\deg(z)))$. If the dimension of $\text{St}(z)$ is a fixed constant, then $\text{Enc_Star}(\text{St}(z))$ gives labels of logarithmic length. For a vertex x of $\text{St}(z)$ labeled by the set $X := \varphi(x)$, the vertex of $\text{St}(z)$ labeled by the value $\min\{i : i \in X\}$ is called the *1st* of x , and the one labeled by $\max\{i : i \in X\}$ is called the *2nd* of x .

For simplicity, we assume that for a vertex x labeled X and a vertex x' labeled $X' = X \setminus \{i\}$, $i \in X$, we have $\text{port}(x, x') = \text{port}(x', x) = i$. Since φ is an isometric embedding, it is easy to see that for any two vertices x and y encoded by the sets $X := L_B(x) = \varphi(x)$ and $Y := L_B(y) = \varphi(y)$, the distance $d_{\text{St}(z)}(x, y)$ between x and y is $|X \Delta Y|$. This is exactly the value returned by $\text{Dist_Star}(X, Y)$. Routing decisions follow from the same property. Assume that $|X| \leq |Y|$. If $X \subseteq Y$, $\text{Rout_Star}(X, Y)$ returns the port to any vertex labeled by $X \cup \{i\}$ with $i \in Y \setminus X$ (say the minimal i). If $X \not\subseteq Y$, then $\text{Rout_Star}(X, Y)$ returns the port of any vertex labeled by $X \setminus \{i\}$ for $i \in X \setminus Y$ (say the minimal i).

FIGURE 6. A labeled star and the routing decisions to go from vertex 123 to vertex 345. The distance between them is $|\{1, 2, 4, 5\}| = 4$.

6.2. **Encoding.** Let $G = (V, E)$ be a cube-free median graph with n vertices. We describe now how DISTANCE_ENCODING constructs for every vertex u of G a distance label $\text{LD}(u)$. This is done recursively and every depth of the recursion will be called a *step*. Initially, we suppose that every vertex u of G is given a unique identifier $\text{id}(u)$. We define this naming step as Step 0 and we denote the corresponding part of $\text{LD}(u)$ by $\text{LD}_0(u)$ (i.e., $\text{LD}_0(u) := \text{id}(u)$). At Step 1, DISTANCE_ENCODING computes a median vertex m of G , the star $\text{St}(m)$ of m , and the partition $\mathcal{F}_m := \{F(x) : x \in \text{St}(m)\}$ of V into fibers. Every vertex u of G (m included) receives the identifier $\text{id}(m)$ of m and its distance $d_G(u, m)$ to m . After that, every vertex x of $\text{St}(m)$ receives a special identifier $L_{\text{St}(m)}(x)$ of size $O(\log |V|)$ consisting of a labeling for the star $\text{St}(m)$, as described in previous Subsection 6.1. Then, DISTANCE_ENCODING computes the gate $u^\downarrow$ in $\text{St}(m)$ of every vertex u of G and adds its identifier $L_{\text{St}(m)}(u^\downarrow)$ to $\text{LD}(u)$. Notice that the identifiers $L_{\text{St}(m)}(x)$ of the vertices

of $\text{St}(m)$ can also be used to distinguish the fibers of $\text{St}(m)$. This triple $(\text{id}(m), d_G(u, m), L_{\text{St}(m)}(u^\downarrow))$ contains the necessary information relative to $\text{St}(m)$ and is thus referred as the *part “star”* of the information $\text{LD}_1(u)$ given to u at Step 1. We denote this part by $\text{LD}_1^{\text{St}}(u)$. We also set $\text{LD}_1^{\text{St}[\text{Med}]}(u) := \text{id}(m)$, $\text{LD}_1^{\text{St}[\text{Dist}]}(u) := d_G(u, m)$ and $\text{LD}_1^{\text{St}[\text{gate}]}(u) := L_{\text{St}(m)}(u^\downarrow)$ for the three components of the label $\text{LD}_1^{\text{St}}(u)$.

Afterwards, at Step 1, the algorithm considers each fiber $F(x)$ of $\mathcal{F}_m$. If $F(x)$ is a panel, then the algorithm computes the total boundary $\partial^* F(x)$ of $F(x)$, which is an isometric quasigated tree. The vertices v of this tree $\partial^* F(x)$ are given special identifiers $\text{LD}_{\partial^* F(x)}(v)$ of size $O(\log^2 |V|)$ consisting of a distance labeling scheme for trees described in Subsection 6.1. For each vertex u of the panel $F(x)$, the algorithm computes the two imprints u_1 and u_2 of u in $\partial^* F(x)$ (it may happen that $u_1 = u_2$) and stores $(\text{LD}_{\partial^* F(x)}(u_1), d_G(u, u_1))$ and $(\text{LD}_{\partial^* F(x)}(u_2), d_G(u, u_2))$ in $\text{LD}_1^{\text{1st}}(u)$ and $\text{LD}_1^{\text{2nd}}(u)$.

If $F(x)$ is a cone and $F(w_1)$ and $F(w_2)$ are the two panels neighboring $F(x)$, then for each vertex u of $F(x)$, the algorithm computes the gates u_1^+ and u_2^+ of u in $F(w_1)$ and $F(w_2)$, respectively. Since $u_i^+ \in \partial_x F(w_i) \subset \partial^* F(x)$, $i = 1, 2$, the labels $\text{LD}_{\partial^* F(w_1)}(u_1^+)$ and $\text{LD}_{\partial^* F(w_2)}(u_2^+)$ in the distance labelings of trees $\partial^* F(w_1)$ and $\partial^* F(w_2)$ are well-defined. Therefore, the algorithm stores $(\text{LD}_{\partial^* F(w_1)}(u_1^+), d_G(u, u_1^+))$ and $(\text{LD}_{\partial^* F(w_2)}(u_2^+), d_G(u, u_2^+))$ in $\text{LD}_1^{\text{1st}}(u)$ and $\text{LD}_1^{\text{2nd}}(u)$. This ends Step 1.

FIGURE 7. Illustration of $\text{LD}_0(u)$ and of the information added to $\text{LD}(u)$ at step i .

Since $\mathcal{F}_m$ partitions V into gated median subgraphs, the label $\text{LD}_2(u)$ added to $\text{LD}(u)$ at Step 2 is constructed as $\text{LD}_1(u)$ replacing G by the fiber $F(u^\downarrow)$ containing u , and so on. Since each fiber contains no more than half of the vertices of the current graph, at Step $\lceil \log |V| \rceil$, the fiber containing any vertex consists solely of this vertex, and the algorithm stops. Therefore, for each pair of vertices u and v of G , there exists a step of the recursion after which u and v are no longer roommates. For an illustration of the parts of $\text{LD}_i(u)$, see Fig.7.

For a vector $L(v) := (t_1, \dots, t_k)$ of vectors $t_1, \dots, t_k$ and an arbitrary vector t , we will denote by $L(v) \circ t := (t_1, \dots, t_k, t)$ the *concatenation* of $L(v)$ and t .

Algorithm 1: DISTANCE_ENCODING($G, \text{LD}(V)$)

Input: A cube-free median graph $G = (V, E)$ and a labeling $\text{LD}(V)$, initially consisting of a unique identifier $\text{id}(u)$ for every $u \in V$

- 1 **if** $V = \{v\}$ **then stop**;
- 2 Find a median vertex m of G ;
- 3 $\text{L}_{\text{St}(m)}(\text{St}(m)) \leftarrow \text{Enc_Star}(\text{St}(m))$;
- 4 **foreach** panel $F(x) \in \mathcal{F}_m$ **do**
- 5 $\text{LD}_{\partial^* F(x)}(\partial^* F(x)) \leftarrow \text{Dist_Enc_Tree}(\partial^* F(x))$;
- 6 **foreach** $u \in F(x)$ **do**
- 7 Find the gate $u^\downarrow$ of u in $\text{St}(m)$;
- 8 Find the imprints u_1 and u_2 of u on $\partial^* F(x)$;
- 9 $(d, d_1, d_2) \leftarrow (d_G(u, m), d_G(u, u_1), d_G(u, u_2))$;
- 10 $L_{\text{St}} \leftarrow (\text{id}(m), d, \text{L}_{\text{St}(m)}(u^\downarrow))$;
- 11 $L_{1\text{st}} \leftarrow (\text{LD}_{\partial^* F(x)}(u_1), d_1)$;
- 12 $L_{2\text{nd}} \leftarrow (\text{LD}_{\partial^* F(x)}(u_2), d_2)$;
- 13 $\text{LD}(u) \leftarrow \text{LD}(u) \circ (L_{\text{St}}, L_{1\text{st}}, L_{2\text{nd}})$;
- 14 DISTANCE_ENCODING($F(x), \text{LD}(V)$) ;
- 15 **foreach** cone $F(x) \in \mathcal{F}_m$ **do**
- 16 **foreach** $u \in F(x)$ **do**
- 17 Find the gate $u^\downarrow$ of u in $\text{St}(m)$;
- 18 Find the panels $F(w_1)$ and $F(w_2)$ neighboring $F(x)$;
- 19 Find the gates u_1^+ and u_2^+ of u in $F(w_1)$ and $F(w_2)$;
- 20 $(d, d_1, d_2) \leftarrow (d_G(u, m), d_G(u, u_1^+), d_G(u, u_2^+))$;
- 21 $L_{\text{St}} \leftarrow (\text{id}(m), d, \text{L}_{\text{St}(m)}(u^\downarrow))$;
- 22 $L_{1\text{st}} \leftarrow (\text{LD}_{\partial^* F(w_1)}(u_1^+), d_1)$;
- 23 $L_{2\text{nd}} \leftarrow (\text{LD}_{\partial^* F(w_2)}(u_2^+), d_2)$;
- 24 $\text{LD}(u) \leftarrow \text{LD}(u) \circ (L_{\text{St}}, L_{1\text{st}}, L_{2\text{nd}})$;
- 25 DISTANCE_ENCODING($F(x), \text{LD}(V)$).

6.3. Distance queries. Let u and v be two vertices of a cube-free median graph $G = (V, E)$ and let $\text{LD}(u)$ and $\text{LD}(v)$ be their labels returned by DISTANCE_ENCODING. Here we describe how the algorithm DISTANCE can compute the information about the relative positions of u and v with respect to each other and how, using it, to compute the distance $d_G(u, v)$.

6.3.1. The algorithm. We continue with the formal description of the algorithm DISTANCE. The functions Distance_1-Neighboring, Distance_2-Neighboring, and Distance_Separated, used in this algorithm, are given in the next subsection (the function Dist_Star is described in Subsection 6.1).

Algorithm 2: DISTANCE(LD(u), LD(v))

Input: The labels LD(u) and LD(v) of two vertices u and v of G

Output: The distance between u and v in G

```
1 if LD0( $u$ ) = LD0( $v$ ) /*  $u = v$  */ then return 0 ;
2 Let  $i$  be the largest integer such that LD $i$ St[Med]( $u$ ) = LD $i$ St[Med]( $v$ ) ;
3  $d \leftarrow \text{Dist\_Star}(\text{LD}_i^{\text{St[gate]}}(u), \text{LD}_i^{\text{St[gate]}}(v))$  ; //  $d_G(u^\downarrow, v^\downarrow)$ 
4  $d_u \leftarrow \text{Dist\_Star}(\text{LD}_i^{\text{St[gate]}}(u), 0)$  ; //  $d_G(u^\downarrow, m)$ 
5  $d_v \leftarrow \text{Dist\_Star}(\text{LD}_i^{\text{St[gate]}}(v), 0)$  ; //  $d_G(v^\downarrow, m)$ 
6 if  $d = 1$  and  $d_u = 1$  then return Distance_1-Neighboring (LD $i$ ( $u$ ), LD $i$ ( $v$ )) ;
7 if  $d = 1$  and  $d_v = 1$  then return Distance_1-Neighboring (LD $i$ ( $v$ ), LD $i$ ( $u$ )) ;
8 if  $d = 2$  and  $d_u = d_v = 2$  then return Distance_2-Neighboring (LD $i$ ( $u$ ), LD $i$ ( $v$ )) ;
9 return Distance_Separated (LD $i$ ( $u$ ), LD $i$ ( $v$ )).
```

6.3.2. *Description and functions.* Given the vertices u and v , the first thing the algorithm has to do is to detect if u and v coincide or not. This is done in line 1 of DISTANCE. If $u \neq v$, then DISTANCE finds the largest integer i such that LD _{i} ^{St[Med]}(u) = LD _{i} ^{St[Med]}(v) (line 2). This corresponds to the first time the vertices u and v belong to different fibers in a partition. Let m be the median vertex of the current median graph. In lines 3,4,5, the algorithm DISTANCE retrieves the distances d , d_u , and d_v between the gates $u^\downarrow$ and $v^\downarrow$ of u and v in the star St(m), and the distances from $u^\downarrow$ and $v^\downarrow$ to m , respectively. This is done by using the identifiers LD _{i} ^{St[gate]}(u) and LD _{i} ^{St[gate]}(v) and the distance decoder for distance labeling in stars. One can easily see that, with this information at hand, one can decide for each of the vertices u and v if it belongs to a cone or to a panel, and, moreover, to decide if the vertices u and v are 1-neighboring, 2-neighboring, or separated. In each of these cases, a call to an appropriate function is done in lines 6-9.

First suppose that the vertices u and v are 1-neighboring ($d = 1$ and one of d_u, d_v is 1 and another is 2), i.e., one of the vertices u, v belongs to a cone and another one belongs to a panel, and the cone and the panel are neighboring. The function Distance_1-Neighboring returns the distance $d_G(u, v)$ in the assumption that u belongs to a panel and v belongs to a cone (if v belongs to a panel and u to a cone, it suffices to swap the names of the vertices u and v before using Distance_1-Neighboring). The function finds the gate v^+ of v in the panel of u by looking at LD _{i} ^{St[gate]}(v) (it also retrieves the distance $d_G(v, v^+)$). It then retrieves the imprint u^* of u (and the distance $d_G(u, u^*)$) on the total boundary of the panel that minimizes the distance of u to one of the two imprints plus the distance from this imprint to the gate v^+ using their tree distance labeling scheme. Finally, Distance_1-Neighboring returns $d_G(u, u^*) + d_G(u^*, v^+) + d_G(v^+, v)$ as $d_G(v, u)$.

```
function Distance_1-Neighboring(LD $i$ ( $u$ ), LD $i$ ( $v$ )):
  dir  $\leftarrow$  1st ; // If LD $i$ St[gate]( $u$ ) = max{ $i : i \in \text{LD}_i^{\text{St[gate]}}(v)$ }
  if LD $i$ St[gate]( $u$ ) = min{ $i : i \in \text{LD}_i^{\text{St[gate]}}(v)$ } then
    dir  $\leftarrow$  2nd ;
   $d_1 \leftarrow \text{Dist\_Tree}(\text{LD}_i^{\text{dir[gate.LDT]}}(v), \text{LD}_i^{\text{1st[imp.LDT]}}(u))$  ; // The distance from the gate
    to the first imprint
   $d_2 \leftarrow \text{Dist\_Tree}(\text{LD}_i^{\text{dir[gate.LDT]}}(v), \text{LD}_i^{\text{2nd[imp.LDT]}}(u))$  ; // The distance from the gate
    to the second imprint
  return LD $i$ dir[Dist]( $v$ ) + min {  $d_1 + \text{LD}_i^{\text{1st[Dist]}}(u), d_2 + \text{LD}_i^{\text{2nd[Dist]}}(u)$  }.
```

Now suppose that the vertices u and v are 2-neighboring (i.e., $d = d_u = d_v = 2$). Then both u and v belong to cones. By inspecting $\text{LD}_i^{\text{St[gate]}}(u)$ and $\text{LD}_i^{\text{St[gate]}}(v)$, the function **Distance_2-Neighboring** determines the panel $F(w)$ sharing a border with the cones $F(u^\downarrow)$ and $F(v^\downarrow)$. Then the function retrieves the respective gates u^+ and v^+ of u and v in this panel $F(w)$ and the distances $d_G(u, u^+)$ and $d_G(v, v^+)$. The distance between the gates u^+ and v^+ is retrieved using the distance decoder for trees. The algorithm returns $d_G(u, u^+) + d_G(u^+, v^+) + d_G(v^+, v)$ as $d_G(u, v)$.

```

function Distance_2-Neighboring( $\text{LD}_i(u)$ ,  $\text{LD}_i(v)$ ):
  foreach  $x \in \{u, v\}$  do
     $\text{dir}_x \leftarrow \text{1st}$  ; // The common panel is the 1st of the cone of  $x$ 
    if  $\text{LD}_i^{\text{St[gate]}}(u) \cap \text{LD}_i^{\text{St[gate]}}(v) = \min\{i : i \in \text{LD}_i^{\text{St[gate]}}(x)\}$  then
       $\text{dir}_x \leftarrow \text{2nd}$  ; // The common panel is the 2nd of the cone of  $x$ 
   $d \leftarrow \text{Dist\_Tree}(\text{LD}_i^{\text{dir}_u[\text{gate\_LDT}]}(u), \text{LD}_i^{\text{dir}_v[\text{gate\_LDT}]}(v))$  ;
  return  $\text{LD}_i^{\text{dir}_u[\text{Dist}]}(u) + \text{LD}_i^{\text{dir}_v[\text{Dist}]}(v) + d$ .

```

In the remaining cases, the vertices u and v are separated. By Lemma 11, there exists a shortest path between u and v passing via m . Both u and v have stored the median vertex m and their distances to m . Therefore, **Distance_Separated** simply returns the sum of those two distances.

```

function Distance_Separated( $\text{LD}_i(u)$ ,  $\text{LD}_i(v)$ ):
  return  $\text{LD}_i^{\text{St[Dist]}}(u) + \text{LD}_i^{\text{St[Dist]}}(v)$ .

```

6.4. Correctness and complexity. The correctness of the algorithm **DISTANCE_ENCODING** results from the following properties of cube-free median graphs: stars and fibers are gated (Lemmas 3 and 4); total boundaries of fibers are quasigated (Corollary 3) isometric trees with gated branches (Lemma 9); and from the formulae for computing the distance between separated, 1-neighboring, and 2-neighboring vertices (Lemmas 11, 12, and 13).

Now we consider the length of the labels given by **DISTANCE_ENCODING** and the time complexity of the construction of encoding and decoding functions. For the encoding time complexity of a graph G , we suppose that we are given access to the distance matrix of G . To analyze the distance decoder, we consider a *RAM model* in which standard arithmetical operations on words of size $O(\log n)$ (such as additions, comparisons, etc.) are supposed made in constant time.

Lemma 14. **DISTANCE_ENCODING** runs in time $O(n^2 \log n)$.

Proof. Computing the median vertex of any graph G on n vertices assuming available its distance matrix can be done in time $O(n^2)$. Encoding a tree-subgraph of G on $O(n)$ vertices using the procedure **Dist_Enc_Tree** requires $O(n \log n)$ operations. A star of G can have size $O(\min\{\Delta(G)^2, n\})$, encoding it with **Enc_Star** has linear time complexity. Finding the gates/imprints of all vertices of G on any gated/quasigated subgraph H of G of size $O(n)$ can be done in time $O(n^2)$. According to those considerations, a single call of **DISTANCE_ENCODING** has quadratic complexity. Denote by p_i the number of parts created up to step i , and denote by $n_{i,j}$ the number of vertices occurring in the j th part created at step i . For every step i ,

$$\sum_{k=1}^{p_i} n_{i,k} = n - p_i = O(n) \quad (1)$$

Consequently, the time complexity of a step is $O(\sum_{k=1}^{p_i} n_{i,k}^2)$. Since $\sum_{k=1}^{p_i} n_{i,k}^2 \leq (\sum_{k=1}^{p_i} n_{i,k})^2 = O(n^2)$ and since the number of steps is $O(\log n)$ by Lemma 8, **DISTANCE_ENCODING** has total time complexity $O(n^2 \log n)$. $\square$

Lemma 15. `DISTANCE_ENCODING` gives to every vertex of an n -vertex cube-free median graph $G = (V, E)$ a label of length $O(\log^3 n)$.

Proof. Since at each division step we select a median vertex, by Lemma 8 every vertex $v \in V$ will appear in at most $\lceil \log |V| \rceil$ different fibers. For each of these fibers, $\text{LD}(v)$ will receive $O(\log^2 n)$ new bits. Indeed, the information stored correspond to Lines 10, 11 and 12 (or 21, 22 and 23) of Algorithm 1. L_{St} clearly has size $O(\log n)$ because so does $L_{\text{St}(m)}(v^\perp)$ as seen in Subsection 6.1 for stars, and $L_{1\text{st}}$ and $L_{2\text{nd}}$ both have size $O(\log^2 n)$ because the tree labeling they contain has size $O(\log^2 n)$ as seen in Subsection 6.1 for trees. $\square$

A label of size $O(\log^3 n)$ can be read in time $O(\log^2 n)$ assuming a RAM model. Given two labels $\text{LD}(u)$ and $\text{LD}(v)$, `DISTANCE` finds the last common median of u and v by reading their labels once and with no additional computation. Once this step is done, `DISTANCE` has to call `Dist_Star` on labels of size $O(\log n)$ which requires a constant number of steps. After that, either the information necessary to compute $d_G(u, v)$ is directly encoded in $\text{LD}(u)$ or $\text{LD}(v)$, or `DISTANCE` needs to call `Dist_Tree` on labels of size $O(\log^2 n)$, which requires an additional time $O(\log n)$. Consequently `DISTANCE` has a “reading time” of $O(\log^2 n)$ and a “computation time” of $O(\log n)$ for a total time complexity of $O(\log^2 n)$. The fact that `DISTANCE`($\text{LD}(u)$, $\text{LD}(v)$) returns $d_G(u, v)$ follows from Lemmas 11, 12 and 13. The following theorem (which is the main result of this section and the main result of the paper) thus holds:

Theorem 6.1. `DISTANCE_ENCODING` constructs in total time $O(n^2 \log(n))$ labels of size $O(\log^3(n))$ of the vertices of a cube-free median graph $G = (V, E)$. Given the labels of two vertices u and v of G , `DISTANCE` computes in time $O(\log^2(n))$ the exact distance $d_G(u, v)$ between u and v .

7. ROUTING LABELING SCHEMES FOR CUBE-FREE MEDIAN GRAPHS

In this section we adapt the labeling scheme for distance queries to obtain the encoding and decoding algorithms for routing in cube-free median graphs. Recall that, given the labels of a source u and a destination v obtained by the encoding, the decoding algorithm has to decide via which port of u to route the message to get closer to v . In other words, the algorithm has to return a neighbor u' of u in $I(u, v)$.

7.1. The idea. The idea of encoding is the same as the one for the distance labeling scheme in Section 6: the graph is partitioned recursively into fibers with respect to median vertices. At every step, the labels of the vertices are given a vector of three parts, named “St”, “1st”, and “2nd” as before. However, the information stored in these parts is not completely the same as for distances. This is due in part to the fact that we need to keep the information specific for routing but also because, at the difference of distance queries, the routing queries are not commutative. For instance, in the function `Distance_1-Neighboring` we assumed that u belongs to a panel and v to a cone. The case when u belongs to a cone and v belongs to a panel is reduced to the first case by calling `Distance_1-Neighboring` with the first argument v and the second one u . This is no longer possible in the routing queries: routing from a panel to a cone is different than routing from a cone to a panel.

As in the distance labeling scheme, the routing decision is taken the first time u and v belong to different fibers of the current partition. Let m be a median vertex of the current graph under partition and let $F(x)$ and $F(y)$ be the two fibers containing u and v , respectively. If u and v are separated, then $d_G(u, v) = d_G(u, m) + d_G(m, v)$, thus routing from u to v can be done by routing from u to m (unless $u = m$). Therefore, the encoding scheme must keep in the label of u the identifier of some neighbor of u in $I(u, m)$. If $u = m$, then it suffices to route from $u = m$ to the gate y of v in $\text{St}(z)$. This is done by using the routing scheme for stars.

If u and v are 2-neighboring, then $F(x)$ and $F(y)$ are cones having a common neighboring panel $F(w)$. Similarly to distance schemes, the routing scheme finds $F(w)$. Since the gates u^+ of u and v^+ of v in $F(w)$ belong to a common shortest (u, v) -path, it suffices to route the message from u to u^+ . Therefore the encoding must keep in the label of u the identifier of a neighbor of u in $I(u, u^+)$ (to which the message from u will be sent). The same information is required when u and v are 1-neighboring and $F(x)$ is a cone and $F(y)$ is a panel. Indeed, in this case there exists a shortest (u, v) -path passing via the gate u^+ of u in $F(y)$ and one of the imprints of v in $\partial^* F(y)$. Therefore, to route from u to v it suffices to route from u to u^+ .

Finally, suppose that u and v are 1-neighboring, however now $F(x)$ is a panel and $F(y)$ is a cone. Recall that in this case there exists a shortest (u, v) -path passing via one of the imprints u_1 or u_2 of u on $\partial^* F(x)$ and the gate v^+ of v in $F(x)$. Therefore, if u is different from v^+ then it suffices to route the message from u to a neighbor of u in $I(u, u_1)$ or $I(u, u_2)$ (depending of the position of v). Therefore, in the label of u we have to keep the identifiers of those two neighbors of u . To decide to which of them we have to route the message from u , we need to compare $d_G(u, u_1) + d_G(u_1, v^+)$ and $d_G(u, u_2) + d_G(u_2, v^+)$. Therefore, at the difference of the routing scheme in trees, our routing scheme for cube-free median graphs must incorporate the distance scheme. On the other hand, if u coincides with v^+ , then necessarily we have to route the message to a neighbor of u in $I(u, v)$, which necessarily belong to the cone $F(y)$ and not to $F(x)$ (because v^+ is the gate of v in $F(x)$). There exists a unique vertex $\text{twin}(v^+)$ of $F(y)$ adjacent to v^+ . We cannot keep the identifier of $\text{twin}(v^+)$ in the label of $u = v^+$ because a vertex in a panel may have arbitrarily many neighbors in the neighboring cones. Instead, we can keep the identifier of $\text{twin}(v^+)$ in the label of v (recall that a cone has only two neighboring panels).

7.2. Encoding. We present now the encoding algorithm in details. Let $G = (V, E)$ be a cube-free median graph and let u be any vertex of G . Let i be any step of the algorithm ROUTING_ENCODING applied to G and let m be a median vertex of the current median subgraph containing u at step i .

The “St” part $\text{LR}_i^{\text{St}}(u)$ of the label of u is composed of the identifier of m , a port from u to m , a port from m to u , and the identifier of gate $x := u^\downarrow$ of u to $\text{St}(m)$ (i.e., of the fiber containing u). Notice that m cannot store the ports to other vertices in order to answer routing queries from m . This is why the label of every vertex u contains the port $\text{LR}_i^{\text{St}[\text{fromMed}]}(u)$ from m to u . Here are the components of $\text{LR}_i^{\text{St}}(u)$:

- (1) $\text{LR}_i^{\text{St}[\text{Med}]}(u) := \text{id}(m)$ is the unique identifier of m ;
- (2) $\text{LR}_i^{\text{St}[\text{toMed}]}(u)$ consists of a port to take from u in order to reach m ;
- (3) $\text{LR}_i^{\text{St}[\text{fromMed}]}(u)$ consists of a port to take from m in order to reach u ;
- (4) $\text{LR}_i^{\text{St}[\text{gate}]}(u)$ contains the identifier of the fiber containing u (i.e., the star labeling of $u^\downarrow$).

The 1st and 2nd parts of the label of u contain similar information but they depend of whether u belongs to a panel or to a cone. If u belongs to a panel $F(x)$ (recall that $x = u^\downarrow$), then $\text{LR}_i^{\text{1st}}(u)$ is composed of the following four components:

- (1) $\text{LR}_i^{\text{1st}[\text{imp.LDT}]}(u)$ is the tree distance labeling of the first imprint u_1 of u on the total boundary $\partial^* F(x)$;
- (2) $\text{LR}_i^{\text{1st}[\text{imp.LRT}]}(u)$ is the tree routing labeling of u_1 in the tree $\partial^* F(x)$;
- (3) $\text{LR}_i^{\text{1st}[\text{toImp}]}(u)$ is $\text{port}(u, u_1)$;
- (4) $\text{LR}_i^{\text{1st}[\text{Dist}]}(u)$ is the distance $d_G(u, u_1)$.

The 2nd part $\text{LR}_i^{\text{2nd}}(u)$ of the label of u is defined in a similar way with respect to the second imprint u_2 of u on $\partial^* F(x)$.

If u belongs to a cone $F(x)$, then $F(x)$ has two neighboring panels $F(w_1)$ and $F(w_2)$. The components $\text{LR}_i^{1\text{st}}(u)$ and $\text{LR}_i^{2\text{nd}}(u)$ of the 1st and 2nd parts of the label of u , each consists of four components. For example, $\text{LR}_i^{1\text{st}}(u)$ is composed of the following data:

- (1) $\text{LR}_i^{1\text{st}[\text{gate_LDT}]}(u)$ consists of a tree distance labeling of the gate u_1^+ of u in the panel $F(w_1)$;
- (2) $\text{LR}_i^{1\text{st}[\text{gate_LRT}]}(u)$ is a tree routing labeling of u_1^+ in the tree $\partial^*F(w_1)$;
- (3) $\text{LR}_i^{1\text{st}[\text{toGate}]}(u)$ contains the port(u, u_1^+);
- (4) $\text{LR}_i^{1\text{st}[\text{fromGate}]}(u)$ is the port port($u_1^+, \text{twin}(u_1^+)$) from u_1^+ to $\text{twin}(u_1^+)$.

The 2nd part $\text{LR}_i^{2\text{nd}}(u)$ of the label of u is defined in a similar way with respect to the gate u_2^+ of u in the panel $F(w_2)$.

We assume that no port is given the number 0. If ROUTING returns 0 or if a label stores a port equal to 0, it means that there is no need to move.

Here is the encoding algorithm:

Algorithm 3: ROUTING_ENCODING($G, \text{LR}(V)$)

Input: A cube-free median graph $G = (V, E)$ and a labeling $\text{LR}(V)$ initially consisting on a unique identifier $\text{id}(v)$ for every $v \in V$

- 1 **if** $V = \{v\}$ **then stop**;
- 2 Find a median vertex m of G ;
- 3 $\text{L}_{\text{St}(m)}(\text{St}(m)) \leftarrow \text{Enc_Star}(\text{St}(m))$;
- 4 $\mathcal{F}_m \leftarrow \{F(x) : x \in \text{St}(m)\}$;
- 5 **foreach** panel $F(x) \in \mathcal{F}_m$ **do**
- 6 $\text{LD}_{\partial^*F(x)}(\partial^*F(x)) \leftarrow \text{Dist_Enc_Tree}(\partial^*F(x))$;
- 7 $\text{LR}_{\partial^*F(x)}(\partial^*F(x)) \leftarrow \text{Rout_Enc_Tree}(\partial^*F(x))$;
- 8 **foreach** $u \in F(x)$ **do**
- 9 Find the gate $u^\downarrow$ of u in $\text{St}(m)$;
- 10 Find the two imprints u_1 and u_2 of u on $\partial^*F(x)$;
- 11 $\{d_1, d_2\} \leftarrow \{d_G(u, u_1), d_G(u, u_2)\}$;
- 12 $\text{L}_{\text{St}} \leftarrow (\text{id}(m), \text{port}(u, m), \text{port}(m, u), \text{L}_{\text{St}(m)}(u^\downarrow))$;
- 13 $\text{L}_{1\text{st}} \leftarrow (\text{LD}_{\partial^*F(x)}(u), \text{LR}_{\partial^*F(x)}(u), \text{port}(u, u_1), d_G(u, u_1))$;
- 14 $\text{L}_{2\text{nd}} \leftarrow (\text{LD}_{\partial^*F(x)}(u), \text{LR}_{\partial^*F(x)}(u), \text{port}(u, u_2), d_G(u, u_2))$;
- 15 $\text{LR}(u) \leftarrow \text{LR}(u) \circ (\text{L}_{\text{St}}, \text{L}_{1\text{st}}, \text{L}_{2\text{nd}})$;
- 16 ROUTING_ENCODING($F(x), \text{LR}(V)$) ;
- 17 **foreach** cone $F(x) \in \mathcal{F}_m$ **do**
- 18 Let $F(w_1)$ be the 1st panel neighboring $F(x)$;
- 19 Let $F(w_2)$ be the 2nd panel neighboring $F(x)$;
- 20 $\text{LD}_{\partial^*F(w_1)}(\partial^*F(w_1)), \text{LD}_{\partial^*F(w_2)}(\partial^*F(w_2)) \leftarrow \text{Dist_Enc_Tree}(\partial^*F(w_1)),$
 $\text{Dist_Enc_Tree}(\partial^*F(w_2))$;
- 21 $\text{LR}_{\partial^*F(w_1)}(\partial^*F(w_1)), \text{LR}_{\partial^*F(w_2)}(\partial^*F(w_2)) \leftarrow \text{Rout_Enc_Tree}(\partial^*F(w_1)),$
 $\text{Rout_Enc_Tree}(\partial^*F(w_2))$;
- 22 **foreach** $u \in F(x)$ **do**
- 23 Find the gate $u^\downarrow$ of u in $\text{St}(m)$;
- 24 Find the gate u_1^+ of u in $F(w_1)$ and let $\text{twin}(u_1^+)$ be the twin of u_1^+ in $F(x)$;
- 25 Find the gate u_2^+ of u in $F(w_2)$ and let $\text{twin}(u_2^+)$ be the twin of u_2^+ in $F(x)$;
- 26 $\text{L}_{\text{St}} \leftarrow (\text{id}(m), \text{port}(u, m), \text{port}(m, u), \text{L}_{\text{St}(m)}(u^\downarrow))$;
- 27 $\text{L}_{1\text{st}} \leftarrow (\text{LD}_{\partial^*F(w_1)}(u), \text{LR}_{\partial^*F(w_1)}(u), \text{port}(u, u_1^+), \text{port}(u_1^+, \text{twin}(u_1^+)))$;
- 28 $\text{L}_{2\text{nd}} \leftarrow (\text{LD}_{\partial^*F(w_2)}(u), \text{LR}_{\partial^*F(w_2)}(u), \text{port}(u, u_2^+), \text{port}(u_2^+, \text{twin}(u_2^+)))$;
- 29 $\text{LR}(u) \leftarrow \text{LR}(u) \circ (\text{L}_{\text{St}}, \text{L}_{1\text{st}}, \text{L}_{2\text{nd}})$;
- 30 ROUTING_ENCODING($F(x), \text{LR}(V)$) ;

7.3. Routing queries. Let u and v be two arbitrary vertices of a cube-free median graph G and let $\text{LR}(u)$ and $\text{LR}(v)$ be their labels returned by the encoding algorithm `ROUTING_ENCODING`. We describe how the routing algorithm `ROUTING` can decide by which port to send the message from u to v to a neighbor of u closer to v than u .

7.3.1. The algorithm. We continue with the formal description of the routing algorithm `ROUTING`. The specific functions ensuring routing from panel to cone, from cone to panel, from cone to cone, or between separated vertices will be described in the next subsection.

Algorithm 4: `ROUTING`($\text{LR}(u)$, $\text{LR}(v)$)

Input: The labels $\text{LR}(u)$ and $\text{LR}(v)$ of two vertices u and v of G , where u is the source and v the target

Output: $\text{port}(u, v)$

```

1 if  $\text{LR}_0(u) = \text{LR}_0(v)$  /*  $u = v$  */ then return 0 ;
2 Let  $i$  be the highest integer such that  $\text{LR}_i^{\text{St}[\text{Med}]}(u) = \text{LR}_i^{\text{St}[\text{Med}]}(v)$ ;
3  $d \leftarrow \text{Dist\_Star}(\text{LR}_i^{\text{St}[\text{gate}]}(u), \text{LR}_i^{\text{St}[\text{gate}]}(v))$  ; //  $d_G(u^\dagger, v^\dagger)$ 
4  $d_u \leftarrow \text{Dist\_Star}(\text{LR}_i^{\text{St}[\text{gate}]}(u), 0)$  ; //  $d_G(u^\dagger, m)$ 
5  $d_v \leftarrow \text{Dist\_Star}(\text{LR}_i^{\text{St}[\text{gate}]}(v), 0)$  ; //  $d_G(v^\dagger, m)$ 
6 if  $d = 1$  and  $d_u = 1$  then return Routing_Panel_to_Cone ( $\text{LR}_i(u)$ ,  $\text{LR}_i(v)$ ) ;
7 if  $d = 1$  and  $d_v = 1$  then return Routing_Cone_to_Panel ( $\text{LR}_i(u)$ ,  $\text{LR}_i(v)$ ) ;
8 if  $d = 2$  and  $d_u = d_v = 2$  then return Routing_Cone_to_Cone ( $\text{LR}_i(u)$ ,  $\text{LR}_i(v)$ ) ;
9 return Routing_Separated ( $\text{LR}_i(u)$ ,  $\text{LR}_i(v)$ ,  $\text{LR}_0(u)$ ).

```

7.3.2. Description and functions. As for distance queries, the first thing to do in order to answer a routing query from u to v is to detect if u and v are 1-neighboring, 2-neighboring or separated, and the type (cone or panel) of the fibers containing them. This is done in the same way as explained in Subsection 6.3. Again, we assume that i is the first step such that u and v are no longer roommates. Denote by m the median vertex used at this step. We denote by $F(x)$ the fiber containing u and by $F(y)$ the fiber containing v (recall that x is the gate of u in $\text{St}(m)$ and y is the gate of v in $\text{St}(m)$).

If u and v are 1-neighboring, the answer is computed differently when the source u is in a cone and when u is in a panel. If u is in a cone $F(x)$ (and thus v is in a panel $F(y)$), we use the function `Routing_Cone_to_Panel`. This function determines which part ($\text{LR}_i^{1\text{st}}(v)$ or $\text{LR}_i^{2\text{nd}}(v)$) of $\text{LR}_i(v)$ contains the information about the gate u^+ of u on $F(y)$. Then the function returns the port $\text{port}(u, u^+)$ to the gate u^+ of u in $F(y)$, stored as $\text{LR}_i^{1\text{st}[\text{toGate}]}(u)$ or $\text{LR}_i^{2\text{nd}[\text{toGate}]}(u)$.

```

function Routing_Cone_to_Panel( $\text{LR}_i(u)$ ,  $\text{LR}_i(v)$ ):
  dir  $\leftarrow$  2nd ;
  if  $\text{LR}_i^{\text{St}[\text{gate}]}(v) = \min\{i : i \in \text{LR}_i^{\text{St}[\text{gate}]}(u)\}$  then
    dir  $\leftarrow$  1st ;
  return  $\text{LR}_i^{\text{dir}[\text{toGate}]}(u)$ .

```

If $F(x)$ is a panel and $F(y)$ is a cone, then u stored the distances to its two imprints u_1 and u_2 on the total boundary $\partial^*F(x)$ and v stored the distance to its gate v^+ in $F(x)$ (v^+ also belongs to $\partial^*F(x)$) and its twin $\text{twin}(v^+)$ in $F(y)$. When u is different from v^+ , the function `Routing_Panel_to_Cone` finds the tree distance labeling of v^+ , computes $\min\{d_G(u, u_1) + d_T(u_1, v^+), d_G(u, u_2) + d_T(u_2, v^+)\}$, and returns the port to the imprint of u minimizing the two distance sums. If u belong to the total boundary $\partial^*F(x)$, then we distinguish two cases. If $u = v^+$, then using the label $\text{LR}_i(v)$ of v the algorithm returns the port from $\text{twin}(v^+)$ to $v^+ = u$. If u belongs to $\partial^*F(x)$ but $u \neq v^+$, since

$LR_i(u)$ and $LR_i(v)$ contain a labeling for routing in trees of u and v^+ , **Routing_Panel_to_Cone** computes $\text{port}(u, v^+)$ using the routing decoder for trees and returns it.

```

function Routing_Panel_to_Cone( $LR_i(u)$ ,  $LR_i(v)$ ):
   $\text{dir}_v \leftarrow 2\text{nd}$  ;
  if  $LR_i^{\text{St}[\text{gate}]}(u) = \min\{i : i \in LR_i^{\text{St}[\text{gate}]}(v)\}$  then
     $\text{dir}_v \leftarrow 1\text{st}$  ;

  if  $LR_i^{1\text{st}[\text{toGate}]}(u) = 0$  or  $LR_i^{2\text{nd}[\text{toGate}]}(u) = 0$  /*  $u$  is on the border */ then
    Let  $\text{dir}_u \in \{1\text{st}, 2\text{nd}\}$  be such that  $LR_i^{\text{dir}_u[\text{toGate}]}(u) = 0$  ;
    if  $LR_i^{\text{dir}_u[\text{gate\_LDT}]}(u) = LR_i^{\text{dir}_v[\text{gate\_LDT}]}(v)$  then
      return  $LR_i^{\text{dir}_v[\text{fromGate}]}(v)$  ; //  $u$  is the gate of  $v$  on the panel  $F(x)$ 
    return  $\text{Rout\_Tree}(LR_i^{\text{dir}_u[\text{gate\_LRT}]}(u), LR_i^{\text{dir}_v[\text{gate\_LRT}]}(v))$  ;

   $d_{1\text{st}} \leftarrow LR_i^{1\text{st}[\text{Dist}]}(u) + \text{Dist\_Tree}(LR_i^{1\text{st}[\text{gate\_LDT}]}(u), LR_i^{\text{dir}_v[\text{gate\_LDT}]}(v))$  ;
   $d_{2\text{nd}} \leftarrow LR_i^{2\text{nd}[\text{Dist}]}(u) + \text{Dist\_Tree}(LR_i^{2\text{nd}[\text{gate\_LDT}]}(u), LR_i^{\text{dir}_v[\text{gate\_LDT}]}(v))$  ;

   $\text{dir}_u \leftarrow 1\text{st}$  ;
  if  $d_{2\text{nd}} = \min\{d_{1\text{st}}, d_{2\text{nd}}\}$  then
     $\text{dir}_u \leftarrow 2\text{nd}$  ; //  $u_{2\text{nd}}$  is the gate of  $u$  on a shortest path to  $v$ 
  return  $LR_i^{\text{dir}_u[\text{toGate}]}(u)$ .

```

If u and v are 2-neighboring, then $F(x)$ and $F(y)$ are cones and the function **Routing_Cone_to_Cone** is similar to the function **Routing_Cone_to_Panel**. The common panel $F(w)$ neighboring $F(x)$ and $F(y)$ can be found by inspecting $LR_i^{\text{St}[\text{gate}]}(u)$ and $LR_i^{\text{St}[\text{gate}]}(v)$. As in the case of **Routing_Cone_to_Panel**, the function **Routing_2-Neighboring** returns the port $\text{port}(u, u^+)$ from u to its gate u^+ in $F(w)$.

```

function Routing_Cone_to_Cone( $LR_i(u)$ ,  $LR_i(v)$ ):
   $\text{dir} \leftarrow 2\text{nd}$  ;
  if  $LR_i^{\text{St}[\text{gate}]}(u) \cap LR_i^{\text{St}[\text{gate}]}(v) = \min\{i : i \in LR_i^{\text{St}[\text{gate}]}(u)\}$  then
     $\text{dir} \leftarrow 1\text{st}$  ;
  return  $LR_i^{\text{dir}[\text{toGate}]}(u)$ .

```

Finally, if u and v are separated, two cases have to be considered depending of whether u is the median vertex m or not. If u is not the median, then u stored $\text{port}(u, m)$. Since a shortest path from u to v passes via m , **Routing_Separated** returns $\text{port}(u, m)$. If u coincides with m , the port $\text{port}(m, v)$ is not stored in $LR_i(u)$ but in $LR_i^{\text{St}[\text{fromMed}]}(v)$, and **Routing_Separated** returns it.

```

function Routing_Separated( $LR_i(u)$ ,  $LR_i(v)$ ,  $\text{id}(u)$ ):
  if  $LR_i^{\text{St}[\text{Med}]}(v) = \text{id}(u)$  then
    return  $LR_i^{\text{St}[\text{fromMed}]}(v)$  ;
  return  $LR_i^{\text{St}[\text{toMed}]}(u)$ .

```

7.4. Correctness and complexity. Similarly to **DISTANCE_ENCODING** it can be shown that **ROUTING_ENCODING** correctly constructs the encoding in total $O(n^2 \log(n))$ time with labels of vertices of size $O(\log^3(n))$. This leads to the main result of this section:

Theorem 7.1. *ROUTING_ENCODING constructs in total time $O(n^2 \log(n))$ labels of size $O(\log^3(n))$ to the vertices of a cube-free median graph $G = (V, E)$. Given the labels of two vertices u and v , ROUTING returns in time $O(\log^2(n))$ a port of u leading to a neighbor of u on a shortest path to v .*

8. CONCLUSION

In this paper we presented distance and routing labeling schemes for cube-free median graphs G with labels of size $O(\log^3 n)$. For that, we considered the partitioning of G into fibers (of size $\leq n/2$) of the star $\text{St}(m)$ of a median vertex m . Each fiber is further recursively partitioned using the same algorithm. We classified the fibers into panels and cones and the pairs of vertices u, v of G into roommates, separated, 1-neighboring, and 2-neighboring pairs. If u and v are roommates, then $d_G(u, v)$ is taken at a later step of the recursion. Otherwise, we showed how to retrieve $d_G(u, v)$ by keeping in the labels of u and v some distances from those vertices to some gates/imprints. Our main ingredient is the fact that the total boundaries of fibers of cube-free median graphs are isometric quasigated trees.

This last property of fibers is an obstacle in generalizing our approach to all median graphs, or even to median graphs of dimension 3. The main problem is that the total boundary is no longer a median graph. Therefore, one cannot apply to this total boundary the distance and routing schemes for cube-free median graphs. Nevertheless, a more brute-force approach works for arbitrary median graphs G of constant maximum degree Δ . In this case, all cubes of G have constant size. Thus, the star $\text{St}(m)$ cannot have more than $O(2^\Delta)$ vertices, i.e., $\text{St}(m)$ has a constant number of fibers. Since every fiber is gated, at every step of the encoding algorithm, every vertex v can store in its label the distance from v to its gates in all fibers of $\text{St}(m)$. Consequently, this leads to distance and routing labeling schemes with labels of (polylogarithmic) length $O(2^\Delta \log^3 n)$ for all median graphs with constant maximum degree Δ . We would like to finish this paper with the following question: *Does there exist a polylogarithmic distance labeling scheme for general median graphs or for median graphs of constant dimension?*

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