Long time dynamics for interacting oscillators on dense graphs

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LONG TIME DYNAMICS FOR INTERACTING OSCILLATORS ON DENSE GRAPHS

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Abstract. The long time dynamics of the stochastic Kuramoto model defined on a graph is analyzed in the subcritical regime. The emphasis is posed on the relationship between the mean field behavior and the connectivity of the underlying graph: we give an explicit deterministic condition on the sequence of graphs such that, for any initial condition, even dependent on the network, the system approaches the unique stable stationary solution and it remains close to it, up to almost exponential times. The condition on the sequence of graphs is expressed through a concentration in $\ell_\infty \to \ell_1$ norm and it is shown to be satisfied by a large class of graphs, random and deterministic, provided that the number of neighbors per site diverges, as the size of the system tends to infinity.

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1. Introduction

1.1. The stochastic Kuramoto model defined on a graph.

For $n \in \mathbb{N}$, consider the family of oscillators $\{\theta^{i,n}_\cdot\}_{i=1,\dots,n}$ on $\mathbb{T}^n := (\mathbb{R}/2\pi\mathbb{Z})^n$, which satisfy:

$$\begin{cases} \mathrm{d}\theta^{i,n}_t = \frac{1}{np_n}\sum_{j=1}^n \xi^{(n)}_{ij}\, J(\theta^{i,n}_t - \theta^{j,n}_t)\,\mathrm{d}t + \mathrm{d}B^i_t, & \text{for } t > 0,\\ \theta^{i,n}_0 = \theta^i_0, & \text{for } i \in \{1,\dots,n\}, \end{cases} \tag{1.1}$$

where $J(\cdot) = -K\sin(\cdot)$ with $K \geqslant 0$, $\xi^{(n)}_{ij}$ take values in $\{0,1,2,\dots\}$ for all $i,j = 1,\dots,n$ and $p_n \in (0,1]$ enters in the normalization of the interaction between the particles. The letter $\mathbf{P}$ denotes the law induced by $\{B^i_\cdot\}_{i\in\mathbb{N}}$ which are IID Brownian motions on $\mathbb{T}$ and $\{\theta^i_0\}_{i\in\mathbb{N}}$ denotes the initial conditions. This setting corresponds to the reversible stochastic Kuramoto model defined on a graph with adjacency matrix $\xi^{(n)} = \{\xi^{(n)}_{ij}\}_{i,j=1,\dots,n}$. We consider both directed and undirected graphs, as well as multigraphs. Whenever it's not crucial, we drop the dependency on n for the variables ξ_{ij} .

System (1.1) is studied by considering the empirical measure μ^n_t associated to $\{\theta^{i,n}_t\}_{i=1,\dots,n}$, defined for all $t \geqslant 0$ by

$$\mu^n_t := \frac{1}{n}\sum_{j=1}^n \delta_{\theta^{j,n}_t} \in \mathcal{P}(\mathbb{T}), \tag{1.2}$$

where $\mathcal{P}(\mathbb{T})$ denotes the space of probability measures on the torus.

On the complete graph, i.e. when $\xi_{ij} = 1$ for all i,j and $p_n \equiv 1$ for all $n \in \mathbb{N}$, it is well known (e.g. [6, Proposition 3.1]) that for all fixed time T, $\mu^n_{t\in[0,T]}$ seen as a continuous function over $\mathcal{P}(\mathbb{T})$, weakly converges in $\mathcal{C}([0,T],\mathcal{P}(\mathbb{T}))$ to the solution of the following

McKean-Vlasov PDE:

$$\begin{cases} \partial_t \mu_t(\theta) = \frac{1}{2}\partial_\theta^2 \mu_t(\theta) - \partial_\theta[\mu_t(\theta)(J * \mu_t)(\theta)], & \text{for } \theta \in \mathbb{T},\ 0 < t \leqslant T, \\ \mu_t|_{t=0} = \mu_0, \end{cases} \tag{1.3}$$

where $*$ stands for the convolution and provided that μ_0^n weakly converges to μ_0.

Equation (1.3) is by now well understood (see Proposition 4.4 and [12]) and it is known that it admits two different regimes depending on K: in the supercritical regime, when $K > 1$, there is a manifold of solutions corresponding to the synchronous states of the oscillators $\{\theta_\cdot^{i,n}\}_{i=1,\dots,n}$; when $K < 1$, the subcritical regime, there is a unique stable stationary solution which corresponds to the incoherent state $\frac{1}{2\pi}$.

The long time dynamics in the supercritical regime of the classical mean field model has been deeply studied in [7] and with random frequencies (the proper stochastic Kuramoto model) in [21]. We consider here the subcritical regime, i.e. $0 \leqslant K < 1$, putting the emphasis on the network structure given by the sequence $\{\xi^{(n)}\}_{n\in\mathbb{N}}$.

1.2. **The graph's perspective.** The aim of this note is to find the minimal assumption on the sequence $\xi = \left\{\xi^{(n)}\right\}_{n\in\mathbb{N}}$, i.e. the interaction network of (1.1), such that the long time behavior of (1.1) is well understood: in other words, whenever system (1.1) is comparable to the classical Kuramoto model or to the PDE formulation (1.3), under a proper scale between size of the system n and some horizon time T_n.

The normalization sequence p_n has to be chosen such that the interaction term in (1.1) makes sense. At least, this requires the assumption that the quantity

$$\frac{1}{np_n}\sum_{j=1}^{n} \xi_{ij}^{(n)} \tag{1.4}$$

is of order one, for each vertex i in the graph. Observe that np_n represents the mean degree in the network $\{\xi^{(n)}\}$ and, whenever (1.4) converges to zero or diverges, either vertices are isolated or the interaction has no more mathematical meaning.

Remark 1.1. *On one hand, conditions on* (1.4) *imply a sort of homogeneity on the graph: namely, a degree homogeneity since each vertex must have the same degree magnitude. On the other hand, they do not require anything on the connectivity: disconnected graphs with homogeneous degree can be easily constructed, but are misleading while studying the empirical measure, we refer the reader to* [8, Remark 1.2] *and* [10, Remark 1.4] *for concrete examples and a precise analysis from this perspective.*

For $n = 2, 3, \dots$, define the normalized adjacency matrix $P^{(n)} = \{P_{ij}^{(n)}\}_{i,j=1,\dots,n}$ by

$$P_{ij}^{(n)} := \frac{\xi_{ij}^{(n)}}{p_n}, \qquad \text{for } i, j = 1, \dots, n. \tag{1.5}$$

Recall that we do not assume any symmetry on $\xi^{(n)}$ and that it can also represent a multigraph.

One would like to compare $P^{(n)}$ to $\mathbf{1}^{(n)}$, the adjacency matrix associated to the classical mean field model, i.e. $\mathbf{1}_{ij}^{(n)} = 1$ for $i, j = 1, \dots, n$. It turns out that a sufficient condition for what we aim at, is given by a control on the difference between $P^{(n)}$ and $\mathbf{1}^{(n)}$ through the $\ell_\infty \to \ell_1$ norm. This norm is defined for a matrix $G = \{G_{ij}\}_{i,j=1,\dots,n}$ as

$$\|G\|_{\infty\to 1} = \sup_{\|s\|_\infty \leqslant 1} \|Gs\|_1 = \sup_{s,t\in\{-1,1\}^n} Gst^\top = \sup_{s_i,t_j\in\{-1,1\}} \sum_{i,j=1}^n G_{ij}s_it_j. \tag{1.6}$$

It has received a lot of attention in the last years: it appears in many applications in computer science (e.g. [15]) and it has been shown to be very useful in graphs concentration (e.g. [14, 19, 22]). Part of this success is because of the equivalence to the cut-norm (e.g. [2]) and of Grothendieck's Inequality, which is recalled here.

Theorem 1.2 (Grothendieck's inequality, [25, Theorem 2.4]). *Let $\{a_{ij}\}_{i,j=1,\dots,n}$ be a $n\times n$ real matrix such that for all $s_i, t_j \in \{-1,1\}$*

$$\sum_{i,j=1}^n a_{ij}s_it_j \leqslant 1. \tag{1.7}$$

Then, there exists a constant $K_R > 0$, such that for every Hilbert space $(H, \langle\cdot,\cdot\rangle_H)$ and for all S_i and T_j in the unit ball of H

$$\sum_{i,j=1}^n a_{ij}\langle S_i, T_j\rangle_H \leqslant K_R. \tag{1.8}$$

It is indeed thanks to this inequality that $\ell_\infty \to \ell_1$ norm turns out to be the natural choice for our setting: an important part of the proof (in particular Lemma 3.2) consists in showing that the fluctuations due to the graph structure can be described by expressions like (1.8), and thus controlled by $\|\cdot\|_{\infty\to 1}$.

From now on, the only condition we require on $\left(\xi^{(n)}, p_n\right)_{n\in\mathbb{N}}$ is to satisfy:

$$\left\|P^{(n)} - \mathbf{1}^{(n)}\right\|_{\infty\to 1} = o(n^2), \tag{1.9}$$

or, in other words,

$$\lim_{n\to\infty} \sup_{s_i,t_j\in\{-1,1\}} \frac{1}{n^2}\sum_{i,j=1}^n \left(\frac{\xi_{ij}^{(n)}}{p_n} - 1\right) s_it_j = 0. \tag{1.10}$$

We will see that Erdős-Rényi random graphs with parameter p_n satisfy condition (1.10) almost surely, provided that $np_n \uparrow \infty$ (see Lemma 5.3). We also provide a class of deterministic graphs, Ramanujan graphs, that satisfies (1.10).

Appendix A presents such results and includes remarks on the relationship between condition (1.10), the degree condition (1.4) and the connectivity of $\{\xi^n\}_{n\in\mathbb{N}}$.

1.3. **Set-up and notations.** The closeness between μ_t^n and μ_t is studied through a norm in an appropriate Hilbert space H_{-1}. This last one is defined as follows.

Denote by $\mathcal{C}_0^1(\mathbb{T})$ the space of $\mathcal{C}^1$ functions on the torus with zero mean and consider

$$\mathcal{L}_0^2 = \left\{ f\in\mathcal{L}^2(\mathbb{T}) : \int_{\mathbb{T}} f = 0\right\}, \tag{1.11}$$

with canonical scalar product $(u,v) := \int_{\mathbb{T}} uv$, for $u, v\in\mathcal{L}_0^2$. Let now V be the closure of $\mathcal{C}_0^1(\mathbb{T})$ with respect to the norm $\|\varphi\|_1 = \left(\int_{\mathbb{T}}(\varphi')^2\right)^{\frac{1}{2}}$ for $\varphi\in\mathcal{C}_0^1(\mathbb{T})$. It is easy to see that V is continuously and densely injected in $\mathcal{L}_0^2$ (thanks to the compactness of $\mathbb{T}$ and Poincaré inequality). Moreover, one can define an inner product on V which makes it an Hilbert space $H_1 := (V, \langle\cdot,\cdot\rangle_1)$ where $\langle\varphi,\psi\rangle_1 = \int_{\mathbb{T}} \varphi'\psi'$ for all $\varphi,\psi \in \mathcal{C}^1_0(\mathbb{T})$.

The dual space of H_1, denoted by H_{-1}, can be described through its Fourier orthonormal basis $\{e_l\}_{l\geqslant 1}$, where $e_l(\theta) = \frac{1}{l}e^{il\theta}$. With this characterization one easily obtains that $\mathcal{P}(\mathbb{T}) - \frac{1}{2\pi} \subset H_{-1}$. Indeed, for $\mu \in \mathcal{P}(\mathbb{T})$,

$$\left\|\mu - \frac{1}{2\pi}\right\|_{-1} = \sqrt{\sum_{l\geqslant 1}\left|\langle\mu, \frac{e^{il\cdot}}{l}\rangle\right|^2} \leqslant \sqrt{\sum_{l\geqslant 1}\frac{1}{l^2}} < \infty. \tag{1.12}$$

In particular, the difference $\mu^n_t - \mu_t$ belongs to H_{-1}.

More information on H_1 as well as the relationship between $\mathcal{P}(\mathbb{T})$ and H_{-1} as metric spaces are given in Appendix B. Hereafter we drop the dependency on $\mathbb{T}$, i.e. we write $\mathcal{C}^1_0$ instead of $\mathcal{C}^1_0(\mathbb{T})$ and so on for the other spaces and integrals.

2. Main result and strategy of the proof

2.1. **Result and discussion.** Recall that we consider the Kuramoto model in the sub-critical regime, i.e. $K < 1$, and we only require $\left(\xi^{(n)}, p_n\right)_{n\in\mathbb{N}}$ to satisfy condition (1.10) and $\mu_0 \in \mathcal{P}(\mathbb{T})$.

Theorem 2.1. *Suppose that for all $\varepsilon_0 > 0$*

$$\lim_{n\to\infty} \mathbf{P}\left(\|\mu^n_0 - \mu_0\|_{-1} \leqslant \varepsilon_0\right) = 1. \tag{2.1}$$

Then, for every positive increasing sequence $\{T_n\}_{n\in\mathbb{N}}$ such that $T_n = \exp(o(n))$, and for all $\varepsilon > 0$ small enough

$$\lim_{n\to\infty} \mathbf{P}\left(\sup_{t\in[0,T_n]} \|\mu^n_t - \mu_t\|_{-1} \leqslant \varepsilon\right) = 1. \tag{2.2}$$

Theorem 2.1 implies the proximity of the empirical measure to the solution of the McKean-Vlasov equation (1.3) for almost exponential times. The result is sharp since large deviation phenomena occur due to the stochastic nature of the system (e.g. [9, 11, 24]) making it escape from the stationary solution.

This result does not depend on the speed of convergence of condition (1.10). The escaping time is indeed only due to the stochastic nature of the system, given by the Brownian motions, and it cannot be improved as explained above. The reason why one can control the perturbation induced by the graph structure for long times (in reality for all times) is because of the exponential stability of the stationary solution, we refer to Lemma 3.2 for a precise statement.

Finally, we would like to point out that no independence between initial conditions and graph is required. This means that even if one accurately assigns the initial conditions for each vertex, the mixing properties of the graph will shuffle all the information and make the empirical measure converge to the stable stationary solution, loosing any memory of the initial state.

Of independent interest, we present a corollary of Theorem 2.1 in the limit case $K = 0$.

Corollary 2.2. *Let $\mu^n_\cdot$ be the empirical measure of n independent Brownian motions $\{B^{j,n}_\cdot\}_{j=1,\dots,n}$ on $\mathbb{T}$. Then, $\mu^n_\cdot$ satisfies the following stochastic differential equation in H_{-1}:*

$$\mu^n_t = e^{t\frac{\Delta}{2}}\mu^n_0 + z^n_t, \quad \textit{for } t \geqslant 0, \tag{2.3}$$

where $e^{t\frac{\Delta}{2}}$ is the semigroup associated to the Laplacian operator and for $h \in H_1$, $z_t^n(h)$ is defined by

$$z_t^n(h) = \frac{1}{n} \sum_{j=1}^{n} \int_0^t \left[\partial_\theta e^{(t-s)\frac{\Delta}{2}} h \right] (B_s^{j,n}) \, \mathrm{d}B_s^{j,n}. \tag{2.4}$$

In particular, there exists $C > 0$ such that for all $T > 0$, the following maximal inequality holds:

$$\mathbf{E}\left[\sup_{t \in [0,T]} \|z_t^n\|_{-1}^2 \right] \leqslant C \log(1+T). \tag{2.5}$$

Corollary 2.2 shows a maximal inequality for the empirical measure of n independent Brownian motions on the torus, establishing an SPDE version of the result for Ornstein-Uhlenbeck processes presented in [13] for the SODE case. This seems like a well known result, yet the author was unable to find it elsewhere.

2.2. **Outline of the proof.** The behavior of μ_t solution to (1.3) is well known (Proposition 4.3): in the subcritical regime ($K < 1$), $\frac{1}{2\pi}$ is the unique stable stationary solution to which μ_t converges for all $\mu_0 \in \mathcal{P}(\mathbb{T})$. We then aim at showing that

(1) starting from $\mu_0^n \in \mathcal{P}(\mathbb{T})$, μ_t^n stays arbitrarily close to μ_t and reaches a neighborhood of $\frac{1}{2\pi}$ in a finite time;
(2) once μ_t^n approaches a neighborhood of $\frac{1}{2\pi}$, it stays in it for long times before large deviation phenomena take over and the closeness between μ_t^n and μ_t is lost.

For the sake of clarity, we will initially suppose that the initial condition μ_0^n is arbitrarily close to $\frac{1}{2\pi}$ (as n tends to infinity and with high probability) and prove (2). For doing this, we first show in Proposition 3.1 that the process $\nu_t^n := \mu_t^n - \frac{1}{2\pi}$ satisfies a stochastic partial differential equation in H_{-1}; then, using the contractive properties of the linear operator associated to the evolution (1.3) around $\frac{1}{2\pi}$, we control the stochastic term in the SPDE (Lemma 3.3) and the perturbation given by the graph structure (Lemma 3.2), obtaining the closeness to $\frac{1}{2\pi}$ for long times (Proposition 4.1). Lemma 3.2 is the fundamental step where we use Grothendieck's inequality and control all the randomness given by $\{\xi^{(n)}\}$.

Concerning (1), we control $\mu_t^n - \mu_t$ with similar estimates as before and, using the fact that μ_t converges to $\frac{1}{2\pi}$, we show that the empirical measure μ_t^n reaches a neighborhood of $\frac{1}{2\pi}$ in finite time which only depends on μ_0. This last result is somehow known whenever the initial conditions are independent of the graph sequence, we present a different proof (Proposition 4.3) which does not require this assumption and allows to more general initial settings.

The proof is concluded combining the two arguments.

2.3. **A glance at the existing literature.** The result presented in Theorem 2.1 is at a crossroads of two different research areas: the long time dynamics of stochastic differential equations and the role of a network in a mean field model.

Concerning the long time behavior of weakly interacting particle systems, Theorem 2.1 represents a very "poor" result, a sort of step zero in this direction, since dealing with the subcritical regime where there is an unique stable stationary solution. For more general, and interesting, results on the long time dynamics, we refer to [7, 21] and the literature therein.

Turning to interacting particle systems on graphs, this subject has become an interesting topic in the mathematical community given the several applications to complex systems, in particular regarding the Kuramoto model and synchronization phenomena (e.g. [1]).

Focusing on mean field systems defined by stochastic differential equations, and neglecting all the results in statistical mechanics, the first articles [10, 3] attacked the problem under a propagation of chaos viewpoint, requiring a strong independence in the initial conditions (and with respect to the graph) and only for finite time scales (or up to times slowly diverging on n, i.e. $T_n = O(\log n)$). Other results in this direction are [20], which extends [10] to graphons, [26] presenting Large Deviations again in the graphon setting, and [18, 23] that address the sparse graph regime. Some effort has been made in [8] to prove convergence of the empirical measure for all initial conditions, even deterministic, but still independent of the sequence of graphs, now restricted to the ER class.

To the author's knowledge, there exists no result studying the long time dynamics of a system defined on graphs and no example (even in finite time) where one can choose the initial conditions dependent on the graph structure.

2.4. **Organisation of the paper.** Proposition 3.1 in Section 3 presents the H_{-1} formulation and Lemmas 3.2 and 3.3 show precise estimates on the perturbations given by the graph structure and the noise term respectively. The proofs for the long time dynamics and the finite time behavior are presented in Section 4, respectively in Subsections 4.1 and 4.2; Subsection 4.3 combines these two results and proves Theorem 2.1.

Appendix A gives a few examples of graph sequences that satisfy condition (1.10), together with remarks on the degrees and connectivity of such sequences. Appendix B contains information about the H_{-1} construction and estimates on the operators used in the previous proofs.

3. The SPDE formulation around the stationary solution

We place ourselves aroud the stationary solution $\frac{1}{2\pi}$. The system evolution is captured by the linear dynamics around $\frac{1}{2\pi}$ and the corresponding linear operator $L_{2\pi}$ is given by

$$L_{2\pi} u \ := \ \tfrac{1}{2}\partial_\theta^2 u - \tfrac{1}{2\pi}(\partial_\theta J) * u, \qquad \text{for } u \in \mathcal{C}2(\mathbb{T}),\ \int_{\mathbb{T}} u(\theta)\,\mathrm{d}\theta = 0. \tag{3.1}$$

The adjoint $L_{2\pi}^*$ of $L_{2\pi}$ in $\mathcal{L}_0^2$ has the following expression

$$L_{2\pi}^* u \ = \ \tfrac{1}{2}\partial_\theta^2 u - \tfrac{1}{2\pi} J * (\partial_\theta u), \tag{3.2}$$

and domain $D(L_{2\pi}^*) = D(L_{2\pi})$. These operators are diagonal in the Fourier basis $\{e_l\}_{l\geqslant 1}$, with eigenvalues denoted by $\{\lambda_l\}_{l\geqslant 1}$. The spectrum is negative and bounded away from 0, let $\gamma_K = \lambda_1 = \frac{1-K}{2} > 0$ denote the spectral gap. The operator $L_{2\pi}$ (resp. $L_{2\pi}^*$) defines an analytic semigroup $e^{tL_{2\pi}}$ (resp. $e^{tL_{2\pi}^*}$) with the following contractive property:

$$\left\| e^{tL_{2\pi}} h \right\|_{-1} \leqslant D_{\gamma,\beta} \frac{e^{-\gamma t/2}}{t^{\beta/2}} \left\| h \right\|_{-1-\beta}, \quad \text{for some } D_{\gamma,\beta} > 0, \tag{3.3}$$

for all $\gamma \in [0, \gamma_K)$, any $0 \leqslant \beta \leqslant 1$ and all $t > 0$, $h \in H_{-1}$. We refer to Appendix B for the definition of the fractional norm $\|\cdot\|_{-1-\beta}$ and the general properties of the semigroups.

3.1. **The formulation in H_{-1}.** Recall μ_t^n is the empirical measure of (1.1), define $\nu_t^n := \mu_t^n - \frac{1}{2\pi}$. We have the following

Proposition 3.1. *The process $\nu_t^n \in H_{-1}$ satisfies the following stochastic partial differential equation in $C([0,T], H_{-1})$:*

$$\nu_t^n = e^{tL_{2\pi}}\nu_0^n - \int_0^t e^{(t-s)L_{2\pi}}\partial_\theta\left[\nu_s^n(J*\nu_s^n)\right] \mathrm{d}s - g_t^n + z_t^n, \tag{3.4}$$

where

$$g_t^n = \frac{1}{n^2}\sum_{i,j=1}^n \int_0^t \left(\frac{\xi_{ij}}{p_n} - 1\right) e^{(t-s)L_{2\pi}}\partial_\theta\left[\delta_{\theta_s^{i,n}}(J*\delta_{\theta_s^{j,n}})\right] \mathrm{d}s, \tag{3.5}$$

and $z_t^n \in H_{-1}$ is defined for $h \in H_1$ by

$$\langle z_t^n, h\rangle_{-1,1} = \frac{1}{n}\sum_{j=1}^n \int_0^t \partial_\theta e^{(t-s)L_{2\pi}^*} h(\theta_s^{j,n})\, \mathrm{d}B_s^j. \tag{3.6}$$

Proof. Let $F = F_t(\theta) \in \mathcal{C}1,2([0,\infty)\times\mathbb{T})$, with $\int F_t = 0$ for all $t \geqslant 0$. For some $t \geqslant 0$, a straightforward application of Ito formula gives

$$\begin{aligned}\langle \mu_t^n - \tfrac{1}{2\pi}, F_t\rangle = \langle \mu_0^n - \tfrac{1}{2\pi}, F_0\rangle + \int_0^t \langle \mu_s^n - \tfrac{1}{2\pi}, \partial_s F_s + L_{2\pi}^* F_s\rangle\, \mathrm{d}s + \\ + \int_0^t \langle (\mu_s^n - \tfrac{1}{2\pi})(J*(\mu_s^n - \tfrac{1}{2\pi})), \partial_\theta F_s\rangle\, \mathrm{d}s + G_t^n(F) + Z_t^n(F),\end{aligned} \tag{3.7}$$

with

$$G_t^n(F) = \frac{1}{n^2}\sum_{i,j=1}^n \int_0^t \left(\frac{\xi_{ij}}{p_n} - 1\right) J(\theta_s^{i,n} - \theta_s^{j,n})\partial_\theta F_s(\theta_s^{i,n})\, \mathrm{d}s, \tag{3.8}$$

$$Z_t^n(F) = \frac{1}{n}\sum_{j=1}^n \int_0^t \partial_\theta F_s(\theta_s^{j,n})\, \mathrm{d}B_s^j. \tag{3.9}$$

The properties of $e^{tL_{2\pi}^*}$ assure that the function

$$F = F_s(\theta) = e^{(t-s)L_{2\pi}^*}h(\theta), \qquad \text{for some } h \in \mathcal{C}2(\mathbb{T}),\ \int h = 0, \tag{3.10}$$

is $\mathcal{C}1,2([0,t]\times\mathbb{T})$. But then $\partial_s F_s = -L_{2\pi}^* F_s$ and one obtains

$$\langle \nu_t^n, F_t\rangle = \langle \nu_0^n, e^{tL_{2\pi}^*}h\rangle + \int_0^t \langle \nu_s^n(J*\nu_s^n), \partial_\theta e^{(t-s)L_{2\pi}^*}h\rangle\, \mathrm{d}s + g_t^n(h) + z_t^n(h), \tag{3.11}$$

where we have used the definition of ν_t^n and the notations

$$g_t^n(h) = \frac{1}{n^2}\sum_{i,j=1}^n \int_0^t \left(\frac{\xi_{ij}}{p_n} - 1\right) J(\theta_s^{i,n} - \theta_s^{j,n})\partial_\theta e^{(t-s)L_{2\pi}^*}h(\theta_s^{i,n})\, \mathrm{d}s, \tag{3.12}$$

$$z_t^n(h) = \frac{1}{n}\sum_{j=1}^n \int_0^t \partial_\theta e^{(t-s)L_{2\pi}^*}h(\theta_s^{j,n})\, \mathrm{d}B_s^j. \tag{3.13}$$

We aim at proving that (3.11) is the weak formulation of the mild equation (3.4) in H_{-1}.

Let $\{\nu_l\}_{l\geqslant 1} \subset \mathcal{L}_0^2$ such that $\nu_l \xrightarrow{l\uparrow\infty} \nu_0^n$ in H_{-1}. Then, for $h \in \mathcal{C}2$

$$\langle \nu_l, e^{tL_{2\pi}^*}h\rangle_{-1,1} = \left(\nu_l, e^{tL_{2\pi}^*}h\right) = \left(e^{tL_{2\pi}}\nu_l, h\right) = \langle e^{tL_{2\pi}}\nu_l, h\rangle_{-1,1}. \tag{3.14}$$

By continuity of the operators, $e^{tL_{2\pi}}\nu_l$ converges in H_{-1} to $e^{tL_{2\pi}}\nu_0^n$ as $l\uparrow\infty$. Taking the limit for $l\uparrow\infty$ in both sides of (3.14), we deduce

$$\langle \nu_0^n, e^{tL^*_{2\pi}}h\rangle_{-1,1} = \langle e^{tL_{2\pi}}\nu_0^n, h\rangle_{-1,1}. \tag{3.15}$$

We now focus on

$$\omega_s^n := \nu_s^n(J*\nu_s^n). \tag{3.16}$$

Consider $\{\nu_{s,l}\}_{l\geqslant 1}\subset \mathcal{L}^2_0$ which converges to ν_s^n in H_{-1} as $l\uparrow\infty$, and define

$$\omega_{s,l} := \nu_{s,l}(J*\nu_s^n). \tag{3.17}$$

For any $l\geqslant 1$, it holds

$$\langle \omega_{s,l}, \partial_\theta e^{(t-s)L^*_{2\pi}}h\rangle_{-1,1} = \left(\omega_{s,l}, \partial_\theta e^{(t-s)L^*_{2\pi}}h\right) = \tag{3.18}$$

$$= -\left(e^{(t-s)L_{2\pi}}\partial_\theta\,\omega_{s,l}, h\right) = -\langle e^{(t-s)L_{2\pi}}\partial_\theta\,\omega_{s,l}, h\rangle_{-1,1}. \tag{3.19}$$

Using the properties of the semigroup, one obtains

$$\left|\langle e^{(t-s)L_{2\pi}}\partial_\theta(\omega_{s,l}-\omega_s^n), h\rangle_{-1,1}\right| \leqslant \|h\|_1 \left\|e^{(t-s)L_{2\pi}}\partial_\theta(\omega_{s,l}-\omega_s^n)\right\|_{-1} \leqslant \tag{3.20}$$

$$\leqslant \|h\|_1\frac{D_{1,1}}{\sqrt{t-s}}\left\|\partial_\theta(\omega_{s,l}-\omega_s^n)\right\|_{-2} = \|h\|_1\frac{D_{1,1}}{\sqrt{t-s}}\left\|\omega_{s,l}-\omega_s^n\right\|_{-1}, \tag{3.21}$$

which implies

$$\left\|e^{(t-s)L_{2\pi}}\partial_\theta(\omega_{s,l}-\omega_s^n)\right\|_{-1} \leqslant \frac{D_{1,1}}{\sqrt{t-s}}\left\|\omega_{s,l}-\omega_s^n\right\|_{-1}. \tag{3.22}$$

Since h is regular and $\omega_{s,l}\xrightarrow{l\uparrow\infty}\omega_s^n$ in H_{-1}, this implies

$$\langle \omega_s^n, \partial_\theta e^{(t-s)L^*_{2\pi}}h\rangle_{-1,1} = -\langle e^{(t-s)L_{2\pi}}\partial_\theta\,\omega_s^n, h\rangle_{-1,1}. \tag{3.23}$$

We now observe from (3.22) that

$$\left\|e^{(t-s)L_{2\pi}}\partial_\theta\omega_s^n\right\|_{-1} \leqslant \frac{D_{1,1}}{\sqrt{t-s}} \tag{3.24}$$

thus the integral in (3.4)

$$\int_0^t e^{(t-s)L_{2\pi}}\partial_\theta\left[\nu_s^n(J*\nu_s^n)\right]\,\mathrm{d}s \tag{3.25}$$

is almost surely finite. Using [27] Theorem 1 p.133, we deduce that (3.25) makes sense as a Bochner integral in H_{-1}. The continuity is a direct consequence of the continuity of $e^{tL_{2\pi}}$.

Assume that $g_t^n(h) = \langle g_t^n, h\rangle_{-1,1}$ and $z_t^n(h) = \langle z_t^n, h\rangle_{-1,1}$ are well defined and continuous with respect to t for all $h\in H_1$; we have shown that

$$\begin{aligned}\langle \nu_t^n, h\rangle_{-1,1} &= \langle e^{tL_{2\pi}}\nu_0^n, h\rangle_{-1,1} + \\ &- \langle \int_0^t e^{(t-s)L_{2\pi}}\partial_\theta\left[\nu_s^n(J*\nu_s^n)\right]\,\mathrm{d}s, h\rangle_{-1,1} - \langle g_t^n, h\rangle_{-1,1} + \langle z_t^n, h\rangle_{-1,1}.\end{aligned} \tag{3.26}$$

Since (3.26) holds for all $h\in H_1$, the identity (3.4) follows. All elements in (3.4) take values in $\mathcal{C}^1([0,T], H_{-1})$ and the proof is then concluded modulo regularity and wellposedness of g_t^n and z_t^n. We refer to Lemma 3.2 and Lemma 3.3 which are presented in the next subsection. □

3.2. **Control on the perturbations.** Two kinds of perturbations are present in the SPDE (3.4): z_t^n given by the stochastic nature of the system and g_t^n given by the presence of a network structure. In this subsection, we exhibit the control over the two perturbations. We start with the control on the graph structure, which uses Grothendieck's Inequality seen in Theorem 1.2.

Lemma 3.2 (Wellposedness and bounds on g_t^n)**.** *For $n \in \mathbb{N}$ and $t \geqslant 0$, let g_t^n be given by*

$$g_t^n = \frac{1}{n^2} \sum_{i,j=1}^n \int_0^t \left(\frac{\xi_{ij}}{p_n} - 1\right) e^{(t-s)L_{2\pi}} \partial_\theta \left[\delta_{\theta_s^{i,n}} (J * \delta_{\theta_s^{j,n}})\right] \mathrm{d}s. \tag{3.27}$$

Then

(1) $g^n \in \mathcal{C}0([0,\infty), H_{-1})$*. In particular, for all $h \in H_1$ and $t \geqslant 0$*

$$\langle g_t^n, h\rangle_{-1,1} = g_t^n(h) = -\frac{1}{n^2} \sum_{i,j=1}^n \int_0^t \left(\frac{\xi_{ij}}{p_n} - 1\right) J(\theta_s^{i,n} - \theta_s^{j,n}) \partial_\theta e^{(t-s)L_{2\pi}^*} h(\theta_s^{i,n})\, \mathrm{d}s. \tag{3.28}$$

(2) *There exists $D > 0$, independent of t, such that*

$$\|g_t^n\|_{-1} \leqslant D\, \frac{\left\|P^{(n)} - \mathbf{1}^{(n)}\right\|_{\infty\to 1}}{n^2}, \qquad \textit{for all } t \geqslant 0. \tag{3.29}$$

Proof. Fix n large. Consider $\{\phi_l\}_{l\geqslant 1} \subset \mathcal{C}\infty$ such that $\phi_l \geqslant 0$, $\phi_l(\theta) = 0$ for $\theta \in [1/l, 2\pi - 1/l]$, $\int \phi_l = 1$ for every $l \geqslant 1$ and $\lim_{l\to\infty} \int F\phi_l = F(0)$ for every $F \in \mathcal{C}0$. For $i = 1, \dots, n$, define

$$\phi_{s,l}^i := \phi_l * \delta_{\theta_s^{i,n}}. \tag{3.30}$$

We start by establishing (3.28). For each $h \in \mathcal{C}2$

$$\langle \frac{1}{n^2} \sum_{i,j=1}^n \left(\frac{\xi_{ij}}{p_n} - 1\right) \phi_{s,l}^i (J * \delta_{\theta_s^{j,n}}), \partial_\theta e^{(t-s)L_{2\pi}^*} h\rangle_{-1,1} = \tag{3.31}$$

$$= \left(\frac{1}{n^2} \sum_{i,j=1}^n \left(\frac{\xi_{ij}}{p_n} - 1\right) \phi_{s,l}^i (J * \delta_{\theta_s^{j,n}}), \partial_\theta e^{(t-s)L_{2\pi}^*} h\right) = \tag{3.32}$$

$$= -\left(\frac{1}{n^2} \sum_{i,j=1}^n \left(\frac{\xi_{ij}}{p_n} - 1\right) e^{(t-s)L_{2\pi}} \partial_\theta \left[\phi_{s,l}^i (J * \delta_{\theta_s^{j,n}})\right], h\right) = \tag{3.33}$$

$$= -\langle \frac{1}{n^2} \sum_{i,j=1}^n \left(\frac{\xi_{ij}}{p_n} - 1\right) e^{(t-s)L_{2\pi}} \partial_\theta \left[\phi_{s,l}^i (J * \delta_{\theta_s^{j,n}})\right], h\rangle_{-1,1} \tag{3.34}$$

But $\frac{1}{n^2} \sum_{i,j=1}^n \left(\frac{\xi_{ij}}{p_n} - 1\right) \phi_{s,l}^i (J * \delta_{\theta_s^{j,n}})$ converges to $\frac{1}{n^2} \sum_{i,j=1}^n \left(\frac{\xi_{ij}}{p_n} - 1\right) \delta_{\theta_s^{i,n}} (J * \delta_{\theta_s^{j,n}})$ since

$$\left\| \frac{1}{n^2} \sum_{i,j=1}^n \left(\frac{\xi_{ij}}{p_n} - 1\right) \left(\phi_{s,l}^i - \delta_{\theta_s^{i,n}}\right) (J * \delta_{\theta_s^{j,n}}) \right\|_{-1} \leqslant \frac{1}{p_n} \sup_{i=1,\dots,n} \left\|\phi_{s,l}^i - \delta_{\theta_s^{i,n}}\right\|_{-1}, \tag{3.35}$$

which tends to zero as l tends to infinity.

Thanks to the properties of the semigroup, the same holds true for

$$\frac{1}{n^2} \sum_{i,j=1}^n \left(\frac{\xi_{ij}}{p_n} - 1\right) e^{(t-s)L_{2\pi}} \partial_\theta \left[\left(\phi_{s,l}^i - \delta_{\theta_s^{i,n}}\right) (J * \delta_{\theta_s^{j,n}})\right]; \tag{3.36}$$

indeed for some $0 < \gamma < \gamma_K$:

$$\left\| \frac{1}{n^2} \sum_{i,j=1}^{n} \left(\frac{\xi_{ij}}{p_n} - 1 \right) e^{(t-s)L_{2\pi}} \partial_\theta \left[\left(\phi^i_{s,l} - \delta_{\theta^{i,n}_s} \right) (J * \delta_{\theta^{j,n}_s}) \right] \right\|_{-1} \leqslant \tag{3.37}$$

$$\leqslant D_{1,1} \frac{e^{-\gamma(t-s)}}{p_n \sqrt{t-s}} \sup_{i=1,\dots,n} \left\| \phi^i_{s,l} - \delta_{\theta^{i,n}_s} \right\|_{-1}. \tag{3.38}$$

A similar argument shows that

$$\left\| \frac{1}{n^2} \sum_{i,j=1}^{n} \left(\frac{\xi_{ij}}{p_n} - 1 \right) e^{(t-s)L_{2\pi}} \partial_\theta \left[\delta_{\theta^{i,n}_s} (J * \delta_{\theta^{j,n}_s}) \right] \right\|_{-1} \leqslant D_{1,1} \frac{e^{-\gamma(t-s)}}{p_n \sqrt{t-s}}, \tag{3.39}$$

which, in turn, implies that

$$\frac{1}{n^2} \sum_{i,j=1}^{n} \int_0^t \left(\frac{\xi_{ij}}{p_n} - 1 \right) e^{(t-s)L_{2\pi}} \partial_\theta \left[\delta_{\theta^{i,n}_s} (J * \delta_{\theta^{j,n}_s}) \right] \mathrm{d}s \tag{3.40}$$

is almost surely finite and continuous with respect to t. We deduce (3.28).

For the second part (3.29), observe that

$$\langle \frac{1}{n^2} \sum_{i,j=1}^{n} \left(\frac{\xi_{ij}}{p_n} - 1 \right) e^{(t-s)L_{2\pi}} \partial_\theta \left[\delta_{\theta^{i,n}_s} (J * \delta_{\theta^{j,n}_s}) \right], h \rangle_{-1,1} = \tag{3.41}$$

$$= -\frac{1}{n^2} \sum_{i,j=1}^{n} \left(\frac{\xi_{ij}}{p_n} - 1 \right) \langle \delta_{\theta^{i,n}_s}, (J * \delta_{\theta^{j,n}_s}) \partial_\theta e^{(t-s)L^*_{2\pi}} h \rangle_{-1,1}. \tag{3.42}$$

We claim that this last term can be controlled by $\left\| P^{(n)} - \mathbf{1}^{(n)} \right\|_{\infty\to 1}$ through Grothendieck's inequality. By choosing $H = H_{-1}$ and

$$a_{ij} = \left(\tfrac{\xi_{ij}}{p_n} - 1 \right), \tag{3.43}$$

$$S_i = \delta_{\theta^{i,n}_s}, \tag{3.44}$$

$$T_j = \frac{\sqrt{t-s}}{D_{1,1} e^{-\gamma(t-s)}} \left(J * \delta_{\theta^{j,n}_s} \right) \partial_\theta e^{(t-s)L^*_{2\pi}} \frac{h}{\|h\|_1}, \tag{3.45}$$

Theorem 1.2 allows us to bound the expression in (3.41) by

$$K_R \frac{D_{1,1} e^{-\gamma(t-s)}}{\sqrt{t-s}} \|h\|_1 \frac{\left\| P^{(n)} - \mathbf{1}^{(n)} \right\|_{\infty\to 1}}{n^2}. \tag{3.46}$$

This shows that

$$\|g^n_t\|_{-1} \leqslant K_R D_{1,1} \frac{\left\| P^{(n)} - \mathbf{1}^{(n)} \right\|_{\infty\to 1}}{n^2} \int_0^t \frac{e^{-\gamma(t-s)}}{\sqrt{t-s}} \mathrm{d}s \leqslant D \frac{\left\| P^{(n)} - \mathbf{1}^{(n)} \right\|_{\infty\to 1}}{n^2}, \tag{3.47}$$

where $D := K_R D_{1,1} \int_0^\infty \frac{e^{-\gamma s}}{\sqrt{s}} \mathrm{d}s > 0$ since the integral converges. The proof is concluded. □

We now turn to the stochastic term z^n_t in (3.4). Recall that $L_{2\pi}$ is diagonal in the Fourier basis $\{e_l\}_{l \geqslant 1}$ of H_{-1}, with eigenvalues denoted by λ_l. Then

Lemma 3.3 (Wellposedness and bounds on z_t^n)**.** *For $n \in \mathbb{N}$ and $t > 0$, let z_t^n be defined by*

$$z_t^n = \sum_{l \geqslant 1} z_t^n(e_l)\, e_l, \tag{3.48}$$

where

$$z_t^n(e_l) = \frac{i}{n} \sum_{j=1}^{n} \int_0^t e^{(t-s)\lambda_l} e^{il\theta_s^{j,n}}\, \mathrm{d}B_s^j. \tag{3.49}$$

Then

(1) *$z^n \in \mathcal{C}0([0,\infty), H_{-1})$ almost surely.*
(2) *There exists $C > 0$ independent of n, such that for all $T > 0$*

$$\mathbf{E}\left[\sup_{t\in[0,T]} \|z_t^n\|_{-1}^2\right] \leqslant C\, \frac{\log(1+2\gamma_K T)}{n}. \tag{3.50}$$

(3) *For every positive increasing sequence $\{T_n\}_{n\in\mathbb{N}}$ such that $T_n = \exp(o(n))$ and for all $\eta > 0$, it holds*

$$\lim_{n\to\infty} \mathbf{P}\left(\sup_{t\in[0,T_n]} \|z_t^n\|_{-1} \leqslant \eta\right) = 1. \tag{3.51}$$

Proof. It easy to see that with this definition of z_t^n, for all $h \in H_1$, $z_t^n(h) = \langle z_t^n, h\rangle_{-1,1}$. We start by proving (2).

For $l \geqslant 1$, let $x_t^l := \sqrt{2\lambda_l n}\, e^{\lambda_l t} \left|z_t^n(e^{il\cdot})\right|$. In particular

$$x_t^l = \left| \frac{\sqrt{2\lambda_l}}{\sqrt{n}} \sum_{j=1}^{n} \int_0^t e^{s\lambda_l} e^{il\theta_s^{j,n}}\, \mathrm{d}B_s^j \right| = \left|a_t^l + i\, b_t^l\right|, \tag{3.52}$$

where a^l and b^l are two continuous real valued martingales. Let $\langle x^l\rangle_t = \langle a^l\rangle_t + \langle b^l\rangle_t$ where $\langle a^l\rangle_t$ and $\langle b^l\rangle_t$ are the quadratic variations of a_t^l and b_t^l respectively, then

$$\langle x^l\rangle_t = \frac{2\lambda_l}{n} \sum_{j=1}^{n} \int_0^t e^{2s\lambda_l} (\cos^2 + \sin^2)(l\theta_s^{j,n})\, \mathrm{d}s = e^{2\lambda_l t} - 1. \tag{3.53}$$

We now use

Lemma 3.4. *Let $Y_t = A_t + i\, B_t$, where A_t and B_t are continuous real valued martingales. Define $X_t = |Y_t|$ and $\langle X\rangle_t = \langle A\rangle_t + \langle B\rangle_t$, where $\langle A\rangle_t$ and $\langle B\rangle_t$ are the quadratic variations of A and B respectively. Then, there exists $C > 0$ such that, for all $T > 0$,*

$$\mathbf{E}\left[\sup_{t\in[0,T]} \frac{X_t^2}{1+\langle X\rangle_t}\right] \leqslant C \log(1 + \log(1 + \langle X\rangle_t)). \tag{3.54}$$

The proof of Lemma 3.4 is presented at the end of the section. By choosing $X_t = x_t^l$, $A_t = a_t^l$ and $B_t = b_t^l$, one obtains that, for $T > 0$,

$$\mathbf{E}\left[\sup_{t\in[0,T]} |z_t^n(e_l)|^2\right] = \frac{1}{2\lambda_l n}\, \mathbf{E}\left[\sup_{t\in[0,T]} \frac{(x_t^l)^2}{1+\langle x^l\rangle_t}\right] \leqslant \frac{C}{2\lambda_l n} \log(1 + 2\lambda_l T). \tag{3.55}$$

It remains to observe that

$$\mathbf{E}\left[\sup_{t\in[0,T]}\|z_t^n\|_{-1}^2\right] \leqslant \mathbf{E}\left[\sum_{l\,\geqslant\,1}\sup_{t\in[0,T]}|z_t^n(e_l)|^2\right] \leqslant C\sum_{l\,\geqslant\,1}\frac{1}{2\lambda_l n}\log(1+2\lambda_l T). \tag{3.56}$$

The conclusion holds by factorizing the first term of the sum and modifying the constant C accordingly: observe that $\sum_{l\,\geqslant\,1}\sup_{T\,\geqslant\,1}\frac{\log(1+2\lambda_l T)}{\lambda_l\log(1+2\lambda_1 T)}<\infty$.

Concerning (1), observe that for $s,t\in[0,T]$ and for some $k\geqslant 2$

$$\|z_t^n - z_s^n\|_{-1}^2 \leqslant \sum_{l=1}^{k}|z_t^n(e_l)-z_s^n(e_l)|^2 + 2\sum_{l>k}\sup_{t\in[0,T]}|z_t^n(e_l)|^2. \tag{3.57}$$

The first term can be make small by using the continuity of $z_t^n(e_l)$; for the second one, observe that we have just proven that $\mathbf{E}\left[\sum_{l\,\geqslant\,1}\sup_{t\in[0,T]}|z_t^n(e_l)|^2\right]<\infty$. This implies that there exists a subsequence $\{k_m\}_{m\in\mathbb{N}}$ such that $\sum_{l>k_m}\sup_{t\in[0,T]}|z_t^n(e_l)|^2$ tends to 0 almost surely as m tends to infinity. The almost sure continuity in (3.57) is then established by choosing s and t close enough and k large enough.

Point (3) is an application of Chebycheff inequality to

$$\mathbf{P}\left(\sup_{t\in[0,T_n]}\|z_t^n\|_{-1}>\eta\right) \leqslant \frac{1}{\eta^2}\mathbf{E}\left[\sup_{t\in[0,T_n]}\|z_t^n\|_{-1}^2\right] \tag{3.58}$$

and the bound presented in (2).

The proof is concluded modulo Lemma 3.4, proven hereafter. □

Proof of Lemma 3.4. Recall that A_t is a martingale, in particular a slight variation of [13, Corollary 2.8] implies that there exists $D>0$ such that

$$\mathbf{E}\left[\sup_{t\in[0,T]}\frac{A_t^2}{1+\langle A\rangle_t}\right] \leqslant D\log(1+\log(1+\langle A\rangle_t)). \tag{3.59}$$

Thus, one can develop

$$\mathbf{E}\left[\sup_{t\in[0,T]}\frac{X_t^2}{1+\langle X\rangle_t}\right] \leqslant \mathbf{E}\left[\sup_{t\in[0,T]}\frac{A_t^2}{1+\langle A\rangle_t}\right] + \mathbf{E}\left[\sup_{t\in[0,T]}\frac{B_t^2}{1+\langle B\rangle_t}\right] \leqslant \tag{3.60}$$

$$\leqslant D\log(1+\log(1+\langle A\rangle_t)) + D\log(1+\log(1+\langle B\rangle_t)) \leqslant \tag{3.61}$$

$$\leqslant 2D\log(1+\log(1+\langle X\rangle_t)), \tag{3.62}$$

and the proof is done by taking $C=2D$. □

4. Proofs

4.1. **Long time behavior around the stable stationary solution.** This subsection is devoted to the proof of the long time behavior around the unique stable stationary solution of (1.3). The main result is given by

Proposition 4.1. *If for all $\varepsilon_0>0$*

$$\lim_{n\to\infty}\mathbf{P}\left(\left\|\mu_0^n - \tfrac{1}{2\pi}\right\|_{-1}\leqslant\varepsilon_0\right)=1. \tag{4.1}$$

Then, there exists $A>0$ such that for every positive increasing sequence $\{T_n\}_{n\in\mathbb{N}}$ such that $T_n=\exp(o(n))$ and for all $0<\varepsilon<A$, it holds

$$\lim_{n\to\infty}\mathbf{P}\left(\sup_{t\in[0,T_n]}\left\|\mu_t^n-\tfrac{1}{2\pi}\right\|_{-1}\leqslant\varepsilon\right)=1. \tag{4.2}$$

Proof. Fix $\varepsilon>0$. From Proposition 3.1 we know that $\nu_t^n:=\mu_t^n-\frac{1}{2\pi}$ satisfies

$$\nu_t^n=e^{tL_{2\pi}}\nu_0^n-\int_0^t e^{(t-s)L_{2\pi}}\partial_\theta\left[\nu_s^n(J*\nu_t^n)\right]\,\mathrm{d}s-g_t^n+z_t^n. \tag{4.3}$$

Taking the norm and using the properties of $e^{tL_{2\pi}}$, for all $0<\gamma<\gamma_K$ one obtains

$$\|\nu_t^n\|_{-1}\leqslant\|\nu_0^n\|_{-1}+D_{1,1}\int_0^t\frac{e^{-\gamma(t-s)}}{\sqrt{t-s}}\|\nu_s^n\|_{-1}^2\,\mathrm{d}s+\|g_t^n\|_{-1}+\|z_t^n\|_{-1}. \tag{4.4}$$

We now use the following result, proven rightafter:

Lemma 4.2. *For some $T>0$ and $\gamma>0$, let $f:[0,T]\to[0,\infty)$ be a continuous function and $g:[0,T]\to[0,\infty)$ be such that for all $0\leqslant t\leqslant T$*

$$f(t)\leqslant f(0)+\int_0^t\frac{e^{-\gamma(t-s)}}{\sqrt{t-s}}f^2(s)\,\mathrm{d}s+g(t). \tag{4.5}$$

There exists $A>0$, depending only on γ, such that if $0<\delta<A$ and if $f(0)<\delta$, $\sup_{t\in[0,T]}g(t)<\delta$, then

$$\sup_{t\in[0,T]}f(t)\leqslant 3\delta. \tag{4.6}$$

Thanks to the contractive properties of $L_{2\pi}$, there exists $D>0$ (Lemma 3.2) such that

$$\sup_{t\geqslant 0}\|g_t^n\|_{-1}<D\,\frac{\left\|P^{(n)}-\mathbf{1}^{(n)}\right\|_{\infty\to 1}}{n^2}. \tag{4.7}$$

Define now $B_1^n(\varepsilon_0)=\{\|\nu_0^n\|\leqslant\varepsilon_0\}$ and $B_2^n(\eta)=\{\sup_{t\in[0,T_n]}\|z_t^n\|_{-1}\leqslant\eta\}$. On $B_1^n(\varepsilon/3)\cap B_2^n(\varepsilon/4)$ and for n large enough, we can apply Lemma 4.2 with

$$\delta=\frac{\varepsilon}{3},\qquad T=T_n, \tag{4.8}$$

$$f(t)=\|\nu_t^n\|_{-1}, \tag{4.9}$$

$$g(t)=\|g_t^n\|_{-1}+\|z_t^n\|_{-1}, \tag{4.10}$$

and obtain

$$\sup_{t\in[0,T_n]}\|\nu_t^n\|_{-1}\leqslant\varepsilon. \tag{4.11}$$

The proof is concluded with A given by Lemma 4.2, since by hypothesis $\mathbf{P}(B_1^n)\to 1$ and Lemma 3.3 implies that $\mathbf{P}(B_2^n)\to 1$ as n tends to infinity. □

Proof of Lemma 4.2. Consider the set $O=\{t\ :\ f(t)\leqslant 3\delta\}\subset[0,T]$. Since f is continuous and $f(0)\leqslant\delta$, O is a non-empty open set in $[0,T]$. Suppose that $\sup(O)=u<T$; we show that $u\in O$, which implies $O=[0,T]$.

Consider

$$f(u) = f(0) + \int_0^u \frac{e^{-\gamma(u-s)}}{\sqrt{u-s}} f^2(s)\,\mathrm{d}s + g(u) \leqslant \tag{4.12}$$

$$\leqslant 2\delta + \delta\left(9\delta \int_0^u \frac{e^{-\gamma(u-s)}}{\sqrt{u-s}}\,\mathrm{d}s\right) \leqslant \tag{4.13}$$

$$\leqslant \delta\left[2 + 9\delta \int_0^\infty \frac{e^{-\gamma s}}{\sqrt{s}}\,\mathrm{d}s\right] \leqslant 3\delta, \tag{4.14}$$

where the last inequality holds for all $\delta \leqslant A := \left(9\int_0^\infty \frac{e^{-\gamma s}}{\sqrt{s}}\right)^{-1}$. Thus $u \in O$ and the proof is concluded. □

4.2. **Finite time behavior.** The first important step is given by

Proposition 4.3. *For all $\varepsilon > 0$ and for all $T > 0$, it holds*

$$\lim_{n\to\infty} \mathbf{P}\left(\sup_{t\in[0,T]} \|\mu_t^n - \mu_t\|_{-1} \leqslant \varepsilon\right) = 1. \tag{4.15}$$

Proof. Fix $\varepsilon > 0$ and $T > 0$ and let $F = F_t(\theta) \in \mathcal{C}1,2\left([0,T]\times\mathbb{T}\right)$, a straightforward application of Ito formula gives

$$\begin{aligned}\langle \mu_t^n - \mu_t, F_t\rangle &= \langle \mu_0^n - \mu_0, F_0\rangle + \int_0^t \langle \mu_s^n - \mu_s, \partial_s F_s + \tfrac{1}{2}\partial_\theta^2 F_s\rangle\,\mathrm{d}s + \\ &+ \frac{1}{n^2}\sum_{i,j=1}^n \int_0^t \frac{\xi_{ij}}{p_n} J(\theta_s^{i,n} - \theta_s^{j,n})\partial_\theta F_s(\theta_s^{i,n})\,\mathrm{d}s - \int_0^t \langle \mu_s, (J*\mu_s)\partial_\theta F_s\rangle\,\mathrm{d}s + \\ &+ \frac{1}{n}\sum_{j=1}^n \int_0^t \partial_\theta F_s(\theta_s^{j,n})\,\mathrm{d}B_s^j.\end{aligned} \tag{4.16}$$

Using the bilinearity of the integration, the equation becomes

$$\begin{aligned}\langle \mu_t^n - \mu_t, F_t\rangle &= \langle \mu_0^n - \mu_0, F_0\rangle + \int_0^t \langle \mu_s^n - \mu_s, \partial_s F_s + \tfrac{1}{2}\partial_\theta^2 F_s\rangle\,\mathrm{d}s + \\ &+ \int_0^t \langle \mu_s^n(J*\mu_s^n) - \mu_s(J*\mu_s), \partial_\theta F_s\rangle\,\mathrm{d}s + G_t^n(F) + Z_t^n(F),\end{aligned} \tag{4.17}$$

with

$$G_t^n(F) = \frac{1}{n^2}\sum_{i,j=1}^n \int_0^t \left(\frac{\xi_{ij}}{p_n} - 1\right) J(\theta_s^{i,n} - \theta_s^{j,n})\partial_\theta F_s(\theta_s^{i,n})\,\mathrm{d}s, \tag{4.18}$$

$$Z_t^n(F) = \frac{1}{n}\sum_{j=1}^n \int_0^t \partial_\theta F_s(\theta_s^{j,n})\,\mathrm{d}B_s^j. \tag{4.19}$$

As already done in Proposition 3.1, one can write the H_{-1} formulation associated to (4.17), which now becomes:

$$\mu_t^n - \mu_t = e^{t\frac{\Delta}{2}}\left(\mu_0^n - \mu_0\right) - \int_0^t e^{(t-s)\frac{\Delta}{2}}\partial_\theta\left[\mu_s^n(J*\mu_s^n) - \mu_s(J*\mu_s)\right]\mathrm{d}s - g_t^n + z_t^n, \tag{4.20}$$

where

$$g_t^n = \frac{1}{n^2} \sum_{i,j=1}^{n} \int_0^t \left(\frac{\xi_{ij}}{p_n} - 1 \right) e^{(t-s)\frac{\Delta}{2}} \partial_\theta \left[\delta_{\theta_s^{i,n}} (J * \delta_{\theta_s^{j,n}}) \right] \mathrm{d}s, \tag{4.21}$$

and z_t^n is denoted for $h \in H_1$ by

$$z_t^n(h) = \frac{1}{n} \sum_{j=1}^{n} \int_0^t \partial_\theta e^{(t-s)\frac{\Delta}{2}} h(\theta_s^{j,n}) \, \mathrm{d}B_s^j. \tag{4.22}$$

Taking the H_{-1} norm in (4.20), one is left with

$$\|\mu_t^n - \mu_t\|_{-1} \leqslant \|\mu_0^n - \mu_0\|_{-1} + \int_0^t \tfrac{C}{\sqrt{t-s}} \|\mu_s^n - \mu_s\|_{-1} \, \mathrm{d}s + \|g_t^n\|_{-1} + \|z_t^n\|_{-1}, \tag{4.23}$$

where we have used the properties of $e^{t\frac{\Delta}{2}}$: continuity and the fact that for $h \in H_{-1}$ one has $\left\| e^{t\frac{\Delta}{2}} h \right\|_{-1} \leqslant \frac{C}{\sqrt{t-s}} \|h\|_{-2}$, see Proposition 6.3 for a general result on $e^{t\frac{\Delta}{2}}$.

The term involving the graph g_t^n can be controlled again by $\left\| P^{(n)} - \mathbf{1}^{(n)} \right\|_{\infty \to 1}$: minor modifications to Lemma 3.2 show that there exists $D > 0$ such that

$$\sup_{t \in [0,T]} \|g_t^n\|_{-1} \leqslant D \, \frac{\left\| P^{(n)} - \mathbf{1}^{(n)} \right\|_{\infty \to 1}}{n^2}. \tag{4.24}$$

For the initial conditions and the stochastic part z_t^n, define the two sets:

$$A_1^n = A_1^n(\varepsilon_0) = \left\{ \|\mu_0^n - \mu_0\|_{-1} \leqslant \varepsilon_0 \right\}; \tag{4.25}$$

$$A_2^n = A_2^n(T, \eta) = \left\{ \sup_{t \in [0,T]} \|z_t^n\|_{-1} \leqslant \eta \right\}. \tag{4.26}$$

On $A_1^n \cap A_2^n$, one obtains

$$\|\mu_t^n - \mu_t\|_{-1} \leqslant \varepsilon_0 + \int_0^t \tfrac{C}{\sqrt{t-s}} \|\mu_s^n - \mu_s\|_{-1} \, \mathrm{d}s + D \, \frac{\left\| P^{(n)} - \mathbf{1}^{(n)} \right\|_{\infty \to 1}}{n^2} + \eta. \tag{4.27}$$

Gronwall-Henry's inequality ([16, Lemma 7.1.1 and Exercice 1]) leads to

$$\sup_{t \in [0,T]} \|\mu_t^n - \mu_t\|_{-1} \leqslant 2 \left(\varepsilon_0 + D \, \frac{\left\| P^{(n)} - \mathbf{1}^{(n)} \right\|_{\infty \to 1}}{n^2} + \eta \right) e^{aT}, \tag{4.28}$$

where a is independent of n, ε_0 and η. Considering ε_0 and η small enough and n large enough, the proof is concluded modulo showing that

$$\lim_{n \to \infty} \mathbf{P} \left(A_1^n \cap A_2^n \right) = 1. \tag{4.29}$$

From the hypothesis on the intial condition (2.1), it is clear that for all ε_0 one has $\mathbf{P}\left(A_1^n(\varepsilon_0) \right) \to 1$ as n tends to infinity. The same conclusion holds for A_2^n by slightly modifying the proof of Lemma 3.3. The proof is concluded. □

The last ingredient of this section comes from the properties of the PDE (1.3): for every initial condition the solution converges to $\frac{1}{2\pi}$, indeed

Proposition 4.4 ([12, Proposition 4.1])**.** *If $K \leqslant 1$, for any $\mu_0 \in \mathcal{P}(\mathbb{T})$, we have*

$$\lim_{t \to \infty} \left\| \mu_t - \frac{1}{2\pi} \right\|_{-1} = 0. \tag{4.30}$$

Proof. See the proof of Proposition 4.1 in [12]: it is in a stronger topology that controls all the derivatives. Namely, it implies the convergence in H_{-1}. □

4.3. **Proof of Theorem 2.1.** Propositions 4.3 and 4.4 assure that for every $\varepsilon_0 > 0$ there exists $T > 0$ such that with high probability as n tends to infinity, one has

$$\left\| \mu_T^n - \frac{1}{2\pi} \right\| \leq \varepsilon_0, \tag{4.31}$$

and obviously $\|\mu_T^n - \mu_T\| \leq \varepsilon_0$. But then one can apply Proposition 4.1 to conclude that there exists $A > 0$, independent of T and ε_0, such that for all $0 < \varepsilon < A$ and for all sequences $\{T_n\}$ it holds

$$\sup_{t\in[T,T_n]} \left\| \mu_t^n - \frac{1}{2\pi} \right\| \leqslant \varepsilon, \tag{4.32}$$

with probability going to one as n tends to infinity. Since μ_t will still be arbitrary close to $\frac{1}{2\pi}$, the proof is concluded.

5. Appendix A: Graphs

5.1. **General properties of the graphs under consideration.** We observe that condition (1.10) implies a weak form of degree homogeneity (recall (1.4)):

Lemma 5.1. *Suppose that* (1.10) *holds. Let* $\delta > 0$*, define*

$$I_n^\delta := \left\{ i \in \{1,\dots,n\} : \lim_{n\to\infty} \left| \frac{1}{n} \sum_{j=1}^n \frac{\xi_{i,j}^{(n)}}{p_n} - 1 \right| \geqslant \delta \right\}. \tag{5.1}$$

Then $|I_n^\delta| = o(n)$.

Proof. Suppose that $\lim_{n\to\infty} \frac{|I_n|}{n} = c$ for some $c > 0$. Then

$$\sup_{s_i,t_j\in\{\pm1\}} \frac{1}{n^2} \sum_{i,j=1}^n \left(\frac{\xi_{ij}^{(n)}}{p_n} - 1 \right) s_i t_j \geqslant \sup_{s_i\in\{\pm1\}} \frac{1}{n} \sum_{i=1}^n \left[\frac{1}{n} \sum_{j=1}^n \left(\frac{\xi_{ij}^{(n)}}{p_n} - 1 \right) \right] s_i \geqslant \tag{5.2}$$

$$\geqslant \frac{1}{n} \sum_{i\in I_n} \left| \frac{1}{n} \sum_{j=1}^n \left(\frac{\xi_{ij}^{(n)}}{p_n} - 1 \right) \right| \geqslant \frac{|I_n|}{n} \inf_{i\in I_n} \left| \frac{1}{n} \sum_{j=1}^n \left(\frac{\xi_{ij}^{(n)}}{p_n} - 1 \right) \right|. \tag{5.3}$$

This last term does not go to zero as n tends to infinity, against (1.10). □

It also implies the existence of an unique giant component.

Lemma 5.2. *Suppose that* (1.10) *holds. Then, there exists a unique sequence of connected components* $\{\mathcal{C}^{(n)}\}$ *in* $\{\xi^{(n)}\}$*, such that* $\left|\mathcal{C}^{(n)}\right| = O(n)$.

Proof. We prove the uniqueness first. Suppose that for every n there exist $\mathcal{C}_1^{(n)}$ and $\mathcal{C}_2^{(n)}$ distinct connected components of $\xi^{(n)}$ such that $\left|\mathcal{C}_i^{(n)}\right| = n_i = O(n)$ for $i = 1, 2$. Without loss of generality, one can suppose $\mathcal{C}_1^{(n)}$ consisting in the first n_1 vertices of $\xi^{(n)}$ and $\mathcal{C}_2^{(n)}$ in the following n_2.

Using the equivalence of $\ell_\infty \to \ell_1$ norm with the cut-norm (e.g. [2]), one obtains

$$\|P_n - \mathbf{1}_n\|_{\infty\to 1} \geqslant \sup_{x_i,y_j\in\{0,1\}} \left| \sum_{i,j=1}^{n} \left(\frac{\xi_{ij}}{p_n} - 1\right) x_i y_j \right| \geqslant \sum_{\substack{1\leqslant i\leqslant n_1\\ n_1\leqslant j\leqslant n_2-n_1}} 1 \;=\; n_1 n_2 = O(n^2). \tag{5.4}$$

For the existence, suppose the connected components of $\xi^{(n)}$ are ordered from the biggest one in size (the first n_1 vertices) to the smallest one (the last vertices). Take the first m components such that $|\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_m| \geqslant n/4$. One easily sees that $|\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_m| \leqslant n/2$. Applying the same reasoning of before with $1 \leqslant i \leqslant n/4$ and $n/2 \leqslant j \leqslant n$, the proof is concluded. □

5.2. **Examples of graph sequences.** We exhibit two classes of graphs, a random and a deterministic one, that satisfy assumption (1.10). The only hypothesis required on p_n is equivalent to asking that the mean degree per site diverges as n tends to infinity, i.e. $np_n \uparrow \infty$.

Erdős-Rényi random graphs. As mentioned in the introduction, $\|\cdot\|_{\infty\to 1}$ has been found very useful for random graph concentration and this is indeed the case of ER graphs (e.g. [14]). We recall the definition and give the result.

For every $n \in \mathbb{N}$, let $\{\xi_{ij}^{(n)}\}_{1\leqslant i\neq j\leqslant n}$ be IID Bernoulli random variables with parameter p_n, $\mathbb{P}$ denoting the associated probability. For every i, $\xi_{ii}^{(n)}$ is set equal to 0, i.e. self loop are not admitted.

Lemma 5.3. *Assume that*

$$\lim_{n\to\infty} np_n = \infty. \tag{5.5}$$

There exists $n_0 \in \mathbb{N}$ *such that*

$$\mathbb{P}\left(\sup_{s_i,t_j} \frac{1}{n^2} \sum_{i,j=1}^{n} \left(\tfrac{\xi_{ij}}{p_n} - 1\right) s_i t_j \geqslant \frac{2}{\sqrt{np_n}} \right) \leqslant e^{-2n}, \quad \textit{for all } n \geqslant n_0. \tag{5.6}$$

Proof. The proof is just an union bound and an application of Bernstein's inequality. Indeed,

$$\mathbb{P}\left(\sup_{s_i,t_j} \frac{1}{n^2} \sum_{i,j=1}^{n} \left(\frac{\xi_{ij}}{p_n} - 1\right) s_i t_j \geqslant \frac{\delta}{\sqrt{np_n}} \right) \leqslant \sum_{s_i,t_j} \mathbb{P}\left(\frac{1}{n^2} \sum_{i,j=1}^{n} \left(\frac{\xi_{ij}}{p_n} - 1\right) s_i t_j \geqslant \frac{\delta}{\sqrt{np_n}} \right). \tag{5.7}$$

Bernstein's inequality ([4, Corollary 2.11]) says that if $X_1, \ldots, X_n$ are independent zero-mean random variables such that $|X_j| \le M$ a.s. for all j, then for all $t \ge 0$

$$\mathbb{P}\left(\sum_{j=1}^{n} X_j > t \right) \;\le\; \exp\left\{ -\frac{t^2}{2\sum_{j=1}^{n} \mathbb{E}[X_j^2] + \frac{2}{3}Mt} \right\}.$$

Let $X_{k(i,j)} = \frac{s_i t_j}{n^2 p_n}(\xi_{ij} - p_n)$ with k some bijection from $\{1,\dots,n\}^2$ to $\{1,\dots,n^2\}$. Then $|X_k| \leqslant \frac{1}{n^2 p_n}$ and $\mathbb{E}\left[X_k^2\right] \leqslant \frac{2}{n^4}$. For n large enough, we thus obtain

$$\mathbb{P}\left(\sum_{k=1}^{n^2} X_k \geqslant \frac{\delta}{\sqrt{np_n}}\right) \leq \exp\left\{-\frac{n\delta^2}{4p_n + \frac{2}{3}\frac{\delta}{\sqrt{np_n}}}\right\} \leqslant \exp\left\{-n\delta^2\right\}. \tag{5.8}$$

The proof is concluded observing that the sum in (5.7) consists in 4^n elements and choosing $\delta = 2$. □

We thus have

Proposition 5.4. *Given* (5.5)*, ER graphs satisfy condition* (1.10) $\mathbb{P}$*-almost surely.*

Proof. It suffices to apply Borel-Cantelli lemma to (5.6). □

Ramanujan graphs. Let $d = 2, 3, \dots$, consider a d-regular graph, i.e. graph where each vertex has exactly d neighbors. We start recalling a well-known result

Lemma 5.5 (Expander mixing lemma)**.** *Let G be a d-regular random graph (G denoting the adjacency matrix itself), it holds*

$$\frac{1}{n^2}\left\|\tfrac{n}{d}G - \mathbf{1}^{(n)}\right\|_{\infty\to 1} \leqslant 4\,\frac{\lambda(d)}{d}, \tag{5.9}$$

where $\lambda(d)$ is the second biggest eigenvalue (in absolute value) associated to G.

Proof. The proof is classical but it is in general formulated in terms of the cut-norm (e.g. [17]). One easily sees that the cut-norm is equivalent (paying a factor 4, e.g. [2]) to the $\ell_\infty \to \ell_1$ norm. □

Ramanujan graphs are d-regular graphs such that $\lambda(d) \leqslant 2\sqrt{d-1}$, they are very well known for their expander properties (e.g. [17]). Condition (1.10) holds whenever d_n diverges; indeed

Proposition 5.6. *Let $d_n = np_n$. Suppose that* (5.5) *holds, i.e.*

$$\lim_{n\to\infty} d_n = \infty. \tag{5.10}$$

Then, every sequence of Ramanujan graphs satisfies condition (1.10)*.*

Proof. Rewriting (5.9) in terms of p_n, it becomes

$$\frac{1}{n^2}\left\|\tfrac{G}{p_n} - \mathbf{1}^{(n)}\right\|_{\infty\to 1} \leqslant \frac{8}{\sqrt{np_n}}. \tag{5.11}$$

The proof is concluded taking the limit for n which tends to infinity. □

6. Appendix B: H_{-1} and Semigroups

6.1. **On the Hilbert space H_{-1}.** Recall the definition of H_1, one has this sequence of continuous and dense inclusions:

$$H_1 \subset \mathcal{L}_0^2 = \mathcal{L}_0^{2*} \subset H_1^* =: H_{-1}, \tag{6.1}$$

where we have chosen the canonical identification for $\mathcal{L}_0^2$.

For any given $u \in H_1$, consider the duality functional $\phi_u : H_1 \to \mathbb{R}$ in H_{-1} defined by $\phi_u(v) = (u, v)$. We can define

$$T : H_1 \to H_{-1} \tag{6.2}$$

$$u \mapsto \phi_u. \tag{6.3}$$

It is known [5, pag. 82] that $T(H)$ is dense in H_{-1} and that T injects H_1 into H_{-1} in a continuous way. This injection allows considering H_1 as a subset of H_{-1} by identifying u and Tu.

The space H_{-1} is again an Hilbert space with inner product given by

$$\langle u, v\rangle_{-1} = \int \mathcal{U}\mathcal{V}, \tag{6.4}$$

where $\mathcal{U}$ and $\mathcal{V}$ are two primitives of u and v respectively, such that $\int \mathcal{U} = 0 = \int \mathcal{V}$. Indeed, one can explicit the isometry between H_1 and H_{-1}:

$$U : H_1 \to H_{-1} \tag{6.5}$$

$$f \mapsto -\partial_\theta^2 f. \tag{6.6}$$

Namely, for $f, g \in \mathcal{C}\infty_0$, it holds

$$\langle Uf, Ug\rangle_{-1} = \int f'g' = \langle f, g\rangle_1. \tag{6.7}$$

In particular, this implies $\|u\|_{-1} = \left\|U^{-1}u\right\|_1 = \sqrt{\int \mathcal{U}^2}$, with $\int \mathcal{U} = 0$.

6.2. **The relationship between H_{-1} and $\mathcal{P}(\mathbb{T})$.** As already shown in (1.12), the difference between probability measures is in H_{-1}. Observe now that H_{-1} induces a distance on $\mathcal{P}(\mathbb{T})$ which controls the bounded-Lipschitz distance d_{bL}, i.e. for all $\mu, \nu \in \mathcal{P}(\mathbb{T})$

$$d_{\mathrm{bL}}(\mu,\nu) = \sup_{\|f\|_{\mathrm{bL}}=1} \int f\,(\mathrm{d}\mu - \mathrm{d}\nu) \leqslant \sup_{h\in\mathcal{C}_0^1, \|h\|_1=1} \int h\,(\mathrm{d}\mu - \mathrm{d}\nu) = \tag{6.8}$$

$$= \sup_{h\in\mathcal{C}_0^1, \|h\|_1=1} \int h'\,(\mathcal{U} - \mathcal{V}) = \sup_{\|h\|_1=1} \langle \mu - \nu, h\rangle_{-1,1} = \tag{6.9}$$

$$= \|\mu - \nu\|_{-1}\,. \tag{6.10}$$

Where we have used the density of $\mathcal{C}_0^1$ in H_1, and denoted by $\mathcal{U}$ and $\mathcal{V}$ the primitives of μ and ν respectively.

6.3. **The linear operator $L_{2\pi}$.** We make use of the fractional norm $\|\cdot\|_{-1-\beta}$ defined for $h \in \mathcal{L}_0^2$ by

$$\|h\|_{-1-\beta} = \left\|(-\Delta)^{\beta/2}h\right\|_{-1} = \sum_{l\,\geqslant\,1} \frac{\left(h, e^{il\cdot}\right)^2}{l^{2+2\beta}}. \tag{6.11}$$

Observe that it is equivalent to

$$\left\|(-\mathrm{Ł}_{2\pi})^{\beta/2}h\right\|_{-1} = \sum_{l\,\geqslant\,1} \frac{\left(h, e^{il\cdot}\right)^2}{\lambda_l^{2+2\beta}}. \tag{6.12}$$

Proposition 6.1. *The operator $L^*_{2\pi}$ (resp. $L_{2\pi}$) is essentially self-adjoint with compact resolvent in H_1 (resp. H_{-1}), its spectrum is given by $\{-\frac{1-K}{2}\} \cup \{-\frac{l^2}{2}\}_{l\geqslant 2}$. Moreover, both $L_{2\pi}$ and $L^*_{2\pi}$ generate a $\mathcal{C}0$-semigroup $t \mapsto e^{tL_{2\pi}}$ (resp. $t \mapsto e^{tL^*_{2\pi}}$) in $\mathcal{L}^2_0$ and $e^{tL^*_{2\pi}} = \left(e^{tL_{2\pi}}\right)^*$.*

Proof. A simple computation shows that the operator $L_{2\pi}$ is diagonal in the Fourier basis. For the continuity and the duality of the semigroups, see for example [16] . □

Denote $\gamma_K = \frac{1-K}{2}$. One has the following estimates.

Proposition 6.2. *For any $\gamma \in [0, \gamma_K)$, any $\beta \in [0, 1]$ and all $t > 0$, $h \in H_1$, there exists a positive constant $D_{\gamma,\beta}$ such that*

$$\left\| e^{tL^*_{2\pi}} h \right\|_{1+\beta} \leqslant D_{\gamma,\beta} \frac{e^{-\gamma t/2}}{t^{\beta/2}} \left\| h \right\|_1 . \tag{6.13}$$

The semigroup $e^{tL_{2\pi}}$ is continuous from H_{-2} to H_{-1} and for all $\gamma \in [0, \gamma_K)$, any $\beta \in [0, 1]$ and all $t > 0$, $u \in H_{-1}$,

$$\left\| e^{tL_{2\pi}} u \right\|_{-1} \leqslant D_{\gamma,\beta} \frac{e^{-\gamma t/2}}{t^{\beta/2}} \left\| u \right\|_{-1-\beta} . \tag{6.14}$$

Proof. Let $\{\lambda_l\}_{l\geqslant 1}$ be the eigenvalues associated to $L^*_{2\pi}$. For $h = \sum_{l\geqslant 1} h_l z_l$, recall the fractional norm

$$\left\| h \right\|_{1+\beta} = \left\| (-L^*_{2\pi})^{\beta/2} h \right\|_1 = \sqrt{\sum_{l\geqslant 1} \lambda_l^\beta h_l^2}. \tag{6.15}$$

Fix $\gamma \in [0, \gamma_K)$, $\beta \in [0, 1]$ and $t > 0$. Then

$$\left\| e^{tL^*_{2\pi}} h \right\|^2_{1+\beta} = \left\| (-L^*_{2\pi})^{\beta/2} e^{tL^*_{2\pi}} h \right\|^2_1 = \sum_{l\geqslant 1} \lambda_l^\beta e^{-t\lambda_l} h_l^2. \tag{6.16}$$

Namely

$$\sum_{l\geqslant 1} \lambda_l^\beta e^{-t\lambda_l} h_l^2 \leqslant \sup_{t\geqslant 0\ l\geqslant 1} \left\{ (t\lambda_l)^\beta e^{-t(\lambda_l - \gamma)} \right\} \frac{e^{-\gamma l}}{t^\beta} \sum_{l\geqslant 1} h_l^2 = D \frac{e^{-\gamma l}}{t^\beta} \|h\|_1^2, \tag{6.17}$$

where $D = \sup_{t\geqslant 0\ l\geqslant 1} \left\{ (t\lambda_l)^\beta e^{-t(\lambda_l-\gamma)} \right\}$. Using the fact that $\gamma < \gamma_K \leqslant -\lambda_l$ for $l = 1, 2, \ldots$ it easy to see that

$$D = \sup_{t\geqslant 0\ l\geqslant 1} t^\beta (\lambda_l - \gamma)^\beta e^{-t(\lambda_l-\gamma)} \left(\frac{\lambda_l}{\lambda_l - \gamma} \right)^\beta < \infty. \tag{6.18}$$

Finally, it suffices to take $D_{\gamma,\delta} = \sqrt{D}$.

The second inequality follows similarly. □

Very similarly, one can prove some well-known properties of the Laplacian operator

Proposition 6.3. *The operator $\frac{\Delta}{2}$ is sectorial and self-adjoint in H_1; its spectrum is given by $\left\{-k^2/2\right\}_{k\geqslant 1}$. Moreover $\frac{\Delta}{2}$ generates a $\mathcal{C}0$-semigroup $t \to e^{t\frac{\Delta}{2}}$ in $\mathcal{L}^2_0$ such that:*

(1) *For all $\alpha \geqslant \beta \geqslant 0$, there exists a constant $C_{\alpha,\beta} > 0$ such that for all $h \in \mathcal{L}^2_0$:*

$$\left\| e^{t\frac{\Delta}{2}} h \right\|_{-1} \leqslant \frac{C_{\alpha,\beta}}{t^{\alpha/2}} \left\| h \right\|_{-1-\beta} , \quad \text{for all } t \geqslant 0. \tag{6.19}$$

(2) *For all* $\alpha \geqslant \beta \geqslant 0$ *and for all* $h \in \mathcal{L}^2_0$ *:*

$$\left\| e^{t\frac{\Delta}{2}} h \right\|_{1+\beta} \leqslant \frac{C_{\alpha,\beta}}{t^{\alpha/2}} \left\| h \right\|_1, \quad \text{for all } t \geqslant 0. \tag{6.20}$$

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