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Abstract: In this paper we propose a monotonicity-based approach for decentralized voltage control of dominantly resistive DC microgrids with ZIP loads. For this purpose, we introduce the notion of set of assignable robust controlled decentralized invariants for the system. Then, upon selection of a desired invariant, an inner decentralized voltage control is designed and a criteria for convergence of the system's trajectories to an equilibrium point—in presence of constant power loads (CPLs)—is established. Interestingly, a simple realization of the proposed controller corresponds to a piece-wise voltage droop control, the gain of which are determined by the control specifications. A discussion on the selection of appropriate invariants is also carried out and the obtained theoretical results are validated on a 8-terminal benchmark.

Keywords: Voltage stability, invariance, monotone systems, decentralized control.

1. INTRODUCTION

In recent years, we have witnessed a tremendous growth in terms of direct-current (DC) supply and demand on the electricity market, due mainly to the widespread diffusion of photovoltaic sources and DC loads (Elsayed et al. (2015)). In this context, DC microgrids offer a desirable choice since they avoid superfluous conversion stages improving the grid efficiency (Dragičević et al. (2016a)).
Voltage stability is one of the fundamental control objectives in DC microgrids, that must be achieved independently from perturbations arising from the volatility of the demand. In addition to the controllers operating at the level of the power converters, it is thus common practice to design an outer decentralized, usually proportional, voltage control scheme that guarantees that bus voltages are maintained within specific bounds (Dragičević et al. (2016a)). While it is widely acknowledged that decentralized controllers fail to simultaneously guarantee an appropriate power distribution (load sharing) and to preserve a tight voltage regulation in presence of non-negligible lines dissipation, it is unclear how to select the controller gains for an optimal, safe regulation of the grid voltages (Meng et al. (2017)). In particular, no guarantees are provided on the trajectories of the voltages during transients, nor on their convergence to constant steady-states in presence of constant power loads (CPLs), especially at load buses. Moreover, saturation effects are usually neglected in the analysis and as a result, cranking up the controller gains might yield below-par performances.

In this work, we address the problem of designing a decentralized control scheme for voltage stability of DC microgrids with ZIP loads. In contrast with traditional power systems literature, we consider a *nonlinear time-varying* model that takes into account voltage safety requirements, limitation of the actuation, volatility of the demand and non-controllability of the load voltages—issues that at the best of the authors' knowledge have been only addressed separately in literature—see for example Zhao and Dörfler (2015), Liu et al. (2018) and De Persis et al. (2018). The model is developed in Section 2, where the control problem is also illustrated. As a preliminary step towards the control design, we exploit the monotonicity of the system to perform an analysis on the invariants that can be assigned via a decentralized control—along the lines of Meyer et al. (2016)—and introduce the notion of set of assignable robust controlled decentralized invariants for a multi-agent system—reminiscent of the notion of set of assignable equilibria. This is done in Section 3. A class of controllers that assigns a desired invariant is then derived in Section 4, where we further establish a criteria for convergence of the trajectories in case of constant load demand, using the result of contraction of trajectories for monotone positive systems developed in Coogan (2016). Interestingly, a trivial implementation of the controller consists in a (piece-wise) voltage droop control—ubiquitous in practical applications (Dragičević et al. (2016b))—whose gains are determined by the upper

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and lower bounds of the desired invariant, thus establishing a link between the desired voltage guarantees and the control design. A thorough discussion on the properties of the set of robust controlled decentralized invariants, such as monotonicity, robustness and plug & play capabilities are further discussed in Section 5. Theoretical results are validated in Section 6, while conclusions and guidelines for future research follow in Section 7.

2. PROBLEM SETUP

2.1 Modeling of DC microgrids

Following the same approach used in Zonetti et al. (2019), we represent a microgrid as a directed graph $\mathcal{G}(\mathcal{N}, \mathcal{E}, \mathcal{B})$, where: $\mathcal{N}$ is the set of nodes, with cardinality n; $\mathcal{E}$ is the set of edges, with cardinality t and $\mathcal{B} \in \mathbb{R}^{n\times t}$ is the incidence matrix capturing the graph topology. The edges correspond to the transmission lines, while the nodes correspond to the buses where the power units are interfaced. Since we focus our attention on dominantly resistive transmission lines, the weighted interconnection topology is equivalently captured by the Laplacian matrix $\mathcal{L} := \mathcal{B} G_T \mathcal{B}^\top \in \mathbb{R}^{n\times n}$, with $G_T := \mathrm{diag}(G_e) \in \mathbb{R}^{t\times t}$, where G_e denotes the conductance associated to the edge $e \in \mathcal{E}$. We further define $\mathcal{N}_S$ as the subset of nodes associated to controllable power units, *i.e.* the generation and energy storage units, with cardinality m, and $\mathcal{N}_L$, as the subset of nodes associated to non-controllable power units, with cardinality $n - m$. Although loads may be connected to both controllable and non-controllable buses, with a little abuse of language we refer to controllable units as *sources* and to non-controllable units as *loads*. The interconnected dynamics of the voltage buses read:

$$C\dot{V} = -(\mathcal{L} + G)V + \sigma, \tag{2.1}$$

where $V := \mathrm{col}(v_i) \in \mathbb{R}^n_{>0}$ denotes the collection of (positive) bus voltages, $\sigma := \mathrm{col}(\sigma_i) \in \mathbb{R}^n$ denotes the collection of input currents and $C := \mathrm{diag}(C_i) \in \mathbb{R}^{n\times n}$, $G := \mathrm{diag}(G_i) \in \mathbb{R}^{n\times n}$ are diagonal matrices denoting the bus capacitances and (small) conductances. Input currents are given by:

$$\sigma_i = (\mathsf{p}_i + b_i u_i)/v_i, \qquad i \in \mathcal{N}, \tag{2.2}$$

with: control input $u_i \in \mathcal{U}_i$, where $\mathcal{U}_i := [\underline{u}_i, \overline{u}_i] \subset \mathbb{R}_{>0}$; $b_i \in \{0, 1\}$, where $b_i = 1$, if $i \in \mathcal{N}_S$ and $b_i = 0$ otherwise; disturbance $\mathsf{p}_i : \mathbb{R}_{>0} \times \mathcal{W}_i \to \mathbb{R}_{<0}$ generated by:

$$\mathsf{p}_i(v_i, w_i) = Z_i v_i^2 + I_i v_i + P_i, \quad w_i := [Z_i \; I_i \; P_i]^\top \in \mathcal{W}_i, \tag{2.3}$$

where $\mathcal{W}_i := [\underline{Z}_i, \overline{Z}_i] \times [\underline{I}_i, \overline{I}_i] \times [\underline{P}_i, \overline{P}_i]$. The input current consists of the difference of a controllable current injection, if any, and the current drawn by the local ZIP loads, see Remark 2.1. By replacing (2.2) into (2.1), the overall system can be rewritten in compact form via the following ordinary differential equations:

$$\dot{V} = f(V, u, w) := -C^{-1}\left[(\mathcal{L} + G)V + (\mathsf{P}(V, w) + Bu) \oslash V\right], \tag{2.4}$$

with state vector $V \in \mathbb{R}^n_{>0}$; control input $u \in \mathcal{U}$, where $\mathcal{U} := \Pi_i \mathcal{U}_i$; disturbance input $w \in \mathcal{W}$, where $\mathcal{W} := \Pi_i \mathcal{W}_i$; input matrix $B = [\mathbb{I}_m \; 0]^\top \in \mathbb{R}^{n\times m}$ and where $\oslash$ denotes the element-wise (Hadamard) division of matrices. An important property of the system (2.4) is that it is monotone with respect to the positive orthant, *i.e.* it is cooperative, a fact can be easily verified via the Kamke-Muller conditions for continuously differentiable vector fields (Smith (2008)), since we have:

$$\frac{\partial f_i}{\partial v_j} \geq 0, \quad \frac{\partial f_i}{\partial w_k} \geq 0, \qquad i, j, k \in \mathcal{N}, \; j \neq i. \tag{2.5}$$

Remark 2.1. To describe the load we used the ZIP model (2.3), which consists of the parallel interconnection of a conductance, a current and a power source. However, the results obtained in this paper hold also for other static models—such as the exponential model—provided that (2.5) is verified.

2.2 Control problem

The control objective is to guarantee safe, decentralized regulation of all grid voltages under limited actuation, both in nominal and perturbed conditions. In nominal conditions, since a precise forecast of the load demand $w^{\mathrm{d}} \in \mathcal{W}$ is available, the vector of grid voltages must be regulated to $V^{\mathrm{d}} \in \mathbb{R}^n$ through a corresponding supply $u^{\mathrm{d}} \in \mathcal{U}$, which verify the power flow (equilibria) equations

$$0 = -V^{\mathrm{d}} \odot (\mathcal{L} + G)V^{\mathrm{d}} + \mathsf{P}(V^{\mathrm{d}}, w^{\mathrm{d}}) + Bu^{\mathrm{d}}, \tag{2.6}$$

where $\odot$ denotes the element-wise product of matrices. In perturbed conditions on the other hand, the grid is exposed to unknown, possibly time-varying, bounded disturbance $w \in \mathcal{W}$ that steers the grid voltages away from their nominal values. In order to ensure a safe operation of the grid, control action must be implemented through the supply $u \in \mathcal{U}$ so that all voltages are kept within a safe, pre-specified region. Moreover, whenever the grid is subject to time-invariant disturbances the voltage trajectories must converge to a constant steady-state value. With no loss of generality, we consider as safety region $\mathcal{V}_\delta := \mathcal{I}_\delta(v^{\mathrm{ref}} 1_n) \subset \mathbb{R}^n$, denoting a ball of radius $\delta > 0$ centered in $v^{\mathrm{ref}} 1_n \in \mathbb{R}^n$, where $v^{\mathrm{ref}} > 0$ corresponds to a voltage reference value and δ is chosen so to ensure that $V^{\mathrm{d}} \in \mathcal{V}_\delta$. We are now ready to present the control problem.

Problem 2.2. Consider the DC microgrid (2.4) and let:

- $\mathcal{V}_\delta := \mathcal{I}_\delta(v^{\mathrm{ref}} 1_n)$, the set of admissible state voltages;
- $\mathcal{U} := [\underline{u} \; \overline{u}] \subset \mathbb{R}^m_{>0}$, the set of admissible control inputs;
- $\mathcal{W} := [\underline{w} \; \overline{w}] \subset \mathbb{R}^n_{<0}$ the set of disturbance inputs.

Find controllers $\mathsf{c}_i : \mathcal{V}_{\delta_i} \to \mathcal{U}_i$, $i \in \mathcal{N}_S$ and a non-empty set $\mathcal{V}^\star \subseteq \mathcal{V}_\delta$ such that the closed-loop trajectories of the DC microgrid satisfy, for any $V(0) \in \mathcal{V}^\star$, the following properties:

C1. $V(t) \in \mathcal{V}^\star, \;\; \forall t > 0$;

C2. if $w(t) \equiv w^\star \in \mathcal{W}$, then
$$\lim_{t\to\infty} V(t) = V^\star, \quad \lim_{t\to\infty} u(t) = u^\star$$
for some *unknown* constant $V^\star \in \mathcal{V}_\delta$, $u^\star \in \mathcal{U}$;

C3. if $w = w^{\mathrm{d}} \in \mathcal{W}$, then
$$\lim_{t\to\infty} V(t) = V^{\mathrm{d}}, \quad \lim_{t\to\infty} u(t) = u^{\mathrm{d}},$$
for some *given* constant $V^{\mathrm{d}} \in \mathcal{V}_\delta$, $u^{\mathrm{d}} \in \mathcal{U}$.

3. ASSIGNABLE DECENTRALIZED INVARIANTS

We are interested in continuous-time control systems described by the nonlinear differential equation:

$$\dot{x} = f(x,u,w), \qquad x \in \mathcal{X}, u \in \mathcal{U}, w \in \mathcal{W}. \tag{3.1}$$

where $\mathcal{X} \subset \mathbb{R}^n$, $\mathcal{U} \subset \mathbb{R}^m$ and $\mathcal{W} \subset \mathbb{R}^p$. The trajectories of (3.1) are denoted $\mathbf{x}(\cdot, x, \mathbf{u}, \mathbf{w})$ where $\mathbf{x}(t, x, \mathbf{u}, \mathbf{w}) \in \mathcal{X}$ is the state reached at time $t > 0$ from the initial state $x \in \mathcal{X}$, under control input $\mathbf{u} : \mathbb{R}_{\geq 0} \to \mathcal{U}$ and disturbance input $\mathbf{w} : \mathbb{R}_{\geq 0} \to \mathcal{W}$. When the control input is generated by a state-dependent feedback controller $\mathsf{c} : \mathcal{X} \rightrightarrows \mathcal{U}$, the dynamics of the closed-loop system is given by:

$$\dot{x} = f_{\mathsf{c}}(x,w) := f(x, \mathsf{c}(x), w), \qquad x \in \mathcal{X}, w \in \mathcal{W},$$

and its trajectories are denoted as $\mathbf{x}_{\mathsf{c}}(\cdot, x, \mathbf{w})$. We introduce the following definition.

Definition 3.1. A set $\mathcal{K} \subseteq \mathcal{X}$ is said to be *robust controlled invariant* for the system (3.1) if there exists a controller $\mathsf{c} : \mathcal{X} \rightrightarrows \mathcal{U}$ such that:

$$\forall x \in \mathcal{K},\ \mathbf{w} : \mathbb{R}_{\geq 0} \to \mathcal{W}, \quad \mathbf{x}_{\mathsf{c}}(t, x, \mathbf{w}) \in \mathcal{K},\ \forall t > 0.$$

The controller c is called a *robust invariance controller.*

We now restrict our attention to multi-agent systems of the form (3.1) constituted by N components, $N \geq 2$, and denote by $\mathcal{N}$ the set of indices. We assume that the sets of states and inputs satisfy $\mathcal{X} = \Pi_i \mathcal{X}_i$, with $\mathcal{X}_i \subseteq \mathbb{R}^{n_i}$, $\mathcal{U} = \Pi_i \mathcal{U}_i$, with $\mathcal{U}_i \subseteq \mathbb{R}^{m_i}$ and $\mathcal{W} = \Pi_i \mathcal{W}_i$ with $\mathcal{W}_i \subseteq \mathbb{R}^{q_i}$. The dynamics associated to agent $k \in \mathcal{N}$ reads:

$$\dot{x}_k = f_k(x_k, u_k, (z_k, w_k)), \tag{3.2}$$

where $x_k \in \mathcal{X}_k$, $u_k \in \mathcal{U}_k$, $w_k \in \mathcal{W}_k$, $z_k := \mathrm{col}_{j \in \mathsf{N}(k)}(x_j) \in \mathcal{Z}_k$, with $\mathcal{Z}_k := \Pi_{j \in \mathsf{N}(k)} \mathcal{X}_j$, $\mathsf{N}(k)$ denoting the collection of neighbors of the k-th agent. The trajectories of (3.2) are denoted $\mathbf{x}_k(\cdot, x_k, \mathbf{u}_k, (\mathbf{z}_k, \mathbf{w}_k))$ where $\mathbf{x}_k(t, x_k, \mathbf{u}_k, (\mathbf{z}_k, \mathbf{w}_k)) \in \mathcal{X}$ is the state reached at time $t > 0$ from the initial state $x_k \in \mathcal{X}_k$, under control input $\mathbf{u}_k : \mathbb{R}_{\geq 0} \to \mathcal{U}_k$ and disturbance input $(\mathbf{z}_k, \mathbf{w}_k) : \mathbb{R}_{\geq 0} \to \mathcal{Z}_k \times \mathcal{W}_k$.

Definition 3.2. Consider a multi-agent system of the form (3.1), where each agent can be described using (3.2). A set $\mathcal{K} = \Pi_i \mathcal{K}_i$, with $\mathcal{K}_i \subseteq \mathcal{X}_i$, is said to be *robust controlled decentralized invariant* for the system, if for any $i \in \mathcal{N}$ there exist controllers $\mathsf{c}_i : \mathcal{X}_i \rightrightarrows \mathcal{U}_i$ such that $\mathcal{K}_i$ is robust controlled invariant for (3.2) with $z_i \in \mathcal{Z}_i$, where $\mathcal{Z}_i = \Pi_{k \in \mathsf{N}(i)} \mathcal{K}_k$ and $w \in \mathcal{W}$. The controller $\mathsf{C} := \mathrm{col}(\mathsf{c}_i)$ is called a *robust decentralized invariance controller.*

Similar to the characterization of assignable equilibria in standard stabilization problems, and recalling Theorem 9 in Meyer et al. (2016), we now provide a characterization of the assignable robust controlled decentralized invariants for the system.

Proposition 3.3. (Meyer et al. (2016)). Consider a multi-agent system of the form (3.1), with $\mathcal{U} = [\underline{u}\ \overline{u}] \subset \mathbb{R}^m$, $\mathcal{W} = [\underline{w}\ \overline{w}] \subset \mathbb{R}^p$ and where each agent is described by (3.2). If the system is positive monotone, then $\mathcal{X}^\star := [\underline{x}\ \overline{x}] \subseteq \mathcal{X}$ is a robust controlled decentralized invariant for the system if and only if the following conditions hold component-wise:

$$f(\underline{x}, \overline{u}, \underline{w}) \geq 0, \qquad f(\overline{x}, \underline{u}, \overline{w}) \leq 0. \tag{3.3}$$

Moreover, the set $\mathcal{A} := \{\mathcal{X}^\star \subseteq \mathcal{X} :\ (3.3) \text{ hold}\}$ is called the *set of assignable robust controlled decentralized invariants.*

Intuitively, the previous result provides a simple criteria to check whether an interval $[\underline{x}\ \overline{x}] \in \mathcal{X}$ is a robust controlled decentralized invariant for the system. This result can be directly applied to DC microgrids modeled by (2.4), as explained in the following corollary.

Corollary 3.4. The set $\mathcal{V}^\star := [\underline{V}\ \overline{V}] \subseteq \mathcal{V}_\delta$ is a robust controlled decentralized invariant for the DC microgrid (2.4) if and only if the following conditions hold component-wise:

$$\begin{aligned} -(\mathcal{L}+G)\underline{V} + \underline{I} + (\mathsf{P}(\underline{V}, \underline{w}) + B\overline{u}) \oslash \underline{V} \geq 0, \\ -(\mathcal{L}+G)\overline{V} + \overline{I} + (\mathsf{P}(\overline{V}, \overline{w}) + B\underline{u}) \oslash \overline{V} \leq 0. \end{aligned} \tag{3.4}$$

The set of assignable robust controlled decentralized invariants is given by $\mathcal{A} := \{\mathcal{V}^\star \subseteq \mathcal{V}_\delta :\ (3.4) \text{ hold}\}$.

A question of interest is how to properly select a decentralized invariant among the assignable ones. We do not dwell any longer here on this subject, for which a more detailed discussion is postponed to Section 5. In the next section instead, under the assumption that an appropriate robust controlled decentralized invariant has been already selected, we provide a solution to Problem 2.2.

Remark 3.5. It is easy to see that whenever we restrict our attention to degenerate invariants $\underline{V} = \overline{V} =: V^{\mathrm{d}}$, $\underline{u} = \overline{u} =: u^{\mathrm{d}}$ and $\underline{w} = \overline{w} =: w^{\mathrm{d}}$ the conditions (3.4) collapse to the equilibria equations (2.6).

Remark 3.6. The conditions (3.4) are inherently distributed since for each $k \in \mathcal{N}$ they read:

$$f_k(\underline{v}_k, \overline{u}_k, (\mathrm{col}(\underline{v}_j), \underline{w}_k)) \geq 0, \quad f_k(\overline{v}_k, \underline{u}_k, (\mathrm{col}(\overline{v}_j), \overline{w}_k)) \leq 0,$$

with $j \in \mathsf{N}(k)$. Similar conditions can be obtained using the assume-guarantee framework developed in Saoud et al. (2018), where local controllers c_k are designed under the assumption that the neighboring agents $j \in \mathsf{N}(k)$ meet the corresponding safety specification $\mathrm{col}(v_j) \in \Pi_{j \in \mathsf{N}(k)}[\underline{v}_j\ \overline{v}_j]$.

4. CONTROL DESIGN

As in Meyer et al. (2016), for a given set that is an assignable desired robust controlled decentralized invariant for the system, it is possible to define necessary and sufficient conditions on the structure of the controller that ensure invariance of such a set. The application of the proposition is straightforward and reported without proof.

Proposition 4.1. Let $\mathcal{V}^\star := \Pi_i[\underline{v}_i\ \overline{v}_i] \in \mathcal{A}$ an assignable robust controlled decentralized invariant for the DC microgrid (2.4) in closed-loop with the controller $\mathsf{c}_i : [\underline{v}_i\ \overline{v}_i] \rightrightarrows [\underline{u}_i\ \overline{u}_i]$:

$$\mathsf{c}_i(v_i) \in \begin{cases} \overline{u}_i, & \text{if } v_i = \underline{v}_i, \\ [\underline{u}_i\ \overline{u}_i], & \text{if } v_i \in (\underline{v}_i\ \overline{v}_i), \\ \underline{u}_i, & \text{if } v_i = \overline{v}_i, \end{cases} \tag{4.1}$$

$i \in \mathcal{N}_S$. Then the control objective C1 of Problem 2.2 is achieved and the controller $\mathsf{C} = \mathrm{col}(\mathsf{c}_i)$ is a robust decentralized invariance controller that assigns $\mathcal{V}^\star$.

Although the invariance controller (4.1) guarantees that the voltages are kept bounded within a safety, predetermined region, an aspect of practical relevance is whether the system's trajectories do converge to stationary values in case of time-invariant load demand. To address this issue, we recall the result of Theorem C in Ji-Fa (1994) for monotone systems, showing that all trajectories originating from an assigned invariant converge to a constant steady-state if and only if such invariant contains a unique equilibrium point.

Theorem 4.2. (Ji-Fa (1994)). Let $\mathcal{V}^\star := [\underline{V}\ \overline{V}] \in \mathcal{A}$ an assigned robust controlled decentralized invariant for the DC microgrid (2.4) in closed-loop with the controller (4.1). Then the control objective C2 of Problem 2.2 with constant load demand $w^\star \in \mathcal{W}$ is achieved if and only if there exists a unique equilibrium point $V^\star \in \mathcal{V}^\star$.

While the proposition gives a necessary and sufficient condition for convergence of the system's trajectories, verifying such a condition analytically is prohibitive in general. In the next proposition then we provide a parameter-dependent sufficient criteria that guarantees convergence of the systems's trajectories to a stationary value—a fact that inherently implies the existence of a unique equilibrium point.

Proposition 4.3. Let $\mathcal{V}^\star := [\underline{V}\ \overline{V}] \in \mathcal{A}$ an assigned robust controlled decentralized invariant for the DC microgrid (2.4), under constant load demand $w^\star := (Z^\star, I^\star, P^\star) \in \mathcal{W}$ and in closed-loop with the controller (4.1). If the following conditions are verified

$$-(G_i - Z_i^\star)v_i^2 - P_i^\star + b_i\left(\frac{\partial \mathsf{c}_i(v_i)}{\partial v_i}v_i - \mathsf{c}_i(v_i)\right) < 0, \quad i \in \mathcal{N}, \tag{4.2}$$

for any $V \in \mathcal{V}^\star$, then the control objective C2 of Problem 2.2 is achieved.

Proof. The DC microgrid in closed-loop with the invariance controller reads:

$$C\dot{V} = -(\mathcal{L} + G)V + [\mathsf{P}(V, w^\star) + B\mathsf{C}(V)] \oslash V,$$

and the Jacobian is thus given by: [1]

$$J(V) = C^{-1}(-\mathcal{L} - G) + + C^{-1}\left[\left(\frac{\partial \mathsf{P}}{\partial V} + B\frac{\partial \mathsf{C}}{\partial V}\right) \odot V - (B\mathsf{C} + \mathsf{P})\right] \oslash V^2.$$

Since $\mathcal{V}^\star$ is invariant, in order to prove the existence of an asymptotically stable equilibrium it suffices to find a component-wise positive vector $\mathrm{w} \in \mathbb{R}^n$ such that $J(V)\mathrm{w} < 0$ ((Coogan, 2016, Th.3)). Now, recalling that $\mathcal{L}$ is a Laplacian matrix, $\mathcal{L}1_n = 0$. Then, by taking $\mathrm{w} = 1_n$, we have that

$$J(V)1_n = -C^{-1}G1_n + + C^{-1}\left[\left(\frac{\partial \mathsf{P}}{\partial V} + B\frac{\partial \mathsf{C}}{\partial V}\right) \odot V - (B\mathsf{C} + \mathsf{P})\right] \oslash V^2,$$

which is component-wise negative if and only if (4.2) hold. The proof is then completed using Theorem 4.2. □

In (Meyer et al. (2016)) the following simple, linear realization of the controller (4.1) is proposed, which applied to the DC microgrid model taken in consideration gives:

$$\mathsf{c}_i(v_i) = -\kappa_i(v_i - \overline{v}_i), \quad \kappa_i := \frac{\overline{u}_i - \underline{u}_i}{\overline{v}_i - \underline{v}_i}, \qquad i \in \mathcal{N}_S. \tag{4.3}$$

However, such a controller is of no practical use for voltage control in DC microgrids. In fact, since the realization (4.3) is independent from the nominal operating conditions, the system would not operate at the desired equilibrium point even in case of perfect knowledge of the demand. We have then the following proposition.

Proposition 4.4. Let $\mathcal{V}^\star := [\underline{V}\ \overline{V}] \in \mathcal{A}$ an assignable robust controlled decentralized invariant for the DC microgrid (2.4), in closed-loop with the controller (4.1). If $u = u^{\mathrm{d}}$ for $V = V^{\mathrm{d}}$ and (4.2) hold, then the control objective C3 of Problem 2.2 is achieved.

Proof. Since $u = u^{\mathrm{d}}$ for $V = V^{\mathrm{d}}$, if $w(t) \equiv w^{\mathrm{d}}$ it immediately follows from (2.6) that $V^{\mathrm{d}} \in \mathcal{V}^\star$ is an equilibrium for the system. The proof is then completed using Proposition 4.3. □

By combining the results of Proposition 4.1, Proposition 4.3 and Propostion 4.4 we formulate in the following corollary a possible realization of the controller (4.1), satisfying the control objectives C1, C2 and C3 of Problem 2.2.

Corollary 4.5. Let $\mathcal{V}^\star := [\underline{V}\ \overline{V}] \in \mathcal{A}$ an assignable robust controlled decentralized invariant for the DC microgrid (2.4), under constant load demand $w^\star := (Z^\star, I^\star, P^\star) \in \mathcal{W}$ and in closed-loop with the controller $\mathsf{c}_i : [\underline{v}_i\ \overline{v}_i] \rightrightarrows [\underline{u}_i\ \overline{u}_i]$:

$$\mathsf{c}_i(v_i) = u_i^{\mathrm{d}} - \kappa_i(v_i)(v_i - v_i^{\mathrm{d}}), \qquad i \in \mathcal{N}_S, \tag{4.4}$$

where

$$\kappa_i(v_i) = \begin{cases} (u_i^{\mathrm{d}} - \underline{u}_i)/(\overline{v}_i - v_i^{\mathrm{d}}), & \text{if } v_i \geq v_i^{\mathrm{d}} \\ (u_i^{\mathrm{d}} - \overline{u}_i)/(\underline{v}_i - v_i^{\mathrm{d}}), & \text{if } v_i < v_i^{\mathrm{d}} \end{cases}$$

and $(V^{\mathrm{d}}, u^{\mathrm{d}}) \in \mathcal{V}^\star \times \mathcal{U}$ verifies (2.6) for a given $w^{\mathrm{d}} \in \mathcal{W}$. Then the control objective C1 of Problem 2.2 is achieved and $\mathsf{C} := \mathrm{col}(\mathsf{c}_i)$ is a robust decentralized invariance controller that assigns $\mathcal{V}^\star$. Then, if the following conditions are satisfied:

$$\begin{aligned} -(G_i - Z_i^\star)\underline{v}_i^2 - P_i^\star - b_i(u_i^{\mathrm{d}} + \kappa_i v_i^{\mathrm{d}}) < 0, &\ \text{ if } (G_i - Z_i^\star) \geq 0, \\ -(G_i - Z_i^\star)\overline{v}_i^2 - P_i^\star - b_i(u_i^{\mathrm{d}} + \kappa_i v_i^{\mathrm{d}}) < 0, &\ \text{ if } (G_i - Z_i^\star) < 0, \end{aligned} \tag{4.5}$$

with $i \in \mathcal{N}$, the control objectives C2 and C3 are also achieved.

Proof. Since the controller (4.4) verifies (4.1) for any $i \in \mathcal{N}_S$, from Proposition 4.1 it follows that C is a robust decentralized invariance controller for the system and then C1 is verified. To prove C2 we show that the conditions (4.2) of Proposition 4.3 are equivalent to (4.5). By replacing (4.4) therein we get

$$-(G_i - Z_i)v_i^2 - P_i^\star - b_i(u_i^{\mathrm{d}} + \kappa_i v_i^{\mathrm{d}}) < 0, \qquad i \in \mathcal{N},$$

which, recalling that $\underline{v}_i \leq v_i \leq \overline{v}_i$, give indeed (4.5). Finally, to prove C3 using Proposition 4.4, it suffices to see that, under nominal load demand w^{d}, $u = u^{\mathrm{d}}$ for $V = V^{\mathrm{d}}$, thus completing the proof. □

Remark 4.6. The controller (4.4) can be interpreted as a piece-wise linear droop control, the gains of which are selected so to assign a pre-specified robust controlled decentralized invariant. Note that other realizations of (4.1) can be employed, provided that they verify conditions (4.2) and preserve the nominal supply in nominal operating conditions, *i.e.* $u_i = u_i^{\mathrm{d}}$, if $V_i = V_i^{\mathrm{d}}$, for any $i \in \mathcal{N}_S$.

Remark 4.7. Conditions (4.2) evaluated at load buses are independent of the controller, *i.e.* $b_i = 0$, and consist in $n - m$ inequalities involving the constant resistive and power components of the ZIP loads. They can be interpreted as the fact that the power dissipated through the resistive component should be greater than the power demand.

Remark 4.8. Conditions (4.2) evaluated at source buses are affected by the control action and can be fulfilled by appropriate realizations of the controller (4.1). This should be contrasted with (Nahata et al. (2018)), where such conditions are independent from the control design.

[1] dependence from V is omitted to improve readability.

5. DECENTRALIZED INVARIANTS COMPUTATION

For monotone systems, the choice of a desired compact invariant set $\mathcal{X}^\star \in \mathcal{A}$ to be assigned via a robust controlled decentralized invariance controller can be seen as a problem analogous to the problem of selecting appropriate references signals to be tracked in standard regulation problems. Likewise, it can be recast as the centralized optimization problem

$$\min_{\mathcal{X}^\star \in \mathcal{A}} \quad \mathsf{F}(\mathcal{X}^\star) \tag{5.1}$$

where $\mathsf{F} : 2^{\mathcal{X}} \to \mathbb{R}$ is an assigned cost function. However, the elements of the set $\mathcal{A}$ are intervals $\mathcal{X}^\star = [\underline{x}\ \overline{x}] \in 2^{\mathcal{X}}$ that verify the conditions (3.3), which are defined on the space of intervals extrema. To facilitate a systematical exploration of $\mathcal{A}$ we thus find convenient to introduce an appropriate parametrization of intervals, by defining a map κ, that for each vector $\lambda \in \mathbb{R}^{2n}$ of real parameters associates an interval $\kappa(\lambda) := [\underline{\kappa}(\lambda)\ \overline{\kappa}(\lambda)] = \Pi_i[\underline{\kappa}_i(\lambda_i)\ \overline{\kappa}_i(\lambda_i)]$. We have then the following property, which stems from the fact that a multi-agent system described by (3.1)-(3.2), where agents are scalar, is monotone if and only if the following logical implication is satisfied (Smith (2008)) for all $i \in [1, N]$:

$$\begin{aligned} x' \le x,\ x'_i = x_i,\ u' \le_u u,\ w' \le_w w& \\ \implies f_i(x', u', w') \le f_i(x, u, w),& \end{aligned} \tag{5.2}$$

where $\le_u$ and $\le_w$ denote the partial orders over the set of control inputs $\mathcal{U}$ and disturbance inputs $\mathcal{W}$, respectively.

Property 5.1. (Monotonicity). Consider the system (3.1)-(3.2), with $n = N$. Let $\kappa : \mathbb{R}^{2N} \to 2^{\mathcal{X}}$ and the set $\Lambda := \{\lambda \in \mathbb{R}^{2N} : \kappa(\lambda) \in \mathcal{A}\}$. If we have:

$$\lambda, \lambda' \in \mathbb{R}^{2N},\ \lambda' \le \lambda \implies \kappa(\lambda') \subseteq \kappa(\lambda),$$

then the following logical implications are satisfied:

$$\begin{aligned} &\lambda \notin \Lambda,\ f_i(\underline{\kappa_i}(\lambda_i), \overline{u}_i, (\underline{z}_i, \underline{w}_i)) < 0 \\ &\qquad\qquad \implies \forall \lambda' \ge \lambda,\ \lambda'_i = \lambda_i,\ \text{ we have } \lambda' \notin \Lambda, \\ &\lambda \notin \Lambda,\ f_i(\overline{\kappa}_i(\lambda_i), \underline{u}_i, (\overline{z}_i, \overline{w}_i)) > 0 \\ &\qquad\qquad \implies \forall \lambda' \ge \lambda,\ \lambda'_i = \lambda_i,\ \text{ we have } \lambda' \notin \Lambda, \end{aligned}$$

where $z_i := \mathrm{col}_{j \in \mathsf{N}(i)}(\kappa_j(\lambda_j))$.

Proof. We provide proof for the first implication only, with the second one following similarly. Assume $\lambda \notin \Lambda$, $f_i(\underline{\kappa}_i(\lambda), \overline{u}_i, (\underline{z}_i, \underline{w}_i)) < 0$ and let $\lambda' \ge \lambda$, $\lambda'_i = \lambda_i$. Then

$$\kappa(\lambda) = [\underline{\kappa}(\lambda)\ \overline{\kappa}(\lambda)] \subseteq \kappa(\lambda') = [\underline{\kappa}(\lambda')\ \overline{\kappa}(\lambda')],$$

from which follows $\underline{\kappa}(\lambda') \le \underline{\kappa}(\lambda)$ and $\underline{\kappa}_i(\lambda_i) = \underline{\kappa}_i(\lambda'_i)$. Hence, using (5.2) we obtain

$$f_i(\underline{\kappa}_i(\lambda'_i), \overline{u}_i, (\underline{z}'_i, \underline{w}_i)) \le f_i(\underline{\kappa}_i(\lambda_i), \overline{u}_i, (\underline{z}_i, \underline{w}_i)) < 0,$$

thus implying that $\lambda' \notin \Lambda$. □

The boundary of the reparametrized region Λ has then the structure of a Pareto front and can therefore be approximated arbitrarily close, from inside and outside, adapting efficient multidimensional binary search algorithms used in multi-objective optimization (Legriel et al. (2010)).

Another interesting property of the set of assignable robust controlled decentralized invariants is that it easily allows for robustification to line uncertainties, a result that follows directly from the conditions of Corollary 3.4.

Property 5.2. (Line uncertainties). Consider the DC microgrid (2.4), where

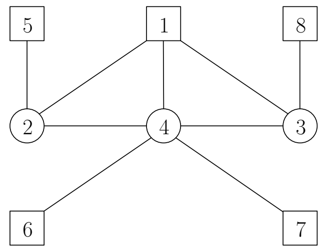

Fig. 1. Topology of the DC microgrid adopted for simulation. Squares denote loads, circles denote sources.

$$\mathcal{L}(\delta_T) = \mathcal{B}G_T^\star(\mathbb{I} + \mathrm{diag}(\delta_{T,i}))\mathcal{B}^\top,$$

with $\delta_T := \mathrm{col}(\delta_{T,i}) \in [\mathbb{1}_n\ \mathbb{1}_n]$. The set $\mathcal{V}^\star := [\underline{V}\ \overline{V}] \subseteq \mathcal{V}_\delta$ is a robust controlled decentralized invariant for the DC microgrid (2.4) under line uncertainties if and only if the following conditions hold component-wise:

$$\begin{aligned} -(\mathcal{L} + G)\underline{V} + (\mathsf{P}(\underline{V}, \underline{w}) + B\overline{u}) \oslash \underline{V} &\ge (|\delta_T|) \odot \underline{V}, \\ -(\mathcal{L} + G)\overline{V} + (\mathsf{P}(\overline{V}, \overline{w}) + B\underline{u}) \oslash \overline{V} &\le -(|\delta_T|) \odot \overline{V}, \end{aligned}$$

where $|\delta_T|$ denotes the component-wise absolute value of δ_T.

Large disturbances resulting from plug & play (PnP) operation—a non-scheduled (dis)connection of a unit at a given bus k—can be naturally modeled by a change in the control input and disturbance sets $\mathcal{U}_k$, $\mathcal{W}_k$. Unfortunately, in these circumstances it might occur that the invariant assigned before PnP operation is not an assignable invariant for the system under new control input and disturbance set. However, we can still exploit the distributed form of (3.4) to generate a new assignable invariant, as explained in the following property, see also Remark 3.6.

Property 5.3. (Plug & play). Let $\mathcal{V}^\star := \Pi_i \mathcal{V}_i^\star \in \mathcal{A}$, an assignable robust controlled decentralized invariant for the DC microgrid (2.4) with control input $\mathcal{U} = \Pi_i \mathcal{U}_i$ and disturbance $\mathcal{W} = \Pi_i \mathcal{W}_i$. Then $\mathcal{V}_{\mathrm{P}}^\star := \Pi_{i \ne k} \mathcal{V}_i^\star \times \mathcal{V}_{\mathrm{p}k}^\star$ is an assignable robust controlled decentralized invariant for (2.4) with control input $\mathcal{U}_{\mathrm{P}} = \Pi_{i \ne k} \mathcal{U}_i \times [\underline{u}_{\mathrm{p}k}, \overline{u}_{\mathrm{p}k}]$ and disturbance $\mathcal{W}_{\mathrm{P}} = \Pi_{i \ne k} \mathcal{W}_i \times [\underline{w}_{\mathrm{p}k}, \overline{w}_{\mathrm{p}k}]$ if

$$\begin{aligned} &f_k(\underline{v}_k, \overline{u}_k, (\mathrm{col}(\underline{v}_j), \underline{w}_k)) \ge 0, \quad f_k(\overline{v}_k, \underline{u}_k, (\mathrm{col}(\overline{v}_j), \overline{w}_k)) \le 0, \\ &f_j(\underline{v}_j, \overline{u}_j, (\mathrm{col}(\underline{v}_h), \underline{w}_j)) \ge 0, \quad f_j(\overline{v}_j, \underline{u}_j, (\mathrm{col}(\overline{v}_h), \overline{w}_j)) \le 0, \end{aligned}$$

with $j \in \mathsf{N}(k)$, $h \in \mathsf{N}(j)$.

Remark 5.4. The optimization problem (5.1) can be ideally paired—up to a reparametrization—to the conventional tertiary control layer employed for the calculation of the system's nominal operating conditions, sharing, unfortunately, analogous difficulties since the set of assignable decentralized invariants is in general non-convex.

6. SIMULATIONS

To validate our results we consider an 8-terminal DC microgrid with three *sources* and five *loads*, as illustrated in Fig. 1. Parameters are provided in Table 1. We take as reference voltage $v^{\mathrm{ref}} = 450\ V$ with admissible voltage deviation of 3%, *i.e.* $\pm 13.5\ V$. The grid is supposed to operate in nominal load conditions $w^{\mathrm{d}} = (Z^{\mathrm{d}}, I^{\mathrm{d}}, P^{\mathrm{d}})$ from 0 to 0.4 s, subject to disturbances from 0.4 s to 1.6 s, when operation is restored to nominal conditions. ZI components are supposed to remain unchanged, while CPL components take different constant values in the range $[-\underline{P}_i\ 0]$ and

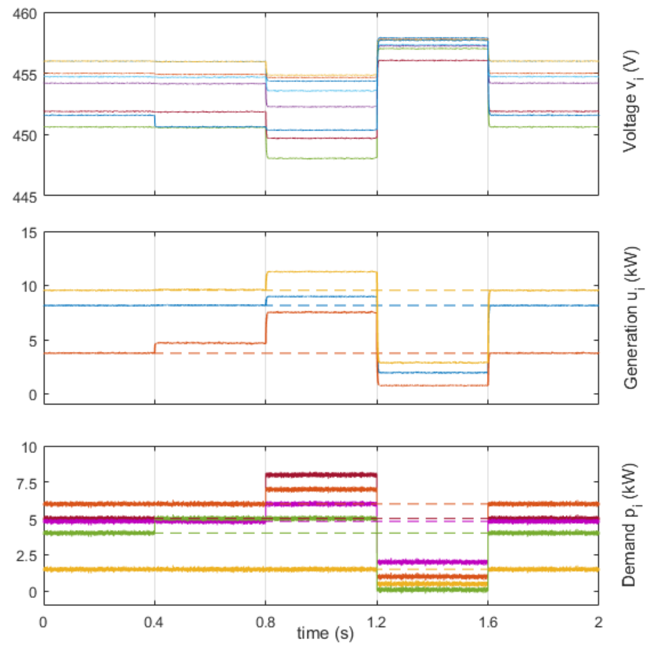

Fig. 2. Voltages, power generation and power demand responses, with dashed lines denoting nominal values.

are affected by a 5% noise. All sources are controlled via the piece-wise droop control (4.4), with actuation limits given by $[0\ \overline{u}_i]$. The gains of the controller are determined by the robust controlled decentralized invariant to be assigned, which is obtained via a parametric reparametrization of the set $\mathcal{A}$, as described in Property 5.1. The

Table 1. Units and lines parameters.

UNITS		L1	S2	S3	S4	L5	L6	L7	L8
C_i(mF)		2.2	1.9	1.5	2	2.2	1.9	2	2
P_i^{d}(kW)		5	-	-	-	6	1.5	4.8	4
$\underline{P}_i$(kW)		8	-	-	-	8	2	6	5
$\overline{u}_i$(kW)		-	15	20	22	-	-	-	-
LINES	12	13	14	24	25	34	38	46	47
$G_{ik}(\Omega^{-1})$	2	2.2	2.4	2.5	2	2.4	1.9	2	2.2

voltage responses and the power injections generated by the controller as a result of the change of the demand are illustrated in Fig. 2. As expected, all voltages are kept within the safety region and no saturation is observed in the power injections. Moreover, nominal conditions are restored after 1.6 s.

7. CONCLUSIONS

In this paper we proposed a decentralized voltage controller for DC microgrids with ZIP loads that guarantees safe operation under bounded demand. The control design is carried out in three steps: first, we characterize the invariants assignable via decentralized control; second, we establish a structure for the controller to guarantee voltage invariance; third, we provide a criteria for convergence of the trajectories. Clear links with common practice are also established, namely the fact that an implementation of the controller consists in a piece-wise voltage droop control, and that the selection of invariants can be recast as an optimization problem similar to the optimal power flow problem. Future works will focus on the trade-off between voltage stability and load sharing and the design of distributed controllers.

REFERENCES

Coogan, S. (2016). Separability of Lyapunov functions for contractive monotone systems. In *2016 IEEE 55th Conference on Decision and Control (CDC)*, 2184–2189.

De Persis, C., Weitenberg, E.R., and Dörfler, F. (2018). A power consensus algorithm for DC microgrids. *Automatica*, 89, 364–375.

Dragičević, T., Lu, X., Vasquez, J.C., and Guerrero, J.M. (2016a). DC microgrids - part II: A review of power architectures, applications, and standardization issues. *IEEE Transactions on power electronics*, 31(5), 3528–3549.

Dragičević, T., Lu, X., Vasquez, J.C., and Guerrero, J.M. (2016b). DC microgrids- part I: A review of control strategies and stabilization techniques. *IEEE Transactions on power electronics*, 31(7), 4876–4891.

Elsayed, A.T., Mohamed, A.A., and Mohammed, O.A. (2015). DC microgrids and distribution systems: An overview. *Electric Power Systems Research*, 119, 407–417.

Ji-Fa, J. (1994). On the global stability of cooperative systems. *Bulletin of the London Mathematical Society*, 26(5), 455–458.

Legriel, J., Le Guernic, C., Cotton, S., and Maler, O. (2010). Approximating the Pareto front of multi-criteria optimization problems. In *Tools and Algorithms for the Construction and Analysis of Systems*, 69–83. Springer Berlin Heidelberg.

Liu, J., Zhang, W., and Rizzoni, G. (2018). Robust stability analysis of DC microgrids with constant power loads. *IEEE Transactions on Power Systems*, 33(1), 851–860.

Meng, L., Shafiee, Q., Trecate, G.F., Karimi, H., Fulwani, D., Lu, X., and Guerrero, J.M. (2017). Review on control of DC microgrids and multiple microgrid clusters. *IEEE Journal of Emerging and Selected Topics in Power Electronics*, 5(3), 928–948.

Meyer, P.J., Girard, A., and Witrant, E. (2016). Robust controlled invariance for monotone systems: application to ventilation regulation in buildings. *Automatica*, 70, 14–20.

Nahata, P., Soloperto, R., Tucci, M., Martinelli, A., and Ferrari Trecate, G. (2018). A passivity-based approach to voltage stabilization in DC microgrids with zip loads. *Under review.*

Saoud, A., Girard, A., and Fribourg, L. (2018). On the composition of discrete and continuous-time assume-guarantee contracts for invariance. In *2018 European Control Conference (ECC)*, 435–440. IEEE.

Smith, H.L. (2008). *Monotone dynamical systems: an introduction to the theory of competitive and cooperative systems.* 41. American Mathematical Soc.

Zhao, J. and Dörfler, F. (2015). Distributed control and optimization in DC microgrids. *Automatica*, 61, 18 – 26.

Zonetti, D., Saoud, A., Girard, A., and Fribourg, L. (2019). A symbolic approach to voltage stability and power sharing in time-varying DC microgrids. European Control Conference 2019.