deg Q Line Bundle

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Abstract

Line bundles of degree $\mathbb{Q}$ and $\mathbb{Z}[1/d]$ are constructed and their cohomology computed.

Contents

1 Introduction

In algebraic topology the degree of the form $\mathbb{Z}[1/d]$ and $\mathbb{Q}$ can be constructed via a direct limit as shown in online version of [Hatcher, 2002, pp 311, 3F.1], this degree is transferred to algebraic geometry in this paper. The section 2 constructs polynomial rings with degree

$\mathbb{Z}[1/d]$ and $\mathbb{Q}$ and the rings are denoted as $R[X_0^{1/d^\infty}, \dots, X_n^{1/d^\infty}]$ and $\mathfrak{R}$ respectively. The monomials of these rings can be ordered by observing the countability of rationals. These rings are Non Notherian, fortunately they are coherent and this is shown in Proposition 2.1.

The grading is put on these ring in the section 3 and the corresponding Proj and localization of these rings constructed. Finally, in section 4 the line bundle $\mathcal{O}(n)$ for $n \in \mathbb{Z}[1/p]$ or $\mathbb{Q}$ is constructed. The corresponding cohomology for various n is computed in Theorems 4.3, 4.4, 4.5, which is similar to standard cohomology except that it is infinite dimensional. The Kunneth formula is introduced in section 4.3.

Finally, the above results are transferred to the perfectoid case in the section 5.

2 Construction of Rational Degrees

The first task at hand is to construct polynomial rings with rational degrees.

2.1 Degree $\mathbb{Z}[1/d]$

Consider the following inclusions with $d \in \mathbb{Z}_{>0}$ not necessarily prime,

$$R[X_0, \dots, X_n] \subset R[X_0^{1/d}, \dots, X_n^{1/d}] \subset \dots \subset R[X_0^{1/d^i}, \dots, X_n^{1/d^i}] \subset \dots, \tag{2.1}$$

the direct limit would be

$$R[X_0^{1/d^\infty}, \dots, X_n^{1/d^\infty}] := \varinjlim_i R[X_0^{1/d^i}, \dots, X_n^{1/d^i}] = \bigcup_i R[X_0^{1/d^i}, \dots, X_n^{1/d^i}] \tag{2.2}$$

and the elements would be polynomials with degree in $\mathbb{Z}[1/d]$. For example, for $d = 5$ an element of the ring could be $X^{2/5} + X^2 + X^3 + X^{36/5} + 1$.

2.2 Degree $\mathbb{Q}$

Start by constructing the following ring R_i which can be ordered by inclusion.

$$\begin{aligned}
R_1 &= R[X_0, \dots, X_n] \\
R_2 &= R[X_0, \dots, X_n, X_1^{1/2}, \dots, X_n^{1/2}] \\
R_3 &= R[X_0, \dots, X_n, X_1^{1/2}, \dots, X_n^{1/2}, X_1^{1/3}, \dots, X_n^{1/3}] \\
\vdots &= \qquad \vdots \\
R_i &= R[X_0, \dots, X_n, X_1^{1/2}, \dots, X_n^{1/2}, \dots, X_1^{1/i}, \dots, X_n^{1/i}] \\
\vdots &= \qquad \vdots \\
&\text{Order via inclusions} \\
&R_1 \subset R_2 \subset \dots \subset R_i \subset \dots \qquad \varinjlim_i R_i = \bigcup_i R_i
\end{aligned} \tag{2.3}$$

The ring $\mathfrak{R} := \varinjlim_i R_i$ contains all polynomials of rational degree, since all denominators of the form $1/n, n \in \mathbb{Z}_{>0}$ can be constructed.

2.3 Open and Closed Sets

As in the standard algebraic geometry closed and open sets can be defined for the ring $A = \mathfrak{R}$ or $R[X_0^{1/d^\infty}, \dots, X_n^{1/d^\infty}]$ as

$$
\begin{aligned}
V(I) &:= \{\mathfrak{p} \in \operatorname{Spec} A \text{ such that } I \subseteq \mathfrak{p}\} \\
D(f) &:= \operatorname{Spec} A \backslash V(fA).
\end{aligned} \tag{2.4}
$$

The structure sheaf can be defined similary on an open set $\mathcal{O}_X(D(f)) = A_f$.

2.4 Coherence

Proposition 2.1. *Let* R *be a notherian ring then* $\mathfrak{R}$ *and* $R[X_0^{1/d^\infty}, \dots, X_n^{1/d^\infty}]$ *are coherent.*

Proof.
1. Let $\mathfrak{I} = (f_1, \dots, f_n)$ be a finitely generated ideal in the ring $R[X_0^{1/d^\infty}, \dots, X_n^{1/d^\infty}]$, then $\mathfrak{I}$ lies in $B_i := R[X_0^{1/d^i}, \dots, X_n^{1/d^i}]$ for some large i, which is a notherian ring. Hence there is a presentation

   $$B_i^m \to B_i^n \to \mathfrak{I} \to 0 \tag{2.5}$$

   Since $\varinjlim_i B_i = R[X_0^{1/d^\infty}, \dots, X_n^{1/d^\infty}]$ applying the direct limit gives presentation for $\mathfrak{I}$.

   $$(\varinjlim_i B_i)^m \to (\varinjlim_i B_i)^n \to \mathfrak{I} \to 0 \tag{2.6}$$

2. Let $\mathfrak{J} = (f_1, \dots, f_n)$ be a finitely generated ideal in the ring $\mathfrak{R}$, then $\mathfrak{J}$ lies in R_i (as in section 2.2) for some large i, which is a notherian ring. Hence there is a presentation

   $$R_i^m \to R_i^n \to \mathfrak{J} \to 0 \tag{2.7}$$

   Since $\varinjlim_i R_i = \mathfrak{R}$ applying the direct limit gives presentation for $\mathfrak{J}$.

   $$(\varinjlim_i R_i)^m \to (\varinjlim_i R_i)^n \to \mathfrak{J} \to 0 \tag{2.8}$$

□

3 Graded Modules

Recall that a ring or a module can be graded over a commutative monoid Δ as in [Bourbaki, 1998, pp 363, Chapter II, §1]. A ring A can be endowed with decomposition $A = \oplus_{d \geqslant 0} A_d$ of abelian groups such that $A_d A_e \subseteq A_{d+e}$ for all $d, e \geqslant 0$ where $d, e \in \Delta$. Similarly graded A modules can be defined with $A_d M_e \subseteq M_{d+e}$ for all $d, e \geqslant 0$ and $d, e \in \Delta$. In this tract Δ could be $\mathbb{Q}$ or $\mathbb{Z}[1/d]$. A homogeneous ideal is of the form $I = \oplus_{d \geqslant 0}(I \cap A_d)$ and the quotient A/I has a natural grading $(A/I)_d = A_d/(I \cap A_d)$. Let Proj A denote the set of prime ideals of A that do not contain $A_+ := \oplus_{d>0} A_d$, then Proj A can be endowed with the structure of a scheme. The closed and open sets for homogeneous ideals I are of the form

$$\begin{aligned} V_+(I) &:= \{\mathfrak{p} \in \text{Proj } A \text{ such that } I \subseteq \mathfrak{p}\} \\ D_+(f) &:= \text{Proj } A \backslash V(fA). \end{aligned} \tag{3.1}$$

3.1 Localization

The localization M_f has a Δ grading where homogeneous elements of degree $d \in \Delta$ are of the form m/f^n where $m \in M, f \in A$ are homogeneous and $d = \deg m - n \deg f$. The degree zero elements are denoted as $M_{(f)} \subset M_f$ and $A_{(f)} \subset A_f$, furthermore, $M_{(f)}$ is an $A_{(f)}$ module.

Example 3.1. The polynomial ring $A := R[X_0^{1/d^\infty}, \dots, X_n^{1/d^\infty}]$ is a graded R algebra and A_d consists of polynomials of degree $d \in \mathbb{Z}[1/d]$. The localization at the affine plane is given as

$$R[X_0^{1/d^\infty}, \dots, X_n^{1/d^\infty}]_{(X_i)} = R\left[\left(\frac{X_0}{X_i}\right)^{1/d^\infty}, \dots, \left(\frac{X_n}{X_i}\right)^{1/d^\infty}\right]. \tag{3.2}$$

Similarly, the ring $\mathfrak{R}$ can be localized at affine plane $X_i = 0$ to get degree zero elements.

4 Twisting Sheaves $\mathcal{O}(n)$

Let $n \in \Delta$ (where Δ is $\mathbb{Z}[1/d]$ or $\mathbb{Q}$) and a Δ graded A module, define a new graded A module $A(n)_d := A_{n+d}$ for all $d \in \Delta$, define $\mathcal{O}_X(n) := \widetilde{A(n)}$. The $\widetilde{\ }$ denotes the localization on the affine open. Thus, on $D_+(f)$ an affine open subset $\mathcal{O}_X(n)|_{D_+(f)} = f^n \mathcal{O}_X|_{D_+(f)}$, furthermore, the usual equality holds $\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) = \mathcal{O}_X(n+m)$.

For the case at hand the set $f^n, n \in \Delta$ should make sense. For example, $X_i^n, n \in \Delta$ will always hold in Proj $\mathfrak{R}$. For R a perfect ring of char p and $\Delta = \mathbb{Z}[1/p]$ it is always possible to take pth power roots and thus $f^n, n \in \mathbb{Z}[1/p]$ always makes sense for $R[X_0^{1/p^\infty}, \dots, X_n^{1/p^\infty}]$.

4.1 $\mathcal{O}(1)$

Let k be a field and consider Proj $k[X_0^{1/d^\infty}, X_1^{1/d^\infty}]$, the affine open sets are $U_0 = D(X_0) = \text{Spec } k[(X_1/X_0)^{1/d^\infty}]$ and $U_1 = D(X_1) = \text{Spec } k[(X_0/X_1)^{1/d^\infty}]$.

For example, for $d = 5$, consider the function

$$\begin{array}{rl} \text{Global Section} & X_0 + X_0^{1/5} X_1^{4/5} + X_1 \\ U_1 & (X_0/X_1) + (X_0/X_1)^{1/5} + 1 \\ U_0 & 1 + (X_1/X_0)^{4/5} + (X_1/X_0) \end{array} \tag{4.1}$$

The transition function from U_1 to U_0 is given as X_1/X_0. The above can be adapted for rational numbers $r \in \mathbb{Q} \cap (0,1)$

$$\begin{array}{rl} \text{Global Section} & X_0 + X_0^r X_1^{1-r} + X_1 \\ U_1 & (X_0/X_1) + (X_0/X_1)^r + 1 \\ U_0 & 1 + (X_1/X_0)^{1-r} + (X_1/X_0) \end{array} \tag{4.2}$$

One immediately sees that the global sections of $\mathcal{O}(1)$ are infinitely generated by monomials of the form $X_0, X_1, X_0^r X_1^{(1-r)}$ where $r \in \mathbb{Z}[1/d] \cap (0,1)$ or $r \in \mathbb{Q} \cap (0,1)$.

The following lemma is adapted from [Liu, 2002, pp 166,Lemma 1.22].

Lemma 4.1. *Let $A = R[X_0^{1/d^\infty}, \dots, X_n^{1/d^\infty}]$ or $A = \mathfrak{R}$, then $\mathcal{O}(n) = A_n$ if $n \geqslant 0$ and $\mathcal{O}(n) = 0$ if $n < 0$. In other words $\oplus_{n\in\Delta}\mathcal{O}(n) = A$ where $\Delta = \mathbb{Z}[1/d]$ or $\mathbb{Q}$.*

Proof. Suppose that $q \geqslant 1$ and f a non zero global section, then it consists of data of local sections of $\mathcal{O}(n)(D_+(X_i))$ which glue (coincide) on the intersections $\mathcal{O}(n)(D_+(X_i) \cap D_+(X_j))$, and thus f can be considered an element of $R[X_0^{1/d^\infty}, \dots, X_q^{1/d^\infty}]_{(X_0,\dots,X_q)}$. Let $1 \leqslant i \leqslant q$, the fact that $f \in X_i^n \mathcal{O}(D_+(X_i))$ implies that f does not have X_0 in its denominator. Since, $f \in X_0^n \mathcal{O}(D_+(X_0))$, this implies f is homogeneous of degree $n \geqslant 0$ and $f = 0$ if $n < 0$.

□

Remark 4.2. A graded module $\oplus_{n\in\Delta}\mathcal{O}(n)$ where $\Delta = \mathbb{R}_{\geqslant 0}$ can be constructed as above, for example, consider the homogeneous polynomial in two variable of degree $\sqrt{2}$,

$$X^{\sqrt{2}} + Y^{\sqrt{2}} + X^{1/\sqrt{2}} \cdot Y^{1/\sqrt{2}} + X.Y^{\sqrt{2}-1}, \tag{4.3}$$

the affine pieces can be constructed by dehomogenisation. The degree with real numbers is useful in homological mirror symmetry. Furthermore, localization can be done at multiplicatively closed sets such as $\{X^r\}, r \in \mathbb{R}_{>0}$. But the coherence of the associated ring $k[X^r], r \in \mathbb{R}_0$ cannot be shown using direct limits as done in Proposition 2.1.

4.2 Computing Cohomology

The global sections of degree 2 of Proj $R[X,Y]$ are generated by X^2, XY, Y^2, where as the global sections of degree 2 of Proj $R[X^{1/d^\infty}, Y^{1/d^\infty}]$ or $\mathfrak{R}$ are given as $X^2, Y^2, XY, X_i^r Y^{1-r_i}$ for $r_i \in \mathbb{Z}[1/d] \cap (0,1)$ or $r_i \in \mathbb{Q} \cap (0,1)$, and are thus infinitely many.

Theorem 4.3. *Let $S = R[X_0^{1/d^\infty}, \dots, X_n^{1/d^\infty}]$ and $X = \text{Proj } S$, then for any $n \in \mathbb{Z}[1/d]$*

1. *There is an isomorphism $S \simeq \oplus_{n\in\Delta} H^0(X, \mathcal{O}_X(n))$.*
2. *$H^n(X, \mathcal{O}_X(-n-1))$ is a free module of infinite rank.*

Proof. 1. Take the standard cover by affine sets $\mathfrak{U} = \{U_i\}_i$ where each $U_i = D(X_i), i = 0, \dots, n$. The global sections are given as the kernel of the following map

$$\prod S_{X_{i_0}} \longrightarrow \prod S_{X_{i_0} X_{i_1}} \tag{4.4}$$

The element mapping to the Kernel has to lie in all the intersections $S = \cap_i S_{X_i}$, as given on [Hartshorne, 1977, pp 118] and is thus the ring S itself.

2. $H^n(X, \mathcal{O}_X(-m))$ is the cokernel of the map

$$d^{n-1} : \prod_k S_{X_0 \cdots \hat{X}_k \cdots X_n} \longrightarrow S_{X_0 \cdots X_n} \tag{4.5}$$

$S_{X_0 \cdots X_n}$ is a free R module with basis $X_0^{l_0} \cdots X_n^{l_n}$ with each $l_i \in \mathbb{Z}[1/d]$. The image of d^{n-1} is the free submodule generated by those basis elements with atleast one $l_i \geqslant 0$. Thus H^n is the free module with basis as negative monomials

$$\{X_0^{l_0} \cdots X_n^{l_n}\} \text{ such that } l_i < 0 \tag{4.6}$$

The grading is given by $\sum l_i$ and there are infinitely many monomials with degree $-n - \epsilon$ where ϵ is something very small and $\epsilon \in \mathbb{Z}[1/d]$. Recall, that in the standard coherent cohomology there is only one such monomial $X_0^{-1} \cdots X_n^{-1}$. For example, in case of $\mathbb{P}^2$ we have $X_0^{-1}X_1^{-1}X_2^{-1}$ but here in addition to above we also have $X_0^{-1/2}X_1^{-1/2}X_2^{-2}$.

Recall that in coherent cohomology of $\mathbb{P}^n$ the dual basis of $X_0^{m_0} \cdots X_n^{m_n}$ is given by $X_0^{-m_0-1} \cdots X_n^{-m_n-1}$ and the operation of multiplication gives pairing. We do not have this pairing here, but we can pair $X_0^{m_0}$ with $X_0^{-m_0}$.

∎

Theorem 4.4. *Let* $S = \mathfrak{R}$ *and* $X = \text{Proj } S$*, then for any* $n \in \mathbb{Q}$

1. *There is an isomorphism* $S \simeq \oplus_{n \in \Delta} H^0(X, \mathcal{O}_X(n))$.
2. $H^n(X, \mathcal{O}_X(-n-1))$ *is a free module of infinite rank.*

Proof. Same proof as above by replacing $\mathbb{Z}[1/d]$ with $\mathbb{Q}$. ∎

Theorem 4.5. *Let* $S = \mathfrak{R}$ *or* $R[X_0^{1/d^\infty}, \dots, X_n^{1/d^\infty}]$ *and* $X = \text{Proj } R$*, then* $H^i(X, \mathcal{O}_X(m)) = 0$ *if* $0 < i < n$.

Proof. The proof from [Vakil, 2017, pp 474-475] is adapted to the case at hand, using the convention that Γ denotes global sections. We will work with $n = 2$ for the sake of clarity, the case for general n is identical. The Čech complex is given in figure 1.

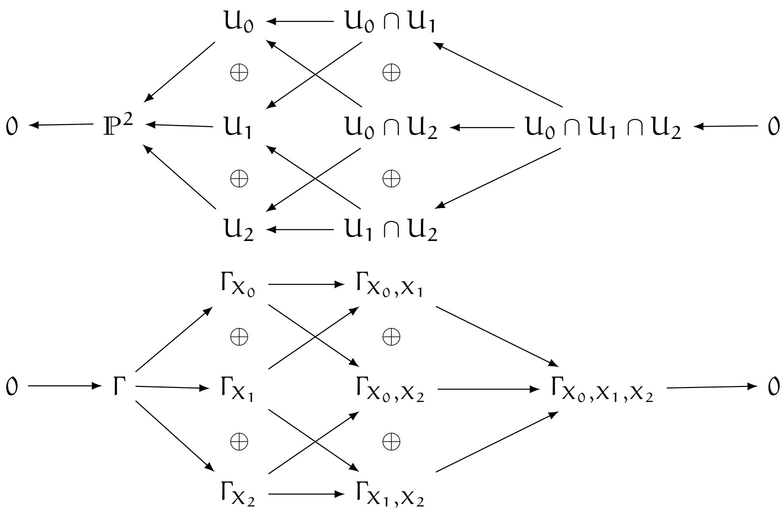

Figure 1: Čech Complex for $\mathbb{P}_K^{n,\mathrm{ad},\mathrm{perf}}, n = 2$

3 **negative exponents** The monomial $X_0^{a_0} \cdot X_1^{a_1} \cdot X_2^{a_2}$ where $a_i < 0$. We cannot lift it to any of the coboundaries (that is lift only to 0 coefficients). If K_{012} denotes the coefficient of the monomial in the complex (Figure 2), we get zero cohomology except for the spot corresponding to $U_0 \cap U_1 \cap U_2$.

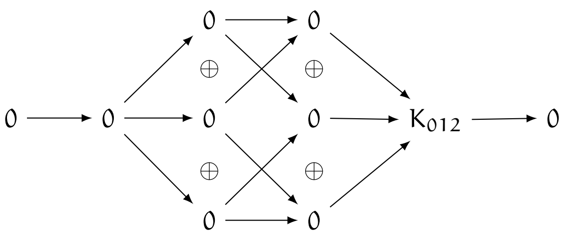

Figure 2: 3 negative exponents

2 negative exponents The monomial $X_0^{a_0} \cdot X_1^{a_1} \cdot X_2^{a_2}$ where two exponents are negative, say $a_0, a_1 < 0$. Then we can perfectly lift to coboundary coming from $U_0 \cap U_1$, which gives exactness.

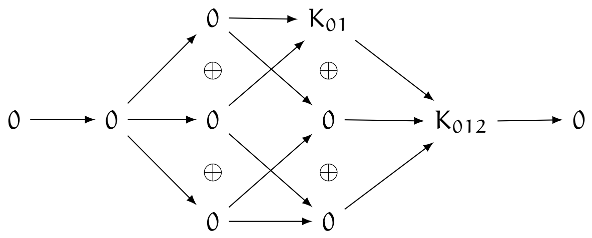

Figure 3: 2 negative exponents

1 negative exponent The monomial $X_0^{a_0} \cdot X_1^{a_1} \cdot X_2^{a_2}$ where one exponents is negative, say $a_0 < 0$, we get the complex (Figure 4). Notice that K_0 maps injectively giving zero cohomology group.

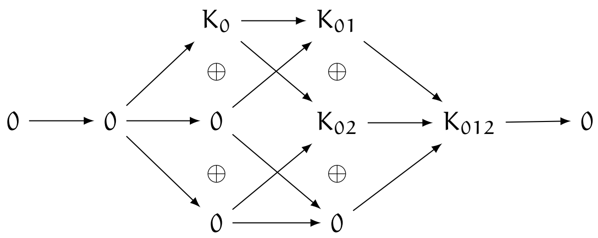

Figure 4: 1 negative exponent

Furthermore, the mapping in the Figure 5 gives Kernel when $f = g$ which is possible for zero only. Again giving us zero cohomology groups.

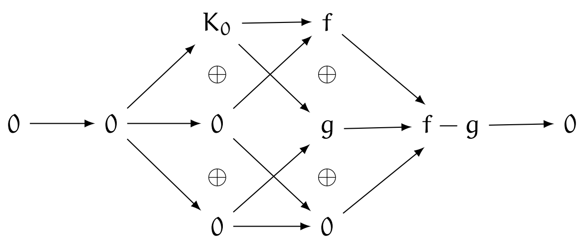

Figure 5: Mapping for 1 negative exponent

0 negative exponent The monomial $X_0^{a_0} \cdot X_1^{a_1} \cdot X_2^{a_2}$ where none of the exponents is negative $a_i > 0$, gives the complex Figure 6.

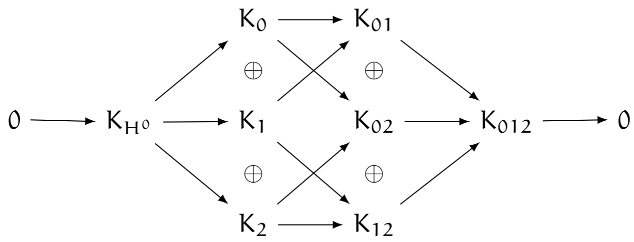

Figure 6: 0 negative exponent

Consider the SES of complex as in Figure 7 . The top and bottom row come from the 1 negative exponent case, thus giving zero cohomology. The SES of complex gives LES of cohomology groups, since top and bottom row have zero cohomology, so does the middle.

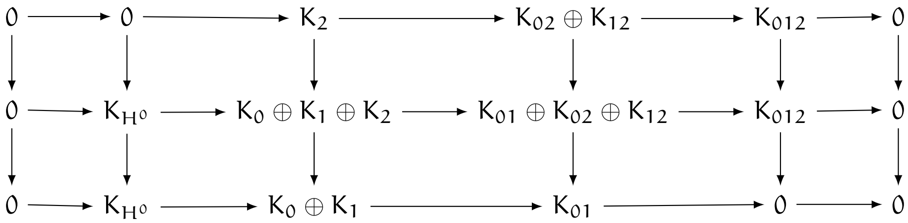

Figure 7: SES of Complex

□

4.3 Kunneth Formula

We can produce a complex for $\mathbb{P}^n \times \mathbb{P}^m$ by taking tensor product of the corresponding Čech complex associated with each space, and by the Theorem of Eilenberg-Zilber we get

$$H^i(\mathbb{P}^n \times \mathbb{P}^m, \mathcal{O}(a,b)) = \sum_{j=0}^{i} H^j(\mathbb{P}^n, \mathcal{O}(a)) \otimes H^{i-j}(\mathbb{P}^m, \mathcal{O}(b)) \quad a, b \in \mathbb{Z}[1/d] \text{ or } \mathbb{Q} \tag{4.7}$$

Furthermore, we can define a cup product following [Liu, 2002, pp 194] to get a homomorphism

$$\smile : H^p(\mathbb{P}^n, \mathcal{O}(a)) \times H^q(\mathbb{P}^m, \mathcal{O}(b)) \to H^{p+q}(\mathbb{P}^n \times \mathbb{P}^m, \mathcal{O}(a,b)) \quad a, b \in \mathbb{Z}[1/d] \text{ or } \mathbb{Q} \tag{4.8}$$

5 Perfectoid Tate Algebra

Let K be a perfectoid field with ring of integers as $\mathfrak{o}_K$ which contains a pseudo uniformizer ϖ. Set $d = p$ and $R = \mathfrak{o}_K$ in section 2.1 to obtain the ring $\mathfrak{o}_K[X_0^{1/p^\infty}, \ldots, X_n^{1/p^\infty}]$. This ring completed with respect to uniformizer ϖ gives

$$\varprojlim_i \mathfrak{o}_K[X_0^{1/p^\infty}, \ldots, X_n^{1/p^\infty}] = \mathfrak{o}_K \left\langle X_0^{1/p^\infty}, \ldots, X_n^{1/p^\infty} \right\rangle \tag{5.1}$$

The perfectoid tate algebra is obtained as

$$\mathcal{R}_n := K \otimes_{\mathfrak{o}_K} \mathfrak{o}_K \left\langle X_0^{1/p^\infty}, \ldots, X_n^{1/p^\infty} \right\rangle = K \left\langle X_0^{1/p^\infty}, \ldots, X_n^{1/p^\infty} \right\rangle \tag{5.2}$$

In [Scholze, 2012] the ring $\mathcal{R}_n$ is obtained from perfection of Tate algebra $K\langle X_0, \ldots, X_n\rangle$, the above approach is followed in [Bedi, 2019]. More concretely, $\mathcal{R}_n$ is a power series given as

$$\begin{aligned} K\left\langle X^{1/p^\infty}\right\rangle &= \sum_{i \in \mathbb{Z}[1/p]_{\geqslant 0}} a_i X^i, \quad a_i \in K, \quad \lim_{i\to\infty} |a_i| = 0 \\ K\left\langle X_0^{1/p^\infty}, \ldots, X_1^{1/p^\infty}\right\rangle &= \sum_{j \in \mathbb{Z}[1/p]^n_{\geqslant 0}} a_{i_1\ldots i_n} X_0^{i_1} \cdots X_n^{i_n}, \quad a_j \in K, \quad \lim_{j\to\infty} |a_j| = 0, \end{aligned} \tag{5.3}$$

where j is a multi index representing the tuple $(i_1, \ldots, i_n)$. The power series is simply a rational degree avatar of Tate Algebras.

5.1 Perfectoid Disc

The projective perfectoid space is obtained by gluing together affine perfectoid spaces and are constructed in [Scholze, 2012]. The case of $\mathbb{P}_K^{1,\text{ad},\text{perf}}$ is done in great detail in [Das, 2016], which closely follows the Tate algebra described in [Fresnel and van der Put, 2012, Chapter 2]. The projective space for Tate analytic case is given on [Bosch et al., 2012, pp. 364] or [Fresnel and van der Put, 2012, pp. 85]. The corresponding perfectoid rings are obtained by perfection of the underlying affinoid rings as shown in [Scholze, 2012]. The underlying space is the standard polydisc given as

$$D_n := \{(x_0, \ldots, x_n) \in K^n \text{ such that } |x_i| \leqslant 1\}. \tag{5.4}$$

Recall that in algebraic geometry two copies of affine line are glued together to get the projective line. The rings corresponding to the affine lines are $K[X]$ and $K[1/X]$, these embed into $K[X, 1/X]$ as a natural injection. In the perfectoid case the rings corresponding to $\mathbb{A}_K^{1,\text{ad},\text{perf}}$ are $K\left\langle X^{1/p^\infty}\right\rangle$ and $K\left\langle X^{-1/p^\infty}\right\rangle$ and these embed into $K\left\langle X^{1/p^\infty}, X^{-1/p^\infty}\right\rangle$ corresponding to $\mathbb{P}_K^{1,\text{ad},\text{perf}}$.

The localisation on affine open subset is given as

$$\mathcal{O}_{\mathbb{P}_K^{n,\text{ad},\text{perf}}}(U_i) = K\left\langle \left(\frac{X_0}{X_i}\right)^{1/p^\infty}, \ldots, \left(\widehat{\frac{X_i}{X_i}}\right)^{1/p^\infty}, \ldots, \left(\frac{X_n}{X_i}\right)^{1/p^\infty} \right\rangle, \tag{5.5}$$

it is obtained as the perfection of the corresponding tate algebra. In the projective perfectoid, we are working with power series which have infinitely many terms instead of finitely many as in the polynomial case.

The sheaf $\mathcal{O}(m)$ is defined similarly but instead of polynomials there is power series with homogeneous elements of deg m.

Theorem 5.1. *1.* $H^0(\mathbb{P}_K^{n,\text{ad,perf}}, \mathcal{O}_{\mathbb{P}_K^{n,\text{ad,perf}}}(m))$ *is a free module of infinite rank.*

2. $H^n(\mathbb{P}_K^{n,\text{ad,perf}}, \mathcal{O}_{\mathbb{P}_K^{n,\text{ad,perf}}}(-m))$ *for* $m > n$ *is a free module of infinite rank.*

3. $H^i(\mathbb{P}_K^{n,\text{ad,perf}}, \mathcal{O}_{\mathbb{P}_K^{n,\text{ad,perf}}}) = 0$ *if* $0 < i < n$.

Proof. Same as in the case of $R[X_0^{1/p^\infty}, \dots, X_1^{1/p^\infty}]$, but work with power series instead. The proof from previous cases carry word for word since finiteness of polynomials is never used in the proofs. □

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