Periodical body's deformations are optimal strategies for locomotion

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Abstract

A periodical cycle of body's deformation is a common strategy for locomotion (see for instance birds, fishes, humans). The aim of this paper is to establish that the auto-propulsion of deformable object is optimally achieved using periodic strategies of body's deformations. This property is proved for a simple model using optimal control theory framework.

1 Introduction

Most of the living organisms self-propel by a periodical cycle of body's deformation. From a bird which flaps theirs wings, a fish which beats its caudal fin, the human walking using a synchronized movement of theirs legs, the motion of living organisms derives from a periodical cycle of shape changing. Starting from this observation, an interesting question is what are the common properties of all these various dynamical systems which imply that the strategy employees for achieving a displacement is to deforming their body in a periodical way. In other words, why all these different systems representing the locomotion in various environments (fluid, air, ground, ...) share this same property.

Understanding theoretically the way that living organisms are able to move is a challenge for many fields [12]. For instance in robotic, trying to adapt natural strategy for the displacement of robot in different environments is an issue. This challenge is so-called bio-inspired locomotion [6]. Many field study this feature with various applications. For instance, let us quote some different issues like in robotic, making robots that are able to walk, run and fly [3, 14, 17, 19], in biomedical, building micro-robot for drug delivery or surgery [18, 22, 26], in biology, understanding the displacement of bacteria as Escherichia coli [24], sperm cell [5].

In what follows, we address if whether the best strategy to move is to reproduce several times an optimal cycle of body's deformation. We attack this problem using a simple toy model and applying an optimal control framework. Even if this paper focuses on a generic simple systems, the same type of dynamics classically governs the displacement of micro-swimmer at low Reynolds number. Numerous models of micro-robot are expressed by a similar dynamics, for instance the Copepod model [4, 25] the spherical one studied in [15], the Three-sphere swimmer [1, 2, 20], ciliate model [16] and others fit this framework. Similarly, the displacements of micro-crawlers, which derive their propulsion capabilities from tangential resistance offered by the substrate, are also governed by simple equations in agreement with our framework [8, 21].

This paper focuses on a generic three-dimensional dynamical system where the rate of the body's deformations is assumed to be the control functions. More specifically, the displacement of a deformable object is solution of an optimal control problem in which the unknowns are the shape deformations. Then, the locomotion derives from the assumption that the body's deformations allow the object to self-propel by maximizing its average speed. Under some regularity and boundedness hypothesis, our main result states that there exists an optimal periodical cycle of body's deformations that allows the object to move optimizing this latter criterium. Even if already numerous studies [4, 9, 16, 27] focusing on the optimal locomotion problem are setting the fact that the strategies of deformation could be periodic, our result implies that this periodical hypothesis makes sense.

This paper is organized as follows. Section 2 is devoted to present the dynamical system and the optimal control problem associated with the auto-propulsion of a deformable body. Section 3 presents the main result which is proved in Section 5. Section 4 gives some regularity properties of the solutions of the considered optimal control problems that are necessary in the proofs in Section 5.

2 Mathematical modeling

Dynamical system In what follows, we focus on simplified model representing a locomotion problem. We consider a deformable object whose configurations are described by a real variable x corresponding to the position of the object and by two shape parameters α_1 and α_2. The displacement of the object derives from a deformation of its own shape $\alpha := (\alpha_1, \alpha_2)$ and is governed by the following equation,

$$\dot{x} = f_1(\alpha_1, \alpha_2)\dot{\alpha}_1 + f_2(\alpha_1, \alpha_2)\dot{\alpha}_2 \,, \tag{1}$$

where f_i, $i = 1, 2$, are smooth functions on $\mathbb{R}^2$. In others words, the speed of the object is decomposed into a sum of two terms, each represents the impact of a rate of deformation $\dot{\alpha}_i$, $i = 1, 2$ on the motion.

Let us note that equation (1) implies that the motion of the object does not depend on its position, which is a typical property of such locomotion models (invariance with respect to a certain displacement group). As a consequence, for a given prescribed absolutely continuous deformation $\alpha(\cdot)$ on $[0, T]$, equation (1) has a unique solution $x(\cdot)$ on $[0, T]$ for each initial position $x(0) \in \mathbb{R}$, and the displacement $\Delta = x(T) - x(0)$ does not depend on the initial position $x(0)$.

We further assume that α_1 and α_2 are bounded parameters. It means that there exists a compact set $\mathcal{K}$ in $\mathbb{R}^2$ such that the shape parameters (α_1, α_2) belongs to it, i.e.,

$$\forall t > 0 \,, \qquad (\alpha_1(t), \alpha_2(t)) \in \mathcal{K} \,.$$

To summarize, the evolution of the moving object is represented by the 3-tuples $X(\cdot) = (\alpha_1(\cdot), \alpha_2(\cdot), x(\cdot))$ which are solutions of the control system

$$\begin{cases} \dot{X} = u_1 F_1(X) + u_2 F_2(X), \\ \text{for a.e. } t \in [0,T], \quad X(t) \in \mathcal{K} \times \mathbb{R}, \quad u(t) \in \mathbb{R}^2, \end{cases} \tag{2}$$

where $u = (u_1, u_2) = \dot{\alpha}$ and the vector fields F_1 and F_2 are defined by

$$F_1(X) = \begin{pmatrix} 1 \\ 0 \\ f_1(\alpha) \end{pmatrix}, \; F_2(X) = \begin{pmatrix} 0 \\ 1 \\ f_2(\alpha) \end{pmatrix}.$$

Note that, although the trajectories $X(\cdot)$ are forced to remain in $\mathcal{K} \times \mathbb{R}$, the smooth vector fields F_1 and F_2 are defined on the whole $\mathbb{R}^3$.

Assumption 1. *We make the following technical hypotheses, that we explain below.*

(A1) *$\mathcal{K}$ is a "nice" compact subset of $\mathbb{R}^2$ (see Definition 12 below). In particular, it is the closure of a connected open subset of $\mathbb{R}^2$ whose boundary $\partial\mathcal{K}$ is a piecewise C^1 curve.*

(A2) *For every $\alpha \in \mathcal{K}$, $g(\alpha) = \frac{\partial f_2}{\partial \alpha_1}(\alpha) - \frac{\partial f_1}{\partial \alpha_2}(\alpha)$ or one of its partial derivative $\frac{\partial^k g}{\partial \alpha_{i_1} \cdots \partial \alpha_{i_k}}(\alpha)$ is nonzero.*

Remark 2.

- Assumption **(A2)** implies that the Lie algebra generated by F_1, F_2 is of dimension 3 at every point of $\mathcal{K} \times \mathbb{R}$. Indeed, for any integer k and any indices $i_1, \ldots, i_k \in \{1, 2\}$, we have

$$[F_1, F_2] = \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}, \qquad [F_{i_1}, \ldots, [F_{i_k}, [F_1, F_2]]] = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial^k g}{\partial \alpha_{i_1} \cdots \partial \alpha_{i_k}} \end{pmatrix}.$$

  Thus **(A2)** ensures that the system (2) is controllable in any connected open subset of $\mathcal{K} \times \mathbb{R}$. Thus, using **(A1)**, the system is controllable in the whole set $\mathcal{K} \times \mathbb{R}$. In particular this property implies that the system is able to move in the x-direction.

- Assumption **(A2)** guarantees that the set $\{g = 0\} \cap \mathcal{K}$ is a finite union of points and 1-dimensional submanifolds of $\mathbb{R}^2$.

- Assumption **(A1)** does not include the case where α_1, α_2 are unconstrained angles, i.e., the case where α belongs to the torus $\mathbb{T}^2$. This will be crucial in Lemma 21 which would not hold in the case $\mathcal{K} = \mathbb{T}^2$.

Optimal control formulations. A fundamental paradigm in locomotion of living organisms is that the derived motions tend to minimize a certain cost. In other terms, the displacement of the moving object derives from an optimal control problem associated with the control system (2). The formulation of this optimal control problem could be expressed in different ways.

Assume that the infinitesimal cost of the motion is defined by a Riemannian metric Q on $\mathbb{R}^2$, i.e., a family of positive definite quadratic forms $Q_\alpha : u \in \mathbb{R}^2 \mapsto Q_\alpha(u) \in \mathbb{R}^+$ depending smoothly on $\alpha \in \mathbb{R}^2$. Let us recall that the functions $\dot{\alpha}_i(\cdot)$ belongs to $L^1([0,T])$, for $i = 1, 2$. We also fix a compact subset $\mathcal{C}$ of $(\mathcal{K} \times \mathbb{R})^2$ which models initial and final constraints.

Definition 3. For $T > 0$, let $\mathcal{X}_T^{\mathcal{C}}$ be the set of absolutely continuous trajectories $X(t)$, $t \in [0,T]$, of (2) whose extremities $(X(0), X(T))$ belong to $\mathcal{C}$. We define the following optimal control problems (OCP).

a. Minimization of time:

$$\min T := \min \left\{ T > 0 \ : \ X(\cdot) \in \mathcal{X}_T^{\mathcal{C}},\ Q_{\alpha(t)}(u(t)) \leq 1 \quad \text{for a.e. } t \in [0,T] \right\}.$$

b. Minimization of energy: defining the energy associated with a trajectory $X(\cdot) \in \mathcal{X}_1^{\mathcal{C}}$ by

$$\mathcal{E}(X(\cdot)) = \int_0^1 Q_{\alpha(t)}(u(t))dt\,,$$

we set

$$\min \mathcal{E} := \min \left\{ \mathcal{E}(X(\cdot)) \ : \ X(\cdot) \in \mathcal{X}_1^{\mathcal{C}} \right\}.$$

c. Minimization of length: defining the length of a trajectory $X(\cdot) \in \mathcal{X}_T^{C}$ by

$$\ell(X(\cdot)) = \int_0^T \sqrt{Q_{\alpha(t)}(u(t))}dt,$$

we set

$$\min \ell := \min \left\{ \ell(X(\cdot)) \ : \ X(\cdot) \in \mathcal{X}_T^{\mathcal{C}}, \quad T > 0 \right\}.$$

Each one of these minimization problems correspond to different modeling choices of the motion: fastest motion with bounded speed of deformation in the first case, lowest energy consumption in fixed time for the second,... However all these problems are equivalent, it is the result of a trivial reparameterization and of the Cauchy-Schwarz inequality (see [23, Sect. 2.1] for instance). It is moreover standard that all these problems admit minimizers (see [13] for instance). The following lemma states this equivalence property. Note that further properties of the energy minimizers are described in Section 4.

Lemma 4. *The optimal control problems **a**, **b**, and **c** admit minimizers. Moreover:*

- $\min T = \min \ell$ *and* $X(\cdot)$ *is a time minimizer if and only if it is a minimizer of* ℓ *and* $Q_{\alpha(t)}(u(t)) = 1$ *for a.e.* $t \in [0,T]$*;*
- $\min \mathcal{E} = (\min \ell)^2$ *and* $X(\cdot)$ *is an energy minimizer if and only if it is a minimizer of* ℓ *such that* $T = 1$ *and* $Q_{\alpha(t)}(u(t)) =$ *constant for a.e.* $t \in [0,1]$*;*
- $X(t)$, $t \in [0,T]$, *is a time minimizer if and only if* $\widetilde{X}(s) = X(sT)$, $s \in [0,1]$, *is an energy minimizer.*

In the sequel, the time minimization formalism will be used which corresponds to a modeling hypothesis of bounded speed of deformation. This makes sense in the context of the locomotion. Hence, unless explicitly stated, a trajectory $X(\cdot)$ satisfies

$$\begin{cases} \dot{X}(t) = u_1(t)F_1(X(t)) + u_2(t)F_2(X(t)), \\ X(t) \in \mathcal{K} \times \mathbb{R} \quad \text{and} \quad Q_{\alpha(t)}(u(t)) \leq 1, \end{cases} \quad \text{for a.e. } t \in [0, T]. \tag{3}$$

The energy minimization formalism will be used only in Section 4 as it is the usual framework to describe regularity properties of the minimizers.

Locomotion strategies The above optimal control problems are well suited to model specific motions such as "move of a quantity Δ in the x-direction". In that case one can choose $\mathcal{C} = \{(X^0 = (\alpha^0, 0), X^1 = (\alpha^1, \Delta)) \ : \ \alpha^0, \alpha^1 \in \mathcal{K}\}$ and solve one of the three (OCP) among the trajectories with extremities in $\mathcal{C}$ (another possible choice is to fix the initial shape parameter α^0, in that case the set $\mathcal{C}$ would be equal to $\{(X^0 = (\alpha^0, 0), X^1 = (\alpha^1, \Delta)) \ : \ \alpha^1 \in \mathcal{K}\}$).

However what we usually understand by locomotion is a less precise motion and can be described as "move in the x-direction". In that case one has to choose a different criterion and the strategy that is usually proposed is to minimize the mean-time $T/x(T)$ (or equivalently to maximize the average velocity $x(T)/T$ or the efficiency $x(T)^2/\mathcal{E}(X(\cdot))$, see [4, 25]. The difficulty with such a criterium is that it does not admit optimal solutions in general, thus we propose to model locomotion strategies for arbitrary large displacements as follows.

Definition 5. The *locomotion strategy* is to solve

$$\liminf_{\Delta \to \infty} \frac{T^\sharp(\Delta)}{\Delta}, \tag{4}$$

where $T^\sharp(\Delta) = \min\{T > 0 \ : \ X(\cdot) \text{ solution of (3) s.t. } x(T) - x(0) = \Delta\}$. If it exists, a curve $X(\cdot)$ realizing the liminf above is called an *optimal locomotion strategy.*

The main issue in this modeling is the existence of optimal locomotion strategy.

3 Main results

The aim of this paper is to prove that optimal locomotion strategies exist and may be achieved using periodic cycles of shape deformation. Let us introduce the following definition.

Definition 6. A *stroke* is a closed curve in the shape parameters, i.e., an absolutely continuous curve $\alpha(t)$, $t \in [0, T]$, such that $\alpha(0) = \alpha(T)$. When the stroke is C^1 and satisfies $\dot{\alpha}(T) = \dot{\alpha}(0)$, it can be extended to a T-periodic C^1 curve $\alpha(t)$, $t \in \mathbb{R}$, which is called a *periodic cycle of shape deformation.*

We introduce also the minimal time needed to move of a quantity $\Delta > 0$ in the x-direction using a stroke as

$$T^*(\Delta) = \min \Big\{ T > 0 \ : \ X(t) = (\alpha(t), x(t)),\ t \in [0,T],\ \text{solution of (3)} \\ \text{s.t. } \alpha(T) = \alpha(0) \text{ and } x(T) - x(0) = \Delta \Big\}.$$

Definition 7. We call *mean-time optimal stroke* a curve $\alpha(\cdot)$ defined on $[0, T_\alpha]$ derived from a solution of (2) associated with a displacement Δ_α such that $T_\alpha = T^*(\Delta_\alpha)$ and

$$\inf_{\Delta>0} \frac{T^*(\Delta)}{\Delta} = \frac{T_\alpha}{\Delta_\alpha}.$$

Our main result states that optimal strategies are obtained by using strokes.

Theorem 8.

(i) There exists mean-time optimal strokes. Moreover there exists mean-time optimal strokes which are simple curves.

(ii) Mean-time optimal strokes are C^1 and extends to periodic C^1 curves.

(iii) Mean-time optimal strokes are optimal locomotion strategies, i.e.,

$$\liminf_{\Delta\to\infty} \frac{T^\sharp(\Delta)}{\Delta} = \inf_{\Delta>0} \frac{T^*(\Delta)}{\Delta}.$$

As a consequence, the periodic cycles of shape deformation defined by the mean-time optimal strokes are optimal locomotion strategies solution of (4). In other terms, the optimal strategy for the moving object is to achieve a displacement by deforming periodically its body with a continuous rate of deformation.

Remark 9. Note that the minimal value $\inf_{\Delta>0} \frac{T^*(\Delta)}{\Delta}$ may be reached by several mean-time optimal strokes corresponding to different values Δ^* of the displacement. However Lemma 19 (next section) implies that these values have a positive lower bound, i.e. that the period $T^*(\Delta^*)$ of a mean-time optimal stroke is bounded by below away from 0.

Remark 10. The fact that the value of $\inf_{\Delta>0} \frac{T^*(\Delta)}{\Delta}$ is non zero is obvious. Indeed, since α belongs to a compact $\mathcal{K}$ and $\dot\alpha$ is bounded, any trajectory admissible for $T^*(\Delta)$ satisfies $||\dot x||_{\infty,[0,T]} \le C$ for some constant C, and then by integrating equation (1), we get

$$|x(T) - x(0)| \le CT. \tag{5}$$

As a consequence, $\inf_{\Delta>0} \frac{T^*(\Delta)}{\Delta} \ge 1/C > 0$.

Remark 11. In the minimization problems (4) and (14) we consider only positive displacement Δ. However the problems would be exactly the same without this positivity hypothesis. Indeed, reverting the time along a trajectory $X(\cdot)$ with $x(T) - x(0) = \Delta$, we obtain a trajectory $\check X(t) = X(T-t)$ with $\check x(T) - \check x(0) = -\Delta$. Hence there holds

$$\liminf_{\Delta\to\infty} \frac{T^\sharp(\Delta)}{\Delta} = \liminf_{|\Delta|\to\infty} \frac{T^\sharp(\Delta)}{|\Delta|} \quad \text{and} \quad \inf_{\Delta>0} \frac{T^*(\Delta)}{\Delta} = \inf_{\Delta\neq0} \frac{T^*(\Delta)}{|\Delta|}.$$

Before entering the core of the proof of Theorem 8, we need to establish regularity properties of the solutions of some energy minimization problems, that we study separately in the next Section 4.

4 Regularity properties for energy minimizers

In this section we give results on the regularity of the minimizers. For this we rely heavily on [7], so we adopt the presentation and notations of that paper. In particular we use for the state constraints the following definition, which specifies Assumption **(A1)**.

Definition 12. A compact subset $\mathcal{K}$ of $\mathbb{R}^2$ is said to be *nice* if it is the closure of a connected open subset of $\mathbb{R}^2$ and may be written as $\mathcal{K} = \{h_i(\alpha) \le 0,\ i = 1, \dots, r\}$, where r is a positive integer and $h_1, \dots, h_r$ are smooth functions such that, for any $\alpha \in \mathcal{K}$,

$$\sum_{j \in \mathcal{J}(\alpha)} \beta_j \nabla h_j(\alpha) \neq 0, \quad \text{where } \mathcal{J}(\alpha) = \{j \in [0, r] \,:\, h_j(\alpha) = 0\},$$

for every set of non-negative and not all zero numbers $\{\beta_j\}_{j \in \mathcal{J}(\alpha)}$.

We consider first the energy minimization problem for a fixed $\Delta > 0$ that we write similarly to $(\mathcal{P})$ in [7] as

$$\mathcal{E}^*(\Delta) = \begin{cases} \text{Minimize } \mathcal{E} = \int_0^1 Q_{\alpha(t)}(u(t))dt \\ \text{over absolutely continuous functions } X = (\alpha, x) : [0,1] \to \mathbb{R}^3 \\ \text{ and measurable } u : [0,1] \to \mathbb{R}^2 \text{ satisfying} \\ \dot{X}(t) = u_1(t) F_1(X(t)) + u_2(t) F_2(X(t)) \text{ for a.e. } t \in [0,1], \\ \tilde{h}_j(X(t)) := h_j(\alpha(t)) \le 0 \text{ for all } t \in [0,1],\ j = 1, \dots, r, \\ u(t) \in \mathbb{R}^2 \text{ for a.e. } t \in [0,1], \\ (X(0), X(1)) \in \mathcal{C} = \{(X^0, X^1) \ :\ \alpha^1 = \alpha^0 \text{ and } x^1 - x^0 = \Delta\}. \end{cases}$$

As stated in Lemma 4, $\mathcal{E}^*(\Delta)$ admits minimizers for any $\Delta > 0$. Moreover, the value function with respect to Δ is regular as follows.

Lemma 13. *The function $\Delta \mapsto \mathcal{E}^*(\Delta)$ is lower semi-continuous.*

Proof. We have to prove that, for every $\Delta_0 > 0$, $\liminf_{\Delta \to \Delta_0} \mathcal{E}^*(\Delta) \ge \mathcal{E}^*(\Delta_0)$. Take any sequence $(\Delta^n)_{n \in \mathbb{N}}$ converging to Δ_0 and choose for each n a minimizer $X^n(\cdot)$ of $\mathcal{E}^*(\Delta^n)$. By [28, Th. 2.5.3] the sequence $(X^n(\cdot))_{n \in \mathbb{N}}$ converges uniformly to a trajectory $X^0(\cdot)$ which is admissible for the minimization problem $\mathcal{E}^*(\Delta_0)$. As a consequence,

$$\lim_{n \to \infty} \mathcal{E}^*(\Delta^n) = \mathcal{E}(X^0(\cdot)) \ge \mathcal{E}^*(\Delta_0),$$

and the lemma is proved. □

We introduce now a second energy minimization problem including a negative penalization of the final displacement. For $\delta > 0$, we set

$$\mathcal{E}_\delta^* = \begin{cases} \text{Minimize } \mathcal{E}_\delta = \int_0^1 Q_{\alpha(t)}(u(t))dt - \delta(x(1)-x(0))^2 \\ \text{over absolutely continuous functions } X = (\alpha, x) : [0,1] \to \mathbb{R}^3 \\ \text{ and measurable } u : [0,1] \to \mathbb{R}^2 \text{ satisfying} \\ \dot{X}(t) = u_1(t)F_1(X(t)) + u_2(t)F_2(X(t)) \text{ for a.e. } t \in [0,1], \\ \tilde{h}_j(X(t)) := h_j(\alpha(t)) \le 0 \text{ for all } t \in [0,1],\ j = 1, \dots, r, \\ u(t) \in \mathbb{R}^2 \text{ for a.e. } t \in [0,1], \\ (X(0), X(1)) \in \mathcal{C}' = \{(X^0, X^1) \ : \ \alpha^1 = \alpha^0\}. \end{cases}$$

The only differences between $\mathcal{E}^*(\Delta)$ and $\mathcal{E}_\delta^*$ lie in the change of costs ($\mathcal{E}_\delta$ instead of $\mathcal{E}$) and in the change of constraints at the extremities ($\mathcal{C}'$ instead of $\mathcal{C}$). Note however that the existence of minimizers is not guaranteed for $\mathcal{E}_\delta^*$ (the existence of a particular δ for which $\mathcal{E}_\delta^*$ admits minimizers is actually equivalent to the main result of this paper).

Since $F_1(\cdot), F_2(\cdot)$ and $\tilde{h}_1(\cdot), \dots, \tilde{h}_r(\cdot)$ are smooth and Q is a smooth Riemannian metric on $\mathbb{R}^2$, hypothesis (H1)-(H3) of [7] are trivially satisfied by both problems. As a consequence, any minimizing solution $\bar{X}(\cdot)$ of $\mathcal{E}^*(\Delta)$ or $\mathcal{E}_\delta^*$ and its associated control $\bar{u}(\cdot)$ satisfy the following state constrained Maximum Principle.

There exists "multipliers" $(p(\cdot), \mu_1(\cdot), \dots, \mu_r(\cdot), \lambda)$, where $p(\cdot)$ is an absolutely continuous function from $[0,1]$ to $\mathbb{R}^3$, $\mu_j(\cdot)$ for $j = 1, \dots, r$, are non-negative Borel measures on $[0,1]$, and $\lambda \ge 0$ is a real number, such that, writing

$$q(t) = p(t) + \sum_{j=1}^r \int_{[0,t)} \nabla \tilde{h}_j(\bar{X}(s))\mu_j(ds), \tag{6}$$

$$\text{and} \qquad \mathcal{H}(X, p, u, \lambda) = \langle p, u_1F_1(X) + u_2F_2(X)\rangle - \lambda Q_\alpha(u),$$

we have

$$(p, \mu, \lambda) \ne (0,0,0), \tag{7}$$

$$\dot{p}(t) = -\nabla_X \mathcal{H}(\bar{X}(t), q(t), \bar{u}(t), \lambda) \quad a.e.\ t \in [0,1], \tag{8}$$

$$\mathcal{H}(\bar{X}(t), q(t), \bar{u}(t), \lambda) = \max_{u \in \mathbb{R}^2} \mathcal{H}(\bar{X}(t), q(t), u, \lambda) \quad a.e.\ t \in [0,1], \tag{9}$$

$$\operatorname{supp}\{\mu_j\} \subset \{t \ : \ \tilde{h}_j(\bar{X}(t)) = 0\} \quad \text{for } j = 1, \dots, r,$$

with the following transversality conditions on $p = (p_1, p_2, p_x)$:

- *if $\bar{X}(\cdot)$ is a minimizing solution of $\mathcal{E}^*(\Delta)$, then*

$$p(0) = p(1) + \sum_{j=1}^r \int_{[0,1]} \nabla \tilde{h}_j(\bar{X}(s))\mu_j(ds),$$

- *if $\bar{X}(\cdot)$ is a minimizing solution of $\mathcal{E}_\delta^*$, then*

$$p_i(0) = p_i(1) + \sum_{j=1}^r \int_{[0,1]} \frac{\partial h_j}{\partial \alpha_i}(\bar{\alpha}(s))\mu_j(ds), \quad i = 1,2,$$

$$p_x(0) = p_x(1) = 2\delta(x(1) - x(0)). \tag{10}$$

We say that $(\bar{X}(\cdot), \bar{u}(\cdot))$ is a *normal extremal* if there exists multipliers $p(\cdot), \mu(\cdot)$, and $\lambda = 1$, and that it is an *abnormal extremal* if there exists multipliers p, μ, and $\lambda = 0$. From the Maximum Principle above, any minimizer must be either a normal or an abnormal extremal.

Lets us study now the abnormal extremals of $\mathcal{E}^*(\Delta)$ and $\mathcal{E}^*_\delta$.

Definition 14. The *singular set* is the subset $\mathcal{S}$ of $\mathcal{K} \subseteq \mathbb{R}^2$ defined as

$$\mathcal{S} = \partial\mathcal{K} \cup \{\alpha \in \mathcal{K} \ : \ g(\alpha) = 0\}, \quad \text{where} \quad g(\alpha) = \frac{\partial f_2}{\partial \alpha_1}(\alpha) - \frac{\partial f_1}{\partial \alpha_2}(\alpha).$$

Assumptions **(A1)** and **(A3)** guarantee that $\mathcal{S}$ is a finite union of points and of 1-dimensional C^1 submanifolds of $\mathbb{R}^2$.

Lemma 15. *If $(\bar{X}(\cdot), \bar{u}(\cdot))$ is an abnormal extremal of $\mathcal{E}^*(\Delta)$, then $\bar{\alpha}([0,1]) \subset \mathcal{S}$.*

Proof. Assume by contradiction that $(\bar{X}(\cdot), \bar{u}(\cdot))$ is an abnormal extremal of $\mathcal{E}^*(\Delta)$ and that $\bar{\alpha}(t_0) \notin \mathcal{S}$ for some time $t_0 \in [0,1)$. Let $(p(\cdot), \mu(\cdot), 0)$ be a multiplier associated with this extremal, $q(\cdot)$ be the function defined by (6), and set $p = (p_1(\cdot), p_2(\cdot), p_x(\cdot))$ and $q(\cdot) = (q_1(\cdot), q_2(\cdot), q_x(\cdot))$. Since $\tilde{h}_j(X) = h_j(\alpha)$, the third component of $\nabla\tilde{h}_j(X)$ is always zero, hence $q_x(\cdot) = p_x(\cdot)$. Moreover, the Hamiltonian being independent of x, by the third component of (8) there holds $\dot{p}_x(t) = 0$ for a.e. t, thus $q_x(\cdot)$ is a constant.

Now, (9) implies $\langle q(t), F_i(\bar{X}(t))\rangle = 0$ for a.e. t, and $i = 1, 2$, that is

$$q_1(t) + q_x f_1(\bar{\alpha}(t)) = q_2(t) + q_x f_2(\bar{\alpha}(t)) = 0 \quad \text{for a.e. } t \in [0,1]. \tag{11}$$

And the first two components of (8) can be written as

$$\begin{cases} \dot{p}_1 = -q_x \left(\frac{d}{dt}(f_1(\bar{\alpha})) - \bar{u}_2 g(\bar{\alpha})\right), \\ \dot{p}_2 = -q_x \left(\frac{d}{dt}(f_2(\bar{\alpha})) + \bar{u}_1 g(\bar{\alpha})\right). \end{cases} \tag{12}$$

By assumption $\bar{\alpha}(t_0) \not\subset \partial\mathcal{K}$, hence $h_j(\bar{\alpha}(t)) < 0$ for any $t \in [t_0, t_0+\varepsilon]$ and any j for some $\varepsilon > 0$. This implies $q(\cdot) = p(\cdot) + c$ where c is a constant, and then $\dot{q}(t) = \dot{p}(t)$, for $t \in [t_0, t_0+\varepsilon]$. Taking the derivative of (11) w.r.t. the time on $[t_0, t_0+\varepsilon]$ and comparing with (12) we obtain

$$\bar{u}_1(t) q_x g(\bar{\alpha}(t)) = \bar{u}_2(t) q_x g(\bar{\alpha}(t)) = 0 \qquad \text{for a.e. } t \in [t_0, t_0+\varepsilon].$$

The vector $\bar{u}(t)$ is a.e. nonzero since any minimizer satisfies $Q_{\bar{\alpha}(t)}(\bar{u}(t)) =$ *constant* $\neq 0$ a.e., and $g(\bar{\alpha}(t)) \neq 0$ on $[t_0, t_0+\varepsilon]$ since $\bar{\alpha}([t_0, t_0+\varepsilon]) \not\subset \mathcal{S}$. Hence q_x is null. This implies that $\dot{p}$ vanishes by (12). Using (11) we get $p(\cdot) = q(\cdot) = 0$, and so $\mu = 0$. As a consequence the multiplier is $(0,0,0)$, which contradicts (7). This ends the proof. $\square$

Lemma 16. *The minimization problem $\mathcal{E}^*_\delta$ admits no abnormal extremal, i.e. any minimizer of $\mathcal{E}^*_\delta$ must be a normal extremal.*

Proof. Let $(\bar{X}(\cdot), \bar{u}(\cdot))$ be an abnormal extremal of $\mathcal{E}^*_\delta$ and $(p, \mu, 0)$ an associated multiplier. Arguing as in the proof of Lemma 15, we obtain that $q_x(\cdot) = p_x(\cdot)$ is a constant, and that equations (11) and (12) hold.

Now since $\lambda = 0$, the transversality condition (10) implies $q_x(\cdot) = p_x(\cdot) = 0$. We then obtain $\dot{p}_1(\cdot) = \dot{p}_2(\cdot) = 0$ by (12), and $q_1(\cdot) = q_2(\cdot) = 0$ by (11). We deduce that $p(\cdot) = q(\cdot) = 0$, and so $\mu = 0$. As a consequence the multiplier is $(0,0,0)$, which is impossible. This ends the proof. $\square$

Let us finish by showing that all normal extremals are regular.

Lemma 17. *If* $(\bar{X}(\cdot), \bar{u}(\cdot))$ *is a normal extremal of* $\mathcal{E}^*(\Delta)$ *or* $\mathcal{E}^*_\delta$*, then* $\bar{X}(\cdot)$ *is* $C^1([0,1])$ *and* $\bar{u}(\cdot)$ *is Lipschitz continuous on* $[0,1]$*. Moreover,* $\dot{\bar{\alpha}}(1) = \dot{\bar{\alpha}}(0)$*.*

Proof. The first statement of the lemma is a direct consequence of [7, Theorem 3.1], hence it suffices to show that hypothesis (H4) of that paper holds along $(\bar{X}(\cdot), \bar{u}(\cdot))$. In our setting, this hypothesis writes as follows.

(H4) For $Y \in \mathbb{R}^3$, we set $\mathcal{J}(Y) = \{j : \tilde{h}_j(Y) = 0\}$. For every $t \in [0,1]$ and every set of non-negative numbers $\{\beta_j\}_{j \in \mathcal{J}(\bar{X}(t))}$, not all zero, we have

$$\sum_{j \in \mathcal{J}(\bar{X}(t))} \beta_j \begin{pmatrix} F_1(\bar{X}(t)) & F_2(\bar{X}(t)) \end{pmatrix}^T \nabla \tilde{h}_j(\bar{X}(t)) \neq 0.$$

Due to the form of F_1 and F_2 and of $\tilde{h}(X(t)) = h(\alpha(t))$ for all $t \in [0,1]$, the above condition is equivalent to

$$\sum_{j \in \mathcal{J}(\bar{X}(t))} \beta_j \nabla h_j(\bar{\alpha}(t)) \neq 0,$$

which is ensured by Definition 12. Thus [7, Theorem 3.1] applies, $\bar{u}(\cdot)$ is Lipschitz continuous and then $\bar{X}(\cdot)$ is C^1 on $[0,1]$.

The second statement of the lemma is a direct consequence of the fact that any time translation of a normal extremal is still a normal extremal. Indeed, let $\bar{X}(\cdot)$ be a normal extremal of $\mathcal{E}^*(\Delta)$ (resp. $\mathcal{E}^*_\delta$). Fix any $t_0 \in (0,1)$ and define the trajectory $\widetilde{X}(\cdot)$ by

$$(\widetilde{X}, \widetilde{u})(s) = \begin{cases} (\bar{X}, \bar{u})(t_0 + s) & \text{if } s \in [0, 1 - t_0], \\ (\bar{X}, \bar{u})(s - 1 + t_0) & \text{if } s \in (1 - t_0, 1]. \end{cases}$$

Obviously $(\widetilde{X}(\cdot), \widetilde{u}(\cdot))$ is admissible for the minimization problem $\mathcal{E}^*(\Delta)$ (resp. $\mathcal{E}^*_\delta$) and, if $(p(\cdot), \mu(\cdot), 1)$ is a multiplier associated with $(\bar{X}(\cdot), \bar{u}(\cdot))$, then $(\widetilde{X}(\cdot), \widetilde{u}(\cdot))$ admits as a multiplier $(\widetilde{p}(\cdot), \widetilde{\mu}(\cdot), 1)$, where

$$\widetilde{\mu}(s) = \begin{cases} \mu(t_0 + s) & \text{if } s \in [0, 1 - t_0], \\ \mu(s - 1 + t_0) & \text{if } s \in (1 - t_0, 1], \end{cases}$$

$$\widetilde{p}(s) = \begin{cases} p(t_0 + s) & \text{if } s \in [0, 1 - t_0], \\ p(s - 1 + t_0) - \sum_{j=1}^r \int_{[0,1]} \nabla \tilde{h}_j(\bar{X}(s)) \mu_j(ds) & \text{if } s \in (1 - t_0, 1]. \end{cases}$$

As a consequence, $(\widetilde{X}(\cdot), \widetilde{u}(\cdot))$ is a normal extremal and thus is of class C^1. We conclude by noticing that $\dot{\bar{\alpha}}(1) = \dot{\tilde{\alpha}}(1 - t_0)$ and $\dot{\bar{\alpha}}(0) = \lim_{s \to (1-t_0)^+} \dot{\tilde{\alpha}}(s)$ are equal since $1 - t_0 \in (0,1)$. $\square$

Corollary 18.

(i) For any $\Delta > 0$*, any minimizer of* $\mathcal{E}^*(\Delta)$ *is a piecewise* C^1 *curve.*

(ii) If $(\bar{X}(\cdot), \bar{u}(\cdot))$ *is a minimizer of* $\mathcal{E}^*_\delta$*, then* $\bar{X}(\cdot)$ *is* $C^1([0,1])$ *and* $\bar{u}(\cdot)$ *is Lipschitz continuous on* $[0,1]$*. Moreover,* $\dot{\bar{\alpha}}(1) = \dot{\bar{\alpha}}(0)$*.*

Proof. The second point is a direct consequence of Lemma 16 and Lemma 17. As for the first point, we have the following alternative for a minimizer $\bar{X}(\cdot)$ of a $\mathcal{E}^*(\Delta)$: either it is a normal extremal, and then it is C^1 by Lemma 17, or it is an abnormal extremal, and then $\bar{\alpha}(t) \in \mathcal{S}$ for every $t \in [0,1]$ by Lemma 15. In the latter case, using that $\mathcal{S}$ is a finite union of points and 1-dimensional submanifolds of $\mathbb{R}^2$ and that $Q_{\bar{\alpha}(t)}(\dot{\bar{\alpha}}(t)) = \textit{constant}$ for a.e. t, we obtain that $\bar{\alpha}(\cdot)$, and so $\bar{X}(\cdot)$, is piecewise C^1. □

5 Proof of Theorem 8

This section is devoted to prove Theorem 8 in which the main difficulty is to establish Point *(i)*, i.e. the existence of mean-time optimal strokes.

5.1 Existence of mean-time optimal strokes

The proof of Point *(i)* is based on the fact that $\frac{T^*(\Delta)}{\Delta}$ reaches its minimum for Δ in a certain compact set $[m, M]$. It requires several intermediate results: Lemma 19 studies the behavior of $T^*(\Delta)/\Delta$ as $\Delta \to 0$; then, Lemma 20 and Lemma 21 stand for the behavior of $T^*(\Delta)/\Delta$ as $\Delta \to \infty$.

Lemma 19.

$$\lim_{\Delta \to 0} \frac{T^*(\Delta)}{\Delta} = +\infty.$$

Proof. Let $(\Delta^n)_{n\in\mathbb{N}}$, $\Delta^n \to 0$, be a minimizing sequence of the inferior limit of $T^*(\Delta)/\Delta$ at 0, i.e. $\lim_{n\to\infty} \frac{T^*(\Delta^n)}{\Delta^n} = \liminf_{\Delta\to 0} \frac{T^*(\Delta)}{\Delta}$. For each $n \in \mathbb{N}$, let $X^n(\cdot) = (\alpha^n(\cdot), x^n(\cdot))$ be a minimizer of $T^*(\Delta^n)$. We can assume that $x^n(0) = 0$. In particular, for each $n \in \mathbb{N}$, $X^n(\cdot)$ has to minimize the time among all trajectories of (2) in $\mathcal{K} \times \mathbb{R}$ with bounded velocity, $Q_{\alpha(t)}(u(t)) \leq 1$ joining $X^n(0) = (\alpha^n(0), 0)$ to $X^n(T^*(\Delta^n)) = (\alpha^n(0), \Delta^n)$.

Let us introduce the sub-Riemannian distance d on $\mathbb{R}^3$ induced by the control system (2) and the Riemannian metric Q: for any pair of points $X^0, X^1 \in \mathbb{R}^3$, $d(X^0, X^1)$ is defined as the infimum of the length $\ell(X(\cdot))$ among all trajectories $X(\cdot)$ of (2) in $\mathbb{R}^3$ joining X^0 to X^1. Noticing on the one hand that minimization of length and minimization of time with bounded velocity coincide (Lemma 4); and on the other hand that d is defined without taking into account the state constraint $\alpha \in \mathcal{K}$, we get, for every $n \in \mathbb{N}$,

$$T^*(\Delta^n) \geq d\left((\alpha^n(0), 0), (\alpha^n(0), \Delta^n)\right).$$

Let us estimate the above sub-Riemannian distance. We will use [11, Th. 2.4] (see also [10, Th. 2]). For a multi-index $I = (i_1, \dots, i_s)$, $i_j = 1$ or 2, $s \in \mathbb{N}$, we set $|I| = s$ and

$$g_I = \frac{\partial^s g}{\partial \alpha_{i_1} \cdots \partial \alpha_{i_s}}.$$

Up to extracting a subsequence, we assume that $\alpha^n(0)$ converges to a point $\alpha^* \in \mathcal{K}$ and we denote by k the smallest integer for which there exists a multi-index I, $|I| = k$, such that $g_I(\alpha^*) \neq 0$. Then, from [11, Th. 2.4] there exists a constant $c' > 0$ such that, for n large enough, there holds

$$|\Delta^n| \leq c' \max_{|I|\leq k} |g_I(\alpha^n(0))| \varepsilon_n^{|I|+2} \quad \text{where } \varepsilon_n = d\left((\alpha^n(0), 0), (\alpha^n(0), \Delta^n)\right).$$

Thus $|\Delta^n| \leq c\varepsilon_n^2$ for some constant $c > 0$. Summarizing, for n large enough we have

$$\frac{T^*(\Delta^n)}{\Delta^n} \geq \frac{d\left((\alpha^n(0), 0), (\alpha^n(0), \Delta^n)\right)}{\Delta^n} \geq \frac{1}{\sqrt{c\Delta^n}},$$

and then

$$\liminf_{\Delta \to 0} \frac{T^*(\Delta)}{\Delta} = \lim_{n\to\infty} \frac{T^*(\Delta^n)}{\Delta^n} = +\infty,$$

which ends the proof. □

The next step focuses on building minimizer of problem (14) which component α is a closed simple curve, i.e., $\alpha : [0, T] \to \mathbb{R}^2$ satisfies $\alpha(t) \neq \alpha(t')$ for any $t \neq t'$ except 0 and T. We will then prove in the final step that such minimizers produce bounded x-displacements.

Lemma 20. *Fix $T > 0$. Then the minimization problem*

$$\inf \left\{ \frac{T^*(\Delta)}{\Delta} \;:\; \Delta > 0 \quad s.t. \quad T^*(\Delta) \leq T \right\} \tag{13}$$

has a solution Δ_T such that $T^*(\Delta_T)$ admits a minimizer $\bar{X}(\cdot) = (\bar{\alpha}(\cdot), \bar{x}(\cdot))$ where $\bar{\alpha}(\cdot)$ is a closed, simple and piecewise C^1 curve.

Proof. Note first that since $T^*(\Delta) = \sqrt{\mathcal{E}^*(\Delta)}$ by Lemma 4, $\Delta \mapsto T^*(\Delta)$ is a lower semi-continuous function from Lemma 13 . As a consequence $\{\Delta \in [0, \infty) : T^*(\Delta) \leq T\}$ is a closed subset of $[0, \infty)$, and actually a compact one since it is bounded by (5). Using Lemma 19 and once more the lower semi-continuity of T^*, we obtain that the problem (13) admits a minimum.

Let Δ_T be a minimum of (13) and consider a minimizer $\bar{X}(\cdot) = (\bar{\alpha}(\cdot), \bar{x}(\cdot))$ of $T^*(\Delta_T)$. The curve $\bar{\alpha}(\cdot)$ is closed by definition of $T^*(\Delta_T)$ and it is piecewise C^1 because it is a constant time-reparameterization of a minimizer of $\mathcal{E}^*(\Delta)$ and Corollary 18 applies. Thus, if $\bar{\alpha}(\cdot)$ is moreover simple, the lemma is proved. Assume now that $\bar{\alpha}(\cdot)$ is not simple. Up to a translation of time, we can assume that $\bar{\alpha}(\cdot)$ is C^1 at $t = 0$, hence there exists $\tau \in (0, T^*(\Delta_T))$ such that $\bar{\alpha}(\cdot)$ is simple on $[0, \tau]$ and

$$\bar{\alpha}(\tau) = \bar{\alpha}(0).$$

Define the trajectory $X^1(\cdot)$ as $\bar{X}(\cdot)$ restricted to $[0, \tau]$ and the trajectory $X^2(\cdot)$ as $\bar{X}(\cdot)$ restricted to $[\tau, T^*(\Delta_T)]$. Let $\ell(X^1(\cdot)) = \tau$ be the length of $X^1(\cdot)$, $\ell(X^2(\cdot)) = T^*(\Delta_T) - \tau$ be the length of $X^2(\cdot)$, $\Delta^1 = \bar{x}(\tau) - \bar{x}(0)$ the x-displacement along $X^1(\cdot)$, and $\Delta^2 = \bar{x}(T^*(\Delta_T)) - \bar{x}(\tau)$ the x-displacement along $X^2(\cdot)$. By construction both components $\alpha^1(\cdot), \alpha^2(\cdot)$ of respectively $X^1(\cdot), X^2(\cdot)$ are closed curves and $\alpha^1(\cdot)$ is simple (see Fig 1). Moreover the time optimality of $\bar{X}(\cdot)$ implies that, for $i = 1, 2$, $X^i(\cdot)$ is a minimizer of $T^*(\Delta^i)$, i.e. $\ell(X^i(\cdot)) = T^*(\Delta^i)$ see Lemma 4. Let us quote that we do not suppose that Δ^i is positive but $T^*(\Delta)$ is defined for any Δ. And finally there holds

$$T^*(\Delta_T) = T^*(\Delta^1) + T^*(\Delta^2) \quad \text{and} \quad \Delta_T = \Delta^1 + \Delta^2.$$

Note first that both Δ^1 and Δ^2 must be positive. Indeed, if one of them is non positive, then the other one, Δ^i, is positive and satisfies

$$\Delta^i > \Delta_T \quad \text{and} \quad T^*(\Delta^i) < T^*(\Delta_T) \leq T, \quad \text{thus} \quad \frac{T^*(\Delta^i)}{\Delta^i} < \frac{T^*(\Delta_T)}{\Delta_T},$$

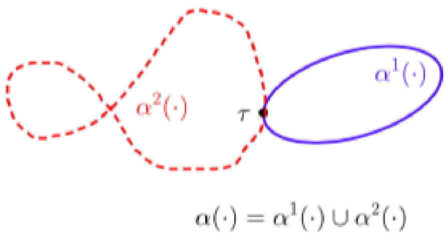

Figure 1: The red dotted curve (resp. the blue one) represents $\alpha^2(\cdot)$ (resp. $\alpha^1(\cdot)$). The union of the two curves is $\bar{\alpha}(\cdot)$.

which contradicts the fact that Δ_T is a solution of (13).

Thus, the decomposition

$$\frac{T^*(\Delta_T)}{\Delta_T} = \frac{\Delta^1}{\Delta_T}\frac{T^*(\Delta^1)}{\Delta^1} + \frac{\Delta^2}{\Delta_T}\frac{T^*(\Delta^2)}{\Delta^2},$$

is a convex combination, which implies that

$$\frac{T^*(\Delta_T)}{\Delta_T} \in \left[\frac{T^*(\Delta^1)}{\Delta^1}, \frac{T^*(\Delta^2)}{\Delta^2}\right].$$

Since both $T^*(\Delta^1)$ and $T^*(\Delta^2)$ are smaller than T, the fact that Δ_T is a solution of (13) implies that neither $\frac{T^*(\Delta^1)}{\Delta^1}$ nor $\frac{T^*(\Delta^2)}{\Delta^2}$ can be smaller than $\frac{T^*(\Delta_T)}{\Delta_T}$, and then

$$\frac{T^*(\Delta_T)}{\Delta_T} = \frac{T^*(\Delta^1)}{\Delta^1} = \frac{T^*(\Delta^2)}{\Delta^2}.$$

Hence Δ^1 is a minimum of (13) and, since $T^*(\Delta^1)$ admits as a minimizer $X^1(\cdot)$ whose component $\alpha^1(\cdot)$ is a closed, simple and piecewise C^1 curve, the lemma is proved. □

Lemma 21. *There exists a constant $M > 0$ (depending on $\mathcal{K}$) such that, for any $T > 0$ and any trajectory $X(\cdot)$ of (2) on $[0, T]$ whose component $\alpha(\cdot)$ is a closed, simple and piecewise C^1 curve, we have*

$$|x(T) - x(0)| \leq M.$$

Proof. From the definition (1) of the dynamics we have

$$x(T) - x(0) = \int_0^T \sum_{i=1}^2 f_i(\alpha(t))\dot{\alpha}_i(t)dt = \int_\alpha \sum_{i=1}^2 f_i(\alpha)d\alpha_i.$$

Applying Green's theorem to the piecewise curve $\alpha(\cdot)$, we get

$$\int_\alpha \sum_{i=1}^2 f_i(\alpha)d\alpha_i = -\int_\Omega g(\alpha)d\alpha_1 \wedge d\alpha_2,$$

where Ω is the domain enclosed by the curve $\alpha(\cdot)$. This domain Ω is contained in the convex hull $\mathrm{Conv}(\mathcal{K})$ of $\mathcal{K}$, which is itself a compact subset of $\mathbb{R}^2$. Thus we obtain

$$|x(T) - x(0)| \leq \underbrace{\int_{\mathrm{Conv}(\mathcal{K})} |g(\alpha)| d\alpha_1 \wedge d\alpha_2}_{:=M},$$

which concludes the proof. $\square$

We are now in a position to prove Theorem 8.

Proof of Theorem 8-(i). Let us write the infimum (14) as

$$\inf_{\Delta>0} \frac{T^*(\Delta)}{\Delta} = \lim_{T\to\infty} \inf \left\{ \frac{T^*(\Delta)}{\Delta} \ : \ \Delta > 0 \quad \text{s.t.} \quad T^*(\Delta) \leq T \right\}.$$

Lemmas 20 and 21 imply that the right-hand term is equal to the infimum of $\frac{T^*(\Delta)}{\Delta}$ on $(0, M]$. Using Lemma 19 and the lower semi-continuity of T^* we get the existence of mean-time optimal strokes, which can moreover be chosen as simple curves by Lemma 20. $\square$

5.2 End of proof of Theorem 8

Let us first prove that mean-time optimal strokes extends to periodic C^1 curves.

Proof of Theorem 8-(ii). Let $\bar{\alpha}(t)$, $t \in [0, T^*(\Delta^*)]$ be a mean-time optimal stroke and $\bar{X}(\cdot)$ the corresponding trajectory of (3). Set $\widetilde{X}(s) = \bar{X}(sT^*(\Delta^*))$, $s \in [0, 1]$. Using Lemma 4, we obtain that $\widetilde{X}(\cdot)$ is a minimizer of

$$\inf_{\Delta>0} \frac{\sqrt{\mathcal{E}^*(\Delta)}}{\Delta} = \frac{T^*(\Delta^*)}{\Delta^*}. \tag{14}$$

Setting $\delta = \frac{T^*(\Delta^*)}{\Delta^*}$, we obtain that $\mathcal{E}_{\delta^2}(X(\cdot)) = \mathcal{E}(X(\cdot)) - \delta^2(x(1) - x(0))^2$ is always nonnegative for solutions $X(t)$, $t \in [0, 1]$, of (2) such that $\alpha(1) = \alpha(0)$. Since $\mathcal{E}_{\delta^2}(\widetilde{X}) = 0$, $\widetilde{X}(\cdot)$ realizes the minimum of $\mathcal{E}_{\delta^2}$ among such $X(\cdot)$, i.e. it is a minimizer of

$$\mathcal{E}^*_{\delta^2} = \inf \left\{ \mathcal{E}_{\delta^2}(X) \ : \ X(t), \ t \in [0, 1], \ \text{solution of (2) s.t.} \ \alpha(1) = \alpha(0) \right\}.$$

The conclusion follows from Corollary 18. $\square$

Finally, Point *(iii)* is a consequence of a relevant bound using Riemannian distance in $\mathbb{R}^2$.

Proof of Theorem 8-(iii). Since $T^\sharp(\Delta) \leq T^*(\Delta)$ for any $\Delta > 0$, it is sufficient to prove that

$$\liminf_{\Delta\to\infty} \frac{T^\sharp(\Delta)}{\Delta} \geq \liminf_{\Delta\to\infty} \frac{T^*(\Delta)}{\Delta} = \frac{T^*(\Delta^*)}{\Delta^*}. \tag{15}$$

Let $(\Delta^n)_{n\in\mathbb{N}}$, $\Delta^n \to \infty$, be a minimizing sequence of the left-hand side, i.e.,

$$\lim_{n\to\infty} \frac{T^\sharp(\Delta^n)}{\Delta^n} = \liminf_{\Delta\to\infty} \frac{T^\sharp(\Delta)}{\Delta}.$$

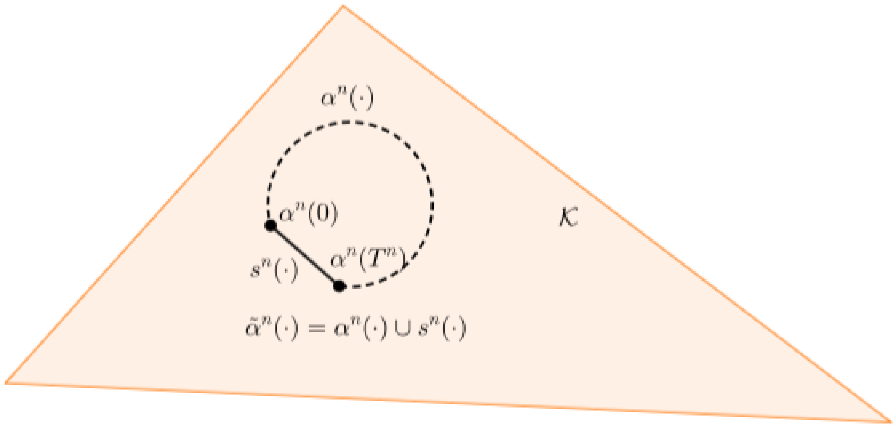

Figure 2: The construction of the curve $\tilde{\alpha}^n(\cdot)$ in function of $\alpha^n(\cdot)$ (represented in dotted line).

For each $n \in \mathbb{N}$, let $X^n(\cdot) = (\alpha^n(\cdot), x^n(\cdot))$ be a minimizer of $T^\sharp(\Delta^n)$. Thus $X^n(\cdot)$ is defined on $[0, T^n]$, where $T^n = T^\sharp(\Delta^n)$, and $x^n(T^n) - x^n(0) = \Delta^n$. The curve $\alpha^n(\cdot)$ is not closed but we have the estimate

$$d_Q(\alpha^n(T^n), \alpha^n(0)) \leq \operatorname{diam}(\mathcal{K}),$$

where d_Q is the Riemannian distance on $\mathcal{K}$ defined by the Riemannian metric Q and $\operatorname{diam}(\mathcal{K})$ is the diameter of $\mathcal{K}$ with respect to this distance. Hence there is an absolutely continuous curve $s^n : [T^n, \tilde{T}^n] \to \mathcal{K}$ such that:

- $s^n(\cdot)$ joins $\alpha^n(T^n)$ to $\alpha^n(0)$, i.e., $s^n(T^n) = \alpha^n(T^n)$, $s^n(\tilde{T}^n) = \alpha^n(0)$,
- $\tilde{T}^n - T^n \leq \operatorname{diam}(\mathcal{K})$,
- $s^n(\cdot)$ is arclength parameterized, i.e., $Q_{s(t)}(\dot{s}(t)) \leq 1$ a.e.

Extend the curve $\alpha^n(\cdot)$ to a curve $\tilde{\alpha}^n(\cdot)$ on $[0, \tilde{T}^n]$ by setting $\tilde{\alpha}^n(t) = \alpha^n(t)$ for $t \in [0, T^n]$ and $\tilde{\alpha}^n(t) = s^n(t)$ for $t \in [T^n, \tilde{T}^n]$ (see Figure 2). This curve is closed and defines a trajectory $\tilde{X}^n(\cdot) = (\tilde{\alpha}^n(\cdot), \tilde{x}^n(\cdot))$ on $[0, \tilde{T}^n]$, with

$$\tilde{x}^n(\tilde{T}^n) - \tilde{x}^n(0) = \Delta^n + \tilde{\Delta}^n \quad \text{and} \quad |\tilde{\Delta}^n| \leq C\operatorname{diam}(\mathcal{K}),$$

thanks to (5).

As a consequence,

$$\frac{\tilde{T}^n}{\Delta^n + \tilde{\Delta}^n} \geq \frac{T^*(\Delta^n + \tilde{\Delta}^n)}{\Delta^n + \tilde{\Delta}^n} \geq \frac{T^*(\Delta^*)}{\Delta^*}.$$

On the other hand, we have

$$\frac{\tilde{T}^n}{\Delta^n + \tilde{\Delta}^n} \leq \frac{T^n + \operatorname{diam}(\mathcal{K})}{\Delta^n - C\operatorname{diam}(\mathcal{K})} \xrightarrow[n\to\infty]{} \liminf_{\Delta\to\infty} \frac{T^\sharp(\Delta)}{\Delta},$$

which implies (15) and concludes the proof. □

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