VALUE-BASED DISTANCE BETWEEN THE INFORMATION STRUCTURES

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Abstract. We define the distance between two information structures as the largest possible difference in the value across all zero-sum games. We provide a tractable characterization of the distance, as the minimal distance between 2 polytopes. We use it to show various results about the relation between games and single-agent problems, the value of additional information, informational substitutes, complements, etc. We show that approximate knowledge is similar to approximate common knowledge with respect to the value-based distance. Nevertheless, contrary to the weak topology, the value-based distance does not have a compact completion: there exists a sequence of information structures, where players acquire more and more information, and $\varepsilon > 0$ such that any two elements of the sequence have distance at least ε. This result answers by the negative the second (and last unsolved) of the three problems posed by J.F. Mertens in his paper "Repeated Games", ICM 1986.

1. Introduction

The role of information is of fundamental importance for the economic theory. It is well known that even small differences in information may lead to significant differences in the behavior (Rubinstein (1989)). A recent literature on the strategic (dis)-continuities has studied this observation very intensively and in full generality. The approach is to typically consider all possible information structures, modeled as elements of an appropriately defined universal information structure, and study the differences in the strategic behavior across all games.

A similar methodology has not been applied to study the relationship between the information, and the agent's bottom line, their payoffs. There are perhaps few reasons for this. First, following Dekel et al. (2006), Weinstein and Yildiz (2007) and others, the literature has focused on the interim rationalizability as the solution concept. Compared with the equilibrium, this choice has several advantages: it is easier to analyze, it is more robust from the decision-theoretic perspective, it can be factorized through the Mertens-Zamir hierarchies of beliefs (Dekel et al. (2006), Ely and Peski (2006)), and, it does not suffer from the existence problems (unlike the equilibrium - see Simon (2003)). However, the value of information is typically measured in the ex ante sense, where solution concepts like the Bayesian Nash equilibrium being more appropriate. Second, the multiplicity of solutions

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necessitates that the literature takes the set-based approach. This, of course, makes the quantitative comparison of the value of information difficult. Last but not least, the freedom in choosing games without any restriction makes the payoff comparison between information structures useless (see Section 7.1 for a detailed discussion of this point).

Nevertheless, we find the questions about the value of a general-formulated information as important and fascinating. How to measure the value of information on the universal type space? How much a player can gain (or lose) from an additional information? Which information structures are similar, in the sense that they always lead to the same payoffs? In order to address these questions, we propose to restrict the analysis to zero-sum games. It is a natural restriction for both conceptual and methodological reasons. The question of the value of information is of special importance when the players' interests are opposing. With zero-sum games, the information has natural comparative statics: a player is better off when her information improves and/or the opponent's information worsens (Peski (2008)). Such comparative statics are intuitive, they hold in the single-agent decision problems (Blackwell (1953)), but they do not hold for general games, where better information may worsen a player's strategic position. Moreover, the restriction avoids some of the problems mentioned in the previous paragraph. Finite zero-sum games have always an equilibrium on common prior information structures (Mertens et al. (2015)) that depends only on the distribution over hierarchies of beliefs. The equilibrium has decent decision-theoretic foundations (Brandt (2019)). Even if the equilibrium is not unique, the ex ante payoff always is and it is equal to the value of the zero-sum game. Finally, the restriction leads to novel issues comparing to the earlier literature, where many of the results rely on either coordination games, or betting games (Rubinstein (1989), Morris (2002), Ely and Peski (2011), Chen and Xiong (2013) among others).We believe that it is important to reconsider various phenomena to clarify their relevance for different classes of games.

We define the distance between two information structures as the largest possible difference in the value across all zero-sum payoff functions that are bounded by a constant. This has a straightforward interpretation as a tight upper bound on the gain or loss from moving from one information structure to another. The first result provides a characterization of the distance as the total variation distance between two sets of information structures: those that are better for player one than one of the original information structures and those that are worse than the other one. This distance can be computed as a solution to a finitely dimensional convex optimization problem.

The characterization is tractable in applications. In particular, we use it to describe the conditions under which the distance between information structures is maximized in single-agent problems (which are special class of zero-sum games). We recover the characterization of a comparison of information structures from Peski (2008), which generalized to 2 players the Blackwell's comparison of experiments via garblings. We also discuss the role of the marginal distribution over the state.

An important aspect of our approach is its natural and simple interpretation. It allows us to define the value of an additional piece of information as the distance between two type spaces, in one of which one or two players have an access to new information. The above characterization of the distance allows us to prove numerous results about the value. We give conditions when the value of new information is maximized in the single-agent problems. Next, we describe the situations when the value of one piece of information decreases when the other piece of information becomes available, in other words, when the two pieces of information are substitutes. Similarly, we show that, under some conditions, the value of one piece of information increases when the other player gets an additional information, or in other words, that the two pieces of information, one for each player, are complements. Finally, we show that the new information matters only if it is valuable to at least one of the players individually. The joint information contained in the correlation between player's signals is in itself not valuable in the zero-sum games.

Apart from its quantitative metric aspect, our distance contains interesting topological information. In a striking example, we show that any information structure in which with a large (close to 1) probability, each player assigns a large (close to 1) probability to some state is similar to a structure in which the state is publicly revealed. In other words, the distinction between approximate knowledge and approximate common knowledge (Monderer and Samet (1989)) is not relevant for the value-based distance. There is a simple intuition for this. The common knowledge is a statement about the joint information. In order to benefit from it, the players' interests need to be somehow aligned, like in coordination games. That is of course impossible in a situation of conflict.

More generally, we show that any sequence of countable information structures converges to a countable structure if and only if the associated hierarchies of beliefs converge in the induced weak topology of Mertens and Zamir (1985). Thus, at least in the neighborhoods of countable structures, the higher order beliefs matter less and less in zero-sum games. This leads to a question whether the higher order beliefs matter at all. Perhaps surprisingly in the light of the above results, the answer is affirmative. We demonstrate it by constructing an infinite sequence of information structures u^n, such that all the information structures $n' \geq n$ have the same n-the order hierarchies of beliefs. We show that there exists $\varepsilon > 0$ such that the value-based distance between each pair of structures is at least ε. In the proof, we construct a Markov chain with the first element of the chain is correlated with the state of the world. We construct an information structure μ^n so that player 1 observes the first n odd elements of the sequence and the other player observes the first n even elements. Our construction implies that in information structure μ^{n+1} each player gets an extra signal. Thus, having more and more information may lead... nowhere. This is unlike the single-player case, where more and more signals is a martingale and the value converges for each decision problem. We conclude that our distance is not robust with respect to the correct specification of higher order beliefs.

The last result sounds similar to the results from the strategic (dis)continuities literature. However, we emphasize that our result is entirely novel. In particular, and it has to deal with at least two major difficulties. The first difficulty is that all earlier constructions heavily relied on non-zero sum games: with either coordination games, or betting elements. Such constructions do not work in zero-sum games. Another difficulty is that when we construct games in which the players' payoffs depend on their higher-order beliefs, we are constrained by an uniform payoff bound.

An important contribution of our result is that it leads to an answer to the last open problem posed in Mertens (1986)[1]. Specifically, his Problem 2 asks about the equicontinuity of the family of value functions over information structures across all (uniformly bounded) zero-sum game. The positive answer would have imply the equicontinuity of the discounted and the average value in repeated games, and it would have consequences for the convergence in the limits theorems[2]. One can show that the problem 2 is equivalent to a question whether the value-based distance is totally bounded[3] on countable information structures. Unfortunately, our results imply that the answer to the problem is negative. In particular, it is not possible to approximate the universal information structure with finitely many well-chosen information structures.

Our paper adds to the literature on the topologies of information structures. This literature was spurred by an observation in Rubinstein (1989) that solution concepts are highly sensitive to higher-order beliefs. Dekel et al. (2006) introduce *strategic* and *uniform-strategic topologies*. In the latter, two types are close if, for any (not necessarily zero-sum) game, the sets of (almost) rationalizable outcomes are (almost) equal (see also Morris (2002)). [4] Chen et al. (2010) and Chen et al. (2016) provide a characterization of the uniform-strategic topology in terms of the uniform weak topology on belief hierarchies. There are two key differences between that and our approach. First, the uniform strategic topology applies to all (including non-zero sum) games. Our restriction allows us to clarify which of the results established in the literature hold in situations of pure conflict. Second, we work with *ex ante* information structures and the equilibrium solution concept, whereas the uniform strategic topology is designed to work on the *interim* level, with rationalizability. The ex ante equilibirum approach is more appropriate for the value comparison and other related questions. For instance, in the information design context, the quality of he information structure is typically evaluated *before* players receive any information.

The value of information literature studies the impact of information on the payoffs in various classes of games. Examples include single agent problems (Blackwell (1953), Athey and Levin (2018)), zero-sum games (Gossner and Mertens (2001), Shmaya (2006)), common interest games Lehrer et al. (2010, 2013), non-zero-sum static games (Gossner (2000)), Markov games (Renault (2006), Pęski and Toikka (2017)), among many others. Our paper contributes with the characterization of the tight upper bound on the loss/gain from moving from one information structure to another. The characterization allows us to discuss various results about the value of additional information. In particular, we characterize the situations when the two pieces of information are substitutes or complements in zero-sum games

[1] Problem 1 asked about the convergence of the value, and it was proved false in Ziliotto (2016). Problem 3 asked about the equivalence between the existence of the uniform value and the uniform convergence of the value functions, it was proved to be false by Monderer and Sorin (1993) and Lehrer and Monderer (1994).

[2] Equicontituity of value functions is used to obtain limit theorems in several works as e.g. Mertens and Zamir (1971), Forges (1982), Rosenberg and Sorin (2001), Rosenberg (2000), Rosenberg and Vieille (2000), Rosenberg et al. (2004), Renault (2006), Gensbittel and Renault (2015), Venel (2014), Renault and Venel (2016). See section 6.3 for a more detailed discussion

[3] Recall that a metric space if for all $\varepsilon > 0$, it can be covered by finitely many balls of radius ε.

[4] Dekel et al. (2006) focus mostly on the weaker notion of *strategic topology* that differs from the uniform strategic in the same way that the pointwise convergence differs from uniform convergence.

(Hellwig and Veldkamp (2009) study when the information acquisition decisions are complements or substitutes in a beauty contest game).

Finally, this paper contributes to a recent but rapidly growing field of information design (Kamenica and Gentzkow (2011), Ely (2017), Bergemann and Morris (2015), to name a few). In that literature, a principal designs an information structure which the agents use to play a game with incomplete information. The objective is to maximize the principal's payoff from the equilibrium outcome of some game. Sometimes, the design of information may be divorced from the game itself. For example, a bank may acquire a software to process and analyze large amounts of financial information before knowing what stock they are going to trade on, or, a spy master allocates resources to different tasks or regions before she understands the nature of future conflicts. Our result shows that the choice space of information structures is large: there is no natural way in approximating the space of choices with a finite set of structures.

Section 2 defines the value-based distance. Section 3 provides and discusses the characterization of the distance as a total variation distance between two sets of measures. Section 4 lists various results about the value of additional information. We discuss the topological content of the value-based distance in Section 5. Section 6 shows that the space of countable information structures is not totally bounded for the value-based distance. Section 7 discusses the relation to other topologies on the space of hierarchies of beliefs. The last section concludes. The proofs are contained in the Appendix.

2. Model

Let K be a finite set, $|K| \geq 2$. A (countable) *information structure* is an element $u \in \Delta(K \times \mathbb{N} \times \mathbb{N})$ of the space of probabilities over tuples $(k, c, d) \in K \times \mathbb{N} \times \mathbb{N}$, with the interpretation that k is a state of the world, and c and d are the signals of, respectively, player 1 and player 2. In other words, an information structure is a 2-player common prior Harsanyi type space over K with at most countably many types. The set of information structures is denoted by $\mathcal{U} = \Delta(K \times \mathbb{N} \times \mathbb{N})$.

We will identify any probability over $K \times C \times D$ where C, D are at most countable sets with an element of $\mathcal{U}$, where we interpret C and D as subsets of $\mathbb{N}$[5] . For $L = 1, 2, ...$, let $\mathcal{U}(L)$ be the subset of information structures where each player receives a signal smaller or equal to $L - 1$ with probability 1, so that each player has at most L different signals.

Whereas previous papers in the literature restrict attention[6] to a particular subset of $\mathcal{U}$ (independent information, lack of information on one side, fixed support...), we will study the general case of information structures in $\mathcal{U}$.

We evaluate information structures via the values of associated zero-sum Bayesian games. A *payoff function* is a map $g : K \times I \times J \to [-1, 1]$, where I, J are finite sets of actions. The set of payoff functions with action sizes of cardinality $\leq L$

[5] More precisely, we associate with every set C which is at most countable an enumeration, i.e. a bijective map ϕ_C between C and $\{0, ..., |C| - 1|\}$ when C is finite or $\mathbb{N}$ when C is infinite and identify the information structure $u \in \Delta(K \times C \times D)$ with the distribution of $(k, \phi_C(c), \phi_D(d))$ induced by u. All our results are independent of the choice of these enumerations

[6] For instance, one can read in De Meyer et al. (2010) "We leave open the question of what happens when the components of the state on which the players have some information fail to be independent.... In this situation the notion of monotonicity is unclear, and the duality method is not well understood."

is denoted by $\mathcal{G}(L)$, and let $\mathcal{G} = \bigcup_{L\geq 1} \mathcal{G}(L)$ be the set of all payoff functions. A information structure u and a payoff function g together define a zero-sum Bayesian game $\Gamma(u,g)$ played as follows: First, (k,c,d) is selected according to u, player 1 learns c and player 2 learns d. Then simultaneously player 1 chooses $i \in I$ and player 2 chooses $j \in J$, and finally the payoff of player 1 is $g(k,i,j)$. The zero-sum game $\Gamma(u,g)$ has the value (the unique equilibrium, or the minmax payoff of player 1) which we denote by $\operatorname{val}(u,g)$. We sometimes refer to player 1 as the maximizer, and to her opponent as the minimizer.

We define *the value-based distance* between two information structures as the largest possible difference in the value across all payoff functions:

$$d(u,v) = \sup_{g\in\mathcal{G}} |\operatorname{val}(u,g) - \operatorname{val}(v,g)|. \tag{2.1}$$

The distance (2.1) satisfies two axioms of a metric: the symmetry, and the triangular inequality. However, it is possible that $d(u,v) = 0$ for $u \neq v$. For instance, if we start from an information structure u and relabel the signals of the players, we obtain an information structure u' which is formally different from u, but "equivalent" to u. Say that u and v are equivalent, and write $u \sim v$, if for all game structures g in $\mathcal{G}$, $\operatorname{val}(u,g) = \operatorname{val}(v,g)$. We let $\mathcal{U}^* = \mathcal{U}/\sim$ be the set of equivalence classes. Thus, d is a pseudo-metric on $\mathcal{U}$ and a metric on $\mathcal{U}^*$.

In order to state our main results, we will also use the total variation norm given for each $u, v \in \mathcal{U}$ by:

$$\|u - v\| = \sum_{k,c,d} |u(k,c,d) - v(k,c,d)|.$$

Since all payoffs are in $[-1,1]$ it is easy to see that $d(u,v) \leq \|u-v\| \leq 2$. It is also convenient to always identify I and J with the sets $\{0,...,|I|-1\}$ and $\{0,...,|J|-1\}$ using some enumeration and to extend g on $K\times\mathbb{N}\times\mathbb{N}$ by letting $g(k,i,j) = -1$ if $i \geq |I|$ and $g(k,i,j) = 1$ if $i < |I|$ and $j \geq |J|$. Note that the value of the game $\Gamma(u,g)$ is not modified and that optimal strategies remain optimal if we allow the players to take any action in $\mathbb{N}$. With this convention, each set $\mathcal{G}(L)$ and thus also $\mathcal{G}$ is identified with a subset of the set of maps from $K\times\mathbb{N}\times\mathbb{N}$ to $[-1,1]$.

3. Computing $d(u,v)$

We give here a tractable characterization of $d(u,v)$ and we illustrate it with some applications.

3.1. **Characterization of $d(u,v)$.** We start with the notion of garbling, used by Blackwell to compare statistical experiments Blackwell (1953). A *garbling* is an element $q : \mathbb{N} \to \Delta(\mathbb{N})$, and the set of all garblings is denoted by $\mathcal{Q}$. Given a garbling g and an information structure u, we define the information structures $q.u$ and $u.q$ so that for each k,c,d,

$$q.u(k,c,d) = \sum_{c'} u(k,c',d)q(c'|c), \text{ and}$$
$$u.q(k,c,d) = \sum_{d'} u(k,c,d')q(d'|d).$$

We also denote by $\mathcal{Q}(L)$ the subset of garblings $q : \mathbb{N} \to \Delta(\{0,...,L-1\})$. There are two interpretations of a garbling. First, the garbling can be seen as an information loss: suppose that (k,c,d) is selected according to u, c' is selected according to the probability $q(c)$, and player 1 learns c' (and player 2 learns d). The new information structure is exactly equal to $q.u$, where the signal received by player 1 has been deteriorated through the garbling q. Similarly, $u.q$ corresponds to the dual situation where the signal of player 2 has been deteriorated. Further, the garbling q can also be seen as a behavior strategy of a player in a Bayesian game $\Gamma(u,g)$: if the signal received is c, play the mixed action $q(c)$. The relation between the two interpretations plays an important role in the proof of Theorem 1 below.

Theorem 1. *For each $u, v \in \mathcal{U}$ for*

$$\sup_{g\in\mathcal{G}} (\mathrm{val}(v,g) - \mathrm{val}(u,g)) \quad = \quad \min_{q_1\in\mathcal{Q}, q_2\in\mathcal{Q}} \|q_1.u - v.q_2\|. \tag{3.1}$$

If $u, v \in \mathcal{U}(L)$ for $L \in \mathbb{N}$, then all the optima can be attained by L-based structures (games and garblings). Hence,

$$d(u,v) = \max\left\{ \min_{q_1\in\mathcal{Q}, q_2\in\mathcal{Q}} \|q_1.u - v.q_2\|, \min_{q_1\in\mathcal{Q}, q_2\in\mathcal{Q}} \|u.q_1 - q_2.v\| \right\}.$$

The Theorem provides a characterization of the value-based distance between two information structures u and v for each player as a total variation distance between two sets obtained as garblings of the original information structures $\{q.u : q \in \mathcal{Q}\}$ and $\{v.q : q \in \mathcal{Q}\}$. If the two information structures are in $\mathcal{U}(L)$, then all the suprema in the first line can be achieved by payoffs functions in $\mathcal{G}(L)$, and garblings in $\mathcal{Q}(L)$.

The Theorem simplifies the problem of computing the value-based distance in two ways. First, it reduces the dimensionality of the optimization domain from payoff functions and strategy profiles (to compute value) to a pair of garblings. More importantly, notice that the solution to the original problem (2.1) is typically a saddle-point as it involves finding optimal strategies in a zero-sum game. On the other hand, the function $\|q_1.u - v.q_2\|$ is convex in garblings (q_1, q_2), and the domains $\mathcal{Q}$ are convex compact sets. Thus, the right-hand side of (3.1) is a convex optimization problem. Moreover, if $u, v \in \mathcal{U}(L)$ for some $L \in \mathbb{N}$, this convex problem is equivalent to a finite dimensional polyhedral convex problem. The characterization of the distance is quite tractable. We show it in numerous applications in this and subsequent sections.

3.2. Comments and applications.

3.2.1. *Single-agent problems.* A special case of a zero-sum game is a single-agent problem. Formally, a payoff function $g \in \mathcal{G}(L)$ is a *single-agent (player 1) problem* if $g(k,i,j) = g(k,i,j')$ for any $k \in K$ and any $i, j, j' \in \{0, ..., L-1\}$ (or alternatively, if the set of actions of player 2 is a singleton, $J = \{*\}$). Let $\mathcal{G}_1 \subset \mathcal{G}$ be the set of player 1 problems. Then, for each $g \in \mathcal{G}_1$, each information structure u, $\mathrm{val}(g,u)$ is the maximal expected payoff of player 1 in problem g. Let

$$d_1(u,v) := \sup_{g\in\mathcal{G}_1} |\mathrm{val}(u,g) - \mathrm{val}(v,g)| \leq d(u,v). \tag{3.2}$$

Analogously, we can define $d_2(u,v)$ as the distance measured using only player 2 single agent problems.

A simple consequence of the definition is that the single-agent distance depends only on the distribution of the posterior beliefs of player 1 about the state. For each $u \in \mathcal{U}$, let $\tilde{u} \in \Delta(\Delta(K))$ denote the distribution of the posterior beliefs of player 1 about the state induced by u, and D be the set of suprema of affine functions from $\Delta(K)$ to $[-1,1]$. Then, one shows that

$$d_1(u,v) = \sup_{f\in D}\left|\int_{p\in\Delta(K)} f(p)d\tilde{u}(p) - \int_{p\in\Delta(K)} f(p)d\tilde{v}(p)\right|. \tag{3.3}$$

A natural question is when the two distances are equal, or, alternatively, when the maximum in (2.1) is attained by the single-agent problems. The next result provides a partial answer to this question. For any structure $u \in \Delta(K \times C \times D)$, we say that the players information is *conditionally independent*, if under u, signals c and d are conditionally independent given k.

Proposition 2. *Suppose that $u, v \in \Delta(K \times C \times D)$ are two information structures with conditionally independent information such that $\operatorname{marg}_{K\times D} u = \operatorname{marg}_{K\times D} v$. Then,*

$$d(u,v) = d_1(u,v).$$

Proposition 2 says that, given two conditionally independent structures, if the information of player 2 is the smae, the distance is equal to value of the difference between the information of player 1. The proof of the Proposition relies on the characterization from Theorem 1 and it shows that the minimum in the optimization problem is attained by the same pair of garblings as in the single-agent version of the problem.

3.2.2. *Comparison of information structures.* We partially order the information structures by how good they are for player 1: given u, v in $\mathcal{U}$, say that *player 1 prefers u to v*, write $u \succeq v$, if for all g in $\mathcal{G}$, $\operatorname{val}(u,g) \geq \operatorname{val}(v,g)$. The order $\preceq$ is reflexive and transitive (and it is also antisymmetric on $\mathcal{U}^*$). One shows that the value is monotonic in the information of the players: for any garbling q, player 1 always weakly prefers u to $q.u$, and $u.q$ to u.

Theorem 1 implies that

$$u \succeq v \Longleftrightarrow \exists_{q_1,q_2\in\mathcal{Q}} q_1.u = v.q_2.$$

(To see why, notice that $u \succeq v$ is equivalent to the left-hand side of the first equality in (3.1) must be weakly negative, which is equivalent to the second claim above.) This characterization was initially obtained in Peski (2008) who generalized the Blackwell characterization of more informative experiment to the multi-player setting.

Additionally, we obtain the following Corollary to Theorem 1:

Corollary 3. *For all information structures u, v,*

$$\sup_{g\in\mathcal{G}} (\operatorname{val}(v,g) - \operatorname{val}(u,g)) = \inf_{u'\preceq u, v'\succeq v} \|u' - v'\|. \tag{3.4}$$

This observation provides an additional interpretation to the characterization from Theorem 1: the maximum gain from replacing information structure u by v is equal to the minimum total variation distance between the set of information structures that are worse than u and those that are better than v.

3.2.3. *The impact of the marginal over K.* Among many ways that two information structures can differ, the most obvious one is that they may have different distributions over the states k. In order to capture the impact of such differences, the next result provides tight bounds on the distance between two type spaces with a fixed distribution of a state:

Proposition 4. *For each $p, q \in \Delta K$,*

$$\min_{u,v:\operatorname{marg}_K u=p,\operatorname{marg}_K v=q} d\left(u,v\right) = \sum_k \left|p_k - q_k\right|, \tag{3.5}$$

$$\max_{u,v:\operatorname{marg}_K u=p,\operatorname{marg}_K v=q} d\left(u,v\right) = 2\left(1 - \max_{p',q'\in\Delta K}\sum_k \min\left(p_k q'_k, p'_k q_k\right)\right).$$

If $p=q$, the upper bound is equal to

$$\max_{u,v:\operatorname{marg}_K u=\operatorname{marg}_K v=p} d(u,v) = 2(1-\max_k p_k).$$

The lower bound in (3.5) is reached when the two information structures do not provide any information to none of the players. The upper bound is reached with information structures where one player knows the state perfectly, and the other player does not know anything.

Example 5. Suppose that $K=\{0,1\}$. Fix $p,q\in\Delta K$. In such a case, one easily checks that the maximum in the right hand side is attained by either $p'=q'=(1,0)$, or $p'=q'=(0,1)$, or $p'=p, q'=q$. It follows that for any two information structures u, v,

$$d\left(u,v\right) \le 2\left(1-\max\left(\min\left(p_0,q_0\right),\min\left(p_1,q_1\right),p_0q_0+p_1q_1\right)\right).$$

The bound is attained when, for example, $u=\bar{u}, v=\underline{v}$ and $\bar{u}\left(k,c,d\right)=p_k\,\mathbb{1}_{c=k}\,\mathbb{1}_{d=0}$ and $\underline{v}\left(k,c,d\right)=q_k\,\mathbb{1}_{c=0}\,\mathbb{1}_{d=k}$ for each $k,c,d\in\{0,1\}$.

3.2.4. *Optimal strategies.* Another useful property of the characterization is that the garblings that solve (3.1) can be used to transform optimal strategies in one structure to approximately optimal strategies on another structure. More precisely, we say that strategy σ of player 1 is ε-optimal in game g on structure u if for any strategy τ of player 2, the payoff of σ against τ is no smaller than $\operatorname{val}\left(u,g\right)-\varepsilon$. We similarly define ε-optimal strategies for player 2.

For a strategy $\sigma\in\mathcal{Q}$ and a garbling $q_1\in\mathcal{Q}$, define $\sigma.q_1$ in $\mathcal{Q}$ by $\sigma.q_1(c)=\sum_{c'} q_1(c'|c)\sigma(c')$ for each signal c: player 1 receives signal c, then selects χ according to $q_1(c)$ and plays $\sigma(\chi)$. We have

Proposition 6. *Fix finite u,v and suppose that q_1 and q_2 are solutions to (3.1). Then, if σ is an optimal strategy in g on v, then $\sigma.q_1$ is a $2d\left(u,v\right)$-optimal strategy in g on u. Similarly, if τ is optimal for player 2 in g on u, then $\tau.q_2$ is $2d\left(u,v\right)$-optimal for player 2 in g on v.*

4. Value of additional information

An important advantage of our approach is that the distance $d\left(u,v\right)$ has a straightforward interpretation as the maximum willingness to pay to move from one information structure to another. This allows us to discuss the value of additional information, which we define as the the distance between structures with or without an extra piece of information.

4.1. **Value of additional information: games vs. single agent.** Consider two information structures

$$u \in \Delta\left(K \times \left(C \times C'\right) \times D\right),$$
$$v = \operatorname{marg}_{K \times C \times D} u.$$

The structure u is the same as v, but where player 1 can observe an additional signal c'. Because u represents more information, u is (weakly) more valuable, and the value of the additional information is defined as $d\left(u,v\right)$, i.e., the tight upper bound on the gain from the additional signal. A corollary to Proposition 2 shows that if the signals of the two players ate independent conditionally on the state, the gain from the new information is the largest in the single-agent problems.

Corollary 7. *Suppose that information in u (and therefore in v) is conditionally independent. Then,*

$$d\left(u,v\right) = d_1\left(u,v\right).$$

The next example illustrates that the thesis of Proposition 7 does not hold without a conditional independence.

Example 8. Suppose that $C = \{*\}, K = D = C' = \{0,1\}$, and

$$u\left(k,d,c'\right) = \begin{cases} \frac{1}{4}\frac{k+c'}{2}, & \text{if } d = 1, \\ \frac{1}{4}\left(1 - \frac{k+c'}{2}\right), & \text{if } d = 0. \end{cases}$$

Then, the new information c' is independent from k, but d is both correlated with k and c'. Because c' is independent from the state, the value of new information in player 1 single-agent problem is 0:

$$d_1\left(u,v\right) = 0.$$

However, signal c' provides non-trivial information about the signal of the other player, hence it is valuable in some games. It is easily seen from 1 that $d(u,v) > 0$. Indeed, since u gives player 1 more information, we have

$$d(u,v) = \min_{q_1,q_2} \|u.q_2 - q_1.v\|,$$

where $q_2 : \{0,1\} \to \Delta\{0,1\}$ and $q_1 \in \Delta\{0,1\}$. The existence of a pair (q_1,q_2) such that $\|u.q_2 - q_1.v\| = 0$ is equivalent to the system of equations

$$\forall (k,d,c') \in \{0,1\}^3,\ u.q_2(k,d,c') = v.q_1(k,d,c'),$$

where the unknowns are q_1, q_2, and one can check that this system does not admit any solution. In other words, the information that would be useless in a single-agent decision problem is valuable in a strategic setting.

A special case of the conditionally independent information is where players receive multiple samples of independent Blackwell experiments. In an online Appendix, we show how to compute the value of additional Blackwell experiments.

4.2. **Informational substitutes.** Next, we ask two questions about the impact of a piece of information on the value of another piece of information. Suppose that

$$\begin{aligned} u &\in \Delta\left(K \times (C \times C_1 \times C_2) \times D\right), \\ v &= \operatorname{marg}_{K \times (C \times C_1) \times D} u, \\ u' &= \operatorname{marg}_{K \times (C \times C_2) \times D} u, \\ v' &= \operatorname{marg}_{K \times C \times D} u. \end{aligned}$$

When moving from v' to u' or v to u, player 1 gains an additional signal c_2. The difference is that in the latter case, player 1 has more information that comes from signal c_1. The next result shows the impact of an additional signal on the value of information.

Proposition 9. *Suppose that, under u, c_1 is conditionally independent from (c, c_2, d) given k. Then,*

$$d\left(u', v'\right) \geq d\left(u, v\right).$$

Given the assumptions, the marginal value of signal c_2 decreases when signal c_1 is also present. In other words, the two pieces of information are substitutes.

The conditional assumption is equivalent to two simpler assumptions (a) c_1 is conditionally independent from (c, c_2) given (k, d), and (b) c_1 and d are conditionally independent given k. Both (a) and (b) are important as it is illustrated in the two subsequent examples.

Example 10. Violation of (a). Suppose that $C = D = \{*\}$, $K = C_1 = C_2 = \{0, 1\}$, c_1 and c_2 are uniformly and independently distributed, and $k = c_1 + c_2 \mod 2$. Then, signal c_2 is itself independent from the state, hence useless without c_1. Knowing c_1 and c_2 means knowing the state, which is, of course, very valuable. Thus, the value of c_2 increases when c_1 is also present.

Example 11. Violation of (b). Suppose that $C = \{*\}$, $K = C_1 = C_2 = D = \{0, 1\}$, c_1 and d are uniformly and independently distributed, $c_2 = d$, and $k = c_1 + d \mod 2$. Notice that part (a) of the assumption holds (given (k, d), both signals c_1 and c_2 are constant, hence, independent), but part (b) is violated. Again, signal c_2 is useless alone, but together with c_1 it allows the determine the state and the information of the other player.

4.3. **Informational complements.** Another question is about the impact of an information of the other player on the value of information. Suppose that

$$\begin{aligned} u &\in \Delta\left(K \times (C \times C_1) \times (D \times D_1)\right), \\ v &= \operatorname{marg}_{K \times C \times (D \times D_1)} u, \\ u' &= \operatorname{marg}_{K \times (C \times C_1) \times D} u, \\ v' &= \operatorname{marg}_{K \times C \times D} u. \end{aligned}$$

When moving from v' to u' or v to u, in both cases, player 1 gains an additional signal c_1. However, in the latter case, player 2 has an additional information that comes from signal d_1. The next result shows the impact of the opponent's signal on the value of information.

Proposition 12. *Suppose that* (c, c_1) *and* d *are conditionally independent given* k. *Then,*

$$d(u', v') \leq d(u, v).$$

Given the assumptions, signal c_1 becomes more valuable when the opponent has also an access to additional information. Hence, the two pieces of information are complements.

Example 13. The independence assumptions are important. Suppose that $C = \{*\}$ and $K = D = D_1 = C_1 = \{0, 1\}$. The state is drawn uniformly. Signal d is equal to the state with probability $\frac{2}{3}$ and signal d_1 is equal to the state for sure. Finally, $c_1 = 1$ iff $d = k$. In other words, player 2's signal d is an imperfect information about the state. Signal c_1 carries information about the quality of the signal of player 1. Such signal is valuable in some games, hence $d(u', v') > 0$. In the same time, if player 2 learns the state perfectly, signal c becomes useless, and $d(u, v) = 0$. (The last claim can be formally proven using Proposition 14 below.)

4.4. **Value of joint information.** Finally, we consider a situation where two players receive additional information simultaneously. We ask when the joint information contained in signals of two players is (not) valuable. We need the following definition. Consider a distribution $\mu \in \Delta(X \times Y \times Z)$ over countable spaces. We say that random variables x and y are ε-conditionally independent given z if

$$\sum_z \mu(z) \sum_{x,y} |\mu(x, y|z) - \mu(x|z)\,\mu(y|z)| \leq \varepsilon.$$

Let

$$u \in \Delta(K \times (C \times C_1) \times (D \times D_1)),$$
$$v = \mathrm{marg}_{K \times C \times D} u.$$

Proposition 14. *Suppose that* d_1 *is* ε*-conditionally independent from* $k \times c$ *given* d*, and* c_1 *is* ε*-conditionally independent from* (k, d) *given* c. *Then,*

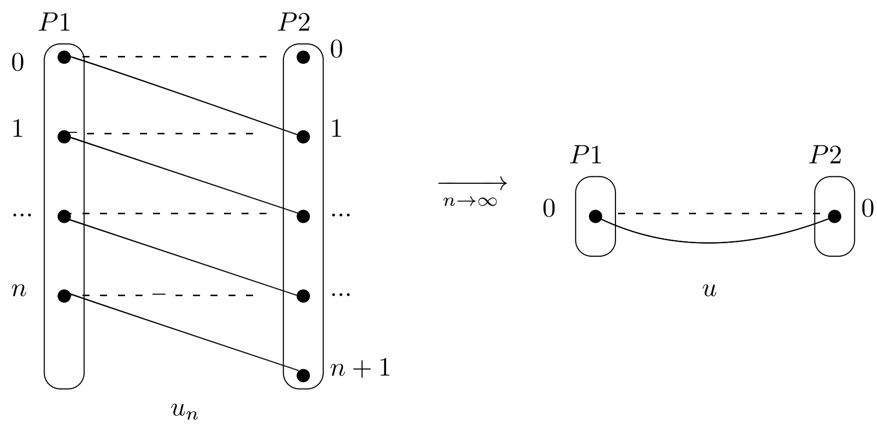
$$d(u, v) \leq \varepsilon.$$

Suppose that the new signal of each player is not providing any significant information about the state of the world and original information of the other player. Such a signal would be useless in a single-decision problem. In principle, the new signals could be useful in a strategic setting as the new signals may be correlated with each other, and jointly correlated with the old information. (In fact, it is possible that the new signals, if learned by both players, would completely reveal the original information. See the example below.) Nevertheless, the Proposition says that the information that is jointly shared by the two players is not valuable in the zero-sum games.

Example 15. Let $K = \{0, 1\}$. Consider a sequence of information structures $u_n \in \Delta(K \times \{0, ..., n\} \times \{0, ...n+1\})$ and structure $u \in \Delta(K \times \{0\} \times \{0\})$ such that

$$u_n(0, l, l) = u_n(1, l, l+1) = \frac{1}{2(n+1)} \text{ for each } l = 0, ..., n,$$
$$u(k, 0, 0) = \frac{1}{2} \text{ for each } k = 0, 1.$$

This is illustrated on the Figure below (all lines have equal probability; the dashed line corresponds to state 0 and the solid line corresponds to state 1).



The idea is that when n is large, with high probability the players will receive signals far from 0 and n. These signals convey very little information to the players and only differ for high-order beliefs.

Notice that the signal of each player is $\frac{1}{n+1}$-independent from the state of the world. The Lemma implies that $d(u_n, u) \leq \frac{2}{n+1} \to 0$. In particular, the information structures u_n converge to the structure u, where no player receive any information.

5. Value-based topology

The two previous sections discussed the quantitative aspect of the value-based distance. Now, we analyze its qualitative aspect: the topological information contained in the distance.

5.1. **Universal space of information structures.**

We begin with recalling relevant facts about the universal type spaces. A (Mertens-Zamir) *universal type space* is a pair of compact metric spaces Θ_1 and Θ_2 with respect to the following property: for each i, there exists a homeomorphism $\phi_i : \Theta_i \to \Delta(K \times \Theta_{-i})$. The universal type space is unique (up to homeomorphism). The spaces Θ_i are constructed as sets of coherent sequences of finite hierarchies of beliefs. The topology on the universal type space is also is referred to as the product topology; the latter name comes from the fact that it can be defined as the Tichonof's topology on the sequences of finite hierarchies.

For each Harsanyi type space (even uncountable), there is a belief-preserving mapping that uniquely maps the types into the associated hierarchies. And so, if $u \in \Delta(K \times C \times D)$ is an information structure, the signals c and d are mapped into hierarchies $\tilde{c} \in \Theta_1$ and $\tilde{d} \in \Theta_2$. The mapping induces a probability distribution $\tilde{u} \in \Delta\Theta_1$ over the hierarchies of beliefs of player 1 (and an analogous object for player 2). Let $\Pi_0 = \{\tilde{u} : u \in \mathcal{U}\}$ be the space of all such distributions. The closure of Π_0 (in the weak topology) is denoted as Π. Using the Mertens-Zamir's terminology, Π is the set of consistent (Borel) probabilities over the universal type space, and Π_0 is the subset of elements of Π with at most countable support. The space Π is compact metric and Π_0 is dense in Π (see corollary III.2.3 and theorem III.3.1 in Mertens et al. (2015)).

On the other hand, each $P \in \Pi$ can be treated as a common prior type space with types in sets Θ_i. The common prior is generated as follows. First, a type θ_1 of player 1 is drawn from distribution P. Then, a state k and a type θ_2 of player 2 is drawn from distribution $\phi_1(\theta_1)$. For each zero-sum game $g \in \mathcal{G}$, Proposition III.4.2 in Mertens et al. (2015) shows that one can find its value on P that we denote as $\mathrm{val}(P, g)$. In particular, we can define the value based distance (2.1) over elements of Π.

There are two fundamental observations. First, by Proposition III.4.4 in Mertens et al. (2015),

$$\mathrm{val}(u, g) = \mathrm{val}(\tilde{u}, g) \text{ for each } u \in \mathcal{U}.$$

In particular, the value on information structure u depends only on the induced distribution over hierarchies of beliefs. Second, by lemma 41 in Gossner and Mertens (2001)), the value separates the elements of Π: we have

$$d(P, Q) > 0 \text{ for any } P, Q \in \Pi, P \neq Q.$$

The two fundamental observations imply that there is a bijection between the set of Π_0 and the set $\mathcal{U}^*$ of equivalence classes of information structures.

5.2. **The weak vs. the value-based topology.** The next result says the spaces Π_0 and the set $\mathcal{U}^*$ have the same topologies.

Theorem 16. *A sequence (u_n) in $\mathcal{U}^*$ converges to $u \in \mathcal{U}^*$ for the value-based distance if and only if the sequence $(\tilde{u}_n)$ converges weakly to $\tilde{u}$ in Π_0.*

The result says that a convergence in value-based topology to a countable structure is equivalent to the convergence in distribution of finite-order hierarchies of beliefs. Informally, at least around countable structures, the higher-order beliefs have diminishing importance.

We describe the idea of the proof when u is finite. In such a case, we surround the hierarchies $\tilde{c}$ for $c \in C$ by sufficiently small and disjoint neighborhoods, so that all hierarchies in the neighborhood of $\tilde{c}$ have similar beliefs about the state and the opponent. We do alike for the other player. The weak convergence ensures that the converging structures assign large probability to the neighborhoods. Finally, we show that any information about players hierarchy beyond the neighborhood to which it belongs is almost conditionally independent (in the sense of Section 4) from the information about the state and the opponents neighborhoods. This allows us to utilize Proposition 14 (or, more precisely, a part of its proof) to show that only the information about neighborhoods matter, and the latter is similar to the information in the limit structure u. If u is countable, we also show that it can be appropriately approximated by finite structures.

A reader may find such a result surprising, given the message of the strategic discontinuities literature (Rubinstein (1989), Dekel et al. (2006), Weinstein and Yildiz (2007), Ely and Peski (2011), etc.). In that literature, the convergence of finite-order hierarchies does not imply strategic convergence even around finite structures (for instance, see Example 18 below). There are multiple ways in which our setting differs. First, we rely on ex ante equilibrium concept, rather than interim rationalizability. We are also interested in the payoff comparison rather than the behavior. Second, we restrict attention to common prior type spaces. Finally, we restrict attention to zero-sum games.

We believe that the last restriction makes the key difference. As we explain in Section 7.1 below, the ex ante focus and payoff comparison but without restriction to zero-sum games lead to a topology that is significantly finer than the weak topology. The role of common prior is less clear. On one hand, Lipman (2003) imply that, at least from the interim perspective, common prior does not generate significant restrictions on finite order hierarchies. On the other hand, we rely on the ex ante perspective, and common prior is definitely important for Proposition 14 that plays an important role in the proof.

Finally, we want to emphasize that Theorem 16 does not mean that the higher-order beliefs do not matter at all. In fact, this result does not hold anymore if the limit u is an uncountable information structure, as illustrated in Section 6.

5.2.1. *Approximate knowledge and approximate common knowledge.* Here, we describe an important application of Theorem 16. Recall that one of the initial impulses of the literature on the higher-order beliefs was the realization of the difference between the approximate knowledge and the approximate common knowledge. Here, we are going to show that, at least when it comes to the knowledge of the payoff state, the difference is not important for zero-sum games.

An information structure $u \in \Delta(K \times C \times D)$ exhibits ε-knowledge of the state if there is a mapping $\kappa : C \cup D \to K$ such that

$$u\Big(\{u(\{k = \kappa(c)\}|c) \geq 1-\varepsilon\}\Big) \geq 1-\varepsilon \text{ and } u\Big(\{u(\{k = \kappa(d)\}|d) \geq 1-\varepsilon\}\Big) \geq 1-\varepsilon.$$

In other words, the probability that any of the player player assigns at least $1-\varepsilon$ to some state is at least $1-\varepsilon$.

Proposition 17. *Suppose that u exhibits ε-knowledge of the state and that $v \in \Delta(K \times K_C \times K_D)$, where $K_C = K_D = K$ and $\operatorname{marg}_K v = \operatorname{marg}_K u$, and $v(k = k_C = k_D) = 1$. (In other words, v is a common knowledge structure with the only information about the state.) Then,*

$$d(u,v) \leq 20\varepsilon.$$

Thus, approximate knowledge structures are close to common knowledge structures. The convergence of approximate knowledge type spaces to the approximate common knowledge is a consequence of Theorem 16. The metric bound stated in the Proposition requires a separate (simple) proof based on Proposition 14.

Example 18. Consider a Rubinstein email-game information structures $u^{p,\alpha}$ defined as follows: The state is $k=1$ with probability p and $k=0$ with the remaining probability. If the state is 0, both players receive signal 0. If the state is 1, player 1 learns that and sends a message to the opponent, who after receiving it, immediately sends it back. The message travels back and forth. Each round, the message may get lost before reaching the target with probability α, in which case the process stops. The signal of the player is the number of the times she sends a message away.

Player 1 always knows the state. Player 2's first order belief attach the probability of at least $\frac{1}{1+\alpha\frac{p}{1-p}}$ to one of the states. Proposition 17 implies that, when $\alpha \to 0$, the Rubinstein's information structures converge to the common knowledge of the state.

6. Large space of information structures

In this section, we show that the value-based distance does not have a compact completion, or more precisely, it is not totally bounded on $\mathcal{U}^*$. Informally, the space of information structures is large: it cannot be approximated by finitely many structures. We use this result to provide a negative answer to a problem posed by J.F. Mertens.

6.1. $(\mathcal{U}^*, d)$ **is not totally bounded.** To focus attention, we assume that $K = \{0,1\}$. This is without loss of generality, as our negative result for $K = \{0,1\}$ clearly implies a negative result for larger K.

The next result finds $\varepsilon > 0$ and an infinite sequence of (finite) information structures such that all of them are at least ε-away from each other in the value-based distance.

Theorem 19. *There exists $\varepsilon > 0$, even $N < \infty$, and a Markov chain of distribution μ over sequences*

$$k, c_1, d_1, c_2, d_2, c_3, ...$$

where $C_l = D_l = \{1, ..., N\}$ and $c_l \in C_l, d_l \in D_l$ for each $l \geq 1$, such that, if we define sets of signals $C^l = \prod_{p=1}^{l} C_p, D^l = \prod_{p=1}^{l} D_p$ and information structures

$$u^l = \operatorname{marg}_{K \times C^l \times D^l} \mu,$$

then, for any $p \neq l$,

$$d\left(u^l, u^p\right) > \varepsilon.$$

Recall that a consistent probability over the universal type space is a probability distribution over hierarchies of beliefs that can be induced by some (potentially uncountable) common prior type space, and that the set Π of consistent probabilities over the universal belief space is compact under the topology of the weak convergence. Theorem 19 implies that (Π_0, d) and therefore (Π, d) are not totally bounded metric spaces, so that (Π, d) is not a compact metric space. It follows that the value-based topology on Π is different from the weak topology, even if these topologies coincide on Π_0 by Theorem 16. (We elaborate further on this point in section 7.2.)

The proof, with an exception of one step that we describe below, is constructive. The first part of the theorem describes the properties of the construction. We define a Markov chain that starts with a state followed by alternating signals for each player. In structure u^n, player 1 observes signals $(c_1, c_2, ..., c_n)$ and player 2 observes $(d_1, d_2, ..., d_n)$. Thus, the sequence of structures u^n can be understood as fragments of a larger information structure, where progressively, more and more information is revealed to each player. The Theorem shows that the larger structure is not the limit of its fragments in the value-based distance. In particular, there is no analog of the martingale convergence theorem for the value-based distance for such sequences.

This has to be contrasted with two other settings, where the martinagle convergence holds. First, in the 1-player case, any sequence of information structures in which the player is receiving more and more signals converges for the distance d_1 (which can be proved using formula (3.3)). Second, using martingale convergence theorems and arguing in the proof of the denseness of Π_0 in Π for the weak topology (see e.g. Thm III.3.1 and Lemma III.3.2 in Mertens et al. (2015)), one shows that $\tilde{u}^n$ converges $\tilde{\mu}$ in Π in the weak topology.

The Markov property means that (a) the state is independent from all players information conditionally on c^1, and (b) each new piece of information is independent from the previous pieces of information conditionally on the most recent information of the other player. As we show in the Appendix D.2, this ensures that the n-th level hierarchy of beliefs of any type in structure u^n is preserved by all consistent types in structures u^m for $m \geq n$. Theorem 19 exhibits therefore a situation in which higher-order beliefs do not have diminishing importance for the value-based distance. In particular, it shows that the knowledge of the n-th level hierarchy of beliefs for any arbitrarily high n is not sufficient to play ε-optimally in any finite zero-sum game $g \in \mathcal{G}$.

6.2. **Comments on the proof.** We briefly sketch the main ideas behind the proof. We fix $\alpha < \frac{1}{25}$. We show that we can find even N high enough and a set $S \subseteq \{1, ..., N\}^2$ with certain mixing properties:

$$\begin{aligned}
|\{j : (i,j) \in S\}| &= \frac{N}{2}, \text{ for each } i, \\
|\{i : (i,j) \in S\}| &\sim \frac{N}{2}, \text{ for each } j, \\
|\{i : (i,j), (i,j') \in S\}| &\sim \frac{N}{4}, \text{ for each } j, j', \\
|\{i : (i,j), (i,j'), (l,i) \in S\}| &\sim \frac{N}{8}, \text{ for each } j, j', l,
\end{aligned}$$

etc. The "$\sim$" means that the left-hand side is within α-related distance to the right-hand side. Altogether, there are 8 properties of this sort (see Appendix D.4 for details.) The properties essentially mean that various sections of S are "uncorrelated" with each other.

Although it might be possible to directly construct S with the required properties, we are unable to do so. Instead, we show the existence of set S using the probabilistic method of P. Erdős (for a general overview of the method, see Alon and Spencer (2008)). Suppose that the sets $S(i)$ for $i = 1, ..., N$ are chosen independently and uniformly from all $\frac{N}{2}$-element subsets of $\{1, ..., N\}$. We show that if N is sufficiently large, then the set $S = \{(i,j) : j \in S(i)\}$ satisfies the required properties with positive probability, proving thus that a set satisfying these properties exists. Our method of the proof is not particularly careful about the optimal N and a rough estimate suggests that it needs $N \geq 10^8$.

Given S, we construct the Markov chain of distribution μ. First, c_1 is chosen from the uniform distribution on $\{1, ..., N\}$. Next, we choose $k = 1$ with probability $\frac{c_1}{N+1}$ and $k = 0$ with the remaining probability. Next, inductively, for each $l \geq 1$, we choose

- d_l uniformly from set $S(c_l) = \{j : (c_l, j) \in S\}$ and conditionally independently from $k, ..., d_{l-1}$ given c_l, and
- c_{l+1} uniformly from set $S(d_l)$ and conditionally independently from $k, ..., c_l$ given d_l.

Thus, the Markov chain becomes time-homogenuous after the choice of state k and c_1. This finishes the construction of the information structures.

To provide a lower bound on the distance between different information structures, we construct a sequence of games. In game g^n, player 1 is supposed to reveal the first n pieces of her information; player 2 reveals the first $n-1$ pieces. The payoffs are such that it is a dominant strategy for player 1 to precisely reveal his first order belief about the state, which amounts to report truthfully c_1. Further than that, we verify whether the sequence of reports

$$\left(\hat{c}_1, \hat{d}_1, ..., \hat{c}_{n-1}, \hat{d}_{n-1}, \hat{c}_n\right)$$

belongs to the support of the distribution of the Markov chain μ. If it does, then player 1 receives payoff $\varepsilon \sim \frac{1}{10(N+1)^2}$. If it doesn't, we identify the first report in the sequence that deviates from the support. The responsible player is punished with payoff -5ε (and the opponent receives 5ε).

The payoffs and the mixing properties of matrix S ensure that players have incentives to report their information truthfully. We check it formally, and we show that that if $l > p$, then

$$d\left(u^l, u^p\right) \geq \text{val}\left(u^l, g^{p+1}\right) - \text{val}\left(u^p, g^{p+1}\right) \geq 2\varepsilon.$$

6.3. **An open problem raised by Mertens.** Because the n-th level hierarchies of beliefs become constant as we move along the sequence u^n, it must be[7] that the sequence $\tilde{u}^n$ converges weakly in Π to the limit

$$\tilde{u}^n \to \tilde{\mu}.$$

(The limit is the consistent probability obtained from the type space over infinite sequences with common prior μ.) Despite the convergence, our Theorem shows that

$$\limsup_n \sup_{g \in \mathcal{G}} \left|\text{val}\left(\mu, g\right) - \text{val}\left(u^n, g\right)\right| = \limsup_n d(\tilde{u}^n, \tilde{\mu}) \geq \varepsilon.$$

In particular, the family of functions $(u \mapsto \text{val}(u, g))_{g \in \mathcal{G}}$ is not equicontinuous on Π for the weak topology.

This answers negatively the second of the three problems[8] posed by Mertens (1986) in his Repeated Games survey from ICM: "This equicontinuity or Lispchitz property character is crucial in many papers...". Precisely, if this family was equicontinuous, a direct application of Ascoli's theorem would have implied that the family $(u \mapsto \text{val}(u, g))_{g \in \mathcal{G}}$ is totally bounded in the space of bounded functions over Π endowed with the uniform norm.

The importance of the Mertens question comes from its application to limit theorems in the repeated games. Consider a general zero-sum repeated game (stochastic game, with incomplete information, and imperfect monitoring), given by a transition $q : K \times I \times J \longrightarrow \Delta(K \times C \times D)$, a payoff function $g : K \times I \times J \longrightarrow [-1, 1]$ and an initial probability u_0 in $\Delta(K \times C \times D)$, where K, I, J, C and D are finite sets. Before stage 1, an initial state k_1 in K and initial private signals $c_1 \in C$ for player 1, and $d_1 \in D$ for player 2, are selected according to u_0. Then at each stage

[7] As mentioned earlier, this convergence can also be proved by using martingale convergence theorems.

[8] Problem 1 asked for the convergence of the value functions $(v_\lambda)_\lambda$ and $(v_n)_n$ in a general zero-sum repeated game with finitely many states, actions and signals, and was disproved in Ziliotto (2016). Problem 3 asks if the existence of a uniform value follows from the uniform convergence of (v_λ), and was disproved in Lehrer and Monderer (1994) for 1-player games, see also Monderer and Sorin (1993).

t, simultaneously player 1 chooses an action i_t in I and player 2 chooses and action $j - t$ in J, the stage payoff is $g(k_t, i_t, j_t)$, an element $(k_{t+1}, c_{t+1}, d_{t+1})$ is selected according to $q(k_t, i_t, j_t)$, the new state is k_{t+1}, player 1 receives the signal c_{t+1}, player 2 the signal d_{t+1}, and the play proceeds to stage $t+1$.

An appropriate state variable is $u_t \in \Delta\left(K \times \left(I^{t-1} \times C^t\right) \times \left(J^{t-1} \times D^t\right)\right)$, representing the current state in K and the finite sequence of private actions and signals previously received by each player. As a consequence, a recursive formula can be explicitly written as follows: for all discount factors $\delta < 1$ and all $u \in \Delta\left(K \times C_0 \times D_0\right)$,

$$\begin{aligned} v_\delta(u; g, q) &= \max_{\sigma_1 : C_0 \to \Delta(I)} \min_{\sigma_2 : D_0 \to \Delta(J)} (1-\delta)\, \gamma_{u,g}(\sigma_1, \sigma_2) + \delta v_\delta(F(u, \sigma_1, \sigma_2; q); g, q), \\ &= \min_{\sigma_2 : D_0 \to \Delta(J)} \max_{\sigma_1 : C_0 \to \Delta(I)} (1-\delta)\, \gamma_{u,g}(\sigma_1, \sigma_2) + \delta v_\delta(F(u, \sigma_1, \sigma_2; q); g, q), \end{aligned}$$

where σ_1, σ_2 are strategies in the game, $\gamma_{u,g}(\sigma_1, \sigma_2)$ is the stage game payoff, and $F(u, \sigma_1, \sigma_2; q) \in \Delta\left(K \times (C_0 \times I \times C) \times (D_0 \times J \times D)\right)$ is the information structure obtained tomorrow, given today's strategies, the state transition and signal function q. Formally,

$$F\left(u, \sigma_1, \sigma_2; q\right)\left(k, c_0, i, c, d_0, j, d\right) = \sum_{k' \in K} u\left(k', c_0, d_0\right) \sigma_1\left(i|c_0\right) \sigma_2\left(j|d_0\right) q\left(k|k', i, j\right).$$

Proposition 20. *Suppose that the family of information structures $\mathcal{U}_0 \subseteq \mathcal{U}$ is totally bounded for the value-based distance d. Suppose that for some stochastic game (g, q), any strategies σ_1, σ_2, we have*

$$u \in \mathcal{U}_0 \Longrightarrow F\left(u, \sigma_1, \sigma_2; q\right) \in \mathcal{U}_0\,.$$

Then, the family $\left(v_\delta\left(.; g, q\right)\right)_{\delta \in [0,1)}$ is totally bounded in the space of bounded functions on $\mathcal{U}_0$ endowed with the uniform norm.

Proof. Let $\mathcal{U}_0^* \subset \mathcal{U}^*$ be the set of equivalence classes of information structures in $\mathcal{U}_0$. The metric space $(\mathcal{U}_0^*, d)$ being totally bounded, its completion $(\overline{\mathcal{U}}_0^*, d)$ is a compact metric space. The functions v_δ are, by construction, 1-Lipschitz from $(\mathcal{U}_0^*, d)$ to $[-1, 1]$. Hence any function v_δ admits a unique 1-Lipschitz extension on $\overline{\mathcal{U}}_0^*$. Ascoli's theorem implies that the family $(v_\delta)_{\delta \in [0,1)}$ is totally bounded in the space of bounded functions on $(\overline{\mathcal{U}}_0, d)$ endowed with the uniform norm, which implies the result. □

The above result might seem quite weak, but it is the first necessary step for proving that v_δ converges uniformly when δ goes to 1. This condition was shown to be sufficient for general Markov Decision Problems in Renault (2011), and his main theorem (together with the Tauberian theorem of Lehrer and Sorin (1992)) implies quite directly the following corollary.

Corollary 21. *In addition to the assumptions of Proposition 20, assume that the transition q of the stochastic game does not depend on the action of player 2, then the family v_δ converges uniformly to a limit v on $\mathcal{U}_0$ as δ goes to 1.*

The existence of a limit value attracted a lot of attention since the first results by Aumann and Maschler (1995) and Mertens and Zamir (1971) for repeated games, and by Bewley and Kohlberg (1976) for stochastic games. Once the fact that the family of value functions is totally bounded is established, the existence of the limit value is typically obtained by showing that there is at most one accumulation point of the family (v_δ), e.g. by showing that any accumulation point satisfies a system of variational inequalities admitting at most one solution (see e.g. the survey Laraki and Sorin (2015) and footnote 2 for related works).

Even if the answer to Mertens' problem is negative, identifying totally bounded subsets of $(\mathcal{U}^*, d)$ might have applications for limit theorems for repeated games for which the hypothesis of Proposition 20 applies. The next section illustrate with our notation several known such subsets.

6.4. **Totally bounded subsets.** In the 1-player case, the characterization 3.3 implies that the set of equivalence classes in $\mathcal{U}$ inducing the same distribution of first order belief of player 1, equipped with the distance d_1 can be identified with the subset of $(\Delta(\Delta(K)), \tilde{d}_1)$ of probabilities with countable support, where $\tilde{d}_1$ is the distance given by right-hand side of (3.3), which metricizes the weak topology on $\Delta(\Delta(K))$. The latter being weakly compact, we deduce that $(\mathcal{U}, d_1)$ is totally bounded.

For 2 players, every particular subspace $\mathcal{U}_i$ for $i = 1, ..., 4$ listed below has been shown to be totally bounded, each time by identifying $\mathcal{U}_i^*$ with a subset of a weakly compact space of probabilities. We only mention the relevant variables used for the identification and the corresponding space of probabilities.

• Set $\mathcal{U}_1$ of information structures where both players receive the same signal: $\mathcal{U}_1^*$ can be identified with $\Delta_c(\Delta(K))$[9]. Here given u in $\mathcal{U}_1$, what matters is the induced law on the common a posteriori of the players on K. Another characterization of $d(u, v)$ has been obtained in Renault and Venel (2016). let D_1 be the subset of 1-Lipschitz functions from $\Delta(K)$ to $\mathbb{R}$ satisfying

$$\forall p, q \in \Delta(K), \forall a, b \geq 0,\ af(p) - bf(q) \leq \|ap - bq\|_1.$$

We have :

$$\forall u, v \in \mathcal{U}_1,\ d(u, v) = \sup_{f \in D_1} \left(\int_{p \in \Delta(K)} f(p) d\tilde{u}(p) - \int_{p \in \Delta(K)} f(p) d\tilde{v}(p) \right).$$

• Set $\mathcal{U}_2$ of information structures $u \in \Delta(K \times C \times D)$ where player 1 knows the signal of player 2, i.e. there exists a map $\kappa : C \to D$ such that $u(\{d = \kappa(c)\}) = 1$: $\mathcal{U}_2^*$ can be identified with $\Delta_c(\Delta_c(\Delta(K)))$ (see Mertens (1986), Gensbittel et al. (2014)). Given $u \in \mathcal{U}_2$, the relevant variable is the induced distribution of the second order beliefs of player 2.

• Set $\mathcal{U}_3$ of independent information structures: $\mathcal{U}_3$ is the set of $u \in \mathcal{U}$ with conditionally independent information. $\mathcal{U}_3^*$ can be identified with $\Delta_c(\Delta(K) \times \Delta(K))$. Given u in $\mathcal{U}_3$, the relevant variable is the pair of first order beliefs of the players.

7. Relation to other topologies on $\mathcal{U}$

7.1. **Payoff-based distance with non-zero-sum games.** It seems natural to ask about a version of distance (2.1) where we take supremum over all bounded games, not necessarily zero-sum. In this subsection, we are going to show that such an approach cannot be useful because so defined measure is equal to its maximum possible value for almost every pair of information structures.

Formally, a *non-zero sum payoff function* is a map $g : K \times I \times J \to [-1, 1]^2$ where I, J are finite sets. Let $\mathrm{Eq}\,(u, g) \subseteq \mathbb{R}^2$ be the set of Bayesian Nash payoffs in

[9] We use the notation $\Delta_c(X)$ for the probabilities with at most countable support on X

game g on information structure u. Assume that the space $\mathbb{R}^2$ is equipped with the maximum norm $d_{\max}(x,y) = \max_{i=1,2}|x_i - y_i|$ and the space of compact subsets of $\mathbb{R}^2$ with the induced Hausdorff distance $d^H_{\max}$. Let

$$d_{NZS}(u,v) = \sup_{g \text{ is a non-zero-sum payoff function}} d^H_{\max}\left(\mathrm{Eq}(u,g), \mathrm{Eq}(v,g)\right).$$

Then, clearly as in our original definition (2.1), $0 \le d_{NZS}(u,v) \le 2$.

Contrary to the value in the zero-sum game, the Bayesian Nash Equilibrium payoffs on information structure u cannot be factorized through the distribution $\tilde{u} \in \Delta(\Theta^1)$ over the hierarchies of beliefs induced by u. For this reason, we only restrict our analysis to information structures that are non-redundant, or equivalently information structures induced by a coherent probability with countable support in Π_0. That is not because we think that the redundant information is not relevant (it is relevant for Bayesian Nash equilibrium, see multiple examples in Dekel et al. (2006), Ely and Peski (2006)), but because the dependence of the BNE on the redundant information is not yet well-understood (see Sadzik (2008))[10].

Let $u \in \Delta(K \times C \times D)$ be an information structure. A subset $A \subseteq K \times I \times J$ is a *proper common knowledge event* if $u(A) \in (0,1)$ and for each signal $s \in C \cup D$, $u(A|s) \in \{0,1\}$. An information structure is *simple* if it does not have a proper common knowledge component. Each finite non-redundant information structure u has a representation as a convex combination of (non-redundant) simple information structures $u = \sum_\alpha p_\alpha u_\alpha$, where $\sum p_\alpha = 1, p_\alpha \ge 0$, and $p_\alpha > 0$ for at most finitely many α.

Theorem 22. *Suppose that u, v are finite and non-redundant information structures. If u and v are simple, then*

$$d_{NZS}(u,v) = \begin{cases} 0, & \text{if } \tilde{u} = \tilde{v}, \\ 2 & \text{otherwise.} \end{cases}$$

More generally, suppose that $u = \sum p_\alpha u_\alpha$ and $v = \sum q_\alpha v_\alpha$ are the decompositions into simple information structures. We can always choose the decompositions so that $\tilde{u}_\alpha = \tilde{v}_\alpha$ for each α. Then,

$$d_{NZS}(u,v) = \sum_\alpha |p_\alpha - q_\alpha|.$$

The distance between the two non-redundant simple information structures is binary, either 0 if the information structures are equivalent, or 2 if they are not. In particular, the distance between most of the simple information structures is equal to its maximum possible value 2. The distance d_{NZS} between two non-redundant, but not necessarily simple information structures depends on how similar is their decomposition into the simple components.

The proof in the case of two non-redundant and simple structures u and v is very simple. Let $\tilde{u} \neq \tilde{v}$. First, it is well-known that there exist a finite game $g : K \times I \times J \to [-1,1]^2$ in which each type of player 1 in the support of $\tilde{u}$ and $\tilde{v}$ reports her hierarchy of beliefs as the unique rationalizable action. Second, Lemma III.2.7 in Mertens et al. (2015) (or corollary 4.7 in Mertens and Zamir (1985)) shows that the supports of distributions $\tilde{u}$ and $\tilde{v}$ must be disjoint (it is also a consequence of

[10] An alternative approach would be to take an equilibrium solution concept that can be factorized through the hierarchies of beliefs. An example is Bayes Correlated Equilibrium from Bergemann and Morris (2015).

the result by Samet (1998)). Thus, we can construct a game, in which, additionally to the first game, player 2 chooses between two actions $\{u, v\}$ which influences her payoff only, in such a way that it is optimal for her to match the information structure to which player 1's reported type belongs. Finally, we multiply the so obtained game by $\varepsilon > 0$ and construct a new game, in which, additionally, player 1 receives payoff $1 - \varepsilon$ if player 2 chooses u and a payoff of $-1 + \varepsilon$ if player 2 chooses v. Hence, the payoff distance between the two information structures is at least $2 - \varepsilon$, where ε is arbitrary small.

7.2. **Pointwise topology.** An alternative way to define the topology (but not the distance) on the space of information structures would be through the convergence of values. Say that a sequence of information structures u_n converges to u pointwise if for all payoff structures $g \in \mathcal{G}$, $\lim_{n\to\infty} \mathrm{val}(u_n, g) = \mathrm{val}(u, g)$. Note that pointwise convergence is also well-defined on Π.

The topology of pointwise convergence is the weakest topology that makes the value mappings continuous. This property is not as easy to interpret and it does not seem to be as useful as the value-based metric induced topology. The next result provides its characterization.

Theorem 23. *Sequence u_n of information structures converges pointwise to u if and only if $d(u_n, u) \to 0$. In other words, the topology of pointwise convergence and the value-based topology coincide on $\mathcal{U}^*$ and they are homeomorphic to the weak topology on Π_0.*
The topology of weak convergence on Π coincides with the topology of pointwise convergence.

Proof. The Theorem follows from the observations made in Section 5.1. On one hand, lemma 41 of Gossner and Mertens (2001) implies that the weak topology on Π is coarser than the topology of weak convergence. On the other hand, because the value is continuous (Lemma 2 in Mertens (1986) or Proposition III.4.3. in Mertens et al. (2015)), the two topologies are equivalent. □

Theorem 23 suggests a possible alternative construction of the set Π of consistent probability over the universal belief space. The alternative construction is simply based on the values of finite zero-sum Bayesian games.

7.3. **Strategic topologies.** Following Dekel et al. (2006), the literature studies the robustness of the behavior with respect to beliefs in a systematic way. A typical approach is to define the ε-ICR as the outcome of the iterated elimination procedure, where, at each stage of the elimination, players actions must be ε-best responses to the conjectures that allow for arbitrary correlations with the state of the world and that assign probability 1 to actions that survived the previous stage. This solution concept depends only on the hierarchy of beliefs. One defines two topologies (see Chen et al. (2016)):

- sequence of types (i.e., hierarchies of beliefs) t_n converges to t in the *weak strategic topology* if and only if for each game G, the set of actions that are 0-ICR at type t in game G is equal to the set of actions such that for each $\varepsilon > 0$, there exists N such that for each $n \geq N$, the action is $\varepsilon - ICR$ at type t_n in game G, and
- sequence of types (i.e., hierarchies of beliefs) t_n converges to t in the *uniform strategic topology* if and only if for each $\varepsilon > 0$, there exists Nsuch that for

each game G (with payoffs bounded in norm by 1), the set of actions that are 0-ICR at type t in game G is equal to the set of actions that are $\varepsilon - ICR$ at type t_n for $n \geq N$.

The two topologies differ by the order of quantifiers; the uniform topology corresponds more to the value-based distance topology and the weak topology corresponds to the topology introduced in Section 7.2. Both topologies are metric. Chen et al. (2016) provide a full characterization of the two topologies directly in terms of hierarchies of beliefs. In particular, it is well-known that the two topologies are strictly finer than the product topology on the universal type space.

It is not entirely obvious how to compare the value-based and the strategic topologies as, due to the difference between ex ante and interim approaches, their domains are different.

One way to do is to equip Θ^1 with the strategic or uniform strategic topology, and to consider the induced weak topology (i.e. of convergence in distribution) on $\Pi \subset \Delta(\Theta^1)$.

One can easily see that the two topologies are strictly finer than the weak topology on Π_0 (this is demonstrated by, for instance the Rubinstein's email game - see Example 18). Given Theorem 16, the two strategic topologies are strictly finer than the value-based topology on Π_0. We do not know whether the two topologies remain finer when the value-based topology is extended to Π.

8. Conclusion

In this paper, we have introduced and analyzed the value-based distance on the space of information structures. The main advantage of the definition is that it has a simple and useful interpretation as the tight upper bound on the loss or gain from moving between two information structures. This allows us to directly apply it to numerous questions about the value of information, the relation between the games and single-agent problems, comparison of information structures, etc. Additionally, we show that the distance contains an interesting topological information. On one hand, the topology induced on the countable information structures is equivalent to the topology of weak convergence of consistent probabilities over coherent hierarchies of beliefs. On the other hand, the set of countable information structures is not totally bounded for the value-based distance, which solves negatively the last open question raised in Mertens (1986), with deep implications for stochastic games.

By restricting our attention to zero-sum games, we were able to re-examine the relevance of many phenomena observed and discussed in the strategic discontinuities literature. On one hand, the distinction between the approximate knowledge and the approximate common knowledge is not important in situations of conflict. On the other hand, the higher order beliefs matter on some, potentially uncountably large structures. More generally, we believe that the discussion of the strategic phenomena on particular classes of games can be fruitful line of future research. It is not the case that each problem must involve coordination games. Interesting classes of games to study could be common interest games, potential games, etc. [11]

[11] As an example of work in this direction, Kunimoto and Yamashita (2018) studies an order on hierarchies and types induced by payoffs in supermodular games.

Appendix A. Proofs of Section 3

A.1. **Proof of Theorem 1.** The proof of Theorem 1 relies on two main aspects: the two interpretations of a garbling (deterioration of signals, and strategy), and the minmax theorem.

Part 1. We start with general considerations. Recall that any $g \in \mathcal{G}(L)$ is (identified with) a map from $K \times \mathbb{N} \times \mathbb{N} \to [-1,1]$ such that $g(k,i,j) = -1$ if $i \geq L$ and $g(k,i,j) = 1$ if $i < L$ and $j \geq L$. For $u \in \mathcal{U}$ and $g \in \mathcal{G}(L)$, we denote by $\gamma_{u,g}(q_1,q_2)$ the payoff of player 1 in the zero-sum game $\Gamma(u,g)$ when player 1 plays $q_1 \in \mathcal{Q}$ and player 2 plays $q_2 \in \mathcal{Q}$. Extending as usual g to mixed actions, we have: $\gamma_{u,g}(q_1,q_2) = \sum_{k,c,d} u(k,c,d) g(k, q_1(c), q_2(d))$. For $u \in \mathcal{U}$ and $g \in \mathcal{G}$, the scalar product $\langle g, u \rangle = \sum_{k,c,d} g(k,c,d) u(k,c,d)$ is well defined, and corresponds to the expectation of g with respect to u, and to the payoff $\gamma_{u,g}(Id, Id)$, where $Id \in \mathcal{Q}$ is the strategy which plays with probability one the signal received. The map $g \in \mathcal{G}(L)$ has been extended to $K \times \mathbb{N} \times \mathbb{N}$ in such a way that

$$\forall q_1 \in \mathcal{Q}(L), \ \min_{q_2 \in \mathcal{Q}(L)} \gamma_{u,g}(q_1,q_2) = \min_{q_2 \in \mathcal{Q}} \gamma_{u,g}(q_1,q_2),$$

and a similar result for player 1. Therefore, we have for all $g \in \mathcal{G}(L)$

$$\begin{aligned} \mathrm{val}(u,g) &= \max_{\sigma_1 \in \mathcal{Q}(L)} \min_{\sigma_2 \in \mathcal{Q}(L)} \gamma_{u,g}(\sigma_1,\sigma_2) = \min_{\sigma_2 \in \mathcal{Q}(L)} \max_{\sigma_1 \in \mathcal{Q}(L)} \gamma_{u,g}(\sigma_1,\sigma_2) \\ &= \max_{\sigma_1 \in \mathcal{Q}} \min_{\sigma_2 \in \mathcal{Q}} \gamma_{u,g}(\sigma_1,\sigma_2) = \min_{\sigma_2 \in \mathcal{Q}} \max_{\sigma_1 \in \mathcal{Q}} \gamma_{u,g}(\sigma_1,\sigma_2) \end{aligned}$$

Let us now compute the payoff $\gamma_{u,g}(q_1,q_2)$, for any q_1, q_2:

$$\begin{aligned} \gamma_{u,g}(q_1,q_2) &= \sum_{k,c,d} u(k,c,d) g(k, q_1(c), q_2(d)) \\ &= \sum_{k,c,d} u(k,c,d) \sum_{c',d')} q_1(c)(c') q_2(d)(d') g(k,c',d') \\ &= \sum_{k,c',d'} g(k,c',d') \sum_{c,d} u(k,c,d) q_1(c)(c') q_2(d)(d') \\ &= \sum_{k,c',d'} g(k,c',d') \left((q_1.u.q_2)(k,c',d') \right) \\ &= \langle g, q_1.u.q_2 \rangle. \end{aligned}$$

Consequently, $\mathrm{val}(u,g) = \max_{q_1 \in \mathcal{Q}} \min_{q_2 \in \mathcal{Q}} \langle g, q_1.u.q_2 \rangle = \min_{q_2 \in \mathcal{Q}} \max_{q_1 \in \mathcal{Q}} \langle g, q_1.u.q_2 \rangle$. Since both players can play the Id strategy in $\Gamma(u,g)$, we obtain for all $u \in \mathcal{U}$ and $g \in \mathcal{G}(L)$:

$$\inf_{q_2 \in \mathcal{Q}} \langle g, u.q_2 \rangle \leq \mathrm{val}(u,g) \leq \sup_{q_1 \in \mathcal{Q}} \langle g, q_1.u \rangle.$$

Notice also that for all u, v in $\mathcal{U}$, $\|u - v\| = \sup_{g \in \mathcal{G}} \langle g, u - v \rangle$.

Part 2. We now prove Theorem 1. Take $g \in \mathcal{G}$ and $q_1, q_2 \in \mathcal{Q}$. Then, $\mathrm{val}(v.q_2, g) \geq \mathrm{val}(v,g)$ and $\mathrm{val}(u,g) \geq \mathrm{val}(q_1.u, g)$, which implies

$$\mathrm{val}(v,g) - \mathrm{val}(u,g) \leq \mathrm{val}(v.q_2,g) - \mathrm{val}(q_1.u,g) \leq \|q_1.u - v.q_2\|.$$

Because g and q_1, q_2 are arbitrary, we obtain

$$\sup_{g \in \mathcal{G}} \left(\mathrm{val}(v,g) - \mathrm{val}(u,g) \right) \leq \inf_{q_1 \in \mathcal{Q}, q_2 \in \mathcal{Q}} \|q_1.u - v.q_2\|.$$

Thus, it remains to prove that

$$\inf_{q_1\in\mathcal{Q},q_2\in\mathcal{Q}} \|q_1.u - v.q_2\| \le \sup_{g\in\mathcal{G}} (\mathrm{val}(v,g) - \mathrm{val}(u,g)) \tag{A.1}$$

and that the minimum is achieved in the left-hand side. We have

$$\inf_{q_1\in\mathcal{Q},q_2\in\mathcal{Q}} \|q_1.u - v.q_2\| = \inf_{q_1\in\mathcal{Q},q_2\in\mathcal{Q}} \sup_{g\in\mathcal{G}} \langle g, v.q_2 - q_1.u\rangle.$$

The sets $\mathcal{Q}$ are convex and compact in the product topology, the set $\mathcal{G}$ is a convex subset of some linear vector space, the map

$$(g,(q_1,q_2)) \to \langle g, v.q_2 - q_1.u\rangle,$$

is bilinear and continuous with respect to $(q_1,q_2)\in\mathcal{Q}^2$ for any fixed $g\in\mathcal{G}$, and by Sion's theorem (see e.g. Sorin Sorin (2002) Proposition A.8):

$$\min_{q_1\in\mathcal{Q},q_2\in\mathcal{Q}} \sup_{g\in\mathcal{G}} \langle g, v.q_2 - q_1.u\rangle = \sup_{g\in\mathcal{G}} \min_{q_1\in\mathcal{Q},q_2\in\mathcal{Q}} \langle g, v.q_2 - q_1.u\rangle.$$

Inequality (A.1) now follows from :

$$\begin{aligned}
\sup_{g\in\mathcal{G}} \inf_{q_1\in\mathcal{Q},q_2\in\mathcal{Q}} \langle g, v.q_2 - q_1.u\rangle &= \sup_{g\in\mathcal{G}} \left(\inf_{q_2\in\mathcal{Q}} \langle g, v.q_2\rangle - \sup_{q_1\in\mathcal{Q}} \langle g, q_1.u\rangle \right) \\
&\le \sup_{g\in\mathcal{G}} (\mathrm{val}(v,g) - \mathrm{val}(u,g)).
\end{aligned}$$

Finally, if $u,v\in\mathcal{G}(L)$ for some $L\in\mathbb{N}$, the argument of part 2 can be repeated by replacing $\mathcal{Q}$ by $\mathcal{Q}(L)$ and $\mathcal{G}$ by $\mathcal{G}(L)$ since for any $q_1,q_2\in\mathcal{Q}(L)$, we have $\|q_1.u-v.q_2\| = \sup_{g\in\mathcal{G}(L)}\langle g, q_1.u-v.q_2\rangle$. Moreover, $\mathcal{G}(L)$ is compact for the topology of pointwise convergence on $K\times\{0,...,L-1\}^2$, and the map $(g,(q_1,q_2))\to\langle g, v.q_2 - q_1.u\rangle$ is continuous with respect to $g\in\mathcal{G}(L)$ for any fixed $(q_1,q_2)\in\mathcal{Q}(L)$, implying that

$$\max_{g\in\mathcal{G}(L)} (\mathrm{val}(v,g) - \mathrm{val}(u,g)) = \min_{q_1\in\mathcal{Q}(L),q_2\in\mathcal{Q}(L)} \|q_1.u - v.q_2\|. \tag{A.2}$$

Clearly,

$$\sup_{g\in\mathcal{G}(L)} (\mathrm{val}(v,g) - \mathrm{val}(u,g)) \le \sup_{g\in\mathcal{G}} (\mathrm{val}(v,g) - \mathrm{val}(u,g)), \text{ and}$$
$$\inf_{q_1\in\mathcal{Q},q_2\in\mathcal{Q}} \|q_1.u - v.q_2\| \le \inf_{q_1\in\mathcal{Q}(L),q_2\in\mathcal{Q}(L)} \|q_1.u - v.q_2\|.$$

and using (3.1) and (A.2), these inequalities are equalities, which concludes the proof.

A.2. **Proof of Proposition 2.** Let us start with general properties of d_1. Let us define the set of single agent information structures as $\mathcal{U}_1 = \Delta(K\times\mathbb{N})$ using the same convention that countable sets are identified with subsets of $\mathbb{N}$. Note that given $u\in\Delta(K\times C\times D)$, $\mathrm{marg}_{K\times C}\, u \in \mathcal{U}_1$. Let

$$\mathcal{G}'_1 = \{g' : K\times\mathbb{N}\to[-1,1],\ \exists L\ s.t. \forall i\ge L,\ g'(k,i) = -1\}$$

be the set of single-agent decision problems, and define for $u',v'\in\mathcal{U}_1$, $d'_1(u',v') = \sup_{g'\in\mathcal{G}'_1} |\mathrm{val}(v',g') - \mathrm{val}(u',g')|$. It is easily seen that for any $u,v\in\Delta(K\times C\times D)$,

$$d_1(u,v) = d'_1(u',v') = \max\{\min_{q\in\mathcal{Q}} \|u' - q.v'\|, \min_{q\in\mathcal{Q}} \|q.u' - v'\|\} \tag{A.3}$$

where $u' = \operatorname{marg}_{K\times C} u$, $v' = \operatorname{marg}_{K\times C} v$, $q.u'(k,c) = \sum_{s\in C} u'(k,s)q(s)(c)$ and where the last equality can be obtained by mimicking (and simplifying) the arguments of the proof of Theorem 1.

We now prove Theorem 1. Using the assumptions, we have $u(k) = v(k)$, $u(c,d|k) = u(c|k)\,u(d|k)$ and $v(c',d|k) = v(d|k)\,v(c'|k) = u(d|k)v(c'|k)$. For any pair of garblings q_1, q_2

$$\begin{aligned}
\|u.q_2 - q_1.v\| &= \sum_{k,c,d}\left|\sum_{\beta} u(k,c,\beta)\,q_2(d|\beta) - \sum_{\alpha} v(k,\alpha,d)\,q_1(c|\alpha)\right| \\
&= \sum_{k,c} u(k) \sum_{d}\left|u(c|k)\sum_{\beta} u(\beta|k)\,q_2(d|\beta) - \left(\sum_{\alpha} v(\alpha|k)\,q_1(c|\alpha)\right)u(d|k)\right| \\
&= \sum_{k,c} u(k) \sum_{d}\left|u(d|k)\,\Gamma(k,c) + \Delta(k,d)\,u(c|k)\right|,
\end{aligned}$$

where

$$\begin{aligned}
\Delta(k,d) &= u(d|k) - \sum_{\beta} u(\beta|k)\,q_2(d|\beta), \\
\Gamma(k,c) &= \sum_{\alpha} v(\alpha|k)\,q_1(c|\alpha) - u(c|k).
\end{aligned}$$

Because $|x+y| \geq |x| + \operatorname{sgn}(x)y$ for each $x, y \in \mathbb{R}$, we have

$$\begin{aligned}
&\sum_{d}\left|u(d|k)\,\Gamma(k,c) + \Delta(k,d)\,u(c|k)\right| \\
&\geq \sum_{d} u(d|k)\,|\Gamma(k,c)| + \operatorname{sgn}(\Gamma(k,c))\,u(c|k)\sum_{d}\Delta(k,d) \\
&= \sum_{d} u(d|k)\,|\Gamma(k,c)|.
\end{aligned}$$

where the last equality comes from the fact that $\sum_d \Delta(k,d) = 0$. Thus, we obtain

$$\begin{aligned}
\|u.q_2 - q_1.v\| &\geq \sum_{k,c,d} u(k)\,|u(d|k)\,\Gamma(k,c)| \\
&= \sum_{k,c,d} u(k)\left|u(d|k)\,u(c|k) - \sum_{\alpha} u(d|k)\,v(\alpha|k)\,q_1(c|\alpha)\right| \\
&= \|u - q_1.v\|.
\end{aligned}$$

We deduce that

$$\min_{q_1,q_2}\|u.q_2 - q_1.v\| = \min_{q_1}\|u - q_1.v\|.$$

Inverting the roles of the players, we also have

$$\min_{q_1,q_2}\|v.q_2 - q_1.y\| = \min_{q_1}\|v - q_1.u\|.$$

We conclude that

$$\begin{aligned} d(u,v) &= \max\{\min_{q_1,q_2} \|u.q_2 - q_1.v\| \,;\, \min_{q_1,q_2} \|v.q_2 - q_1.y\|\} \\ &= \max\{\min_{q_1} \|u - q_1.v\| \,;\, \min_{q_1} \|v - q_1.u\|\} \\ &= d_1(u,v), \end{aligned}$$

where the last equality follows from (A.3) together with the fact that $\operatorname{marg}_{K\times D} u = \operatorname{marg}_{K\times D} v$.

A.3. **Proof of Corollary 3.** By Theorem 1,

$$\sup_{g\in\mathcal{G}} (\mathrm{val}(v,g) - \mathrm{val}(u,g)) = \min_{q_1\in\mathcal{Q}, q_2\in\mathcal{Q}} \|q_1.u - v.q_2\|.$$

Since $q_1.u \preceq u$ and $v.q_2 \succeq v$, we have $\sup_{g\in\mathcal{G}} (\mathrm{val}(v,g) - \mathrm{val}(u,g)) \geq \inf_{u'\preceq u, v'\succeq v} \|u' - v'\|$. Consider now $u' \preceq u$ and $v' \succeq v$. We have for all $g \in \mathcal{G}$:

$$\|u' - v'\| \geq \mathrm{val}(v',g) - \mathrm{val}(u',g) \geq \mathrm{val}(v,g) - \mathrm{val}(u,g),$$

so $\sup_{g\in\mathcal{G}} (\mathrm{val}(v,g) - \mathrm{val}(u,g)) \leq \inf_{u'\preceq u, v'\succeq v} \|u' - v'\|$. Hence

$$\sup_{g\in\mathcal{G}} (\mathrm{val}(v,g) - \mathrm{val}(u,g)) = \inf_{u'\preceq u, v'\succeq v} \|u' - v'\|$$

A.4. **Proof of Proposition 4.** We prove the lower bound of (3.5). Let $g(k) = \mathbb{1}_{p_k > q_k} - \mathbb{1}_{p_k \leq q_k}$. Then,

$$d(u,v) \geq \mathrm{val}(u,g) - \mathrm{val}(v,g) = \sum_{k\in K} (p_k - q_k)\, g(k) = \sum_{k\in K} |p_k - q_k| \,.$$

Let us prove the upper bound of (3.5). Define $\bar{u}$ and $\underline{v}$ in $\Delta(K \times K_C \times K_D)$ with $K = K_C = K_D$ such that $\bar{u}(k,c,d) = p_k \,\mathbb{1}_{c=k}\,\mathbb{1}_{d=k_0}$ for some fixed $k_0 \in K$ (complete information for player 1, trivial information for player 2, and the same prior about k as u) and $\underline{v}(k,c,d) = q_k \,\mathbb{1}_{c=k_0}\,\mathbb{1}_{d=k}$ for all (k,c,d) (trivial information for player 1, complete information for player 2, and the same beliefs about k as v). Since the value of a zero-sum game is weakly increasing with player 1's information and weakly decreasing with player 2's information, we have $u \preceq \bar{u}$ and $\underline{v} \preceq v$. Therefore,

$$\sup_{g\in\mathcal{G}} (\mathrm{val}(u,g) - \mathrm{val}(v,g)) \leq \sup_{g\in\mathcal{G}} (\mathrm{val}(\bar{u},g) - \mathrm{val}(\underline{v},g)) = \min_{q_1\in\mathcal{Q}, q_2\in\mathcal{Q}} \|\bar{u}.q_2 - q_1.\underline{v}\|,$$

where, according to Theorem 1, the minimum in the last expression is attained for garblings with values in ΔK. Since player 2 has a unique signal in $\bar{u}$, only $q_2(.|k_0) \in \Delta K$ matters. We denote it by $q' = q_2(.|k_0)$. Similarly, we define $p' = q_1(.|k_0) \in \Delta(K)$. Then,

$$
\begin{aligned}
\|\bar{u}.q_2 - q_1.\underline{v}\| &= \sum_{(k,c,d)\in K^3} |p_k\, \mathbb{1}_{c=k}\, q'_d - q_k\, \mathbb{1}_{d=k}\, p'_c| \\
&= \sum_{k\in K} |p_k q'_k - q_k p'_k| + p_k(1-q'_k) + q_k(1-p'_k) \\
&= 2 + \sum_{k\in K} |p_k q'_k - q_k p'_k| - p_k q'_k - q_k p'_k \\
&= 2\left(1 - \sum_{k\in K} \min\left(p_k q'_k, q_k p'_k\right)\right).
\end{aligned}
$$

A similar inequality holds by inverting the roles of u and v, and the upper bound follows from the fact that one can choose arbitrary p', q'.

If $p = q$, then

$$
\sum_{k\in K} \min\left(p_k q'_k, q_k p'_k\right) = \sum_{k\in K} p_k \min\left(q'_k, p'_k\right) \le \sum_{k\in K} p_k p'_k \le \max_{k\in K} p_k,
$$

where the latter is attained by $p'_k = q'_k = \mathbb{1}_{\{k=k^*\}}$ for some $k^* \in K$ such that $p_{k^*} = \max_{k\in k} p_k$.

A.5. **Proof of Proposition 6.** We prove the first claim. Consider any strategy τ of player 2 in $\mathcal{Q}$. Notice that $(\sigma.q_1).u = \sigma.(q_1.u)$ and $(v.q_2).\tau = v.(\tau.q_2)$. Using the notation of the proof of theorem 1, we have

$$
\begin{aligned}
\gamma_{u,g}(\sigma.q_1, \tau) &= \langle g, (\sigma.q_1).u.\tau\rangle \\
&= \langle g, \sigma.(q_1.u).\tau\rangle \\
&\ge \langle g, \sigma.(v.q_2).\tau\rangle - \|q_1.u - v.q_2\| \\
&\ge \langle g, \sigma.v.(\tau.q_2)\rangle - d(u,v) \\
&\ge \operatorname{val}(v,g) - d(u,v) \\
&\ge \operatorname{val}(u,g) - 2d(u,v),
\end{aligned}
$$

so $\sigma.q_1$ is $2d(u,v)$-optimal in g on u.

Appendix B. Proofs of Section 4

B.1. **Proof of Proposition 9.** Because u has more information than v,

$$
d\left(u,v\right) = \min_{q_2\in\mathcal{Q}} \min_{q_1\in\mathcal{Q}} \|u.q_2 - q_1.v\| \le \min_{q_2\in\mathcal{Q}} \min_{q_1: C\to\Delta(C\times C_1\times C_2)} \|u.q_2 - q_1.v\|,
$$

where in the second inequality we use a restricted set of player 1's garblings that do not depend on c_1. Further, for any such q_1 and an arbitrary garbling q_2, we have

$$
\begin{aligned}
\|u.q_2 - q_1.v\| &= \sum_{k,c,c_1,c_2,d} \left| \sum_{\beta} u\left(k,c,c_1,c_2,\beta\right) q_2\left(d|\beta\right) - \sum_{\alpha} u\left(k,c,c_1,d\right) q_1\left(c,c_2|\alpha\right)\right| \\
&= \sum_{k,c,c_1,c_2,d} u\left(k,c_1\right) \left| \sum_{\beta} u\left(c,c_2,\beta|k,c_1\right) q_2\left(d|\beta\right) - \sum_{\alpha} u\left(c,d|k,c_1\right) q_1\left(c,c_2|\alpha\right)\right|.
\end{aligned}
$$

Because of the conditional independence assumption, the above is equal to

$$\begin{aligned}
&= \sum_{k,c,c_2,d} \left(\sum_{c_1} u\left(k,c_1\right)\right) \left|\sum_{\beta} u\left(c,c_2,\beta|k\right) q_2\left(d|\beta\right) - \sum_{\alpha} u\left(c,d|k\right) q_1\left(c,c_2|\alpha\right)\right| \\
&= \sum_{k,c,c_2,d} \left|\sum_{\beta} u\left(k,c,c_2,\beta\right) q_2\left(d|\beta\right) - \sum_{\alpha} u\left(k,c,d\right) q_1\left(c,c_2|\alpha\right)\right| \\
&= \left\|u'.q_2 - q_1.v'\right\|.
\end{aligned}$$

Hence,

$$d\left(u,v\right) \leq \min_{q_2} \min_{q_1 : C \to \Delta(C\times C_1 \times C_2)} \left\|u'.q_2 - q_1.v'\right\| = d\left(u',v'\right).$$

B.2. **Proof of Proposition 12.** We have

$$d\left(u',v'\right) = d_1\left(u',v'\right) = d_1\left(u,v\right) \leq d\left(u,v\right).$$

The first equality comes from Proposition 7, and the second from the fact that u and u' (resp. v and v') induce the same distribution on player 1 first order beliefs, and the inequality from the definition of the two distances.

B.3. **Proof of Proposition 14.** We have the following Lemma:

Lemma 24. *Suppose that c_1 is ε-conditionally independent from (k,d) given c. Then,*

$$\sup_{g\in\mathcal{G}} \operatorname{val}\left(u,g\right) - \operatorname{val}\left(v,g\right) \leq \varepsilon.$$

Proof. Let $q_2 : D\times D_1 \to D$ be defined as $q_2\left(d,d_1\right)\left(d'\right) = \mathbb{1}_{d'=d}$. Let $q_1 : C \to C\times C_1$ be defined as $q_1\left(c,c_1|c\right) = u\left(c_1|c\right)$. Then,

$$\begin{aligned}
\left\|u.q_2 - q_1.v\right\| &= \sum_{k,c,c_1,d} \left|u\left(k,c,c_1,d\right) - u\left(k,c,d\right) u\left(c_1|c\right)\right| \\
&= \sum_{c} u\left(c\right) \sum_{k,c_1,d} \left|u\left(k,c_1,d|c\right) - u\left(k,d|c\right) u\left(c_1|c\right)\right| \leq \varepsilon.
\end{aligned}$$

The claim follows from Theorem 1. □

An analogous argument shows that, if d_1 is ε-conditionally independent from (k,c) given d, then

$$\sup_{g\in\mathcal{G}} \operatorname{val}\left(v,g\right) - \operatorname{val}\left(u,g\right) \leq \varepsilon.$$

Appendix C. Proofs of Section 5

C.1. **Theorem 16: the weak topology is contained in the value-based topology.** Because the weak topology is metrisable, it is enough to show that if $u_n \in \Delta\left(K\times C_n \times D_n\right)$ and $u \in \Delta\left(K\times C\times D\right)$ are information structures such that $d\left(u_n,u\right) \to 0$, then $\tilde{u}_n$ converges to $\tilde{u}$ in the weak topology. Let $\tilde{u}^*$ be an accumulation point of sequence $\tilde{u}_n$. Such a point exists because Π is compact.

Suppose that $\tilde{u}^* \neq \tilde{u}$. Then, by lemma 41 in Gossner Mertens Gossner and Mertens (2001)), there is a game g such that $\left|\operatorname{val}\left(\tilde{u},g\right) - \operatorname{val}\left(\tilde{u}^*,g\right)\right| > 0$. For a fixed game, $\operatorname{val} : \Pi \to [-1,1]$ is continuous in the weak topology (see Lemma 2 in Mertens (1986) or Proposition III.4.3. in Mertens et al. (2015)). It follows that

$\lim_n |\mathrm{val}\,(\tilde{u},g) - \mathrm{val}\,(\tilde{u}_n,g)| = |\mathrm{val}\,(\tilde{u},g) - \mathrm{val}\,(\tilde{u}^*,g)| > 0$. But this contradicts $d\,(u_n,u) = d\,(\tilde{u}_n,\tilde{u}) \to 0$.

C.2. **Theorem 16: the value-based topology is contained in the weak topology.** Because the weak topology is metrisable, it is enough to show that if $u_n \in \Delta\,(K \times C_n \times D_n)$ and $u \in \Delta\,(K \times C \times D)$ are information structures such that $\tilde{u}_n$ converges to $\tilde{u}$ in the weak topology, then $d\,(u_n,u) \to 0$. We will show that for each $\varepsilon > 0$,

$$\limsup_{n\to\infty} \sup_{g\in\mathcal{G}} (\mathrm{val}\,(u_n,g) - \mathrm{val}\,(u,g)) \leq 0. \tag{C.1}$$

Because we can switch the roles of players, this will suffice to establish our claim.

Partitions of unity. We can without loss of generality assume that u is non-redundant and all signals c and d have positive probability. We can associate signals $c \in C \subseteq \mathbb{N}$ and $d \in D \subseteq \mathbb{N}$ with the corresponding hierarchies of beliefs in Θ_1 and Θ_2. In other words, we identify $C \subseteq \Theta_1$ as the (countable) support of $\tilde{u}$ and $D \subseteq \Theta_2$ as the smallest countable set such that for each $c \in C$, $\phi_1\,(K \times D|c) = 1$ (i.e., D is the union of countable supports of all beliefs of hierarchies in C). For each $c \in C$ and $d \in D$, we denote the corresponding hierarchies under u as $\tilde{c}$ and $\tilde{d}$. Also, let $C^m = C \cap \{0,...,m\}$ and $D^m = D \cap \{0,...,m\}$.

Because Θ_2 is Polish, for each $m \in \mathbb{N}$ and each $d \in D^m$, we can find continuous functions $\kappa_d^m : \Theta_2 \to [0,1]$ for $m \in \mathbb{N}, d \in \{0,...,m\}$ such that

$$\kappa_d^m\left(\tilde{d}\right) = 1 \text{ for each } d \in D^m,$$
$$\kappa_d^m \equiv 0 \text{ if } d \notin D, \text{ and}$$
$$\sum_{d=0}^{m} \kappa_d^m\,(\theta_2) = 1 \text{ for each } \theta_2 \in \Theta_2.$$

In other words, for each m, $\{\kappa_d^m\}_{0\leq d\leq m}$ is a continuous partition of unity on space Θ_2 with the property that for each $d \in D^m$, κ_d^m peaks at hierarchy $\tilde{d}$.

Notice that for each $c \in C$, each $d \in D^p$, we have

$$\mathbb{E}_{\phi_1(\tilde{c})}[\mathbb{1}_{\{k\}}(.)\kappa_d^p(.)] \geq u\,(k,d|c)\,,$$

and

$$\sum_{k\in K}\sum_{d=0}^{p} \left|\mathbb{E}_{\phi_1(\tilde{c})}[\mathbb{1}_{\{k\}}(.)\kappa_d^p(.)] - u\,(k,d|c)\right| = u(D \setminus D^p|c).$$

Because all hierarchies $\tilde{c}, c \in C$ are distinct, for each m, there exists $p^m < \infty$ and $\varepsilon^m \in \left(0,\frac{1}{m}\right)$ such that for any $c,c' \in C^m$ such that $c \neq c'$,

$$\sum_{k\in K}\sum_{d=0}^{p^m} \left|\mathbb{E}_{\phi_1(\tilde{c})}[\mathbb{1}_{\{k\}}\,\kappa_d^{p^m}] - \mathbb{E}_{\phi_1(\tilde{c}')}[\mathbb{1}_{\{k\}}\,\kappa_d^{p^m}]\right| \geq 2\varepsilon^m.$$

Let

$$h_c^m\,(\theta_1) = \sum_{k}\sum_{d=0}^{p^m} \left|\mathbb{E}_{\phi_1(\theta_1)}[\mathbb{1}_{\{k\}}\,\kappa_d^{p^m}] - \mathbb{E}_{\phi_1(\tilde{c})}[\mathbb{1}_{\{k\}}\,\kappa_d^{p^m}]\right|.$$

Then, h_c^m is a continuous function such that $h_c^m\,(\tilde{c}) = 0$, and such that if $h_c^m\,(\theta_1) \leq \varepsilon^m$ for some $c \in C^m$, then $h_{c'}^m\,(\theta_1) \geq \varepsilon^m$ for any $c' \in C^m$ such that $c' \neq c$. Define continuous functions for $0 \leq c \leq m+1$,

$$\begin{aligned}\kappa_c^m(\theta_1) &= \max\left(1 - \frac{1}{\varepsilon^m} h_c^m(\theta_1), 0\right) \text{ for } c \in C_m, \\ \kappa_c^m &\equiv 0 \text{ if } c \notin C, \text{ and} \\ \kappa_{m+1}^m(\theta_1) &= 1 - \sum_{c=0}^{m} \kappa_c^m(\theta_1).\end{aligned}$$

Then, for each m, $\sum_{c=0}^{m+1} \kappa_c^m \equiv 1$, and $\kappa_c^m(\theta_1) \in [0,1]$ for each $c = 0, ..., m+1$, which implies that $\{\kappa_c^m\}_{0 \leq c \leq m+1}$ is a continuous partition of unity on space Θ_1 such that for each $c \in C^m$, $\kappa_c^m(\tilde{c}) = 1$.

Conditional independence. For each information structure $v \in \Delta(K \times C' \times D')$, define an information structure

$$K^m v \in \Delta(K \times C' \times \{0, ..., m+1\} \times D' \times \{0, ..., p^m\})$$

so that

$$K^m v\left(k, c', \hat{c}, d', \hat{d}\right) = v(k, c', d') \kappa_{\hat{c}}^m(\tilde{c}') \kappa_{\hat{d}}^{p^m}\left(\tilde{d}'\right).$$

Let $\delta^m v = 2\varepsilon^m + K^m v(\hat{c} = m+1)$.

We are going to show that, under $K^m v$, signal c' is $\delta^m v$-conditionally independent from $\left(k, \hat{d}\right)$ given $\hat{c}$. Notice first that, if $K^m v\left(k, d', \hat{d}, c', \hat{c}\right) > 0$ for some $\hat{c} \in C^m$, then $h_{\hat{c}}^m(\tilde{c}') \leq \varepsilon^m$. It follows that

$$\begin{aligned}&\sum_k \sum_{\hat{d}=0}^{p^m} \left| K^m v\left(k, \hat{d} | \hat{c}, c'\right) - \mathbb{E}_{\phi_1(\tilde{\hat{c}})}[\mathbb{1}_{\{k\}} \kappa_{\hat{d}}^{p^*}] \right| \\ &= \sum_k \sum_{\hat{d}=0}^{p^m} \left| K^m v\left(k, \hat{d} | c'\right) - \mathbb{E}_{\phi_1(\tilde{\hat{c}})}[\mathbb{1}_{\{k\}}[\kappa_{\hat{d}}^{p^m}] \right| \\ &= \sum_k \sum_{\hat{d}=0}^{p^m} \left| \mathbb{E}_{\phi_1(\tilde{c}')}[\mathbb{1}_{\{k\}} \kappa_{\hat{d}}^{p^m}] - \mathbb{E}_{\phi_1(\tilde{\hat{c}})}[\mathbb{1}_{\{k\}} \kappa_{\hat{d}}^{p^m}] \right| = h_{\hat{c}}^m(\tilde{c}') \leq \varepsilon^m.\end{aligned}$$

On the other hand

$$\begin{aligned}&\sum_k \sum_{\hat{d}=0}^{p^m} \left| K^m v\left(k, \hat{d} | \hat{c}\right) - \mathbb{E}_{\phi_1(\tilde{\hat{c}})}[\mathbb{1}_{\{k\}} \kappa_{\hat{d}}^{p^*}] \right| \\ &= \sum_k \sum_{\hat{d}=0}^{p^m} \left| \frac{1}{K^m v(\hat{c})} \sum_{c' \in C'} K^m v(c', \hat{c}) K^m v\left(k, \hat{d} | \hat{c}, c'\right) - \mathbb{E}_{\phi_1(\tilde{\hat{c}})}[\mathbb{1}_{\{k\}}[\kappa_{\hat{d}}^{p^m}] \right| \\ &\leq \sum_{c' \in C'} \frac{K^m v(c', \hat{c})}{K^m v(\hat{c})} \sum_k \sum_{\hat{d}=0}^{p^m} \left| K^m v\left(k, \hat{d} | \hat{c}, c'\right) - \mathbb{E}_{\phi_1(\tilde{\hat{c}})}[\mathbb{1}_{\{k\}}[\kappa_{\hat{d}}^{p^m}] \right| = h_{\hat{c}}^m(\tilde{c}') \leq \varepsilon^m.\end{aligned}$$

Hence,

$$\begin{aligned}&\sum_{\hat{c}=1}^{m+1}\sum_{c'}K^m v\left(\hat{c},c'\right)\sum_{k,\hat{d}}\left|K^m v\left(k,\hat{d}|\hat{c},c'\right)-K^m v\left(k,\hat{d}|\hat{c}\right)\right|\\ \leq&2\varepsilon^m\sum_{\hat{c}=1}^{m}K^m v\left(\hat{c}\right)+K^m v\left(\hat{c}=m+1\right)\leq\delta^m v.\end{aligned}$$

Define the information structure

$$L^m v=\operatorname{marg}_{K\times\{0,\ldots,p^m\}\times\{0,\ldots,m+1\}}K^m v.$$

Then, because $d\left(K^m v,v\right)=0$, Lemma 24 implies that

$$\sup_{g\in\mathcal{G}}\left(\operatorname{val}\left(v,g\right)-\operatorname{val}\left(L^m v,g\right)\right)\leq\delta^m v.$$

Proof of claim (C.1). Observe that for each $k,\hat{c},\hat{d}$,

$$\left(L^m u_n\right)\left(k,\hat{c},\hat{d}\right)=\mathbb{E}_{\tilde{u}_n}\left(\kappa_{\hat{c}}^m\left(\theta_1\right)\mathbb{E}_{\phi_1\left(\theta_1\right)}[\mathbb{1}_{\{k\}}\,\kappa_{\hat{d}}^{p^m}]\right).$$

Because all the functions in the brackets above are continuous, the weak convergence $\tilde{u}_n\to\tilde{u}$ implies that

$$\left(L^m u_n\right)\left(k,\hat{c},\hat{d}\right)\to\left(L^m u\right)\left(k,\hat{c},\hat{d}\right).$$

for each $k,\hat{c},\hat{d}$. Because the information structures $L^m u_n$ and $L^m u$ are described on the same and finite spaces of signals, the pointwise convergence implies

$$d\left(L^m u_n,L^m u\right)\leq\left\|L^m u_n-L^m u\right\|\to 0\text{ as }n\to\infty.$$

Moreover, if $\hat{c}\in C^m$ and $\hat{d}\in D^{p^m}$, the definitions imply that

$$\left(L^m u\right)\left(k,\hat{c},\hat{d}\right)\geq u\left(k,\hat{c},\hat{d}\right).$$

Thus,

$$d\left(L^m u,u\right)\leq\left\|L^m u-u\right\|\leq 2\left(u\left(C\backslash C^m\right)+u\left(D\backslash D^{p^m}\right)\right)\underset{n\to\infty}{\longrightarrow}0.$$

It follows that

$$\delta^m u_n=\left(K^m u_n\right)\left(\hat{c}=m+1\right)\underset{n\to\infty}{\longrightarrow}\left(L^m u\right)\left(\hat{c}=m+1\right),$$

and

$$\left(L^m u\right)\left(\hat{c}=m+1\right)=1-(L^m u)(C^m\times D^{p^m})\leq 1-u(C^m\times D^{p^m})\leq u\left(C\backslash C^m\right)+u\left(D\backslash D^{p^m}\right).$$

Together, we obtain for each m,n

$$\begin{aligned}&\sup_{g\in\mathcal{G}}\left(\operatorname{val}\left(u_n,g\right)-\operatorname{val}\left(u,g\right)\right)\\ \leq&\sup_{g\in\mathcal{G}}\left(\operatorname{val}\left(u_n,g\right)-\operatorname{val}\left(L^m u_n,g\right)\right)\\ &+\sup_{g\in\mathcal{G}}\left(\operatorname{val}\left(L^m u_n,g\right)-\operatorname{val}\left(L^m u\right)\right)\\ &+\sup_{g\in\mathcal{G}}\left(\operatorname{val}\left(L^m u\right)-\operatorname{val}\left(u,g\right)\right)\\ \leq&\delta^m u_n+\left\|L^m u_n-L^m u\right\|+\left(u\left(C\backslash C^m\right)+u\left(D\backslash D^{p^m}\right)\right).\end{aligned}$$

Hence,

$$\lim_{n\to\infty}\sup_{g\in\mathcal{G}}\sup\left(\mathrm{val}\left(v,g\right)-\mathrm{val}\left(L^{m}v,g\right)\right)\leq 3\left(u\left(C\backslash C^{m}\right)+u\left(D\backslash D^{p^{m}}\right)\right).$$

When $m\to\infty$, the right hand side converges to 0 as well.

C.3. **Proof of Proposition 17.** Let $u'\in\Delta\left(K\times\left(K_{C}\times C\right)\times\left(K_{D}\times D\right)\right)$ be defined so that $u=\mathrm{marg}_{K\times c\times D}u'$ and $u'(\{k_{C}=\kappa\left(c\right),k_{D}=\kappa\left(d\right)\})=1$. Because u' does not have any new information, we verify (for instance using Lemma 14) that $d\left(u,u'\right)=0$. We are going to show that C is 16ε-conditionally independent from $K\times K_{D}$ given K_{C}. Notice that because u exhibits ε-knowledge,

$$\begin{aligned}u'\left\{k_{C}\neq k\text{ or }k_{D}\neq k\right\}&\leq u'\left\{k_{C}\neq k\right\}+u'\left\{k_{D}\neq k\right\}\\&\leq 2\varepsilon+2\varepsilon=4\varepsilon.\end{aligned}$$

Thus,

$$\begin{aligned}&\sum_{k,k_{C},k_{D}}u'\left(k_{C}\right)\sum_{c}\left|u'\left(k,k_{D},c|k_{C}\right)-u'\left(k,k_{D}|k_{C}\right)u'\left(c|k_{C}\right)\right|\\
=&\sum_{k,k_{C},k_{D}}u'\left(k,k_{C},k_{D}\right)\sum_{c}\left|u'\left(c|k,k_{C},k_{D}\right)-\sum_{k',k_{D}'}u'\left(c|k',k_{C},k'_{D}\right)u'\left(k',k'_{D}|k_{C}\right)\right|\\
\leq&\sum_{k}u'\left(k,k,k\right)\sum_{c}\left|u'\left(c|k,k,k\right)-\sum_{k',k_{D}'}u'\left(c|k',k_{C}=k,k'_{D}\right)u'\left(k',k'_{D}|k_{C}=k\right)\right|\\
&+2u'\left\{k_{C}\neq k\text{ or }k_{D}\neq k\right\}\\
\leq&\sum_{k}u'\left(k,k,k\right)\sum_{c}\left|u'\left(c|k,k,k\right)-u'\left(c|k,k,k\right)\frac{u'\left(k,k,k\right)}{u'\left(k_{C}=k\right)}\right|\\
&+\sum_{k}u'\left(k,k,k\right)\sum_{c}\sum_{k'\neq k,\text{ or }k'_{D}\neq k}\left|u'\left(c|k,k_{C}=k,k_{D}\right)u'\left(k',k'_{D}|k_{C}=k\right)\right|\\
&+2u'\left\{k_{C}\neq k\text{ or }k_{D}\neq k\right\}\\
\leq&\sum_{k}u'\left(k,k,k\right)\left|1-\frac{u'\left(k,k,k\right)}{u'\left(k_{C}=k\right)}\right|+3u'\left\{k_{C}\neq k\text{ or }k_{D}\neq k\right\}\\
\leq&\sum_{k}\left|u'\left(k_{C}=k\right)-u'\left(k,k,k\right)\right|+3u'\left\{k_{C}\neq k\text{ or }k_{D}\neq k\right\}\\
\leq&4u'\left\{k_{C}\neq k\text{ or }k_{D}\neq k\right\}\leq 16\varepsilon.\end{aligned}$$

Because an analoguous result applies to the information of the other player, Lemma 14 shows that

$$d\left(u',v'\right)\leq 16\varepsilon,$$

where $v'=\mathrm{marg}_{K\times K_{C}\times K_{D}}$. Because

$$\begin{aligned}d\left(v,v'\right)\leq&\sum_{k,k_{C},k_{D}}\left|v\left(k,k_{C},k_{D}\right)-v'\left(k,k_{C},k_{D}\right)\right|\\
&\leq 2v'\left\{k_{C}\neq k\text{ or }k_{D}\neq k\right\}=2u'\left\{k_{C}\neq k\text{ or }k_{D}\neq k\right\}\leq 4\varepsilon,\end{aligned}$$

the triangle inequality implies that

$$d\left(u,v\right)\leq d\left(u,u'\right)+d\left(u',v'\right)+d\left(v,v'\right)\leq 20\varepsilon.$$

Appendix D. Proof of Theorem 19.

N is a very large even integer to be fixed later, and we write $A = C = D = \{1, ..., N\}$, with the idea of using C while speaking of actions or signals of player 1, and using D while speaking of actions and signals of player 2. We fix ε and α, to be used later, such that

$$0 < \varepsilon < \frac{1}{10(N+1)^2}, \text{ and } \alpha = \frac{1}{25}.$$

We will consider a Markov chain on A, satisfying:

- the law of the first state of the Markov chain is uniform on A,
- for each a in A, there are exactly $N/2$ elements b in A such that $\nu(b|a) = 2/N$: given that the current state of the Markov chain is a, the law of the next state is uniform on a subset of states of size $N/2$,
- and few more conditions, to be be defined later.

The rest of the proof is split in 4 parts: we first define the information structures $(u^l)_{l\geq 1}$ and some payoff structures $(g^p)_{p\geq 1}$. Next, we show that the Markov property of the construction ensures that the hierarchies of beliefs are preserved. Then we define two conditions $UI1$ and $UI2$ on the information structures and show that they imply the conclusions of theorem 19. Finally, we show, via the probabilistic method, the existence of a Markov chain ν satisfying all our conditions.

D.1. Information and payoff structures $(u^l)_{l\geq 1}$ and $(g^l)_{l\geq 1}$.

For $l \geq 1$, define the information structure $u^l \in \Delta(K \times C^l \times D^l)$ by: for each state k in K, signal $c = (c_1, ..., c_l)$ in C^l of player 1 and signal $d = (d_1, ..., d_l)$ in D^l for player 2,

$$u^l(k, c, d) = \nu(c_1, d_1, c_2, d_2, ..., c_l, d_l) \left(\frac{c_1}{N+1} \mathbf{1}_{k=1} + \frac{1 - c_1}{N+1} \mathbf{1}_{k=0} \right).$$

The following interpretation of u^l holds: first select $(a_1, a_2, ..., a_{2l}) = (c_1, d_1, ..., c_l, d_l)$ in A^{2l} according to the Markov chain ν (i.e. uniformly among the nice sequences of length $2l$), then tell $(c_1, c_2, ..., c_l)$ (the elements of the sequence with odd indices) to player 1, and $(d_1, d_2, ..., d_l)$ (the elements of the sequence with even indices) to player 2. Finally choose the state $k = 1$ with probability $c_1/(N+1)$, and state $k = 0$ with the complement probability $1 - c_1/(N+1)$.

Notice that the definition is not symmetric among players, the first signal c_1 of player 1 is uniformly distributed and plays a particular role. The marginal of u^l on K is uniform, and the marginal of u^{l+1} over $(K \times C^l \times V^l)$ holds : condition *2*) of theorem 19 is satisfied.

A sequence $(a_1, ..., a_l)$ of length $l \geq 1$ is said to be *nice* it it is in the support of the Markov chain: $\nu(a_1, ..., a_l) > 0$. For instance any sequence of length 1 is nice, and $N^2/2$ sequences of length 2 are nice. Consider a sequence $(a_1, ..., a_l)$ of elements of A which is not nice, i.e. such that $\nu(a_1, ..., a_l) = 0$. We say that the sequence is *not nice because of player 1* if $\min\{t \in \{1, ..., l\}, \nu(a_1, ..., a_t) = 0\}$ is odd, and *not nice because of player 2* if $\min\{t \in \{1, ..., l\}, \nu(a_1, ..., a_t) = 0\}$ is even. A sequence $(a_1, ..., a_l)$ is now either nice, or not nice because of player 1, or not nice because of player 2. A sequence of length 2 is either nice, or not nice because of player 2.

For $p \geq 1$, define the payoff structure $g^p : K \times C^p \times D^{p-1} \to [-1, 1]$ such that for all k in K, $c' = (c'_1, ..., c'_p)$ in C^p, $d' = (d'_1, ..., d'_{p-1})$ in D^{p-1} :

$$\begin{aligned}
g^p(k, c', d') &= g_0(k, c'_1) + h^p(c', d'), \ \ \textit{with} \\
g_0(k, c'_1) &= -\left(k - \frac{u'_1}{N+1}\right)^2 + \frac{N+2}{6(N+1)}, \\
h^p(c', d') &= \begin{cases} \varepsilon & \text{if} \quad (c'_1, d'_1, ..., c'_p) \textit{ is nice}, \\ 5\varepsilon & \text{if} \quad (c'_1, d'_1, ..., c'_p) \textit{ is not nice because of player } 2, \\ -5\varepsilon & \text{if} \quad (c'_1, d'_1, ..., c'_p) \textit{ is not nice because of player } 1. \end{cases}
\end{aligned}$$

One can check that $|g^p| \leq 5/6 + 5\varepsilon \leq 8/9$. Regarding the g_0 part of the payoff, consider a decision problem for player 1 where: c_1 is selected uniformly in A and the state is selected to be $k = 1$ with probability $c_1/(N+1)$ and $k = 0$ with probability $1 - c_1/(N+1)$. Player 1 observes c_1 but not k, and he choose c'_1 in A and receive payoff $g_0(k, c'_1)$. We have $\frac{c_1}{N+1} g_0(1, c'_1) + (1 - \frac{c_1}{N+1}) g_0(0, c'_1) = \frac{1}{(N+1)^2}(c'_1(2c_1 - c'_1) + (N+1)((N+2)/6 - c_1))$. To maximize this expected payoff, it is well known that player 1 should play his belief on k, i.e. $c'_1 = c_1$. Moreover, if player 1 chooses $c'_1 \neq c_1$, its expected loss from not having chosen c_1 is at least $\frac{1}{(N+1)^2} \geq 10\varepsilon$. And the constant $\frac{N+2}{6(N+1)}$ has been chosen such that the value of this decision problem is 0.

Consider now $l \geq 1$ and $p \geq 1$. By definition, the Bayesian game $\Gamma(u^k, g^p)$ is played as follows: first, $(c_1, d_1, ..., c_l, d_l)$ is selected according to the law ν of the Markov chain, player 1 learns $(c_1, ..., c_l)$, player 2 learns $(d_1, ..., d_l)$ and the state is $k = 1$ with probability $c_1/(N+1)$ and $k = 0$ otherwise. Then *simultaneously* player 1 chooses c' in C^p and player 2 chooses d' in D^{p-1}, and finally the payoff to player 1 is $g^p(k, c', d')$. Notice that by the previous paragraph about g_0, it is always strictly dominant for player 1 to report correctly his first signal, i.e. to choose $c'_1 = c_1$. We will show in the next section that if $l \geq p$ and player 1 simply plays the sequence of signals he received, player 2 can not do better than also reporting truthfully his own signals, leading to a value not lower than the payoff for nice sequences, that is ε. On the contrary in the game $\Gamma(u^l, g^{l+1})$, player 1 has to report not only the l signals he has received, but also an extra-signal c'_{l+1} that he has to guess. In this game we will prove that if player 2 truthfully reports his own signals, player 1 will incur the payoff -5ε with probability at least (approximately) 1/2, and this will result in a low value. These intuitions will prove correct in the next section, under some conditions $UI1$ and $UI2$.

D.2. **Higher order beliefs.** Recall that n-order beliefs are defined inductively as conditional laws. Precisely, the first order beliefs θ^i_1 of player i is the conditional law of k given her signal. The n-order belief θ^i_n of player i is the conditional law of $(\omega, \theta^{-i}_{n-1})$ given her signal. In this construction, conditional laws are seen as random variables taking values in space of probability measures.

Lemma 25. *For all $l > p$, the joint distribution of $(\omega, \theta^1_{2p}, \theta^2_{2p})$ induced by the information structure u^l is independent of l.*

Proof. We use the notation $\mathcal{L}(X|Y)$ for the conditional law of X given Y, and the identification $(a_1, ..., a_{2l}) = (c_1, d_1, ..., c_l, d_l)$. At first, note that by construction k and $(a_2, ..., a_{2l})$ are conditionally independent given a_1, so that the sequence

$(k, a_1, a_2, ..., a_{2l})$ is a Markov process. It follows that $\theta_1^1 = \mathcal{L}(k|c_1, ..., c_l) = \mathcal{L}(k|c_1)$. The Markov property implies that

$$\theta_1^2 = \mathcal{L}(k|d_1, ..., d_l) = \mathcal{L}(k|d_1),\ \theta_2^2 = \mathcal{L}(d, \theta_1^1(c_1)|d_1, ..., d_l) = \mathcal{L}(k, \theta_1^1(c_1)|d_1),$$

and therefore we have

$$\theta_2^1 = \mathcal{L}(k, \theta_1^2(d_1)|c_1, ..., c_l) = \mathcal{L}(k, \theta_1^2(d_1)|c_1, c_2).$$

By induction, and applying the same argument (future and past of a Markov process are conditionally independent given the current position), we deduce that for all $n \geq 1$,

$$\theta_{2n}^1 = \mathcal{L}(k, \theta_{2n-1}^2|c_1, ..., c_{\min(l,n+1)}), \quad \theta_{2n+1}^1 = \mathcal{L}(k, \theta_{2n}^2|c_1, ..., c_{\min(l,n+1)}),$$

$$\theta_{2n-1}^2 = \mathcal{L}(k, \theta_{2n-2}^1|d_1, ..., d_{\min(l,n)}), \quad \theta_{2n}^2 = \mathcal{L}(k, \theta_{2n-1}^1|d_1, ..., d_{\min(l,n)}).$$

As a consequence, for all $n \leq p$, these conditional laws do not depend on which u^l we are using as soon as $l > p$. □

D.3. **Conditions UI and values.** To prove that the intuitions of the previous paragraph are correct, we need to ensure that players have incentives to report their true signals, so we need additional assumptions on the Markov chain.

Notations and definition: Let $l \geq 1$, $m \geq 0$, $c = (c_1, ..., c_l)$ in C^l and $d = (d_1, ..., d_m)$ in D^m. We write :

$$a^{2q}(c,d) = (c_1, d_1, ..., c_q, d_q) \in A^{2q} \quad \text{for each } q \leq \min\{l, m\},$$
$$a^{2q+1}(c,d) = (c_1, d_1, ..., c_q, d_q, c_{q+1}) \in A^{2q+1} \quad \text{for each } q \leq \min\{l-1, m\}.$$

For $r \leq \min\{2l, 2m+1\}$,
we say that c and d are *nice at level* r, and we write $c \smile_r d$, if $a^r(c,d)$ is nice.

In the next definition we consider an information structure $u^l \in \Delta(K \times C^l \times D^l)$ and denote by $\tilde{c}$ and $\tilde{d}$ the respective random variables of the signals of player 1 and 2.

Definition 26. We say that the *conditions* $UI1$ *are satisfied* if for all $l \geq 1$, all $c = (c_1, ..., c_l)$ in C^l and $c' = (c'_1, ..., c'_{l+1})$ in C^{l+1} such that $c_1 = c'_1$, we have

$$u^l\left(c' \smile_{2l+1} \tilde{d} \mid \tilde{c} = c, c' \smile_{2l} \tilde{d}\right) \in [1/2 - \alpha, 1/2 + \alpha] \tag{D.1}$$

and for all $m \in \{1, ..., l\}$ such that $c_m \neq c'_m$, for $r = 2m-2, 2m-1$,

$$u^l\left(c' \smile_{r+1} \tilde{d} \mid \tilde{c} = c, c' \smile_r \tilde{d}\right) \in [1/2 - \alpha, 1/2 + \alpha]. \tag{D.2}$$

We say that the *conditions* $UI2$ *are satisfied* if for all $1 \leq p \leq l$, for all $d \in D^l$, for all $d' \in D^{p-1}$, for all $m \in \{1, ..., p-1\}$ such that $d_m \neq d'_m$, for $r = 2m-1, 2m$

$$u^l\left(\tilde{c} \smile_{r+1} d' | \tilde{d} = d, \tilde{c} \smile_r d'\right) \in [1/2 - \alpha, 1/2 + \alpha]. \tag{D.3}$$

To understand the conditions $UI1$, consider the Bayesian game $\Gamma(u^l, g^{l+1})$, and assume that player 2 truthfully reports his sequence of signals and that player 1 has received the signals $(c_1, ..., c_l)$ in C^l. (D.1) states that if the sequence of reported signals $(c'_1, \tilde{d}_1, ..., c'_l, \tilde{d}_l)$ is nice at level $2l$, then whatever the last reported signal c'_{l+1}, the conditional probability that $(c'_1, \tilde{d}_1, ..., c'_l, \tilde{d}_l, c'_{l+1})$ is still nice is in $[1/2 - \alpha, 1/2 + \alpha]$, i.e. close to 1/2. Regarding (D.2), first notice that if $c' = c$, then by construction $(c'_1, \tilde{d}_1, ..., c'_l, \tilde{d}_l)$ is nice and $u^l\left(c' \smile_{r+1} \tilde{d} \mid \tilde{c} = c, c' \smile_r \tilde{d}\right) = u^l\left(c \smile_{r+1} \tilde{d} \mid \tilde{c} = c\right) = 1$ for each $r = 1, ..., 2l-1$. Assume now that for some $m = 1, ..., l$, player 1 misreports his m^{th}-signal, i.e. reports $c'_m \neq c_m$. (D.2) requires that given that the reported signals were nice so far (at level $2m-2$), the conditional probability that the reported signals are not nice at level $2m-1$ (integrating c'_m) is close to $1/2$, and moreover if the reported signals are nice at this level $2m-1$, adding the next signal $\tilde{d}_m$ of player 2 has probability close to $1/2$ to keep the reported sequence nice. Conditions $UI2$ have a similar interpretation, considering the Bayesian games $\Gamma(u^l, g^p)$ for $p \leq l$, assuming that player 1 reports truthfully his signals and that player 2 plays d' after having received the signals d.

Proposition 27. *Conditions $UI1$ and $UI2$ imply :*

$$\forall l \geq 1, \forall p \in \{1, ..., l\}, \qquad \text{val}(u^l, g^p) \geq \varepsilon. \tag{D.4}$$

$$\forall l \geq 1, \quad \text{val}(u^l, g^{l+1}) \leq -\varepsilon. \tag{D.5}$$

As a consequence of this proposition, under conditions $UI1$ and $UI2$ we easily obtain condition 1) of theorem 19 :

Corollary 28. *If $l \neq p$ then $d(u^l, u^p) \geq 2\varepsilon$.*

Proof. Assume $l > p$, then $d(u^l, u^p) \geq \text{val}(u^l, g^{p+1}) - \text{val}(u^p, g^{p+1}) \geq \varepsilon - (-\varepsilon)$. □

Proof of proposition 27. We assume that $UI1$ and $UI2$ hold. We fix $l \geq 1$, work on the probability space $K \times C^l \times D^l$ equipped with the probability u^l, and denote by $\tilde{c}$ and $\tilde{d}$ the random variables of the signals received by the players.

1) We first prove (D.4), and consider the game $\Gamma(u^l, g^p)$ with $p \in \{1, ..., l\}$. We assume that player 1 chooses the truthful strategy. Fix $d = (d_1, ..., d_l)$ in D^l and $d' = (d'_1, ..., d'_{p-1})$ in D^{p-1}, and assume that player 2 has received the signal d and chooses to report d'.

Define the non-increasing sequence of events:

$$A_n = \{\tilde{c} \smile_n d'\}.$$

We will prove by backward induction that:

$$\forall n = 1, ..., p, \quad \mathbb{E}[h^p(\tilde{c}, d') | \tilde{d} = d, A_{2n-1}] \geq \varepsilon. \tag{D.6}$$

If $n = p$, $h^p(\tilde{c}, d') = \varepsilon$ on the event A_{2p-1}, implying the result. Assume now that for some n such that $1 \leq n < p$, we have : $\mathbb{E}[h^p(\tilde{c}, d') | \tilde{d} = d, A_{2n+1}] \geq \varepsilon$. Since we have a non-increasing sequence of events, $\mathbb{1}_{A_{2n-1}} = \mathbb{1}_{A_{2n+1}} + \mathbb{1}_{A_{2n-1}} \mathbb{1}_{A^c_{2n}} + \mathbb{1}_{A_{2n}} \mathbb{1}_{A^c_{2n+1}}$, so by definition of the payoffs, $h^p(\tilde{c}, d') \mathbb{1}_{A_{2n-1}} = h^p(\tilde{c}, d') \mathbb{1}_{A_{2n+1}} + 5\varepsilon \mathbb{1}_{A_{2n-1}} \mathbb{1}_{A^c_{2n}} - 5\varepsilon \mathbb{1}_{A_{2n}} \mathbb{1}_{A^c_{2n+1}}$.

First assume that $d'_n = d_n$. By construction of the Markov chain, $u^l(A_{2n+1} | A_{2n-1}, \tilde{d} = d) = 1$, implying that $u^l(A^c_{2n+1} | A_{2n-1}, \tilde{d} = d) = u^l(A^c_{2n} | A_{2n-1}, \tilde{d} = d) = 0$. As a consequence,

$$\begin{aligned}
\mathbb{E}[h^p(\tilde{c}, d') | \tilde{d} = d, A_{2n-1}] &= \mathbb{E}[h^p(\tilde{c}, d') \mathbb{1}_{A_{2n+1}} | \tilde{d} = d, A_{2n-1}] \\
&= \mathbb{E}[\mathbb{E}[h^p(\tilde{c}, d') | \tilde{d} = d, A_{2n+1}] \mathbb{1}_{A_{2n+1}} | \tilde{d} = d, A_{2n-1}] \\
&\geq \varepsilon.
\end{aligned}$$

Assume now that $d'_n \neq d_n$. Assumption UI2 implies that :

$$\begin{aligned} u^l(A^c_{2n}|A_{2n-1}, \tilde{d}=d) &\geq 1/2-\alpha, \\ u^l(A_{2n}\cap A^c_{2n+1}|A_{2n-1}, \tilde{d}=d) &\leq (1/2+\alpha)^2, \\ u^l(A_{2n+1}|A_{2n-1}, \tilde{d}=d) &\geq (1/2-\alpha)^2. \end{aligned}$$

It follows that :

$$\begin{aligned} \mathbb{E}[h^p(\tilde{c}, d'|\tilde{d}) = d, A_{2n-1}] &= \mathbb{E}[\mathbb{E}[h^p(\tilde{c}, d')|\tilde{d}=d, A_{2n+1}]\, \mathbb{1}_{A_{2n+1}}\, |\tilde{d}=d, A_{2n-1}] \\ &\quad + 5\varepsilon u^l(A^c_{2n}|A_{2n-1}, \tilde{d}=d) - 5\varepsilon u^l(A_{2n}\cap A^c_{2n+1}|A_{2n-1}, \tilde{d}=d) \\ &\geq \varepsilon\,(\frac{1}{4}-\alpha+\alpha^2) + 5\,\varepsilon\,(\frac{1}{2}-\alpha) - 5\,\varepsilon\,(\frac{1}{4}+\alpha+\alpha^2) \\ &= \varepsilon\,(\frac{3}{2}-11\alpha-4\alpha^2) \geq \varepsilon, \end{aligned}$$

And (D.6) follows by backward induction.

Since A_1 is an event which holds almost surely, we deduce that $\mathbb{E}[h^p(\tilde{c}, d')|\tilde{d} = d] \geq \varepsilon$. Hence the truthful strategy of player 1 guarantees the payoff ε in $\Gamma(u^l, g^p)$.

2) We now prove (D.5) and consider the Bayesian game $\Gamma(u^l, g^{l+1})$, assuming that player 2 chooses the truthful strategy. Fix $c = (c_1, ..., c_l)$ in C^l and $c' = (c'_1, ..., c'_{l-1})$ in C^{l-1}, and assume that player 1 has received the signal c and chooses to report c'. We will show that the expected payoff of player 1 is not larger than $-\varepsilon$, and assume w.l.o.g. that $c'_1 = c_1$. Consider the non-increasing sequence of events :

$$B_n = \{c' \smile_n \tilde{d}\}.$$

We will prove by backward induction that:

$$\forall n = 1, ..., l, \quad \mathbb{E}[h^{l+1}(c', \tilde{d})|\tilde{c} = c, B_{2n}] \leq -\varepsilon.$$

If $n = l$, we have $\mathbb{1}_{B_{2l}} = \mathbb{1}_{B_{2l+1}} + \mathbb{1}_{B_{2l}}\, \mathbb{1}_{B^c_{2l+1}}$, and $h^{l+1}(c', \tilde{d})\, \mathbb{1}_{B_{2l}} = \varepsilon\, \mathbb{1}_{B_{2l+1}} - 5\varepsilon\, \mathbb{1}_{B_{2l}}\, \mathbb{1}_{B^c_{2l+1}}$. UI1 implies that $|u^l(B_{2l+1}|\tilde{c} = c, B_{2l}) - \frac{1}{2}| \leq \alpha$, and it follows that :

$$\begin{aligned} \mathbb{E}[h^{l+1}(c', \tilde{d})|\tilde{c} = c, B_{2l}] &= \varepsilon\, u^l(B_{2l+1}|\tilde{c} = c, B_{2l}) - 5\varepsilon\, u^l(B^c_{2l+1}|u = \hat{u}, B_{2l}) \\ &\leq \varepsilon\,(\frac{1}{2}+\alpha) - 5\varepsilon\,(\frac{1}{2}-\alpha) \leq -\varepsilon. \end{aligned}$$

Assume now that for some $n = 1, ..., l-1$, we have $\mathbb{E}[h^{l+1}(c', \tilde{d})|\tilde{c} = c, B_{2n+2}] \leq -\varepsilon$. We have $\mathbb{1}_{B_{2n}} = \mathbb{1}_{B_{2n+2}} + \mathbb{1}_{B_{2n}}\, \mathbb{1}_{B^c_{2n+1}} + \mathbb{1}_{B_{2n+1}}\, \mathbb{1}_{B^c_{2n+2}}$, and by definition of h^{l+1},

$$h^{l+1}(c', \tilde{d})\, \mathbb{1}_{B_{2n}} = h^{l+1}(c', \tilde{d})\, \mathbb{1}_{B_{2n+2}} - 5\varepsilon\, \mathbb{1}_{B_{2n}}\, \mathbb{1}_{B^c_{2n+1}} + 5\varepsilon\, \mathbb{1}_{B_{2n+1}}\, \mathbb{1}_{B^c_{2n+2}}.$$

First assume that $c'_{n+1} = c_{n+1}$, then $u^l(B_{2n+2}|B_{2n}, \tilde{c} = c) = 1$. Then :

$$\begin{aligned} \mathbb{E}[h^{l+1}(c', \tilde{d})|\tilde{c} = c, B_{2n}] &= \mathbb{E}[h^{l+1}(c', \tilde{d})\, \mathbb{1}_{B_{2n+2}}\, |\tilde{c} = c, B_{2n}], \\ &= \mathbb{E}[\mathbb{E}[h^{l+1}(c', \tilde{d})|\tilde{c} = c, B_{2n+2}]\, \mathbb{1}_{B_{2n+2}}\, |\tilde{c} = c, B_{2n}] \leq -\varepsilon. \end{aligned}$$

Assume on the contrary that $c'_{n+1} \neq c_{n+1}$, assumption UI1 implies that :

$$\begin{aligned} u^l(B^c_{2n+1}|B_{2n}, \tilde{c} = c) &\geq 1/2-\alpha, \\ u^l(B_{2n+1}\cap B^c_{2n+2}|B_{2n}, \tilde{c} = c) &\leq (1/2+\alpha)^2, \\ u^l(B_{2n+2}|B_{2n}, \tilde{c} = c) &\geq (1/2-\alpha)^2. \end{aligned}$$

It follows that :

$$\begin{aligned}\mathbb{E}[h^{l+1}(c',\tilde{d})|\tilde{c}=c,B_{2n}] &= \mathbb{E}[\mathbb{E}[h^{l+1}(c',\tilde{d})|\tilde{c}=c,B_{2n+2}]\,\mathbb{1}_{B_{2n+2}}\,|\tilde{c}=c,B_{2n}] \\ &\quad -5\,\varepsilon\,u^l(B^c_{2n+1}|B_{2n},\tilde{c}=c)+5\,\varepsilon\,u^l(B_{2n+1}\cap B^c_{2n+2}|B_{2n},\tilde{c}=c) \\ &\leq -\varepsilon\,(\frac{1}{4}-\alpha+\alpha^2)-5\,\varepsilon\,(\frac{1}{2}-\alpha)+5\,\varepsilon\,(\frac{1}{4}+\alpha+\alpha^2)\leq -\varepsilon.\end{aligned}$$

By induction, we obtain $\mathbb{E}[h^{l+1}(c',\tilde{d})|\tilde{c}=c,B_2]\leq -\varepsilon$. Since B_2 holds almost surely here, we get $\mathbb{E}[h^{l+1}(c',\tilde{d})|\tilde{c}=c]\leq -\varepsilon$, showing that the truthful strategy of player 2 guarantees that the payoff of the maximizer is less or equal to $-\varepsilon$, and concluding the proof.

D.4. **Existence of an appropriate Markov chain.** Here we conclude the proof of Theorem 19 by showing the existence of an even integer N and a Markov chain with law ν on $A=\{1,...,N\}$ satisfying our conditions :

1) the law of the first state of the Markov chain is uniform on A,
2) for each a in A, there are exactly $N/2$ elements b in A such that $\nu(b|a)=2/N$,
3) $UI1$ and $UI2$.

Denoting by $P=(P_{a,b})_{(a,b)\in A^2}$ the transition matrix of the Markov chain, we have to prove the existence of P satisfying 2) and 3). The proof is non constructive and uses the following probabilistic method, where we select independently for each a in A, the set $\{b\in A, P_{a,b}>0\}$ uniformly among the subsets of A with cardinal $N/2$. We will show that when N goes to infinity, the probability of selecting an appropriate transition matrix does not only become positive, but converges to 1.

Formally, denote by $\mathcal{S}_A$ the collection of all subsets $S\subseteq A$ with cardinality $|S|=\frac{1}{2}N$. We consider a collection $(S_a)_{a\in A}$ of i.i.d. random variables uniform distributed over $\mathcal{S}_A$ defined on a probability space $(\Omega_N,\mathcal{F}_N,\mathbb{P}_N)$. For all a, b in A, let

$$X_{a,b}=\mathbb{1}_{\{b\in S_a\}}\text{ and }P_{a,b}=\frac{2}{N}X_{a,b}.$$

By construction, P is a transition matrix satisfying 2). Theorem 19 will now follow directly from the following proposition.

Proposition 29.

$$\mathbb{P}_N\,(\;P\textit{ induces a Markov chain satisfying UI1 and UI2}\;)\xrightarrow[n\to\infty]{}1.$$

In particular, the above probability is strictly positive for all sufficiently large N.

The rest of this section is devoted to the proof of proposition 29.

We start with probability bounds based on Hoeffding's inequality.

Lemma 30. *For any $a\neq b$, each $\gamma>0$*

$$\mathbb{P}_N\left(\left||S_a\cap S_b|-\frac{1}{4}N\right|\geq\gamma N\right)\leq\frac{1}{2}e^4Ne^{-2\gamma^2N}.$$

Proof. Consider a family of i.i.d. Bernoulli variables $(\widetilde{X}_{i,j})_{i=a,b,\,j\in A}$ of parameter $\frac{1}{2}$ defined on a space $(\Omega,\mathcal{F},\mathbb{P})$. For $i=a,b$, define the events $\widetilde{L}_i=\{\sum_{j\in A}\widetilde{X}_{i,j}=\frac{N}{2}\}$ and the set-valued variables $\widetilde{S}_i=\{j\in A\,|\,\widetilde{X}_{i,j}=1\}$. It is straightforward to check that the conditional law of $(\widetilde{S}_a, \widetilde{S}_b)$ given $\widetilde{L}_a \cap \widetilde{L}_b$ under $\mathbb{P}$ is the same as the law of (S_a, S_b) under $\mathbb{P}_N$. It follows that

$$\mathbb{P}_N\left(\left||S_a \cap S_b| - \frac{1}{4}N\right| \geq \gamma N\right) = \mathbb{P}\left(\left||\widetilde{S}_a \cap \widetilde{S}_b| - \frac{1}{4}N\right| \geq \gamma N \,\middle|\, \widetilde{L}_a \cap \widetilde{L}_b\right) \leq \frac{\mathbb{P}\left(\left||\widetilde{S}_a \cap \widetilde{S}_b| - \frac{1}{4}N\right| \geq \gamma N\right)}{\mathbb{P}\left(\widetilde{L}_a \cap \widetilde{L}_b\right)}.$$

Using Hoeffding inequality, we have

$$\mathbb{P}\left(\left||\widetilde{S}_a \cap \widetilde{S}_b| - \frac{1}{4}N\right| \geq \gamma N\right) = \mathbb{P}\left(\left|\sum_{j\in A} \widetilde{X}_{a,j}\widetilde{X}_{b,j} - \frac{1}{4}N\right| \geq \gamma N\right) \leq 2e^{-2\gamma^2 N}.$$

On the other hand, using Stirling approximation[12], we have

$$\mathbb{P}\left(\widetilde{L}_a \cap \widetilde{L}_b\right) = \left(\frac{1}{2^N}\frac{N!}{\left(\frac{N}{2}!\right)^2}\right)^2 \geq \left(\frac{2^{N+1}N^{-\frac{1}{2}}}{2^N e^2}\right)^2 = \frac{4}{Ne^4}.$$

We deduce that $\mathbb{P}_N\left(\left||S_a \cap S_b| - \frac{1}{4}N\right| \geq \gamma N\right) \leq \frac{1}{2}e^4 N e^{-2\gamma^2 N}$. □

Lemma 31. *For each $a \neq b$, for any subset $S \subseteq A$ and any $\gamma \geq \frac{1}{2N-2}$,*

$$\mathbb{P}_N\left(\left|\sum_{i\in S} X_{i,a} - \frac{1}{2}|S|\right| \geq \gamma N\right) \leq 2e^{-2N\gamma^2}, \text{ and } \mathbb{P}_N\left(\left|\sum_{i\in S} X_{i,a}X_{i,b} - \frac{1}{4}|S|\right| \geq \gamma N\right) \leq 2e^{-\frac{1}{2}N\gamma^2}.$$

Proof. For the first inequality, notice that $X_{i,a}$ are i.i.d. Bernoulli random variables with parameter $\frac{1}{2}$. The Hoeffding inequality implies that :

$$\mathbb{P}_N\left(\left|\sum_{i\in S} X_{i,a} - \frac{1}{2}|S|\right| \geq \gamma N\right) \leq 2e^{-2\gamma^2\frac{N^2}{|S|}} \leq 2e^{-2N\gamma^2}.$$

□

For the second inequality, let $Z_i = X_{i,a}X_{i,b}$. Notice that all variables Z_i are i.i.d. Bernoulli random variables with parameter $p = \frac{1}{2}\left(\frac{\frac{N}{2}-1}{N-1}\right) = \frac{1}{4} - \frac{1}{4N-4}$. The Hoeffding inequality implies that

$$\begin{aligned}\mathbb{P}_N\left(\left|\sum_{i\in S} Z_i - \frac{1}{4}|S|\right| \geq \gamma N\right) &\leq \mathbb{P}_N\left(\left|\sum_{i\in S} Z_i - p|S|\right| \geq \frac{1}{2}\gamma N\right)\\ &\leq 2e^{-2\gamma^2\frac{N^2}{|S|}} \leq 2e^{-\frac{1}{2}N\gamma^2},\end{aligned}$$

where we used that $|S||p - \frac{1}{4}| \leq \frac{N}{4N-4} \leq \frac{\gamma N}{2}$ for the first inequality.

Definition 32. For each $a \neq b$ and $c \neq d$, each $\gamma > 0$, define :

$$\begin{array}{lll} Y_a = 2\sum_{i\in A} X_{i,a}, & Y^c = 2\sum_{i\in A} X_{c,i} = N, & \\ Y_{a,b} = 4\sum_{i\in A} X_{i,a}X_{i,b}, & Y_a^c = 4\sum_{i\in A} X_{i,a}X_{c,i}, & Y^{c,d} = 4\sum_{i\in A} X_{c,i}X_{d,i}, \\ Y_{a,b}^c = 8\sum_{i\in A} X_{i,a}X_{i,b}X_{c,i}, & Y_a^{c,d} = 8\sum_{i\in A} X_{i,a}X_{c,i}X_{d,i}, & Y_{a,b}^{c,d} = 16\sum_{i\in A} X_{i,a}X_{i,b}X_{c,i}X_{d,i}. \end{array}$$

[12] We have $n^{n+\frac{1}{2}}e^{-n} \leq n! \leq en^{n+\frac{1}{2}}e^{-n}$ for each n.

Lemma 33. *For each $a \neq b$ and $c \neq d$, each $\gamma \geq 64/N$, each of the variables $Z \in \{Y_a, Y^c, Y_{a,b}, Y^{c,d}, Y_a^c, Y_{a,b}^c, Y_a^{c,d}, Y_{a,b}^{c,d}\}$,*

$$\mathbb{P}_N\left(|Z - N| \geq \gamma N\right) \leq e^4 N e^{-\frac{N}{32}\left(\frac{\gamma}{10}\right)^2}.$$

Proof. In case $Z = Y_a$ or $Y_{a,b}$, the bound follows from Lemma 31 (for $S = A$). If case $Z = Y^c$, the bound is trivially satisfied. If $Z = Y^{c,d}$, the bound follows from Lemma 30.

In case $Z = Y_{a,b}^{c,d}$, notice that

$$Y_{a,b}^{c,d} = 16 \sum_{i \in S_c \cap S_d} Z_i, \quad \text{where} \quad Z_i = X_{i,a} X_{i,b}.$$

All variables Z_i are i.i.d. Bernouilli random variables with parameter $p = \frac{1}{4} - \frac{1}{4N-4}$. Moreover, $\{Z_i\}_{i \neq c,d}$ are independent of $S_c \cap S_d$. Up to enlarge the probability space, we can construct a new collection of i.i.d. Bernoulli random variables Z_i' such that $Z_i' = Z_i$ for all $i \neq c, d$ and such that $\{(Z_i')_{i \in A}, S_c \cap S_d\}$ are all independent. Then,

$$\left| Y_{a,b}^{c,d} - 16 \sum_{i \in S_c \cap S_d} Z_i' \right| \leq 32,$$

and, because $\frac{1}{2}\gamma N \geq 32$, we have

$$\mathbb{P}_N\left(\left|Y_{a,b}^{c,d} - N\right| \geq \gamma N\right) \leq \mathbb{P}_N\left(\left|\sum_{i \in S_c \cap S_d} Z_i' - \frac{1}{16} N\right| \geq \frac{1}{32}\gamma N\right).$$

Define the events

$$A = \left\{\left|\frac{1}{4}\left|S_c \cap S_d\right| - \frac{N}{16}\right| \geq \frac{1}{160}\gamma N\right\}, \quad B = \left\{\left|\sum_{i \in S_c \cap S_d} Z_i' - \frac{1}{4}\left|S_c \cap S_d\right|\right| \geq \frac{1}{40}\gamma N\right\}.$$

Then, the probability can be further bounded by

$$\leq \mathbb{P}_N(A) + \mathbb{P}_N(B) \leq \frac{1}{2} e^4 N e^{-2N\left(\frac{1}{40}\gamma\right)^2} + 2 e^{-\frac{1}{2}N\left(\frac{1}{40}\gamma\right)^2} \leq e^4 N e^{-\frac{N\gamma^2}{3200}}$$

where the first bound comes from Lemma 30, and the second from the second bound in Lemma 31.

The remaining bounds have proofs similar (and simpler) to the case $Z = Y_{a,b}^{c,d}$. $\square$

Finally, we describe an event E that collects these bounds. Recall that $\alpha = 1/25$, and define for each $a \neq b$ and $c \neq d$,

$$\begin{aligned} E_{a,b,c,d} \quad = \quad & \left\{\left|\frac{Y_{a,b}}{Y_a} - 1\right| \leq 2\alpha\right\} \cap \left\{\left|\frac{Y_{a,b}^c}{Y_a^c} - 1\right| \leq 2\alpha\right\} \cap \left\{\left|\frac{Y_a^{c,d}}{Y_a^c} - 1\right| \leq 2\alpha\right\} \cap \left\{\left|\frac{Y_{a,b}^{c,d}}{Y_a^{c,d}} - 1\right| \leq 2\alpha\right\} \\ & \left\{\left|\frac{Y^{c,d}}{Y^c} - 1\right| \leq 2\alpha\right\} \cap \left\{\left|\frac{Y_a^c}{Y^c} - 1\right| \leq 2\alpha\right\} \cap \left\{\left|\frac{Y_a^{c,d}}{Y^{c,d}} - 1\right| \leq 2\alpha\right\}. \end{aligned}$$

Finally, let

$$E = \bigcap_{a,b,c,d: a \neq b \text{ and } c \neq d} E_{a,b,c,d}.$$

Lemma 34. *We have*

$$\mathbb{P}_N(E) > 1 - 7e^4 N^5 e^{-\frac{N}{2163200}} \xrightarrow[n\to\infty]{} 1.$$

Proof. Take $\gamma = \frac{\alpha}{1+\alpha} = \frac{1}{26}$ and let

$$F_{a,b,c,d} = \bigcap_{Z \in \{Y_a, Y_{a,b}, Y^{c,d}, Y^{c,d}, Y_a^c, Y_{a,b}^c, Y_a^{c,d}, Y_{a,b}^{c,d}\}} \{|Z - N| \leq \gamma N\}.$$

It is easy to see that $F_{a,b,c,d} \subseteq E_{a,b,c,d}$. The probability that $F_{a,b,c,d}$ holds can be bounded from Lemma 33 (as soon as $N \geq \frac{64}{\gamma} = 1664$), as

$$\mathbb{P}_N(F_{a,b,c,d}) \geq 1 - 7e^4 N e^{-\frac{N}{32.(260)^2}}.$$

The result follows since there are less than N^4 ways of choosing (a,b,c,d). □

Computations using the bound of lemma 34 show that $N = 52.10^6$ is enough to have the existence of an appropriate Markov chain. So one can take $\varepsilon = 3.10^{-17}$ in the statement of theorem 19. We conclude the proof of proposition 29 by showing that event E implies conditions $UI1$ and $UI2$.

Lemma 35. *If event E holds, then the conditions $UI1, UI2$ are satisfied.*

Proof. We fix the law ν of the Markov chain on A and assume that it has been induced, as explained at the beginning of section D.4, by a transition matrix P satisfying E. For $l \geq 1$, we forget about the state in K and still denote by u^l the marginal of u^l over $C^l \times D^l$. If $c = (c_1, ..., c_l) \in C^l$ and $d = (d_1, ..., d_l) \in D^l$, we have $u^l(c,d) = \nu(c_1, d_1, ..., c_l, d_l)$.

Let us begin with condition UI2 which we recall here: for all $1 \leq p \leq l$, for all $d \in D^l$, for all $d' \in D^{p-1}$, for all $m \in \{1, ..., p-1\}$ such that $d_m \neq d'_m$, for $r = 2m-1, 2m$,

$$u^l\left(\tilde{c} \smile_{r+1} d' | \tilde{d} = d, \tilde{c} \smile_r d'\right) \in [1/2 - \alpha, 1/2 + \alpha], \tag{D.3}$$

where $(\tilde{c}, \tilde{d})$ is a random variable selected according to u^l. The quantity $u^l\left(\tilde{c} \smile_{r+1} d' | \tilde{d} = d, \tilde{c} \smile_r d'\right)$ is thus the conditional probability of the event ($\tilde{c}$ and d' are nice at level $r+1$) given that they are nice at level r and that the signal received by player 2 is d. We divide the problem into different cases.

<u>Case $m > 1$ and $r = 2m-1$.</u>

Note that the events $\{\tilde{c} \smile_{2m} d'\}$ and $\{\tilde{c} \smile_{2m-1} d'\}$ can be decomposed as follows :

$$\begin{aligned} \{\tilde{c} \smile_{2m-1} d'\} &= \{\tilde{c} \smile_{2m-2} d'\} \cap \{X_{d'_{m-1}, \tilde{c}_m} = 1\}, \\ \{\tilde{c} \smile_{2m} d'\} &= \{\tilde{c} \smile_{2m-2} d'\} \cap \{X_{d'_{m-1}, \tilde{c}_m} = 1\} \cap \{X_{\tilde{c}_m, d'_m} = 1\}. \end{aligned}$$

So $u^l\left(\tilde{c} \smile_{2m} d' | \tilde{d} = d, \tilde{c} \smile_{2m-1} d'\right) = u^l\left(X_{\tilde{c}_m, d'_m} = 1 | \tilde{d} = d, \tilde{c} \smile_{2m-1} d'\right)$, and the Markov property gives:

$$\begin{aligned} u^l\left(\tilde{c} \smile_{2m} d' | \tilde{d} = d, \tilde{c} \smile_{2m-1} d'\right) &= u^l\left(X_{\tilde{c}_m, d'_m} = 1 | X_{d'_{m-1}, \tilde{c}_m} = 1, X_{d_{m-1}, \tilde{c}_m} = 1, X_{\tilde{c}_m, d_m} = 1\right), \\ &= \frac{\sum_{i \in U} X_{i, d'_m} X_{d'_{m-1}, i} X_{d_{m-1}, i} X_{i, d_m}}{\sum_{i \in U} X_{d'_{m-1}, i} X_{d_{m-1}, i} X_{i, d_m}}. \end{aligned}$$

This is equal to $\frac{1}{2}\frac{Y_{d_m,d'_m}^{d_{m-1},d'_{m-1}}}{Y_{d_m}^{d_{m-1},d'_{m-1}}}$ if $d'_{m-1} \neq d_{m-1}$, and to $\frac{1}{2}\frac{Y_{d_m,d'_m}^{d_{m-1}}}{Y_{d_m}^{d_{m-1}}}$ if $d'_{m-1} = d_{m-1}$. In both cases, E implies (D.3).

Case $r = 2m$.

We have $u^l\left(\tilde{c} \smile_{2m+1} d' | \tilde{d} = d, \tilde{c} \smile_{2m} d'\right) = u^l\left(X_{d'_m,\tilde{c}_{m+1}} = 1 | \tilde{d} = d, \tilde{c} \smile_{2m} d'\right)$, and by the Markov property :

$$\begin{aligned} u^l\left(\tilde{c} \smile_{2m+1} d' | \tilde{d} = d, \tilde{c} \smile_{2m} d'\right) &= u^l\left(X_{d'_m,\tilde{c}_{m+1}} = 1 | X_{d_m,\tilde{c}_{m+1}} = 1, X_{\tilde{c}_{m+1},d_{m+1}} = 1\right), \\ &= \frac{\sum_{i\in U} X_{d'_m,i} X_{d_m,i} X_{i,d_{m+1}}}{\sum_{i\in U} X_{d_m,i} X_{i,d_{m+1}}} \\ &= \frac{1}{2}\frac{Y_{d_{m+1}}^{d'_m,d_m}}{Y_{d_{m+1}}^{d_m}} \in [1/2-\alpha, 1/2+\alpha]. \end{aligned}$$

Case $m = 1$, $r = 1$.

$$\begin{aligned} u^l\left(\tilde{c} \smile_2 d' | \tilde{d} = d, \tilde{c} \smile_1 d'\right) &= u^l\left(\tilde{c} \smile_2 d' | \tilde{d} = d\right), \\ &= u^l\left(X_{\tilde{c}_1,d'_1} = 1 | X_{\tilde{c}_1,d_1} = 1\right), \\ &= \frac{\sum_{i\in U} X_{i,d'_1} X_{i,d_1}}{\sum_{i\in U} X_{i,d_1}} \\ &= \frac{1}{2}\frac{Y_{d_1,d'_1}}{Y_{d_1}} \in [1/2-\alpha, 1/2+\alpha]. \end{aligned}$$

Let us now consider condition $UI1$: we require that for all $l \geq 1$, all $c = (c_1, ..., c_l)$ in C^l and $c' = (c'_1, ..., c'_{l+1})$ in C^{l+1} such that $c_1 = c'_1$, we have

$$u^l\left(c' \smile_{2l+1} \tilde{d} \mid \tilde{c} = c, c' \smile_{2l} \tilde{d}\right) \in [1/2-\alpha, 1/2+\alpha] \tag{D.1}$$

and for all $m \in \{1, ..., l\}$ such that $c_m \neq c'_m$, for $r = 2m-2, 2m-1$,

$$u^l\left(c' \smile_{r+1} \tilde{d} \mid \tilde{c} = c, c' \smile_r \tilde{d}\right) \in [1/2-\alpha, 1/2+\alpha]. \tag{D.2}$$

We start with (D.1).

$$\begin{aligned} u^l\left(c' \smile_{2l+1} \tilde{d} | \tilde{c} = c, c' \smile_{2l} \tilde{d}\right) &= u^l\left(X_{\tilde{d}_l,c'_{l+1}} = 1 | \tilde{c} = c, c' \smile_{2l} \tilde{d}\right), \\ &= u^l\left(X_{\tilde{d}_l,c'_{l+1}} = 1 | X_{c_l,\tilde{d}_l} = 1, X_{c'_l,\tilde{d}_l} = 1\right), \\ &= \frac{\sum_{i\in V} X_{i,c'_{l+1}} X_{c_l,i} X_{c'_l,i}}{\sum_{i\in V} X_{c_l,i} X_{c'_l,i}}. \end{aligned}$$

This is $\frac{1}{2}\frac{Y_{c_{l+1}}^{c_l,c'_l}}{Y^{c_l,c'_l}}$ if $c'_l \neq c_l$, and $\frac{1}{2}\frac{Y_{c_{l+1}}^{c_l}}{Y^{c_l}}$ if $c'_l = c_l$. In both cases, (D.1) holds.

We finally consider (D.2) and distinguish several case.

Case $r = 2m-1$ and $m = l$.

$$\begin{aligned} u^l\left(c' \smile_{2l} \tilde{d} | \tilde{c} = c, c' \smile_{2l-1} \tilde{d}\right) &= u^l\left(X_{c'_l, \tilde{d}_l} = 1 | \tilde{c} = c, c' \smile_{2l-1} \tilde{d}\right), \\ &= u^l\left(X_{c'_l, \tilde{d}_l} = 1 | X_{c_l, \tilde{d}_l} = 1\right), \\ &= \frac{\sum_{i \in V} X_{c'_l, i} X_{c_l, i}}{\sum_{i \in V} X_{c_l, i}}, \\ &= \frac{1}{2} \frac{Y^{c'_l, c_l}}{Y^{c_l}} \in [1/2 - \alpha, 1/2 + \alpha]. \end{aligned}$$

Case $r = 2m - 1$ and $m < l$.

$$\begin{aligned} u^l\left(c' \smile_{2m} \tilde{d} | \tilde{c} = c, c' \smile_{2m-1} \tilde{d}\right) &= u^l\left(X_{c'_m, \tilde{d}_m} = 1 | \tilde{c} = c, c' \smile_{2m-1} \tilde{d}\right), \\ &= u^l\left(X_{c'_m, \tilde{d}_m} = 1 | X_{c_m, \tilde{d}_m} = 1, X_{\tilde{d}_m, c_{m+1}} = 1\right), \\ &= \frac{\sum_{i \in V} X_{c'_m, i} X_{c_m, i} X_{i, c_{m+1}}}{\sum_{i \in V} X_{c_m, i} X_{i, c_{m+1}}}, \\ &= \frac{1}{2} \frac{Y^{c'_m, c_m}_{c_{m+1}}}{Y^{c_m}_{c_{m+1}}} \in [1/2 - \alpha, 1/2 + \alpha]. \end{aligned}$$

Case $r = 2m - 2$.

$$\begin{aligned} u^l\left(c' \smile_{2m-1} \tilde{d} | \tilde{c} = c, c' \smile_{2m-2} \tilde{d}\right) &= u^l\left(X_{\tilde{d}_{m-1}, c'_m} = 1 | \tilde{c} = c, c' \smile_{2m-1} \tilde{d}\right), \\ &= u^l\left(X_{\tilde{d}_{m-1}, c'_m} = 1 | X_{c'_{m-1}, \tilde{d}_{m-1}} = X_{c_{m-1}, \tilde{d}_{m-1}} = X_{\tilde{d}_{m-1}, c_m} = 1\right), \\ &= \frac{\sum_{i \in V} X_{i, c'_m} X_{i, c_m} X_{c'_{m-1}, i} X_{c_{m-1}, i}}{\sum_{i \in V} X_{i, c_m} X_{c'_{m-1}, i} X_{c_{m-1}, i}}. \end{aligned}$$

This is $\frac{1}{2} \frac{Y^{c'_{m-1}, c_{m-1}}_{c'_m, c_m}}{Y^{c'_{m-1} c_{m-1}}_{c_m}}$ if $c_{m-1} \neq c'_{m-1}$, and $\frac{1}{2} \frac{Y^{c_{m-1}}_{c'_m, c_m}}{Y^{c_{m-1}}_{c_m}}$ if $c_{m-1} = c'_{m-1}$. In both cases, it belongs to $[1/2 - \alpha, 1/2 + \alpha]$, concluding the proofs of lemma 35, proposition 29 and theorem 19. $\square$

Appendix E. Proofs of Section 7

E.1. **Proof of Theorem 22.** Suppose that u and v are two simple, and non-redundant information structures. Let $\tilde{u}$ and $\tilde{v}$ be the associated probability distributions over belief hierarchies of player 1. It is easy to show that if two non-redundant information structures induce the same distributions over hierarchies of beliefs $\tilde{u} = \tilde{v}$, then they are equivalent from any strategic point of view, and, in particular, they induce the same set of ex ante BNE payoffs. Hence, we assume that $\tilde{u} \neq \tilde{v}$.

Let $H_u = \mathrm{supp}\tilde{u}$ and $H_v = \mathrm{supp}\tilde{v}$. Lemma III.2.7 in Mertens et al. (2015) implies that the sets H_u and H_v are disjoint.

It is well known that there exists a non-zero sum payoff function $g^{(0)} : K \times (I \times I_0) \times J \to [-1, 1]^2$ such that $I_0 = H_u \cup H_v$ and such that the set of rationalizable actions for player 1 of type $c \in C$ with hierarchy $h(c)$ is contained in the set $I \times \{h(c)\}$. In particular, in a Bayesian Nash equilibrium, each type of player 1 will report its hierarchy. Construct game $g^{(1)}: K \times (I \times I_0) \times (J \times \{u, v\}) \to [-1,1]^2$ with payoffs

$$g_1^{(1)}(k,i,i_0,j,j_0) = g_1^{(0)}(k,i,i_0,j),$$

$$g_2^{(1)}(k,i,i_0,j,j_0) = \frac{1}{2} g_2^{(0)}(k,i,i_0,j) + \begin{cases} \frac{1}{2}, & \text{if } j_0 = u \text{ and } i_0 \in H_u \\ -\frac{1}{2}, & \text{if } j_0 = u \text{ and } i_0 \notin H_u, \\ 0, & \text{if } j_0 = v. \end{cases}$$

Then, the rationalizable actions of player 2 in game $g^{(1)}$ are contained in $J \times \{u\}$ for any type in type space u and in $J \times \{v\}$ for any type in type space v.

Finally, for any $\varepsilon \in (0,1)$,construct a game $g^{\varepsilon}: K \times (I \times I_0) \times (J \times \{u, v\}) \to [-1,1]^2$ with payoffs

$$g_1^{\varepsilon}(k,i,i_0,j,j_0) = \varepsilon g_1^{(0)}(k,i,i_0,j,j_0) + (1-\varepsilon) \begin{cases} 1, & \text{if } j_0 = u, \\ -1, & \text{if } j_0 = v, \end{cases},$$

$$g_2^{\varepsilon} \equiv g_2^{(1)}.$$

Then, the Bayesian Nash equilibrium payoff of player belongs to $[1-\varepsilon, 1]$ on the structure u and $[-1, -1+\varepsilon]$ on the structure v. It follows that the payoff distance between the two type spaces is at least $2 - 2\varepsilon$, for arbitrary $\varepsilon > 0$.

Next, suppose that u and v are two simple information structures with the decomposition $u = \sum_\alpha p_\alpha u_\alpha$ and $v = \sum_\alpha q_\alpha v_\alpha$ and such that $\tilde{u}_\alpha = \tilde{v}_\alpha$ for each α. Let g be a non-zero sum payoff function. Let σ_α be an equilibrium on u_α with payoffs $g_\alpha \in \mathbb{R}^2$. Let s_α is the associated equilibrium on v_α (that can be obtained by mapping the hierarchies of beliefs through an appropriate bijection) with the same payoffs g_α. The distance between payoffs is bounded my

$$\left|\sum p_\alpha g_a - q_\alpha g_\alpha\right|_{\max} = \left|\sum (p_\alpha - q_\alpha) g_\alpha\right|_{\max} \leq \sum |p_\alpha - q_\alpha|,$$

where the last inequality comes from the fact that payoffs are bounded.

On the other hand, let $A = \{\alpha : p_\alpha > q_\alpha\}$. Using a similar construction as above, we can construct a game $g^{(1)}$ such that player 2's actions have a form $J \times \{u_A, u_B\}$, and his rationalizable actions are contained in set $J \times \{u_A\}$ for any type in type space $u_\alpha, \alpha \in A$ and in $J \times \{u_B\}$ otherwise. Further, we construct a game $g^{(\varepsilon)}$ as above. Then, any player 1's equilibrium $g_{1,\alpha}^{(\varepsilon)}$ payoff is at least $1-\varepsilon$ for any type in type space $u_\alpha, \alpha \in A$, and $-1+\varepsilon$ for any type in type space u_α for $\alpha \notin A$. Denoting the equilibrium payoff of player 2 as $g_{2,\varepsilon}^{\varepsilon}$, the payoff distance in game g^{ε} is at least

$$\begin{aligned} &\max\left(\left|\sum_\alpha (p_\alpha - q_\alpha) g_{1,\alpha}\right|, \left|\sum_\alpha (p_\alpha - q_\alpha) g_{2,\alpha}\right|\right) \\ \geq &\left|\sum_\alpha (p_\alpha - q_\alpha) g_{1,\alpha}\right| \\ \geq &\left[\sum_{\alpha \in A} (p_\alpha - q_\alpha) - \sum_{\alpha \notin A} (p_\alpha - q_\alpha)\right](1-\varepsilon) \\ \geq &(1-\varepsilon) \sum |p_\alpha - q_\alpha|. \end{aligned}$$

Because the $\varepsilon > 0$ is arbitrary, the two above inequalities show that the payoff distance is equal to $\sum |p_\alpha - q_\alpha|$.

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