GEVREY INDEX THEOREM FOR SOME INHOMOGENEOUS SEMILINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

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Abstract. In this article, we are interested in the Gevrey properties of the formal power series solution in time of some partial differential equations with a power-law nonlinearity and with analytic coefficients at the origin of $\mathbb{C}^2$. We prove in particular that the inhomogeneity of the equation and the formal solution are together s-Gevrey for any $s \geqslant s_c$, where s_c is a nonnegative rational number fully determined by the Newton polygon of the associated linear PDE. In the opposite case $s < s_c$, we show that the solution is generically s_c-Gevrey while the inhomogeneity is s-Gevrey, and we give an explicit example in which the solution is s'-Gevrey for no $s' < s_c$.

Contents

1. Setting the problem

For several years, various works have been done on the divergent solutions of some classes of linear partial differential equations or integro-differential equations in two variables or more, allowing thus to formulate many results on Gevrey properties, summability or multisummability (e.g. [1, 3–6, 9, 11, 12, 14, 20, 22, 24, 25, 28–36, 42, 43, 49–51, 59, 61]).

In the case of the nonlinear partial differential equations, the situation is much more complicated. The existing results concern mainly Gevrey properties, especially the convergence (e.g. [10, 16, 18, 19, 21, 26, 37–39, 48, 52–58]), and there are very few results about the summation (see [17, 23, 27, 41, 44]).

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In this article, we propose to investigate the Gevrey properties of the inhomogeneous semilinear partial differential equation

$$\begin{cases} \partial_t^\kappa u - t^v a(t,x)\partial_x^p u - b(t,x)u^m = \widetilde{f}(t,x) \\ \partial_t^j u(t,x)_{|t=0} = \varphi_j(x), j=0,...,\kappa-1 \end{cases} \tag{1.1}$$

in two variables $(t,x)\in\mathbb{C}^2$, where

- κ and p are two positive integers;
- v is a nonnegative integer;
- the coefficients $a(t,x)$ and $b(t,x)$ are analytic on a polydisc $D_{\rho_0}\times D_{\rho_1}$ centered at the origin $(0,0)$ of $\mathbb{C}^2$ (D_ρ denotes the disc with center $0\in\mathbb{C}$ and radius $\rho>0$) and $a(0,x)\not\equiv 0$;
- the degree of the power-law nonlinearity is an integer $\geqslant 2$;
- the inhomogeneity $\widetilde{f}(t,x)$ is a formal power series in t with analytic coefficients in D_{ρ_1} (we denote by $\widetilde{f}(t,x)\in\mathcal{O}(\mathcal{D}_{\rho_1})[[t]]$) which may be smooth, or not[1];
- the initial conditions $\varphi_j(x)$ are analytic on D_{ρ_1} for all $j=0,...,\kappa-1$.

Equation (1.1) is fundamental in many physical, chemical, biological, and ecological problems. For example: for $(\kappa,p)=(1,2)$, eq. (1.1) arises in problems involving diffusion and nonlinear growth such as heat and mass transfer, combustion theory, and spread theory of animal or plant populations (nonlinear heat equation); for $(\kappa,p)=(2,2)$, eq. (1.1) describes the propagation of nonlinear waves in an inhomogeneous medium (nonlinear Klein-Gordon equation); and, for $(\kappa,p)=(2,4)$, eq. (1.1) describes the relationship between the beam's deflection and an applied lateral nonlinear force (nonlinear Bernoulli-Euler equation).

Notation 1.1. In the sequel, we write any formal series $\widetilde{g}(t,x)\in\mathcal{O}(\mathcal{D}_{\rho_1})[[t]]$ on the form

$$\widetilde{g}(t,x)=\sum_{j\geqslant 0} g_{j,*}(x)\frac{t^j}{j!} \text{ with } g_{j,*}(x)\in\mathcal{O}(\mathcal{D}_{\rho_1}) \text{ for all } j.$$

Then, it is easy to check that eq. (1.1) admits a *unique* formal series solution

$$\widetilde{u}(t,x)\sum_{j\geqslant 0} u_{j,*}(x)\frac{t^j}{j!}\in\mathcal{O}(\mathcal{D}_{\rho_1})[[t]],$$

where the coefficients $u_{j,*}(x)\in\mathcal{O}(\mathcal{D}_{\rho_1})$ are recursively determined by the initial conditions $u_{j,*}(x)=\varphi_j(x)$ $(j=0,...,\kappa-1)$ and, for all $j\geqslant 0$, by the relations

$$u_{j+\kappa,*}(x)=f_{j,*}(x)+\sum_{\ell=0}^{j-v}\frac{j!}{\ell!(j-v-\ell)!}a_{\ell,*}(x)\partial_x^p u_{j-v-\ell,*}(x)+ \sum_{\ell=0}^{j}\sum_{\substack{\ell_1+...+\ell_m\\=j-\ell}}\frac{j!}{\ell!\ell_1!...\ell_m!}b_{\ell,*}(x)u_{\ell_1,*}(x)...u_{\ell_m,*}(x) \tag{1.2}$$

with the classical convention that the first sum is zero as soon as $j-v<0$.

Doing that, a natural question arises:

"What relationship exists between the Gevrey order of the solution $\widetilde{u}(t,x)$ and the Gevrey order of the inhomogeneity $\widetilde{f}(t,x)$?"

[1] We denote $\widetilde{f}$ with a tilde to emphasize the possible divergence of the series $\widetilde{f}$.

Indeed, according to the algebraic structure of the s-Gevrey spaces $\mathcal{O}(\mathcal{D}_{\rho_1})[[t]]_s$ (see section 2 for the exact definition of theses spaces), it is classical one has

$$\widetilde{u}(t,x) \in \mathcal{O}(\mathcal{D}_{\rho_1})[[t]]_s \Rightarrow \widetilde{f}(t,x) \in \mathcal{O}(\mathcal{D}_{\rho_1})[[t]]_s.$$

But, what can we say about the converse?

A precise answer was given by the author in the special case of the semilinear heat equation

$$\begin{cases} \partial_t u - \alpha(x)\partial_x^2 u - \beta(x)u^m = \widetilde{f}(t,x) \\ u(0,x) = \varphi(x) \end{cases} \tag{1.3}$$

Proposition 1.2 ([48]). *Let $\widetilde{u}(t,x)$ be the formal solution in $\mathcal{O}(\mathcal{D}_{\rho_1})[[t]]$ of eq. (1.3). Then,*

1. *$\widetilde{u}(t,x)$ and $\widetilde{f}(t,x)$ are together s-Gevrey for any $s \geqslant 1$;*
2. *$\widetilde{u}(t,x)$ is generically 1-Gevrey while $\widetilde{f}(t,x)$ is s-Gevrey with $s < 1$.*

In particular, we observe that this result highlights the special value $s_c = 1$, which is defined as the (inverse of the) positive slope of the Newton polygon at $t = 0$ of the homogeneous linear heat equation $\partial_t u - \alpha(x)\partial_x^2 u = 0$. We call this value the *critical value of* eq. (1.3).

On the other hand, in the linear case

$$\begin{cases} \partial_t^\kappa u - a(t,x)\partial_x^p u = \widetilde{f}(t,x) \\ \partial_t^j u(t,x)_{|t=0} = \varphi_j(x), j = 0, ..., \kappa - 1 \end{cases} \tag{1.4}$$

the author has also proved in [49] that the solution $\widetilde{u}(t,x)$ and the inhomogeneity $\widetilde{f}(t,x)$ are together convergent when $p \leqslant \kappa$ and $1/k$-Gevrey otherwise, where k denotes the positive slope of the Newton polygon at $t = 0$ of the homogeneous associated equation.

The aim of this article is to extend these two results to the very general eq. (1.1). To do this, the organization of the paper is as follows. In section 2, we biefly recall the definition and some basic properties about the s-Gevrey formal power series in $\mathcal{O}(\mathcal{D}_{\rho_1})[[t]]$ which are needed in the sequel. Section 3 is devoted to the main result of the article (theorem 3.1), which states, on one hand, that the solution $\widetilde{u}(t,x)$ and the inhomogeneity $\widetilde{f}(t,x)$ are together s-Gevrey for any s greater than a convenient *critical value* $s_c \geqslant 0$ which is fully determined by the Newton polygon at $t = 0$ of the linear part $L_{\kappa,p} := \partial_t^\kappa - t^v a(t,x)\partial_x^p$ of eq. (1.1), and, on the other hand, that $\widetilde{u}(t,x)$ is generically s_c-Gevrey while $\widetilde{f}(t,x)$ is s-Gevrey with $s < s_c$. A detailed proof of this result is developed in section 4.

2. Gevrey formal series

Before stating our main result (see theorem 3.1 below), let us first recall for the convenience of the reader some definitions and basic properties about the Gevrey formal series in $\mathcal{O}(\mathcal{D}_{\rho_1})[[t]]$, which are needed in the sequel.

All along the article, we consider t as the variable and x as a parameter. Thereby, to define the notion of *Gevrey classes of formal power series in* $\mathcal{O}(\mathcal{D}_{\rho_1})[[t]]$, one extends the classical notion of *Gevrey classes of elements in* $\mathbb{C}[[t]]$ to families parametrized by x in requiring similar conditions, the estimates being however uniform with respect to x. Doing that, any formal power series of $\mathcal{O}(\mathcal{D}_{\rho_1})[[t]]$ can be seen as a formal power series in t with coefficients in a convenient Banach space defined as the space of functions that are holomorphic on a disc $\mathcal{D}_\rho$ $(0 < \rho < \rho_1)$ and continuous up to its boundary, equipped with the usual supremum norm. For a general study of series with coefficients in a Banach space, we refer for instance to [2].

Definition 2.1. Let $s \geqslant 0$ be. A formal series

$$\widetilde{u}(t,x) = \sum_{j\geqslant 0} u_{j,*}(x)\frac{t^j}{j!} \in \mathcal{O}(\mathcal{D}_{\rho_1})[[t]]$$

is said to be *Gevrey of order* s (in short, s*-Gevrey*) if there exist three positive constants $0 < \rho < \rho_1$, $C > 0$ and $K > 0$ such that the inequalities

$$\sup_{|x|\leqslant\rho} |u_{j,*}(x)| \leqslant CK^j\Gamma(1+(s+1)j)$$

hold for all $j \geqslant 0$.

In other words, definition 2.1 means that $\widetilde{u}(t,x)$ is s-Gevrey in t, uniformly in x on a neighborhood of $x = 0 \in \mathbb{C}$.

We denote by $\mathcal{O}(\mathcal{D}_{\rho_1})[[t]]_s$ the set of all the formal series in $\mathcal{O}(\mathcal{D}_{\rho_1})[[t]]$ which are s-Gevrey. Observe that the set $\mathbb{C}\{t,x\}$ of germs of analytic functions at the origin of $\mathbb{C}^2$ coincides with the union $\bigcup_{\rho_1>0}\mathcal{O}(\mathcal{D}_{\rho_1})[[t]]_0$; in particular, any element of $\mathcal{O}(\mathcal{D}_{\rho_1})[[t]]_0$ is convergent and $\mathbb{C}\{t,x\} \cap \mathcal{O}(\mathcal{D}_{\rho_1})[[t]] = \mathcal{O}(\mathcal{D}_{\rho_1})[[t]]_0$. Observe also that the sets $\mathcal{O}(\mathcal{D}_{\rho_1})[[t]]_s$ are filtered as follows:

$$\mathcal{O}(\mathcal{D}_{\rho_1})[[t]]_0 \subset \mathcal{O}(\mathcal{D}_{\rho_1})[[t]]_s \subset \mathcal{O}(\mathcal{D}_{\rho_1})[[t]]_{s'} \subset \mathcal{O}(\mathcal{D}_{\rho_1})[[t]]$$

for all s and s' satisfying $0 < s < s' < +\infty$.

Following proposition 2.2 specifies the algebraic structure of $\mathcal{O}(\mathcal{D}_{\rho_1})[[t]]_s$.

Proposition 2.2 ([2, 49])**.** *Let $s \geqslant 0$. Then, the set $(\mathcal{O}(\mathcal{D}_{\rho_1})[[t]]_s, \partial_t, \partial_x)$ is a $\mathbb{C}$-differential algebra.*

We are now turn to the study of the Gevrey properties of eq. (1.1).

3. Gevrey index theorem

As we said in section 1, the aim of this article is to generalize the results obtained in [48, 49] by making explicit the relationship between the Gevrey order of the solution $\widetilde{u}(t,x)$ and the Gevrey order of the inhomogeneity $\widetilde{f}(t,x)$.

As in [48], this relationship is fully determined by a critical value that depends solely on the Newton polygon $N_t(L_{\kappa,p})$ at $t=0$ of the linear part $L_{\kappa,p} := \partial_t^\kappa - t^v a(t,x)\partial_x^p$ of eq. (1.1)[2].

Before stating our main result (see theorem 3.1 below), let us begin with a brief study of $N_t(L_{\kappa,p})$.

[2]Observe that this fact is well-known in the case of the ODEs: the Gevrey order of the formal solutions of any semilinear meromorphic ordinary differential equations is given by the Newton polygon of its linear part –see [7, 47] for instance.

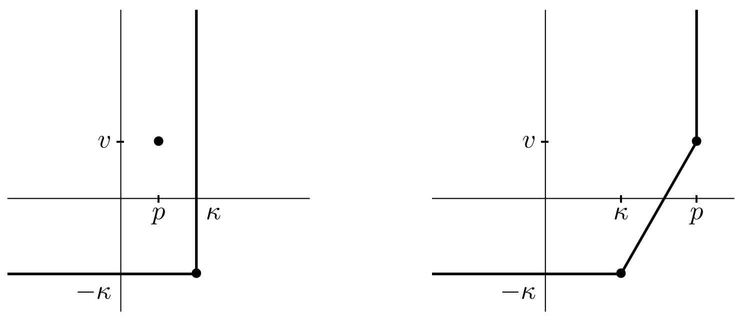
◁ Newton polygon $N_t(L_{\kappa,p})$. As definition of the Newton polygon, we choose the definition of M. Miyake [34] (see also A. Yonemura [61] or S. Ouchi [42]) which is an analogue to the one given by J.-P. Ramis [46] for linear ordinary differential equations. Recall that, H. Tahara and H. Yamazawa use in [59] a slightly different one.

The Newton polygon $N_t(L_{\kappa,p})$ is then defined as the convex hull of $C(\kappa, -\kappa) \cup C(p, v)$, where $C(a, b)$ denotes for any $(a, b) \in \mathbb{R}^2$ the domain

$$C(a, b) = \{(x, y) \in \mathbb{R}^2; x \leqslant a \text{ and } y \geqslant b\}.$$

Hence, the following two cases.

- *First case:* $p \leqslant \kappa$. $N_t(L_{\kappa,p})$ has no side with a positive slope (see fig. 1a).
- *Second case:* $p > \kappa$. $N_t(L_{\kappa,p})$ has just one side with a positive slope and this slope is $k = \dfrac{\kappa + v}{p - \kappa}$ (see fig. 1b).



(A) Case $p \leqslant \kappa$ (B) Case $p > \kappa$

FIGURE 1. The Newton polygon $N_t(L_{\kappa,p})$

◁ Main result. We are now able to state the result in view in this article.

Theorem 3.1 (Gevrey index theorem)**.** *Let s_c be the rational number defined by*

$$s_c := \begin{cases} 0 & \text{if } p \leqslant \kappa \\ \dfrac{1}{k} = \dfrac{p-\kappa}{\kappa+v} & \text{if } p > \kappa \end{cases}$$

Then,

1. *$\widetilde{u}(t, x)$ and $\widetilde{f}(t, x)$ are together s-Gevrey for any $s \geqslant s_c$;*
2. *$\widetilde{u}(t, x)$ is generically s_c-Gevrey while $\widetilde{f}(t, x)$ is s-Gevrey with $s < s_c$.*

Definition 3.2. The number s_c defined in theorem 3.1 is called *the critical value of eq. (1.1).*

Observe, in the case of eq. (1.3), that theorem 3.1 coincides with proposition 1.2. We have indeed $\kappa = 1$, $p = 2$ and $v = 0$; hence, $s_c = 1$.

Observe also that, since no condition is made on the coefficient $b(t, x)$ except it is analytic at the origin $(0, 0) \in \mathbb{C}^2$, theorem 3.1 applies as well to the linear case $b(t, x) \equiv 0$ and generalizes thereby the result already obtained in [49].

The proof of theorem 3.1 is detailed in section 4 below. The first point is the most technical and the most complicated. Its proof is based on the Nagumo norms, a technique of majorant series and a fixed point procedure (see section 4.1). As for the second point, it stems both from the first one and from proposition 4.11 that gives an explicit example for which $\widetilde{u}(t,x)$ is s'-Gevrey for no $s' < s_c$ while $\widetilde{f}(t,x)$ is s-Gevrey with $s < s_c$ (see section 4.2).

4. Proof of theorem 3.1

4.1. **Proof of the first point.** According to proposition 2.2, it is clear that

$$\widetilde{u}(t,x) \in \mathcal{O}(\mathcal{D}_{\rho_1})[[t]]_s \Rightarrow \widetilde{f}(t,x) \in \mathcal{O}(\mathcal{D}_{\rho_1})[[t]]_s.$$

Reciprocally, let us fix $s \geqslant s_c$ and let us suppose that the inhomogeneity $\widetilde{f}(t,x)$ is s-Gevrey. By assumption, its coefficients $f_{j,*}(x) \in \mathcal{O}(\mathcal{D}_{\rho_1})$ satisfy the following condition (see definition 2.1): there exist three positive constants $0 < \rho < \rho_1$, $C > 0$ and $K > 0$ such that the inequalities

$$|f_{j,*}(x)| \leqslant CK^j\Gamma(1+(s+1)j) \tag{4.1}$$

hold for all $j \geqslant 0$ and all $|x| \leqslant \rho$.

We must prove that the coefficients $u_{j,*}(x) \in \mathcal{O}(\mathcal{D}_{\rho_1})$ of $\widetilde{u}(t,x)$ satisfy similar inequalities. The approach we present below is analoguous to the ones already developed in [4, 49–51] in the framework of linear partial and integro-differential equations, and in [48] in the case of the semilinear heat equation. It is based on the Nagumo norms [8, 40, 60] and on a technique of majorant series. However, as we shall see, our calculations appear to be much more technical and complicated, especially because the coefficients $a(t,x)$ and $b(t,x)$ are not constant in the variable t, but also because the valuation v of $a(t,x)$ with respect to t is not necessarily zero. Furthermore, the nonlinear term u^m generates several technical combinatorial situations.

Before starting the calculations, let us first recall for the convenience of the reader the definition of the Nagumo norms and some of their properties which are needed in the sequel.

4.1.1. *Nagumo norms.*

Definition 4.1. Let $f \in \mathcal{O}(\mathcal{D}_{\rho_1})$, $q \geqslant 0$ and $0 < r < \rho_1$ be. Then, the *Nagumo norm* $\|f\|_{q,r}$ *with indices* (q,r) *of* f is defined by

$$\|f\|_{q,r} := \sup_{|x|\leqslant r} |f(x)d_r(x)^q|,$$

where $d_r(x)$ denotes the Euclidian distance $d_r(x) := r - |x|$.

Following proposition 4.2 gives us some properties of the Nagumo norms.

Proposition 4.2. *Let $f, g \in \mathcal{O}(\mathcal{D}_{\rho_1})$, $q, q' \geqslant 0$ and $0 < r < \rho_1$ be. Then,*

1. $\|\cdot\|_{q,r}$ *is a norm on* $\mathcal{O}(\mathcal{D}_{\rho_1})$.
2. $|f(x)| \leqslant \|f\|_{q,r}\, d_r(x)^{-q}$ *for all* $|x| < r$.
3. $\|f\|_{0,r} = \sup_{|x|\leqslant r} |f(x)|$ *is the usual sup-norm on the disc* D_r.
4. $\|fg\|_{q+q',r} \leqslant \|f\|_{q,r}\,\|g\|_{q',r}$.
5. $\|\partial_x f\|_{q+1,r} \leqslant e(q+1)\,\|f\|_{q,r}$.

Inequalities 4–5 of proposition 4.2 are the most important properties of the Nagumo norms. Observe besides that the same index r occurs on their both sides, allowing thus to get estimates for the product fg in terms of f and g and for the derivatives $\partial_x f$ in terms of f without having to shrink the disc D_r.

Let us now turn to the proof of the first point theorem 3.1.

4.1.2. *Some inequalities.* From recurrence relations (1.2), we first get the following identities for all $j \geqslant 0$:

$$\frac{u_{j+\kappa,*}(x)}{\Gamma(1+(s+1)(j+\kappa))} = \frac{f_{j,*}(x)}{\Gamma(1+(s+1)(j+\kappa))} + \sum_{\ell=0}^{j-v} \frac{j!}{\ell!(j-v-\ell)!} \frac{a_{\ell,*}(x)\partial_x^p u_{j-v-\ell,*}(x)}{\Gamma(1+(s+1)(j+\kappa))} + \sum_{\ell=0}^{j} \sum_{\substack{\ell_1+\ldots+\ell_m\\=j-\ell}} \frac{j!}{\ell!\ell_1!\ldots\ell_m!} \frac{b_{\ell,*}(x)u_{\ell_1,*}(x)\ldots u_{\ell_m,*}(x)}{\Gamma(1+(s+1)(j+\kappa))} \tag{4.2}$$

with the initial conditions $u_{j,*}(x) = \varphi_j(x)$ for all $j = 0, \ldots, \kappa-1$.

Let us now define the positive constants $\sigma_s := (s+1)(\kappa+v)$ and

$$A_j := \frac{\|u_{j,*}\|_{j\sigma_s,\rho}}{\Gamma(1+(s+1)j)} = \frac{\|\varphi_j\|_{j\sigma_s,\rho}}{\Gamma(1+(s+1)j)} \tag{4.3}$$

for all $j = 0, \ldots, \kappa-1$.

Remark 4.3. Observe that the condition $s \geqslant s_c$ implies

$$\sigma_s \geqslant \sigma_{s_c} = \begin{cases} \kappa+v & \text{if } p \leqslant \kappa \\ p+v & \text{if } p > \kappa \end{cases},$$

and, therefore,

$$\sigma_s \geqslant p+v. \tag{4.4}$$

By applying the Nagumo norm of indices $((j+\kappa)\sigma_s, \rho)$ to relations (4.2), we derive from property 1 of proposition 4.2 the relations:

$$\frac{\|u_{j+\kappa,*}\|_{(j+\kappa)\sigma_s,\rho}}{\Gamma(1+(s+1)(j+\kappa))} \leqslant \frac{\|f_{j,*}\|_{(j+\kappa)\sigma_s,\rho}}{\Gamma(1+(s+1)(j+\kappa))} + \sum_{\ell=0}^{j-v} \frac{j!}{\ell!(j-v-\ell)!} \frac{\|a_{\ell,*}\partial_x^p u_{j-v-\ell,*}\|_{(j+\kappa)\sigma_s,\rho}}{\Gamma(1+(s+1)(j+\kappa))} + \sum_{\ell=0}^{j} \sum_{\substack{\ell_1+\ldots+\ell_m\\=j-\ell}} \frac{j!}{\ell!\ell_1!\ldots\ell_m!} \frac{\|b_{\ell,*}u_{\ell_1,*}\ldots u_{\ell_m,*}\|_{(j+\kappa)\sigma_s,\rho}}{\Gamma(1+(s+1)(j+\kappa))}.$$

Next, properties 4-5 of proposition 4.2 bring us to the following inequalities:

$$\begin{aligned}(4.5)\quad \frac{\|u_{j+\kappa,*}\|_{(j+\kappa)\sigma_s,\rho}}{\Gamma(1+(s+1)(j+\kappa))} \leqslant \frac{\|f_{j,*}\|_{(j+\kappa)\sigma_s,\rho}}{\Gamma(1+(s+1)(j+\kappa))}+\\ \sum_{\ell=0}^{j-v} A_{j,\ell,s}\frac{\|a_{\ell,*}\|_{(\kappa+v+\ell)\sigma_s-p,\rho}}{\ell!}\frac{\|u_{j-v-\ell,*}\|_{(j-v-\ell)\sigma_s,\rho}}{\Gamma(1+(s+1)(j-v-\ell))}+\\ \sum_{\ell=0}^{j}\sum_{\substack{\ell_1+...+\ell_m\\=j-\ell}} B_{j,\ell,\ell_1,...\ell_m,s}\frac{\|b_{\ell,*}\|_{(\ell+\kappa)\sigma_s,\rho}\|u_{\ell_1,*}\|_{\ell_1\sigma_s,\rho}...\|u_{\ell_m,*}\|_{\ell_m\sigma_s,\rho}}{\ell!\Gamma(1+(s+1)\ell_1)...\Gamma(1+(s+1)\ell_m)},\end{aligned}$$

where the constants $A_{j,\ell,s}$ and $B_{j,\ell,\ell_1,...\ell_m,s}$ are positive and defined by

$$A_{j,\ell,s} := \frac{j!e^p\left(\prod_{\ell'=0}^{p-1}((j-v-\ell)\sigma_s+p-\ell')\right)\Gamma(1+(s+1)(j-v-\ell))}{(j-v-\ell)!\Gamma(1+(s+1)(j+\kappa))}$$

$$B_{j,\ell,\ell_1,...\ell_m,s} := \frac{j!}{\ell_1!...\ell_m!}\frac{\Gamma(1+(s+1)\ell_1)...\Gamma(1+(s+1)\ell_m)}{\Gamma(1+(s+1)(j+\kappa))}.$$

Remark 4.4. Of course, all the norms, especially the norm $\|a_{\ell,*}\|_{(\kappa+v+\ell)\sigma_s-p,\rho}$, are well-defined. Indeed, due to inequality (4.4), we have

$$(\kappa+v+\ell)\sigma_s-p \geqslant (\kappa+v)(p+v)-p = p(\kappa+v-1)+v(\kappa+v)$$

and, consequently, $(\kappa+v+\ell)\sigma_s-p \geqslant 0$ since $v \geqslant 0$ and $p,\kappa \geqslant 1$.

Following propositions 4.5 and 4.8 allows to bound the constants $A_{j,\ell,s}$ and $B_{j,\ell,\ell_1,...\ell_m,s}$.

Proposition 4.5. *Let $j \geqslant v$ and $\ell \in \{0,...,j-v\}$ be. Then,*

$$A_{j,\ell,s} \leqslant (e(\kappa+v))^p.$$

Proof. Proposition 4.5 stems from the two following lemmas 4.6 and 4.7. □

Lemma 4.6. *Let $j \geqslant 0$ and $\ell \in \{0,...,j-v\}$ be. Then,*

$$\frac{j!}{(j-v-\ell)!\Gamma(1+(s+1)(j+\kappa))} \leqslant \frac{1}{\Gamma(1+(s+1)(j+\kappa-\ell)-v)}.$$

Proof. Lemma 4.6 is clear for $\ell+v=0$. Let us now assume $\ell+v \geqslant 1$ and let us write the quotient $j!/(j-v-\ell)!$ on the form

$$(4.6)\qquad \frac{j!}{(j-v-\ell)!} = \prod_{\ell'=0}^{\ell+v-1}(j-\ell').$$

On the other hand, applying $\ell+v$ times the recurrence relation $\Gamma(1+z)=z\Gamma(z)$ to $\Gamma(1+(s+1)(j+\kappa))$, we get:

$$(4.7)\quad \Gamma(1+(s+1)(j+\kappa)) = \Gamma(1+(s+1)(j+\kappa)-\ell-v)\prod_{\ell'=0}^{\ell+v-1}((s+1)(j+\kappa)-\ell').$$

Combinating then (4.6) and (4.7), we obtain

$$\frac{j!}{(j-v-\ell)!\Gamma(1+(s+1)(j+\kappa))} = \frac{\displaystyle\prod_{\ell'=0}^{\ell+v-1} \frac{j-\ell'}{(s+1)(j+\kappa)-\ell'}}{\Gamma(1+(s+1)(j+\kappa))} \leqslant \frac{1}{\Gamma(1+(s+1)(j+\kappa)-\ell-v)}$$

and lemma 4.6 follows from the inequalities

$$\begin{aligned} 1+(s+1)(j+\kappa)-\ell-v &\geqslant 1+(s+1)(j+\kappa-\ell)-v \\ &\geqslant 1+\sigma_s-v \\ &\geqslant 1+p \qquad \text{(relation (4.4))} \\ &\geqslant 2 \end{aligned}$$

and from the increase of the Gamma function on $[2,+\infty[$. □

Lemma 4.7. *Let $j \geqslant 0$ and $\ell \in \{0,...,j-v\}$ be. Then,*

$$\frac{\displaystyle\prod_{\ell'=0}^{p-1}((j-v-\ell)\sigma_s+p-\ell')}{\Gamma(1+(s+1)(j+\kappa-\ell)-v)} \leqslant \frac{(\kappa+v)^p}{\Gamma(1+(s+1)(j-v-\ell))}.$$

Proof. ⊲ Let us first assume $\ell = j-v$. We must prove the inequality

$$\frac{\displaystyle\prod_{\ell'=0}^{p-1}(p-\ell')}{\Gamma(1+(s+1)(\kappa+v)-v)} \leqslant (\kappa+v)^p.$$

Using the relation (4.4), we have

$$1+(s+1)(\kappa+v)-v = 1+\sigma_s-v = 1+p \geqslant 2;$$

hence,

$$\Gamma(1+(s+1)(\kappa+v)-v) \geqslant \Gamma(1+p) = p! = \prod_{\ell'=0}^{p-1}(p-\ell')$$

since the Gamma function is increasing on $[2,+\infty[$. Lemma 4.7 follows then from the inequality $\kappa+v \geqslant 1$.

⊲ Let us now assume $\ell < j-v$. Due to the definition of σ_s, we first have

$$\prod_{\ell'=0}^{p-1}((j-v-\ell)\sigma_s+p-\ell') = (\kappa+v)^p \prod_{\ell'=0}^{p-1}\left((s+1)(j-v-\ell)+\frac{p-\ell'}{\kappa+v}\right). \tag{4.8}$$

On the other hand, applying p times the recurrence relation $\Gamma(1+z) = z\Gamma(z)$ to $\Gamma(1+(s+1)(j+\kappa-\ell)-v)$, we besides have

$$\Gamma(1+(s+1)(j+\kappa-\ell)-v) = \Gamma(1+(s+1)(j+\kappa-\ell)-v-p)\prod_{\ell'=0}^{p-1}((s+1)(j+\kappa-\ell)-v-\ell'). \tag{4.9}$$

Observe that this identity makes since the relation (4.4) implies

$$(s+1)(j+\kappa-\ell)-v-p > \sigma_s-v-p \geqslant 0.$$

Observe also that we have the inequality

$$(s+1)(j-v-\ell)+\frac{p-\ell'}{\kappa+v}\leqslant(s+1)(j+\kappa-\ell)-v-\ell'$$

for all $\ell'\in\{0,...,p-1\}$. Indeed, the relation (4.4) and the inequality $\kappa+v\geqslant 1$ imply

$$\begin{aligned}(s+1)(j-v-\ell)+\frac{p-\ell'}{\kappa+v}-(s+1)(j+\kappa-\ell)+v+\ell'\\ =\frac{p-\ell'}{\kappa+v}-\sigma_s+v+\ell'\leqslant(p-\ell')\left(\frac{1}{\kappa+v}-1\right)\leqslant 0.\end{aligned}$$

Consequently, identities (4.8) and (4.9) provide the following inequality

$$\frac{\prod_{\ell'=0}^{p-1}((j-v-\ell)\sigma_s+p-\ell')}{\Gamma(1+(s+1)(j+\kappa-\ell)-v)}\leqslant\frac{(\kappa+v)^p}{\Gamma(1+(s+1)(j-\ell+\kappa)-v-p)},$$

and lemma 4.7 follows then from the relations

$$\begin{aligned}1+(s+1)(j-\ell+\kappa)-v-p&\geqslant 1+(s+1)(j-\ell+\kappa)-\sigma_s\\ &=1+(s+1)(j-v-\ell)\geqslant 2\end{aligned}$$

and from the increase of the Gamma function on $[2,+\infty[$. Observe that the first inequality stems again from the inequality (4.4). Observe also that, without the condition $j<\ell-v$, the second inequality is no longer valid.

This ends the proof of lemma 4.7. □

Proposition 4.8. *Let $j\geqslant 0$ and $\ell\in\{0,...,j\}$. Then, for all $\ell_1,...,\ell_m\in\mathbb{N}$ such that $\ell_1+...+\ell_m=j-\ell$:*

$$B_{j,\ell,\ell_1,...\ell_m,s}\leqslant 1.$$

Proof. First of all, let us write $B_{j,\ell,\ell_1,...\ell_m,s}$ on the form

$$B_{j,\ell,\ell_1,...\ell_m,s}=B'_{j,\ell,\ell_1,...\ell_m,s}\times B''_{j,\ell,\ell_1,...\ell_m,s}$$

with

$$\begin{aligned}B'_{j,\ell,\ell_1,...\ell_m,s}&:=\frac{j!}{(j-\ell)!}\frac{\Gamma(1+(s+1)(j-\ell))}{\Gamma(1+(s+1)(j+\kappa))}\\ B''_{j,\ell,\ell_1,...\ell_m,s}&:=\frac{(j-\ell)!}{\ell_1!...\ell_m!}\frac{\Gamma(1+(s+1)\ell_1)...\Gamma(1+(s+1)\ell_m)}{\Gamma(1+(s+1)(j-\ell))}.\end{aligned}$$

Since $B''_{j,\ell,\ell_1,...\ell_m,s}\leqslant 1$ (see the proof of [48, Prop. 4.5]), it is sufficient to prove that $B'_{j,\ell,\ell_1,...\ell_m,s}\leqslant 1$.

When $j=0$, this is clear due to the increase of the Gamma function on $[2,+\infty[$ and the condition $\kappa\geqslant 1$.

Let us now assume $j \geqslant 1$. From the recurrence relation $\Gamma(1+z) = z\Gamma(z)$ applied ℓ times, we first derive the following relations:

$$\begin{aligned}\frac{j!}{(j-\ell)!}\Gamma(1+(s+1)(j-\ell)) &= \Gamma(1+(s+1)(j-\ell))\prod_{\ell'=1}^{\ell}(j-\ell+\ell') \\ &\leqslant \Gamma(1+(s+1)(j-\ell))\prod_{\ell'=1}^{\ell}((s+1)(j-\ell)+\ell') \\ &= \Gamma(1+(s+1)(j-\ell)+\ell)\end{aligned}$$

with the classical convention that the product is 1 as soon as $\ell = 0$. Next, since the condition $j \geqslant 1$ implies

$$1+(s+1)(j+\kappa) \geqslant 1+(s+1)j \geqslant 1+(s+1)(j-\ell)+\ell \geqslant 2,$$

we deduce from the increase of the Gamma function on $[2,+\infty[$ the inequalities

$$\Gamma(1+(s+1)(j-\ell)+\ell) \leqslant \Gamma(1+(s+1)j) \leqslant \Gamma(1+(s+1)(j+\kappa)) \tag{4.10}$$

and, consequently, $B'_{j,\ell,\ell_1,\ldots\ell_m,s} \leqslant 1$.

This ends the proof of proposition 4.8. □

Apply propositions 4.5 and 4.8 to inequalities (4.5). We get:

$$\begin{aligned}\frac{\|u_{j+\kappa,*}\|_{(j+\kappa)\sigma_s,\rho}}{\Gamma(1+(s+1)(j+\kappa))} \leqslant{}& \frac{\|f_{j,*}\|_{(j+\kappa)\sigma_s,\rho}}{\Gamma(1+(s+1)(j+\kappa))} + \\ &\sum_{\ell=0}^{j-v}\alpha_{\ell,s}\frac{\|u_{j-v-\ell,*}\|_{(j-v-\ell)\sigma_s,\rho}}{\Gamma(1+(s+1)(j-v-\ell))} + \\ &\sum_{\ell=0}^{j}\sum_{\substack{\ell_1+\ldots+\ell_m\\=j-\ell}}\beta_{\ell,s}\frac{\|u_{\ell_1,*}\|_{\ell_1\sigma_s,\rho}\ldots\|u_{\ell_m,*}\|_{\ell_m\sigma_s,\rho}}{\Gamma(1+(s+1)\ell_1)\ldots\Gamma(1+(s+1)\ell_m)}\end{aligned} \tag{4.11}$$

for all $j \geqslant 0$, where the constants $\alpha_{\ell,s}$ and $\beta_{\ell,s}$ are positive and defined by

$$\alpha_{\ell,s} := (e(\kappa+v))^p\frac{\|a_{\ell,*}\|_{(\kappa+v+\ell)\sigma_s-p,\rho}}{\ell!} \text{ and } \beta_{\ell,s} := \frac{\|b_{\ell,*}\|_{(\ell+\kappa)\sigma_s,\rho}}{\ell!}.$$

We shall now bound the Nagumo norms $\|u_{j,*}\|_{j\sigma_s,\rho}$ for any $j \geqslant 0$. To do that, we shall proceed similarly as in [4, 48–51] by using a technique of majorant series. However, as we shall see, the calculations are much more complicated.

4.1.3. *A Majorant Series.* Let us consider the formal series $v(X) = \sum_{j\geqslant 0} v_j X^j$, where the coefficients v_j are recursively determined by the initial conditions $v_j = A_j$ ($j = 0, \ldots, \kappa-1$; see relations (4.3)) and, for all $j \geqslant 0$, by the relations

$$v_{j+\kappa} = g_j + \sum_{\ell=0}^{j-v}\alpha_{\ell,s}v_{j-v-\ell} + \sum_{\ell=0}^{j}\sum_{\substack{\ell_1+\ldots+\ell_m\\=j-\ell}}\beta_{\ell,s}v_{\ell_1}\ldots v_{\ell_m} \tag{4.12}$$

with

$$g_j := \frac{\|f_{j,*}\|_{(j+\kappa)\sigma_s,\rho}}{\Gamma(1+(s+1)(j+\kappa))}.$$

By construction, we have

$$0 \leqslant \frac{\|u_{j,*}\|_{j\sigma_s,\rho}}{\Gamma(1+(s+1)j)} \leqslant v_j \tag{4.13}$$

for all $j \geqslant 0$ (proceed by induction on j). Following proposition 4.9 allows us to bound the v_j's.

Proposition 4.9. *The formal series $v(X)$ is convergent. In particular, there exist two positive constants $C', K' > 0$ such that $v_j \leqslant C'K'^j$ for all $j \geqslant 0$.*

Proof. It is sufficient to prove the convergence of $v(X)$.

First of all, let us observe that $v(X)$ is the unique formal power series in X solution of the functional equation

$$(1 - X^{\kappa+v}\alpha(X))v(X) = X^{\kappa}\beta(X)(v(X))^m + h(X), \tag{4.14}$$

where $\alpha(X) := \sum_{j\geqslant 0} \alpha_{j,s}X^j$, $\beta(X) := \sum_{j\geqslant 0} \beta_{j,s}X^j$ and

$$h(X) := A_0 + A_1X + ... + A_{\kappa-1}X^{\kappa-1} + X^{\kappa}\sum_{j\geqslant 0} g_jX^j$$

are three convergent power series with nonnegative coefficients. Indeed, according to the inequalities (4.1) and (4.10), and the analyticity of $a(t,x)$ and $b(t,x)$ at the origin $(0,0) \in \mathbb{C}^2$, we have

- $0 \leqslant g_j \leqslant \dfrac{CK^j\Gamma(1+(s+1)j)\rho^{(j+\kappa)\sigma_s}}{\Gamma(1+(s+1)(j+\kappa))} \leqslant C\rho^{\kappa\sigma_s}(K\rho^{\sigma_s})^j$,
- $0 \leqslant \alpha_{j,s} \leqslant \dfrac{(e(\kappa+v))^pC_1K_1^jj!\rho^{(\kappa+v+j)\sigma_s-p}}{j!} = C_1'K_1'^j$,
- $0 \leqslant \beta_{j,s} \leqslant \dfrac{C_1K_1^jj!\rho^{(j+\kappa)\sigma_s}}{j!} = C_1''K_1''^j$

with convenient positive constants C_1, K_1, C_1', K_1', C_1'' and K_1''. We denote in the sequel by $r_\alpha > 0$ (resp. $r_\beta > 0$, $r_h > 0$) the radius of convergence of the series $\alpha(X)$ (resp. $\beta(X)$, $h(X)$). We also denote by $r_\alpha' > 0$ the radius of convergence of the series $1/(1 - X^{\kappa+v}\alpha(X))$.

Next, we proceed through a fixed point method as follows. Let us set

$$V(X) = \sum_{i\geqslant 0} V_i(X)$$

and let us choose the solution of eq. (4.14) given by the system

$$\begin{cases} (1 - X^{\kappa+v}\alpha(X))V_0(X) = h(X) \\ (1 - X^{\kappa+v}\alpha(X))V_{i+1}(X) = X^{\kappa}\beta(X) \displaystyle\sum_{\substack{\ell_1+...+\ell_m \\ =i}} V_{\ell_1}(X)...V_{\ell_m}(X) \quad \text{for } i \geqslant 0. \end{cases}$$

By induction on $i \geqslant 0$, we easily check that

$$V_i(X) = \frac{C_{i,m}X^{\kappa i}(\beta(X))^i(h(X))^{i(m-1)+1}}{(1 - X^{\kappa+v}\alpha(X))^{im+1}}, \tag{4.15}$$

where the $C_{i,m}$'s are the positive constants recursively determined from $C_{0,m} := 1$ by the relations

$$C_{i+1,m} = \sum_{k_1+...+k_m=i} C_{k_1,m}...C_{k_m,m}.$$

Thereby, all the V_i's are analytic functions on the disc with center $0 \in \mathbb{C}$ and radius $\min(r'_\alpha, r_\beta, r_h)$ at least. Moreover, identities (4.15) show us that $V_i(X)$ is of order $X^{\kappa i}$ for all $i \geqslant 0$. Consequently, the series $V(X)$ makes sense as a formal power series in X and we get $V(X) = v(X)$ by unicity.

We are left to prove the convergence of $V(X)$. To do that, let us choose $0 < r < \min(r'_\alpha, r_\alpha, r_\beta, r_h)$. By definition, the constants $C_{i,m}$'s are the generalized Catalan numbers of order m and we have[3]

$$C_{i,m} = \frac{1}{(m-1)i+1}\binom{im}{i} \leqslant 2^{im}$$

for all $i \geqslant 0$ (see [13, 15, 45] for instance). On the other hand, the convergent series $\alpha(X)$, $\beta(X)$ and $h(X)$ define increasing functions on $[0, r]$. Therefore, identities (4.15) imply the inequalities

$$|V_i(X)| \leqslant \frac{h(r)}{1 - r^{\kappa+v}\alpha(r)} \left(\frac{2^m \beta(r)(h(r))^{m-1}}{(1 - r^{\kappa+v}\alpha(r))^m} |X|^\kappa \right)^i$$

for all $i \geqslant 0$ and all $|X| \leqslant r$. Consequently, the series $V(X)$ is normally convergent on any disc with center $0 \in \mathbb{C}$ and radius

$$0 < r' < \min\left(r, \left(\frac{(1 - r^{\kappa+v}\alpha(r))^m}{2^m \beta(r)(h(r))^{m-1}} \right)^{1/\kappa} \right).$$

This proves the analyticity of $V(X)$ at 0 and achieves then the proof of proposition 4.9. □

According to relations (4.13), proposition 4.9 allows us to also bound the Nagumo norms $\|u_{j,*}\|_{j\sigma_s,\rho}$.

Corollary 4.10. *Let $C', K' > 0$ be as in proposition 4.9. Then, the inequalities*

$$\|u_{j,*}\|_{j\sigma_s,\rho} \leqslant C' K'^j \Gamma(1 + (s+1)j)$$

hold for all $j \geqslant 0$.

We are now able to conclude the proof of theorem 3.1.

4.1.4. *Conclusion.* We must prove on the sup-norm of the $u_{j,*}(x)$ estimates similar to the ones on the norms $\|u_{j,*}\|_{j\sigma_s,\rho}$ (see corollary 4.10). To this end, we proceed by shrinking the closed disc $|x| \leqslant \rho$. Let $0 < \rho' < \rho$. Then, for all $j \geqslant 0$ and all $|x| \leqslant \rho'$, we have

$$|u_{j,*}(x)| = \left| u_{j,*}(x) d_\rho(x)^{j\sigma_s} \frac{1}{d_\rho(x)^{j\sigma_s}} \right| \leqslant \frac{|u_{j,*}(x) d_\rho(x)^{j\sigma_s}|}{(\rho - \rho')^{j\sigma_s}} \leqslant \frac{\|u_{j,*}\|_{j\sigma_s,\rho}}{(\rho - \rho')^{j\sigma_s}}$$

and, consequently,

$$\sup_{|x| \leqslant \rho'} |u_{j,*}(x)| \leqslant C' \left(\frac{K'}{(\rho - \rho')^{\sigma_s}} \right)^j \Gamma(1 + (s+1)j)$$

[3] These numbers were named in honor of the mathematician Eugène Charles Catalan (1814-1894). They appear in many probabilist, graphs and combinatorial problems. For example, they can be seen as the number of m-ary trees with i source-nodes, or as the number of ways of associating i applications of a given m-ary operation, or as the number of ways of subdividing a convex polygon into i disjoint $(m+1)$-gons by means of non-intersecting diagonals. They also appear in theoretical computers through the generalized Dyck words. See for instance [13] and the references inside.

by applying corollary 4.10. This ends the proof of the first point of theorem 3.1.

4.2. **Proof of the second point.** Let us fix $s < s_c$[4]. According to the filtration of the s-Gevrey spaces $\mathcal{O}(\mathcal{D}_{\rho_1})[[t]]_s$ (see section 2) and the first point of theorem 3.1, it is clear that we have the following implications:

$$\widetilde{f}(t,x) \in \mathcal{O}(\mathcal{D}_{\rho_1})[[t]]_s \Rightarrow \widetilde{f}(t,x) \in \mathcal{O}(\mathcal{D}_{\rho_1})[[t]]_{s_c} \Rightarrow \widetilde{u}(t,x) \in \mathcal{O}(\mathcal{D}_{\rho_1})[[t]]_{s_c}.$$

To conclude that we can not say better about the Gevrey order of $\widetilde{u}(t,x)$, that is $\widetilde{u}(t,x)$ is *generically* s_c-Gevrey, we need to find an example for which the solution $\widetilde{u}(t,x)$ of eq. (1.1) is s'-Gevrey for no $s' < s_c$. Proposition 4.11 below provides such an example.

Proposition 4.11. *Let us consider the equation*

$$\begin{cases} \partial_t^\kappa u - at^v \partial_x^p u - bu^m = \widetilde{f}(t,x), \ \ a > 0, b \geqslant 0 \\ \partial_t^j u(t,x)_{|t=0} = \varphi(x), j = 0,...,\kappa-1 \end{cases} \tag{4.16}$$

where $\varphi(x)$ is the analytic function on D_1 defined by

$$\varphi(x) = \frac{1}{1-x}.$$

Suppose that the inhomogeneity $\widetilde{f}(t,x)$ satisfies the following conditions:

- *$\widetilde{f}(t,x)$ is s-Gevrey;*
- *$\partial_x^\ell f_{j,*}(0) \geqslant 0$ for all $\ell, j \geqslant 0$.*

Then, the formal solution $\widetilde{u}(t,x)$ of eq. (4.16) is exactly s_c-Gevrey.

Proof. Due to the calculations above, it is sufficient to prove that $\widetilde{u}(t,x)$ is s'-Gevrey for no $s' < s_c$.

First of all, we derive from the general relations (1.2) that the coefficients $u_{j,*}(x)$ of $\widetilde{u}(t,x)$ are recursively determined by the initial conditions $u_{j,*}(x) = \varphi(x)$ ($j = 0,...,\kappa-1$) and, for all $j \geqslant 0$ by the relations

$$u_{j+\kappa,*}(x) = f_{j,*}(x) + \frac{aj!}{(j-v)!}\partial_x^p u_{j-v,*}(x) + b \sum_{\substack{\ell_1+...\ell_m \\ =j}} \frac{j!}{\ell_1!...\ell_m!} u_{\ell_1}...u_{\ell_m}.$$

In particular, we easily check that the coefficients $u_{j(\kappa+v),*}(x)$ read for all $j \geqslant 1$ on the form

$$u_{j(\kappa+v),*}(x) = a^j \partial_x^{jp}\varphi(x) \prod_{\ell=1}^{j} \frac{(\ell v + (\ell-1)\kappa)!}{((\ell-1)v + (\ell-1)\kappa)!} + \text{rem}_{j(\kappa+v)}(x),$$

where $\text{rem}_{j(\kappa+v)}(x)$ is a linear combination with nonnegative coefficients of terms of the form

$$\prod_{\substack{\ell \in \{0,...,jv+(j-1)\kappa\} \\ d_1,d_2 \geqslant 0 \\ p_1,p_2,p_3,p_4 \geqslant 0}} a^{p_1} b^{p_2} \left(\partial_x^{d_1} f_{\ell,*}(x)\right)^{p_3} \left(\partial_x^{d_2}\varphi(x)\right)^{p_4}.$$

[4] Of course, this case only occurs when $p > \kappa$.

Using then our assumptions on the coefficients a and b and on the inhomogeneity $\widetilde{f}(t,x)$, and applying technical lemmas 4.12 and 4.13 below, we finally get the following inequalities:

$$u_{j(\kappa+v),*}(0) \geqslant a^j (jp)! \prod_{\ell=1}^{j} \frac{(\ell v + (\ell-1)\kappa)!}{((\ell-1)v + (\ell-1)\kappa)!} \geqslant \left(\frac{a}{2^{p+v}}\right)^j (j(p+v))!. \tag{4.17}$$

Let us now suppose that $\widetilde{u}(t,x)$ is s'-Gevrey for some $s' < s_c$. Then, definition 2.1 and inequality (4.17) imply

$$1 \leqslant C \left(\frac{2^{p+v}K}{a}\right)^j \frac{\Gamma(1 + j(s'+1)(\kappa+v))}{\Gamma(1+j(p+v))}$$

for all $j \geqslant 0$ and some convenient positive constants C and K independent of j. Proposition 4.11 follows since such inequalities are impossible: applying the Stirling's Formula, we get

$$C \left(\frac{2^{p+v}K}{a}\right)^j \frac{\Gamma(1 + j(s'+1)(\kappa+v))}{\Gamma(1+j(p+v))} \underset{j\to+\infty}{\sim} C' \left(\frac{K'}{j^{\sigma}}\right)^j \tag{4.18}$$

with

- $C' := C\sqrt{\dfrac{(s'+1)(\kappa+v)}{p+v}}$;
- $K' := \dfrac{2^{p+v}K}{a} \dfrac{((s'+1)(\kappa+v))^{(s'+1)(\kappa+v)}}{(p+v)^{p+v}} e^{p+v-(s'+1)(\kappa+v)}$;
- $\sigma := p + v - (s'+1)(\kappa+v)$.

and the right hand-side of (4.18) goes to 0 when j tends to infinity. Indeed, the condition $s' < s_c$ implies

$$\sigma > p + v - (s_c+1)(\kappa+v) = 0.$$

This ends the proof. □

Lemma 4.12. *Let $j \geqslant 1$ be. Then,*

$$\prod_{\ell=1}^{j} \frac{(\ell v + (\ell-1)\kappa)!}{((\ell-1)v + (\ell-1)\kappa)!} \geqslant (jv)!. \tag{4.19}$$

Proof. Lemma 4.12 is clear for $j = 1$. Let us now suppose that inequality (4.19) holds for a certain $j \geqslant 1$. Then,

$$\prod_{\ell=1}^{j+1} \frac{(\ell v + (\ell-1)\kappa)!}{((\ell-1)v + (\ell-1)\kappa)!} \geqslant \frac{((j+1)v + j\kappa)!}{(jv + j\kappa)!}(jv)!$$

and we conclude due to the inequality $\dbinom{(j+1)v + j\kappa}{j\kappa} \geqslant \dbinom{jv + j\kappa}{j\kappa}$. □

Lemma 4.13. *Let $j \geqslant 1$ be. Then,*

$$(jp)!(jv)! \geqslant \frac{(j(p+v))!}{2^{j(p+v)}}.$$

Proof. Lemma 4.13 is direct from the inequality $\dbinom{j(p+v)}{jp} \leqslant 2^{j(p+v)}$. □

This ends the proof of the second point of theorem 3.1; hence, completes the proof of the Gevrey index theorem

References

[1] W. Balser. Divergent solutions of the heat equation: on an article of Lutz, Miyake and Schäfke. *Pacific J. Math.*, 188(1):53–63, 1999.

[2] W. Balser. *Formal power series and linear systems of meromorphic ordinary differential equations*. Universitext. Springer-Verlag, New-York, 2000.

[3] W. Balser. Multisummability of formal power series solutions of partial differential equations with constant coefficients. *J. Differential Equations*, 201(1):63–74, 2004.

[4] W. Balser and M. Loday-Richaud. Summability of solutions of the heat equation with inhomogeneous thermal conductivity in two variables. *Adv. Dyn. Syst. Appl.*, 4(2):159–177, 2009.

[5] W. Balser and M. Miyake. Summability of formal solutions of certain partial differential equations. *Acta Sci. Math. (Szeged)*, 65(3-4):543–551, 1999.

[6] W. Balser and M. Yoshino. Gevrey order of formal power series solutions of inhomogeneous partial differential equations with constant coefficients. *Funkcial. Ekvac.*, 53:411–434, 2010.

[7] B. L. J. Braaksma. Multisummability of formal power series solutions of nonlinear meromorphic differential equations. *Ann. Inst. Fourier (Grenoble)*, 42(3):517–540, 1992.

[8] M. Canalis-Durand, J.-P. Ramis, R. Schäfke, and Y. Sibuya. Gevrey solutions of singularly perturbed differential equations. *J. Reine Angew. Math.*, 518:95–129, 2000.

[9] O. Costin, H. Park, and Y. Takei. Borel summability of the heat equation with variable coefficients. *J. Differential Equations*, 252(4):3076–3092, 2012.

[10] T. Gramchev and G. Lysik. Uniform analytic-Gevrey regularity of solutions to a semilinear heat equation. *Banach Center Publ.*, 81:213–226, 2008.

[11] M. Hibino. Borel summability of divergence solutions for singular first-order partial differential equations with variable coefficients. I & II. *J. Differential Equations*, 227(2):499–563, 2006.

[12] M. Hibino. On the summability of divergent power series solutions for certain first-order linear PDEs. *Opuscula Math.*, 35(5):595–624, 2015.

[13] P. Hilton and J. Pedersen. Catalan numbers, their generalization, and their uses. *Math. Intelligencer*, 13(2):64–75, 1991.

[14] K. Ichinobe. On k-summability of formal solutions for a class of partial differential operators with time dependent coefficients. *J. Differential Equations*, 257(8):3048–3070, 2014.

[15] D. A. Klarner. Correspondences between plane trees and binary sequences. *J. Combinatorial Theory*, 9:401–411, 1970.

[16] A. Lastra and S. Malek. On parametric Gevrey asymptotics for some nonlinear initial value Cauchy problems. *J. Differential Equations*, 259:5220–5270, 2015.

[17] A. Lastra and S. Malek. On parametric multisummable formal solutions to some nonlinear initial value Cauchy problems. *Adv. Differ. Equ.*, 2015:200, 2015.

[18] A. Lastra, S. Malek, and J. Sanz. On Gevrey solutions of threefold singular nonlinear partial differential equations. *J. Differential Equations*, 255:3205–3232, 2013.

[19] A. Lastra and H. Tahara. Maillet type theorem for nonlinear totally characteristic partial differential equations. *Math. Ann.*, https://doi.org/10.1007/s00208-019-01864-x, 2019.

[20] D. A. Lutz, M. Miyake, and R. Schäfke. On the Borel summability of divergent solutions of the heat equation. *Nagoya Math. J.*, 154:1–29, 1999.

[21] G. Lysik and S. Michalik. Formal solutions of semilinear heat equations. *J. Math. Anal. Appl.*, 341:372–385, 2008.

[22] S. Malek. On the summability of formal solutions of linear partial differential equations. *J. Dyn. Control Syst.*, 11(3):389–403, 2005.

[23] S. Malek. On the summability of formal solutions of nonlinear partial differential equations with shrinkings. *J. Dyn. Control Syst.*, 13(1):1–13, 2007.

[24] S. Malek. On the Stokes phenomenon for holomorphic solutions of integrodifferential equations with irregular singularity. *J. Dyn. Control Syst.*, 14(3):371–408, 2008.

[25] S. Malek. On Gevrey functions solutions of partial differential equations with fuchsian and irregular singularities. *J. Dyn. Control Syst.*, 15(2):277–305, 2009.

[26] S. Malek. On Gevrey asymptotic for some nonlinear integro-differential equations. *J. Dyn. Control Syst.*, 16(3):377–406, 2010.

[27] S. Malek. On the summability of formal solutions for doubly singular nonlinear partial differential equations. *J. Dyn. Control Syst.*, 18(1):45–82, 2012.
[28] S. Michalik. Summability of divergent solutions of the n-dimensional heat equation. *J. Differential Equations*, 229:353–366, 2006.
[29] S. Michalik. Summability of formal solutions to the n-dimensional inhomogeneous heat equation. *J. Math. Anal. Appl.*, 347:323–332, 2008.
[30] S. Michalik. On the multisummability of divergent solutions of linear partial differential equations with constant coefficients. *J. Differential Equations*, 249:551–570, 2010.
[31] S. Michalik. Summability and fractional linear partial differential equations. *J. Dyn. Control Syst.*, 16(4):557–584, 2010.
[32] S. Michalik. Multisummability of formal solutions of inhomogeneous linear partial differential equations with constant coefficients. *J. Dyn. Control Syst.*, 18(1):103–133, 2012.
[33] S. Michalik. Corrigendum to "On the multisummability of divergent solutions of linear partial differential equations with constant coefficients"[j. differential equations 249 (3) (2010) 551-570]. *J. Differential Equations*, 255:2400–2401, 2013.
[34] M. Miyake. Newton polygons and formal Gevrey indices in the Cauchy-Goursat-Fuchs type equations. *J. Math. Soc. Japan*, 43(2):305–330, 1991.
[35] M. Miyake. Borel summability of divergent solutions of the Cauchy problem to non-Kovaleskian equations. In *Partial differential equations and their applications (Wuhan, 1999)*, pages 225–239. World Sci. Publ., River Edge, NJ, 1999.
[36] M. Miyake and Y. Hashimoto. Newton polygons and Gevrey indices for linear partial differential operators. *Nagoya Math. J.*, 128:15–47, 1992.
[37] M. Miyake and A. Shirai. Convergence of formal solutions of first order singular nonlinear partial differential equations in the complex domain. *Ann. Polon. Math.*, 74:215–228, 2000.
[38] M. Miyake and A. Shirai. Structure of formal solutions of nonlinear first order singular partial differential equations in complex domain. *Funkcial. Ekvac.*, 48:113–136, 2005.
[39] M. Miyake and A. Shirai. Two proofs for the convergence of formal solutions of singular first order nonlinear partial differential equations in complex domain. *Surikaiseki Kenkyujo Kokyuroku Bessatsu, Kyoto Unviversity*, B37:137–151, 2013.
[40] M. Nagumo. Über das Anfangswertproblem partieller Differentialgleichungen. *Jap. J. Math.*, 18:41–47, 1942.
[41] S. Ouchi. Genuine solutions and formal solutions with Gevrey type estimates of nonlinear partial differential equations. *J. Math. Sci. Univ. Tokyo*, 2:375–417, 1995.
[42] S. Ouchi. Multisummability of formal solutions of some linear partial differential equations. *J. Differential Equations*, 185(2):513–549, 2002.
[43] S. Ouchi. Borel summability of formal solutions of some first order singular partial differential equations and normal forms of vector fields. *J. Math. Soc. Japan*, 57(2):415–460, 2005.
[44] M. E. Pliś and B. Ziemian. Borel resummation of formal solutions to nonlinear Laplace equations in 2 variables. *Ann. Polon. Math.*, 67(1):31–41, 1997.
[45] G. Pólya and G. Szegö. *Aufgaben und Lehrsätze aus der Analysis, Vol. I*, volume 125. Springer-Verlag, Berlin, Göttingen, Heidelberg, 1954.
[46] J.-P. Ramis. Théorèmes d'indices Gevrey pour les équations différentielles ordinaires. *Mem. Amer. Math. Soc.*, 48:viii+95, 1984.
[47] J.-P. Ramis and Y. Sibuya. Hukuhara domains and fundamental existence and uniqueness theorems for asymptotic solutions of Gevrey type. *Asymptotic Anal.*, 2(1):39–94, 1989.
[48] P. Remy. Gevrey index theorem for the inhomogeneous n-dimensional heat equation with a power-law nonlinearity and variable coefficients. *HAL*.
[49] P. Remy. Gevrey order and summability of formal series solutions of some classes of inhomogeneous linear partial differential equations with variable coefficients. *J. Dyn. Control Syst.*, 22(4):693–711, 2016.
[50] P. Remy. Gevrey order and summability of formal series solutions of certain classes of inhomogeneous linear integro-differential equations with variable coefficients. *J. Dyn. Control Syst.*, 23(4):853–878, 2017.
[51] P. Remy. Gevrey properties and summability of formal power series solutions of some inhomogeneous linear Cauchy-Goursat problems. *J. Dyn. Control Syst.*, in press.
[52] A. Shirai. Maillet type theorem for nonlinear partial differential equations and newton polygons. *J. Math. Soc. Japan*, 53:565–587, 2001.

[53] A. Shirai. Convergence of formal solutions of singular first order nonlinear partial differential equations of totally characteristic type. *Funkcial. Ekvac.*, 45:187–208, 2002.
[54] A. Shirai. A maillet type theorem for first order singular nonlinear partial differential equations. *Publ. RIMS. Kyoto Univ.*, 39:275–296, 2003.
[55] A. Shirai. Maillet type theorem for singular first order nonlinear partial differential equations of totally characteristic type. *Surikaiseki Kenkyujo Kokyuroku, Kyoto University*, 1431:94–106, 2005.
[56] A. Shirai. Alternative proof for the convergence or formal solutions of singular first order nonlinear partial differential equations. *Journal of the School of Education, Sugiyama Jogakuen University*, 1:91–102, 2008.
[57] A. Shirai. Gevrey order of formal solutions of singular first order nonlinear partial differential equations of totally characteristic type. *Journal of the School of Education, Sugiyama Jogakuen University*, 6:159–172, 2013.
[58] A. Shirai. Maillet type theorem for singular first order nonlinear partial differential equations of totally characteristic type, part ii. *Opuscula Math.*, 35(5):689–712, 2015.
[59] H. Tahara and H. Yamazawa. Multisummability of formal solutions to the Cauchy problem for some linear partial differential equations. *J. Differential Equations*, 255(10):3592–3637, 2013.
[60] W. Walter. An elementary proof of the Cauchy-Kowalevsky theorem. *Amer. Math. Monthly*, 92(2):115–126, 1985.
[61] A. Yonemura. Newton polygons and formal Gevrey classes. *Publ. Res. Inst. Math. Sci.*, 26:197–204, 1990.

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